On the Peak Sets for Holomorphic Lipschitz Functions

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I. Introduction and notations. Let D denote the open unit disc in the complex plane and let T be its boundary. For $0 < \alpha \leq 1$, Lip α will denote the algebra of complex-valued functions f analytic on D, continuous on $\bar{D}$ and satisfying a Lipschitz condition of order α on $\bar{D}$:

$$|f(z) - f(w)| \leq K|z - w|^{\alpha}, \qquad z, w \in \bar{D}.$$

A closed set $E \subset T$ is said to be a *peak set* for Lip α if there exists $f \in$ Lip α such that $f = 1$ on E and $|f| < 1$ on $\bar{D}\backslash E$ (and in this case we say that f peaks on E).

In [9], W. P. Novinger and D. M. Oberlin studied the peak sets for Lip α. They showed that the peak sets for Lip 1 are just the finite sets. Moreover, if (ε_n) is the sequence of lengths of the complementary intervals of E in T, they proved that

$$(1) \qquad |E| = 0, \qquad \sum_n \varepsilon_n^{(1-\alpha)/(3-\alpha)} < +\infty,$$

is a sufficient condition and that

$$(2) \qquad |E| = 0, \qquad \sum_n \varepsilon_n^{1-\alpha} |\log \varepsilon_n|^{-\delta} < +\infty, \qquad \delta > 1$$

is a necessary condition for E being a peak set for Lip α. They also conjectured that

$$(3) \qquad |E| = 0, \qquad \sum_n \varepsilon_n^{1-\alpha} < +\infty$$

is a necessary and sufficient condition. If $\rho(z)$ denotes the Euclidean distance from z to E, we note that (3) means that $\rho^{-\alpha} \in L^1(T)$, whereas (1) means that $\rho^{-2/(3-\alpha)} \in L^1(T)$.

In this paper we continue this study and we offer some necessary conditions and some sufficient conditions which are very close to the conjecture. In particular, we show (Theorem 2.7) that $\rho^{-\alpha} \in L^{1+\delta}(T)$ for some $\delta > 0$ is a sufficient

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condition and that $\rho^{-\alpha} \in$ weak $L^1(T)$ is a necessary one (Theorem 3.2). Thus we do not reach the complete characterization. Nevertheless, observe that these results imply the following: a closed set is a peak set for some Lip β with $\beta > \alpha$ if and only if $\rho^{-\beta} \in L^1(T)$ for some $\beta > \alpha$. That is, the "union" of the conjectures for $\beta > \alpha$ is true. These results are obtained in Sections II and III.

We are also interested in this paper in the relation between the study of this kind of problems in Lip α and in the a priori nonrelated Gevrey analytic class G_α. This is defined as the class of holomorphic functions in D, C^∞ in $\bar{D}$, such that for some constants C, Q

$$|f^{(n)}(z)| \le CQ^n(n!)^{1+1/\alpha}, \qquad n = 0, 1, \ldots, z \in D.$$

This relation was first remarked by Dyn'kin in [3], where he notes that a closed set $E \subset T$ is an interpolation set for the holomorphic Lipschitz classes (these are the K-sets, see [3], and are common to all regular classes) if and only if it is an interpolation set for some G_α (see [4]).

By introducing a (natural) special class of peak sets for Lip α, which we call the *strong peak sets,* we are able to present another aspect of this relation. Namely, it turns out that the interpolation sets for G_α and the strong peak sets for Lip α are the same, with the same α. In particular every interpolation set for Lip α is a peak set for some Lip β, but not necessarily $\beta = \alpha$ as is shown by an example. Another example shows that conversely, not every peak set in Lip α is an interpolation set, unlike the disc algebra case. Section IV is devoted to proving these results.

As in [1], [3], [5], some of the notations and techniques we use are from real variable theory (BMO, Muckenhoupt's weights, maximal function). But to some extent (necessity of equilibrium conditions) we think that the role of these concepts is more than technical. In the last section we discuss this and we close by posing some open questions.

Now we collect some notations:

As above, E will always denote a closed set of Lebesgue measure zero and $\rho(z)$ the Euclidean distance from z to E. In case $z \in T$, we shall not distinguish between $\rho(z)$ and the distance measured along the arc, so that we shall often proceed as if we were on the real line. The sequence of complementary intervals of E in T is denoted $\{(a_n, b_n)\}$ and we set $\varepsilon_n = b_n - a_n$.

