Invariant Manifolds Associated to Homothetic Orbits in the n-body Problem

JOSEFINA CASASAYAS & JAUME LLIBRE

1. Introduction. Our main goal in this paper is to study the global behavior of the homothetic solutions of the planar and spatial n-body problem which begin and end in a simultaneous collision of all the particles. A similar study for the collinear n-body problem has been made by Devaney [2]. Following Devaney we consider the homothetic solutions as heteroclinic orbits of a dynamical system connecting two hyperbolic fixed points. This system is obtained regularizing the n-body problem using a technique of McGehee [4] where the singularity of total collision is blown up and in its place is glued an invariant total collision manifold. Furthermore, scaling the time, the solutions which previously reached total collision in finite time are made to tend asymptotically toward hyperbolic equilibrium points for the flow on the total collision manifold. We summarize the changes of variable and resulting equations (see [4], [8], [2]).

Let $q = (q_1,\ldots,q_n)^t$, $p = (p_1,\ldots,p_n)^t$ where q_i, $p_i \in R^k$ are the position and momentum of the body with mass m_i. Let

$$M = \operatorname{diag}(\underbrace{m_1,\ldots,m_1}_{k},\ldots,\underbrace{m_n,\ldots,m_n}_{k}),$$

$$B = \operatorname{diag}(\underbrace{I_k,\ldots,I_k}_{n}),$$

$$A = BM,$$

$$T(p) = \frac{1}{2}p^tM^{-1}p \quad \text{(kinetic energy)},$$

$$U(q) = \sum_{1\le i<j\le n} \frac{m_im_j}{|q_i-q_j|} \quad \text{(potential energy)},$$

where I_k is the identity matrix in R^k. The equations of motion are

$$\dot{q} = M^{-1}p,$$
$$\dot{p} = \nabla U(q),$$

with the first integrals:

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$$Aq = 0 \quad \text{(center of mass)},$$
$$Bp = 0 \quad \text{(linear momentum)},$$
$$c = \sum_{1 \le i \le n} q_i \wedge p_i \quad \text{(angular momentum) and}$$
$$h = T(p) - U(q) \quad \text{(total energy)}.$$

From now on we fix the center of mass and the linear momentum at the origin. The changes

$$r = (q^tMq)^{1/2},$$
$$s = q/r,$$
$$v = q^tp/r^{1/2},$$
$$u = M^{-1}pr^{1/2} - vs,$$
$$dt = r^{3/2}d\tau,$$

put the equations of motion in the form

$$\begin{aligned} r' &= rv, \\ v' &= \frac{1}{2}v^2 + u^tMu - V(s), \\ s' &= u, \\ u' &= -\frac{1}{2}vu - (u^tMu)s + \nabla V(s), \end{aligned} \tag{1.1}$$

where $' = d/d\tau$ and V is the restriction of the potential U to the ellipsoid S given by $s^tMs = 1$. The energy integral becomes

$$\frac{1}{2}u^tMu + \frac{1}{2}v^2 - V(s) - rh = 0.$$

Let Δ be the set of partial collisions, i.e. $\Delta = \{s \in S : s_i = s_j \text{ for some } i \neq j\}$. Then the flow given by the system (1.1) may be regarded as a real analytic vector field without singularities on $(r,v,s,u) \in [0,\infty) \times R \times T(S - \Delta) = \mathcal{M}$, where $T(S - \Delta)$ denotes the tangent bundle of $S - \Delta$. In particular, this flow leaves invariant the boundary $r = 0$, the so-called total collision manifold.

The following proposition is due to McGehee [4] and Devaney [2] for the collinear n-body problem and in the general case to Simó [8].

