A Note on a Conjecture of Smale

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Introduction. The full classification of the invariant manifolds I_{hc} of the planar three-body problem (see [5], [2], [3], [1]) for positive masses of the bodies takes into account the order of the values of the normalized potential $\bar{V}$ at the critical points. Let L be a Lagrangian point and E_{ijk} the Eulerian one corresponding to the bodies on a line in the order i, j, k. In [5] (see also [1] page 740) Smale conjectured that for almost all choices of m_1, m_2, m_3 the numbers $\bar{V}(L)$, $\bar{V}(E_{123})$, $\bar{V}(E_{231})$, $\bar{V}(E_{312})$ are distinct. In this note we prove the conjecture and we point out that for a given order of the masses, i.e. $m_1 \geqslant m_2 \geqslant m_3$, we have a related order of the potentials $\bar{V}(L) > \bar{V}(E_{312}) \geqslant \bar{V}(E_{123}) \geqslant \bar{V}(E_{231})$. In a neighbourhood of the limiting cases we prove the inequalities analytically, and in the remaining region we show the results of a numerical computation.

In the planar three-body problem with normalized masses $\sum_{i=1}^{3} m_i = 1$ the knowledge of the relative equilibrium solutions is equivalent to the knowledge of the critical points of the normalized potential $\bar{V}$ (see [5], [3], [1]). The potential $V = -\sum_{i<j} m_i m_j / r_{ij}$, $r_{ij} = |x_i - x_j|$, where $x_i \in R^2$ is the position of m_i with respect to the center of mass, is normalized by keeping the moment of inertia $I = (1/2) \sum_{i<j} m_i m_j r_{ij}^2$ equal to 1.

It is well known that there are exactly five critical points: 2 Lagrangian points with the masses forming an equilateral triangle and 3 Eulerian points with masses on a line. The masses can be viewed as the barycentric coordinates of a point of a triangle T. We shall refer to that triangle as the mass triangle.

Let I_{hc} be the set of points of the phase space with energy h and angular momentum c. We call loosely I_{hc} an invariant manifold because it is invariant under the three-body flow. It is not true that I_{hc} is a manifold for all (h, c) values.

We hope that the knowledge of the topology of I_{hc} can help to fully understand the flow in the three-body problem. The study of I_{hc} for different (h, c) pairs requires to know the order of the normalized potentials. The first objective of this note is to prove the following theorem that was conjectured by Smale [5].

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Theorem 1. *For almost all choices of positive masses m_1, m_2, m_3 the values of the normalized potential at the critical points are distinct.*

Proof. The relation $I = 1$ and the triangle inequality show that the set of values r_{12}, r_{23}, r_{31} we deal with is diffeomorphic to a triangle. The vertices must be excluded because they are related to collisions (see [2]). The sides are related to collinear configurations. It is known that in this triangle there is a Lagrangian point for which $\bar{V}$ is a maximum and that the three Eulerian points are saddles [1]. $\bar{V}$ is always negative and tends to $-\infty$ when points on the triangle tend to a vertex. This is enough to show $\bar{V}(L) > \bar{V}(E_{ijk})$ for any values of i, j, k.

Now we seek relations for the masses which equate two of the potentials at Eulerian points. It is obvious that $\bar{V}(E_{ijk}) = \bar{V}(E_{jik})$ if $m_i = m_j$. From now on we confine our study to the region T' of T defined by $m_1 > m_2 > m_3$ (see Fig. 1). Points P, Q, R, S, O, W are representative, respectively, of the following values of the masses: (1,0,0), (0,1,0), (0,0,1), (1/2,1/2,0), (1/3,1/3,1/3), (0,1/2,1/2).

Let us suppose that on a line the bodies are disposed in the order 3,1,2 and the distances from m_3 to m_1 and from m_1 to m_2 equal a and ax, respectively (see Fig. 2). Then the critical point E_{312} is obtained solving the equation

$$p(x) = (m_1 + m_2) + (2m_1 + 3m_2)x + (3m_2 + m_1)x^2 - (3m_3 + m_1)x^3 - (3m_3 + 2m_1)x^4 - (m_3 + m_1)x^5 = 0.$$

(For the details see [7 page 276] where it is also proved that it has only one positive root.)