Proposition 1. *The vector field (1.1) is of gradient type on the total collision manifold. The equilibrium points of (1.1) are on the total collision manifold and for them* $u = 0$, $s = s_c$ *and* $v = \pm(2V(s_c))^{1/2}$, *where* s_c *is a central configuration (see [13]) or, equivalently, a critical point of* V *(see [10]). The* $+$ *sign in* v *being associated to total ejections and the* $-$ *sign to total collisions.*

Thus, to each central configuration s_c we may associate a pair of equilibrium points, which we denote by s_c^+ and s_c^-. For each $h < 0$ there is a homothetic orbit (which is the unique homographic orbit) connecting s_c^+ to s_c^- (see [2] and [13]). We denote this homothetic orbit by $\gamma_h(s_c)$. Such orbits $\gamma_h(s_c)$ begin in a total ejection, i.e. there is some time t_e such that for $t \downarrow t_e$ all the bodies tend to the same point. They end in a total collision: for some time $t_c > t_e$ we have that for all $t \uparrow t_c$ all the bodies tend to the same point. Since the equilibrium points s_c^+ and s_c^- are hyperbolic (see next section), the orbit $\gamma_h(s_c)$ can be seen as a heteroclinic orbit. We denote by $W^u(s_c^-)$ and $W^s(s_c^-)$ the unstable and stable invariant manifold of the hyperbolic point s_c^-, respectively.

We consider the planar or spatial n-body problem as a mechanical system with symmetry (see [10]) given by the Lie group $G = S^1$ or $G = SO(3)$ acting diagonally, respectively.

Let I_h be the space of points $(r,v,s,u) \in \mathcal{M}$ such that

$$\frac{1}{2} u^t M u + \frac{1}{2} v^2 - V(s) - rh = 0,$$

i.e., the points of $\mathcal{M}$ with total energy equal h. Let $I_{h,c}$ be the space of points of I_h such that

$$r^{1/2} \left(\sum m_i s_i \wedge u_i \right) - c = 0,$$

i.e., the points of $\mathcal{M}$ with total energy and angular momentum equal h and c, respectively. Let $I_{h,0} = I_{h,c=0}$. It is clear that the spaces $I_{h,c}$ are invariant under the flow (1.1). Since for any value of c the isotropy group G_c leaves $I_{h,c}$ invariant, we can define the quotient space $\bar{I}_{h,c} = I_{h,c}/G_c$. For $c = 0$ we obtain immediately $G_c = G$. Therefore, going from $I_{h,0}$ to $\bar{I}_{h,0}$ we reduce the number of parameters by 1 or 3 in the planar or spatial n-body problem, respectively.

Let $\bar{V}$ be the restriction of V to S/G. Recall that a central configuration s_c is a critical point of $\bar{V}$. If $D^2\bar{V}(s_c)$ is the Hessian of $\bar{V}$, then a central configuration s_c is called degenerate (non-degenerate) when $D^2\bar{V}(s_c)$ is so (or not). We denote by $\operatorname{ind}(s_c)$ the index of the critical point s_c, i.e. the number of negative eigenvalues of $D^2\bar{V}(s_c)$.

Our main results are the following theorems.

Theorem A. *Let s_c be a nondegenerate central configuration of the planar (respectively spatial) n-body problem. Then the dimensions of $W^u(s_c^+)$ and $W^s(s_c^-)$ are the same and equal to $2n - 3 - \operatorname{ind}(s_c)$ (respectively $3n - 6 - \operatorname{ind}(s_c)$) in $\bar{I}_{h,0}$. The dimensions of $W^s(s_c^+)$ and $W^u(s_c^-)$ are the same and equal to $2n - 4 + \operatorname{ind}(s_c)$ (respectively $3n - 7 + \operatorname{ind}(s_c)$) in the same space. The dimension of $\bar{I}_{h,0}$ is $4n - 7$ (respectively $6n - 13$).*

We restrict our attention to the invariant space $\bar{I}_{h,0}$ because by Sundman's theorem [11] a necessary condition in order to have a total collision or ejection is that the angular momentum is equal to zero.

Theorem A is a generalization to the planar and spatial n-body problem of a similar theorem of Devaney [2] for the collinear n-body problem. Palmore in [5] gives a bound of the index for a planar central configuration. This implies that we have a bound of the dimensions of the invariant manifolds $W^{u,s}(s_c^{+-})$ for the planar n-body problem.

Theorem B. *Let s_c be a collinear central configuration. Then the manifolds $W^u(s_c^+)$ and $W^s(s_c^-)$ are the same for the collinear, planar and spatial n-body problem in $\bar{I}_{h,0}$. For the collinear n-body problem I_h is equivalent to $\bar{I}_{h,0}$.*

It is known, see [10], that there are precisely $n!$ collinear central configurations and that each one corresponds with a permutation of the n particles. Then using Theorem B the following corollary is immediate.