Then

$$\bar{V}(E_{312}) = -\frac{1}{a}\left(\frac{m_1 m_2}{x} + m_3 m_1 + \frac{m_2 m_3}{1 + x}\right),$$

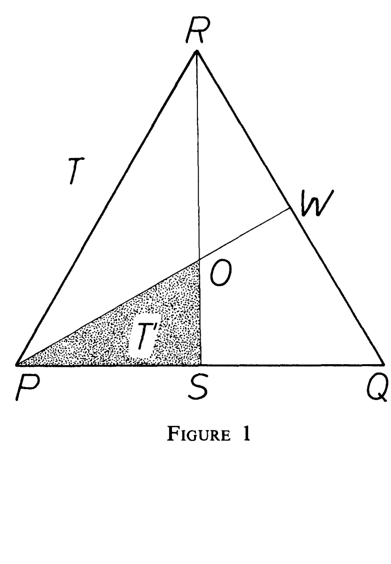

FIGURE 1

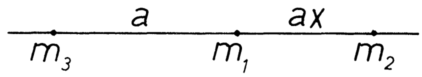

FIGURE 2

where $a^2\,[m_1 m_2 x^2 + m_3 m_1 + m_2 m_3 (1 + x)^2] = 2$.

We introduce the function

$$\begin{aligned} A(m_1, m_2, m_3, x) &= 2\bar{V}^2(E_{312}) \\ &= [m_1 m_2 x^2 + m_3 m_1 + m_2 m_3 (1 + x)^2] \\ &\quad \cdot \left(\frac{m_1 m_2}{x} + m_3 m_1 + \frac{m_2 m_3}{1 + x} \right)^2 . \end{aligned}$$

In a symmetric way, applying circular permutation, we obtain the equations $q(y) = 0$, $r(z) = 0$ and introduce the functions $B(m_1, m_2, m_3, y) = A(m_2, m_3, m_1, y)$, $C(m_1, m_2, m_3, z) = A(m_3, m_1, m_2, z)$, for the cases E_{123} and E_{231}, respectively. As in T' we have $p(1) > 0$, $q(1) < 0$, $r(1) > 0$, we get immediately $x > 1$, $y < 1$, $z > 1$.

The p and q equations can be rewritten as

$$\{(1 - x^3)(x + 1)^2\} m_1 + \{3x^2 + 3x + 1\}\, m_2 + \{- x^3 (x^2 + 3x + 3)\} m_3 = 0\,,$$

$$\{-y^3 (y^2 + 3y + 3)\}\, m_1 + \{(1 - y^3)(y + 1)^2\} m_2 + \{3y^2 + 3y + 1\}\, m_3 = 0\,,$$

from where we can obtain smoothly m_2, m_3 in terms of m_1 for every $x > 1$, $y < 1$. Hence the map $f{:}T' \to f(T') \subset R^2$ defined by $f(m_1, m_2, m_3) = (x,y)$ is a diffeomorphism.

Now we equate $\bar{V}(E_{312})$ and $\bar{V}(E_{123})$, or, equivalently, we put $A = B$. Therefore

$$(*) \qquad \begin{aligned} & y^2(1 + y)^2\, [m_1 m_2 x^2 + m_3 m_1 + m_2 m_3 (1 + x)^2] \\ &\qquad \cdot [m_1 m_2 (1 + x) + m_3 m_1 x (1 + x) + m_2 m_3 x]^2 \\ &= x^2 (1 + x)^2 [m_2 m_3 y^2 + m_1 m_2 + m_3 m_1 (1 + y)^2] \\ &\qquad \cdot [m_2 m_3 (1 + y) + m_1 m_2 y (1 + y) + m_2 m_3 y]^2 . \end{aligned}$$

This is an algebraic curve γ whose algebraic dimension is one and hence the topological dimension is also one. Its measure in R^2 is zero and therefore the measure of $f^{-1}(\gamma \cap f(T'))$ is also zero. By symmetry the same result holds for $B = C$ and $C = A$. In R^3_+ the set of masses for which some of the potentials at Eulerian points coincide is an algebraic cone. □