Corollary C. *For the collinear, planar and spatial n-body problem there are not heteroclinic orbits connecting a total ejection with a total collision associated to different central configurations when one of the central configurations is collinear.*

In general, heteroclinic orbits connecting a total ejection with a total collision associated to different central configurations are possible (see [9]).

Two submanifolds M_1 and M_2 of a manifold M are said to be transverse at x if $x \in M_1 \cap M_2$ and if $T_xM_1 + T_xM_2 = T_xM$, where T_xM denotes the tangent space to M at x. We say that M_1 meets M_2 transversally if either M_1 meets M_2 transversally at x for all $x \in M_1 \cap M_2$, or $M_1 \cap M_2$ is empty.

Theorem D. *In the collinear, planar or spatial n-body problem a necessary condition in order to have a transversal homothetic solution $\gamma_h(s_c)$ in $\bar{I}_{h,0}$ with $h < 0$ is that the function $\bar{V}$ be a nondegenerate minimum at the point s_c associated to the homothetic solution. For the collinear n-body problem $\bar{I}_{h,0}$ is equivalent to I_h.*

That this condition is also sufficient has been proved recently by Simó and Llibre [3]. See also Devaney [2] for the collinear n-body problem. The importance of transversality has been discussed in [2].

In the appendix we give some examples of transversal homothetic orbits.

2. Proof of Theorem A. Recall that the stable manifold of an equilibrium point p consists of all orbits which tend asymptotically toward p. We denote this manifold by $W^s(p)$. The classical stable manifold theorem asserts that $W^s(p)$ is a smooth immersed disk having dimension equal to the number of characteristic exponents with negative real parts. Similarly, the unstable manifold, denoted by $W^u(p)$, consists of orbits tending toward p in backward time and has dimension equal to the number of characteristic exponents with positive real parts.

We begin computing the characteristic exponents for the flow (1.1) of the planar (respectively spatial) n-body problem on M at s_c^+, which are given by the eigenvalues of the differential matrix of the vector field (1.1) at s_c^+ given by

$$C = \begin{pmatrix} v_c^+ & 0 & 0 & 0 \\ 0 & v_c^+ & -s_c^t MV(s_c) & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & D^2V(s_c) & -\frac{1}{2} v_c^+ I \end{pmatrix},$$

where $v_c^+ = (2V(s_c))^{1/2}$. Since $V(s_c) \neq 0$, we have that two of the characteristic exponents equal $v_c^+ \neq 0$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{2n-3}$ (respectively λ_{3n-4}) be the eigenvalues of $D^2V(s_c)$, then

$$\{-v_c^+ \pm [(v_c^+)^2 + 16\lambda_i]^{1/2}\}/4, \tag{2.1}$$

$i = 1, \ldots, 2n - 3$ (respectively $3n - 4$), are eigenvalues of

$$\begin{pmatrix} 0 & I \\ D^2V(s_c) & -\frac{1}{2} v_c^+ I \end{pmatrix}.$$

The proof of (2.1) is identical to the proof for the collinear n-body problem, see [2, page 397].

Due to the invariance of V under G we have that 1 (respectively 3) of the eigenvalues of $D^2V(s_c)$ is zero. Then we assume that $\lambda_{2n-3} = 0$ (respectively $\lambda_{3n-6} = \lambda_{3n-5} = \lambda_{3n-4} = 0$). We also suppose explicitly that s_c is not a degenerate critical point. Then it is easy to prove that eigenvalues of C are:

v_c^+, associated to the variations of the total energy h,

v_c^+, associated to the variations of r,

$-v_c^+/2$ (respectively $-v_c^+/2, -v_c^+/2, -v_c^+/2$), associated to the variations of the angular momentum,

0 (respectively 0,0,0), associated to the invariance under S^1 (respectively $SO(3)$) and

$\{-v_c^+ \pm [(v_c^+)^2 + 16\lambda_i]^{1/2}\}/4$ where λ_i is an eigenvalue of $D^2\bar{V}(s_c)$, $i = 1, \ldots, 2n - 4$ (respectively $3n - 7$).

Then the differential matrix of the vector field (1.1) at s_c^+ restricted to the tangent space to $\bar{I}_{h,0}$ has $2n - 3 - \mathrm{ind}(s_c)$ (respectively $3n - 6 - \mathrm{ind}(s_c)$) eigenvalues with positive real part and $2n - 4 + \mathrm{ind}(s_c)$ (respectively $3n - 7 + \mathrm{ind}(s_c)$) with negative real part. Switch dimensions at the total collision s_c^-. Hence we have proved Theorem A.

3. Proof of Theorem B. Recall that the collinear n-body problem has precisely $n!$ central configurations. Moreover, see [10], each central configuration s_c is given by a nondegenerate minimum of V on $S - \Delta$, i.e. $\mathrm{ind}(s_c) = 0$. The reason that our count of central configurations differs from Smale's [10] by a factor of two is that Smale is interested in relative equilibria and hence identifies a central configuration at s with its negative.

The following theorem is due to Devaney [2].

Theorem 2. *Let s_c be a central configuration of the collinear n-body problem. Then the dimensions of $W^u(s_c^+)$ and $W^s(s_c^-)$ are the same and equal to $n - 1$ in I_h. The dimensions of $W^s(s_c^+)$ and $W^u(s_c^-)$ are the same and equal to $n - 2$ in the same space. The dimension of I_h is $2n - 3$.*

On the other hand, Palmore [5] has shown that the index of a collinear central configuration s_c is $n - 2$ in the planar n-body problem (note the change of sign in the definition of the potential energy). Then, by Theorem A, the dimensions of $W^u(s_c^+)$ and $W^s(s_c^-)$ are $n - 1$ in $\bar{I}_{h,0}$ for the planar n-body problem.

Note that the collinear n-body problem is contained in the planar n-body problem and that the reductions to $\bar{I}_{h,0}$ do not affect the collinear n-body problem. Then Theorem B follows for the collinear and planar n-body problem. By reasons of symmetry Theorem B also follows for the spatial n-body problem.

4. Proof of Theorem D. Let $\gamma_h(s_c)$ be a transversal homothetic solution in $\bar{I}_{h,0}$ with $h < 0$. Since Theorem D has been proved by Devaney [2] in the collinear n-body problem, in what follows we consider the planar n-body problem. Then, by Theorem A, both $W^u(s_c^+)$ and $W^s(s_c^-)$ are $(2n - 3 - \mathrm{ind}(s_c))$-dimensional and $\bar{I}_{h,0}$ is $(4n - 7)$-dimensional. Since $\gamma_h(s_c)$ lies in $W^u(s_c^+) \cap W^s(s_c^-)$ and $\gamma_h(s_c)$ is transversal we have that

$$\dim \bar{I}_{h,0} \leq \dim W^u(s_c^+) + \dim W^s(s_c^-) - 1.$$

That is, $4n - 7 \leq 4n - 7 - 2\,\mathrm{ind}(s_c)$. Therefore, $\mathrm{ind}(s_c) = 0$, i.e. the function $\bar{V}$ has a nondegenerate minimum at s_c. Hence Theorem D is proved for the planar n-body problem. The proof for the spatial n-body problem is similar.

Appendix. In this appendix we give some examples of transversal homothetic orbits. Recall that, as we have said in the introduction, recently it has been proved that the condition of Theorem D is also sufficient in order to have a transversal homothetic solution.

As was stated by Devaney [2] all the homothetic solutions in the collinear n-body problem are transversal in I_h because associated central configurations are nondegenerate minima of the function V on $S - \Delta$.

In the planar 3-body problem the unique central configurations that are nondegenerate minima of the function $\bar{V}$ are the Lagrange central configurations (see [1] for details). Then the homothetic solutions associated to the Lagrange central configurations are transversal in $\bar{I}_{h,0}$. Using [1] and Theorem A we obtain the following table.