More than what has been proved is actually true. A numerical computation shows that in T'(i.e. if $m_1 > m_2 > m_3$) we have $\bar{V}(E_{312}) > \bar{V}(E_{123}) > \bar{V}(E_{231})$. Fig. 3 displays a quantitative representation of the negative potentials obtained through direct computation with a fine mesh (steps of 0.005 in the masses, i.e. 3333 points in T'). The common value at $m_1 = m_2 = m_3$ is $\bar{V}(E_{ijk}) = -5 \times 3^{1/2}/54 \simeq 0.160375$. For a given point in T' the maximum of the differences between the values of the three potentials at that point is $\simeq 0.0058$. Hence the relative differences in Fig. 3 are very small. We prove part of what has been observed numerically in the next statement (we refer to Fig. 1 for the names of the points). For the sake of simplicity we call the functions $\bar{V}(E_{312})$, $\bar{V}(E_{123})$ and $\bar{V}(E_{231})$ defined on T, V_1, V_2 and V_3, respectively.

Theorem 2. *There is a neighbourhood U of the boundary $\Gamma = \partial T'$ such that in $T' \cap U$ we have $V_1 > V_2 > V_3$. On Γ the next table holds*

Set	$\overline{SP}$	O	Int$(\overline{OS})$	Int$(\overline{PO})$
Relations	$V_1 = V_2 = V_3 = -(m_1 m_2)^{3/2}/\sqrt{2}$	$V_1 = V_2 = V_3 = -5\sqrt{3}/54$	$V_1 = V_2 > V_3$	$V_1 > V_2 = V_3$

where Int *means the open segment.*

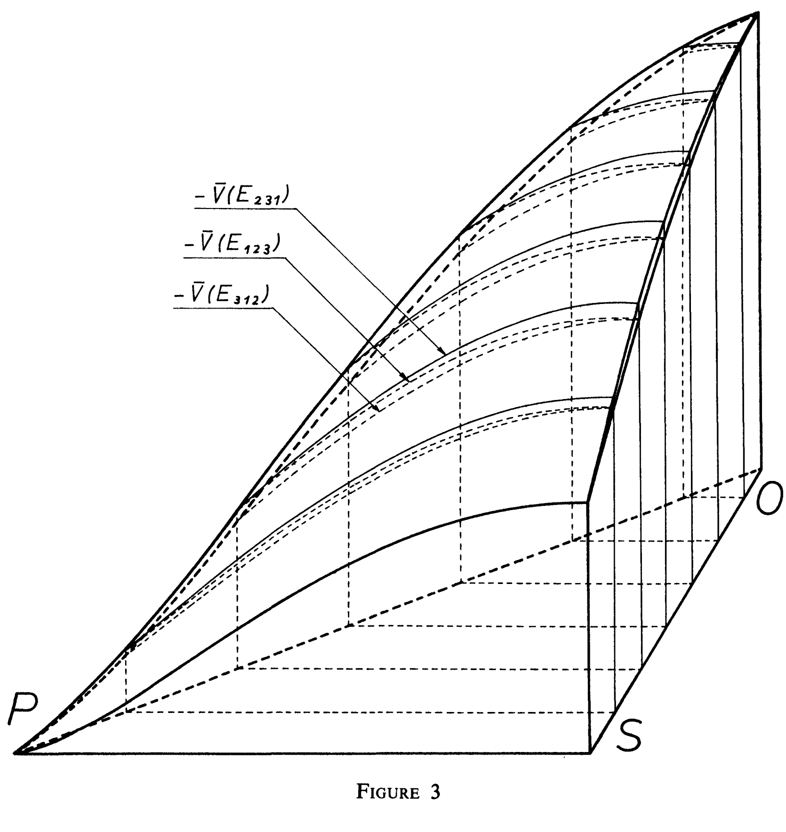

FIGURE 3

Proof. First we study the behaviour in the boundary. The only nontrivial assertions are $V_1 > V_2$ in Int($\overline{PO}$) and $V_2 > V_3$ in Int($\overline{OS}$). We take into account symmetries and the fact that $V_1 > V_2$ in a neighbourhood of P (to be proved below). To prove our assertions it is enough to show that on $\overline{PW}$ the functions V_1 and V_2 only coincide at the points P, O and W and that the function $V_1 - V_2$ has a simple zero at O.