TABLE I

central configuration	number of relative equilibria	index	dim $W^u(s_c^+)$	dim $W^s(s_c^+)$
rectilinear (Euler)	3	1	2	3
equilateral (Lagrange)	2	0	3	2

The dimensions of the stable and unstable invariant manifolds for the Euler and Lagrange central configurations in the planar 3-body problem can also be computed using the Siegel exponents (see [8] and [12]). From [6, page 88] the Siegel exponents at s_c^+ are given by

2/3 associated to the variations of the total energy h,

-1 associated to the variations of the time t,

$-1/3$ associated to the variations of the angular momentum,
0 associated to the invariance under S^1,
$\{-1 \pm [13 \pm 12\mu]^{1/2}\}/6$ where $\mu \in (0,1)$, for the Lagrange central configuration and
$\{-1 \pm [25 + 16\mu]^{1/2}\}/6$ and $\{-1 \pm [1 - 8\mu]^{1/2}\}/6$ where $\mu \in (0,7)$, for the Euler central configuration.

On the other hand, we have computed the eigenvalues of the matrix C in the proof of Theorem A. Since

$$\{-v_c^+ \pm [(v_c^+)^2 + 16\lambda_i]^{1/2}\}/4 = \{-1 \pm [1 + 16\lambda_i/(v_c^+)^2]^{1/2}\}/(4/v_c^+),$$

we obtain that seven of the Siegel exponents correspond (up to a constant equal to $(3/2)v_c^+$) to eigenvalues of the matrix C. The Siegel exponent -1 associated to the variations of time corresponds to the eigenvalue v associated to the variations of r. This is due to the following. Siegel in [6, page 83] makes the following change of time

$$dt = -t d\tau. \tag{5.1}$$

Furthermore, if $t = 0$ corresponds to a triple collision and $I = r^2$ is the moment of inertia, Siegel in [6, page 70] shows that the function $\dot{I}I^{-1/4} = 2v$ approaches a finite limit $\delta \geq 0$ as $t \to 0$, and therefore $I \sim \chi t^{4/3}$ as $t \to 0$, with $\chi = ((3/4)\delta)^{4/3} > 0$. Then $r^{3/2} \sim (3/2)vt$ as $t \to 0$. In this paper, we have made the following change of time

$$dt = r^{3/2} d\tau,$$

that is,

$$dt \sim \frac{3}{2} vt\, d\tau \qquad \text{as } t \to 0.$$

This, according to (5.1) proves the above assertion.

All the central configurations of the planar 4-body problem with equal masses are nondegenerate. They are given in the following table.

TABLE II

central configuration	number of relative equilibria	index	dim $W^u(s_c^+)$	dim $W^s(s_c^+)$
rectilinear	12	2	3	6
square	6	0	5	4
equilateral	8	2	3	6
isosceles	24	1	4	5
scalene	48	1	4	5
	48	0	5	4

This table is obtained from [5] and [7]. Then, the central configurations associated to transversal homothetic solutions are the square and the scalene with index equal to zero.

Using the results of [7] we can construct a table similar to Table II for the general planar 4-body problem. Note, in this case, that for almost all the values of the masses the unique central configuration which has index 0 is related (by evolution of the masses) to square central configuration for equal masses.

In the planar 4-body problem there is a central configuration such that for some values of the masses the associated homothetic solution is transversal and for other ones it is not transversal. In fact, we consider three masses equal to unity on the vertexes of an equilateral triangle and on its center a mass m. For each value of the mass m this configuration, denoted by $s_c(m)$, is central for the planar 4-body problem. Then, $s_c(m)$ is a nondegenerate minimum of $\bar{V}$ if and only if $m < m' = (2 + 3^{4/3})/(18 - 5 \cdot 3^{1/2})$. If $m > m'$, then $\mathrm{ind}(s_c(m)) = 2$ (for more details see [5]). In short we have the following table.

TABLE III

	dim $W^u(s_c(m)^+)$	dim $W^s(s_c(m)^+)$
$m < m'$	5	4
$m > m'$	3	6

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CASASAYAS: UNIVERSITAT DE BARCELONA, BARCELONA, SPAIN
LLIBRE: UNIVERSITAT AUTÒNOMA DE BARCELONA, BARCELONA, SPAIN

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