If $m_2 = m_3$ we get $x = 1$, $m_1 = 1 - 2m_2$. The insertion in the equation marked (*) (i.e. $V_1 = V_2$) and in $q(y) = 0$ produces the system

$$y^2(1 + y)^2(4 - 7m_2)^2 = 2[2 - 4m_2 + (2 - 4m_2)y + (1 - m_2)y^2]$$

$$\cdot\, [m_2 + (2 - 3m_2)y + (1 - 2m_2)y^2]^2$$

$$2m_2 + 5m_2 y + 4m_2 y^2 - (3 - 5m_2)y^3 - (3 - 4m_2)y^4 - (1 - m_2)y^5 = 0\,.$$

We know the solutions $(m_2, y) = (0, 0), (1/3, 1), (1/2, 1.4318316\ \ldots)$ corresponding to points P, O, W, respectively. Last value of y is the solution of $G(y) = 2 + 5y + 4y^2 - y^3 - 2y^4 - y^5 = 0$. The elimination of m_2 gives the polynomial $F(y) = y^4(1 - y)\, G(y)\, K(y)$, where

$$K(y) = 144 + 768y + 1768y^2 + 2224y^3 + 1543y^4 + 391y^5$$

$$- 275y^6 - 311y^7 - 133y^8 - 27y^9 - 2y^{10}\,.$$

As K is the difference of two positive monotonic functions in R_+ and $K(3/2) = 21105/2$, the positive zero is greater than $3/2$. On the other side $m_2 = 1 - (2 + 5y + 4y^2 + 2y^3 + y^4)/(2 + 5y + 4y^2 + 5y^3 + 4y^4 + y^5)$ is an increasing function if $y > 1$. For $y = 3/2$ we get $m_2 = 1053/2023$. Therefore, there is no additional change of sign of $V_1 - V_2$ in $\overline{PW}$.

For the remaining part of the proof we consider four separate cases a), b), c), d) and introduce four neighbourhoods U_1, U_2, U_3, U_4 of P, $\overline{PO}$, $\overline{OS}$ and $\overline{SP}$, respectively. The behaviour in the neighbourhoods, U_5, U_6 of points O and S follows easily from the preceding cases. The assertion of the theorem will be proven in U_i, $i = 1 \div 6$. Then we take $U = \bigcup_{i=1}^{6} U_i$.

a) As m_2, m_3 are small and $m_1 = 1 - m_2 - m_3$ we can expand the potentials in (fractionary) powers of m_2, m_3. We get

$$V_1 = -(m_2 + m_3)^{3/2} + \frac{1}{2}\,(3m_2^2 + m_2 m_3 + 3m_3^2)\,(m_2 + m_3)^{1/2} + O_{7/2}\,,$$

$$V_2 = -(m_2 + m_3)^{3/2} - \frac{3^{4/3}}{2}\,m_2 m_3 (m_2 + m_3)^{1/6} + \frac{3}{2}\,(m_2 + m_3)^{5/2}$$

$$+ \frac{1}{3} m_2 m_3 (m_2 - m_3)(m_2 + m_3)^{-1/2} + O_{17/6}$$

$$V_3 = -(m_2 + m_3)^{3/2} - \frac{3^{4/3}}{2} m_2 m_3 (m_2 + m_3)^{1/6} + \frac{3}{2} (m_2 + m_3)^{5/2}$$

$$- \frac{1}{3} m_2 m_3 (m_2 - m_3)(m_2 + m_3)^{-1/2} + O_{17/6}$$

where O_s means a term of the order of $(m_2 + m_3)^s$. From the preceding expressions a) follows.

b) Due to the symmetry $V_2(m_1, m_2, m_3) = V_3(m_1, m_3, m_2)$ (on $\overline{PO}$) and it is enough to prove

$$D = \left.\frac{\partial V_3}{\partial m_2}\right|_{\substack{m_1 \text{ fixed} \\ m_2 = m_3}} < 0 ,$$

and use a) for points near P.

The function V_3 is given by

$$V_3 = -\frac{1}{a}\left(\frac{m_1 m_2}{1+z} + \frac{m_3 m_1}{z} + m_2 m_3\right) ,$$

a being such that $a^2 [m_1 m_2 (1+z)^2 + m_3 m_1 z^2 + m_2 m_3] = 2$, and z being the unique positive root of $r(z) = 0$. We recall that $z \geq 1$ in $\overline{PO}$.

After some manipulation we obtain

$$D = m_1 \left[\frac{a^2}{4} V_3 (1 + 2z) + \frac{1}{z(1+z)a}\right]$$

$$+ m_1 m_2 \frac{\partial z}{\partial m_2} \left[\frac{a^2}{2} V_3 (1 + 2z) + \frac{1}{a}\left(\frac{1}{(1+z)^2} + \frac{1}{z^2}\right)\right] .$$

We prove that the second bracket is zero. This is equivalent to proving

$$\left(\frac{m_1}{1+z} + \frac{m_1}{z} + m_2\right)(1 + 2z)$$

$$- [m_1 (1+z)^2 + m_1 z^2 + m_2]\left(\frac{1}{(1+z)^2} + \frac{1}{z^2}\right) = 0 .$$

After simplification we have

$$m_1(-3z^2 - 3z - 1) + m_2(2z^5 + 5z^4 + 4z^3 - z^2 - 2z - 1) = 0 ,$$

which is the equation $r(z) = 0$ restricted to $\overline{PO}$.

The proof of b) will be ended if we show that the first bracket in D

is negative. This is equivalent to showing

$$g(z,m_1) = 2 + (2z^3 + 3z^2 + z)(m_1 - 1) < 0 ,$$

on $\overline{PO}$. If $g(z,m_1) = 0$ then $m_1 = 1 - 2/(2z^3 + 3z^2 + z)$. Inserting this value in $r(z)$ we have

$$s(z) = -4z^5 - 10z^4 - 10z^3 - z^2 + 3z + 1 = 0 ,$$

on $\overline{PO}$. Descartes' rule ensures that there is a unique positive root $z < 1$. Since $z \geqslant 1$ on $\overline{PO}$, we have $g(z,m_1) < 0$.

c) Using symmetry again, it is enough to prove $D < 0$ on $\overline{OW}$, and hence only the fact that $g(z,m_1) < 0$ must be verified. Now $m_1 < 1/3$ and $z < 1$.

The solution z of $r(z) = 0$ on $\overline{OW}$ satisfies $z > 3/5$. This is obvious because when m_1 increases z increases and $r(3/5) > 0$. (In fact $r(z) = 0$ has the solution $z \simeq 0.69840614$ for $m_1 = 0$.) Since $s(3/5) < 0$, we have $g(z,m_1) < 0$ on $\overline{OW}$.

d) We rewrite the normalized potentials as

$$V_1 = -\frac{1}{a_1}\left(m_1m_2 + \frac{m_3m_1}{\rho_1} + \frac{m_2m_3}{1+\rho_1}\right),$$

$$V_2 = -\frac{1}{a_2}\left(m_1m_2 + \frac{m_3m_1}{\rho_2} + \frac{m_2m_3}{\rho_2 - 1}\right),$$

$$V_3 = -\frac{1}{a_3}\left(m_1m_2 + \frac{m_3m_1}{\rho_3} + \frac{m_2m_3}{1-\rho_3}\right),$$

where $a_1^2\,[m_1m_2 + \rho_1^2m_3m_1 + (1+\rho_1)^2m_2m_3] = 2$, and a_2 and a_3 satisfy similar relations and ρ_i are the solutions of suitable quintics (see Fig. 4).

Put $m_3 = \varepsilon$, $m_1 = m_1^0 - \varepsilon/2$, $m_2 = m_2^0 - \varepsilon/2$. It is enough to study the behaviour of $\partial V_j/\partial\varepsilon|_{\varepsilon=0}$. We have

$$\left.\frac{\partial V_1}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{1}{a_0}\left(-\frac{1}{2} + \frac{m_1^0}{\rho_1} + \frac{m_2^0}{1+\rho_1}\right)$$

$$-\frac{m_1^0m_2^0a_0}{4}\left(-\frac{1}{2} + m_1^0\rho_1^2 + m_2^0(1+\rho_1)^2\right),$$

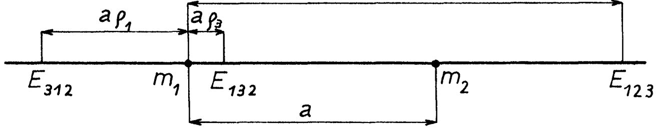

FIGURE 4

$$\left.\frac{\partial V_2}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{1}{a_0}\left(-\frac{1}{2}+\frac{m_1^0}{\rho_2}+\frac{m_2^0}{\rho_2-1}\right) - \frac{m_1^0 m_2^0 a_0}{4}\left(-\frac{1}{2}+m_1^0\rho_1^2+m_2^0(\rho_2-1)^2\right),$$

$$\left.\frac{\partial V_3}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{1}{a_0}\left(-\frac{1}{2}+\frac{m_1^0}{\rho_3}+\frac{m_2^0}{1-\rho_3}\right) - \frac{m_1^0 m_2^0 a_0}{4}\left(-\frac{1}{2}+m_1^0\rho_1^2+m_2^0(1-\rho_3)^2\right),$$

where $a_0 = (m_1^0 m_2^0/2)^{-1/2}$. The proof of d) is reduced to checking whether or not

$$\frac{m_1^0}{\rho_3}+\frac{m_2^0}{1-\rho_3}+\frac{1}{2}\left[m_1^0\rho_3^2+m_2^0(1-\rho_3)^2\right]$$
$$>\frac{m_1^0}{\rho_2}+\frac{m_2^0}{\rho_2-1}+\frac{1}{2}\left[m_1^0\rho_2^2+m_2^0(\rho_2-1)^2\right]$$
$$>\frac{m_1^0}{\rho_1}+\frac{m_2^0}{1+\rho_1}+\frac{1}{2}\left[m_1^0\rho_1^2+m_2^0(1+\rho_1)^2\right].$$

That is known to be true (see [6 page 142, property (5)]). □

We wish to point out that the first and second derivatives of the normalized potential are bounded in $T'' = T - (U_1 \cup U_1' \cup U_1'')$, where U_1', U_1'' are the analogs of U_1 near Q and R. Using the implicit function theorem we see that the only difficulty occurs with the lower bound of $dr/dz|_{z=z_q}$ where z_q is the positive solution of $r(z) = 0$. Descartes' rule ensures that there are unique points z_i, z_m with $0 < z_i < z_m < z_q$ such that $r''(z_i) = 0$, $r'(z_m) = 0$. Therefore $-r'$ increases for every $z > z_m$. Hence

$$|r'(z_q)| > \left|\frac{r(z_q)-r(z_m)}{z_q-z_m}\right| > \left|-\frac{r(0)}{z_q}\right| > \tau\left(\frac{3.3}{\tau}\right)^{-1/3} > 0.67\,\tau^{2/3},$$

where $\tau = 1 - \max_{T''}\{m_i\}$.

Let G be the sets of points in $T' - T' \cap U$ belonging to a mesh of step length h. From the boundedness of the derivatives we see that there is some step length h and some difference δ such that the complete proof of $V_1 > V_2 > V_3$ in T' is reduced to checking (numerically! with controlled error) whether or not $V_1 > V_2 > V_3$ and

$$\inf_{m\in G}\left(|V_1(m)-V_2(m)|+|V_2(m)-V_3(m)|\right) > \delta.$$

From what has been said we obtain the following result.

Assertion. *The relations* $m_1 > m_2 > m_3$ *imply* $\bar{V}(E_{312}) > \bar{V}(E_{123}) > \bar{V}(E_{231})$.

Note: The full computation has not actually been done.

From Theorem 2 and results of Iacob it follows that if $m_1 \geqslant m_2 \geqslant m_3$ and $h^2c < -\bar{V}^2(E_{231})$ then I_{hc} has three connected components. Similar results cannot be established for the problem of n bodies, $n > 3$, because for all the values of the masses, energy and momentum I_{hc} has only one connected component (see [4]).

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