If I is an arc in T we denote by $|I|$ its length, x_I its center and by 2^nI the arc which has the same center as I and 2^n times its length. We say that an arc $I \subset T$ and a point $z \in D$ are *related* if $z/|z|$ is the center of I and $|I| = 1 - |z|$.

For $x \in T\backslash E$, I_x stands for the complementary interval of E in T containing x and we set $J_x = \{e^{it} : |e^{it} - x| \le \rho(x)/4\}$.

For $u \in L^1(T)$ we denote by $I(u)$ the mean of u over I

$$I(u) = \frac{1}{|I|}\int_I u\, dm, \quad dm = \text{Lebesgue measure on } T.$$

The notation u^* stands for the *Hardy-Littlewood maximal function* of u:

$$u^*(x) = \sup\{I(|u|), I \text{ centered at } x\}.$$

BMO is the subspace of $L^1(T)$ of all $u \in L^1(T)$ such that

$$\|u\|_{\text{BMO}} = \sup_I I(|u - I(u)|) < \infty,$$

that is, u has *bounded mean oscillation*. The complex version of BMO is the space BMOA of all $h \in H^1$ whose boundary values belong to BMO. BMOA is contained in the Block space, i.e. for all $h \in$ BMOA

(4) $$|h'(z)| = 0(1 - |z|)^{-1} \qquad z \in D.$$

The *Muckenhoupt weights* (A_1), (A_2) and (B_2) are respectively defined by the conditions

(A_1) $$w^*(x) \leq \text{const}\, w(x), \qquad \text{a.e.}$$

(A_2) $$I(w)I(w^{-1}) \leq \text{const}, \qquad \forall I \subset T$$

(B_2) $$|I| \int_{T\setminus I} \frac{w(t)}{|t - x_I|^2}\, dt \leq \text{const}\, I(w), \qquad \forall I \subset T.$$

Recall (see [6]) that $(A_1) \Rightarrow (A_2) \Rightarrow (B_2)$.

Finally, for two variable quantities a, b, the notation $a \sim b$ will mean that they are of the same order, in the sense that there exist some constants $m, M \geq 0$ such that $m \leq a/b \leq M$.

II. Sufficient conditions. Observe that the existence of a peaking function for E is equivalent to the existence of a function $f \in \text{Lip}\,\alpha$, such that $f = 0$ on E and $\text{Re}\, f(z) > 0$ on $\bar{D}\setminus E$ (use the operations $f \to 1 - f$ and $f \to e^{-f}$).

2.1 Lemma. *If g is a holomorphic function on D with positive real part, then g is an outer function,* $\log|g| \in$ BMO *and*

(5) $$|g'(z)| \leq \text{const}\, |g(z)|(1 - |z|)^{-1}, \qquad z \in D.$$

Proof. g is an outer function, i.e., of the form

$$g(z) = c \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|g(e^{it})|\, dt, \qquad z \in D,\ |c| = 1$$

because both g and g^{-1} belong to H^p with $p < 1$. That $\log|g| \in$ BMO follows from the fact that it is the conjugate of the bounded function Arg g. Then $h = \log g \in$ BMOA, and so satisfies (4), from which (5) follows. □

The following simple but useful lemma is proved in [1] and will be used often, so we state it explicitly. The important point in it, as will become clear later, is that the estimate does not depend on the distance $1 - |z|$ from z to T, but on the length of J, the distance from z to the endpoints of J and the derivatives of ψ on J.

2.2 Lemma. *Let ψ be of class C^{n+1} in some arc $J = [a,b]$ of T, let $c = (a+b)/2$ be the middle point of J and let $A(z)$ be defined by*

$$A(z) = \int_a^b \frac{e^{it}}{(e^{it}-z)^{n+1}} \psi(e^{it})\,dt, \qquad z \in D.$$

Put $\psi^0 = \psi$, $\psi^{(k)}(e^{it}) = e^{it}(d/dt)\psi^{(k-1)}(e^{it})$, $k = 1, \ldots, n+1$, and $M_k =$ $\max\{|\psi^{(k)}(e^{it})|, a \le t \le b\}$. *Then, for $z = re^{ic}$, $0 \le r < 1$,*

$$|A(z)| \le \text{const}\left(\sum_{k=0}^{n-1} \frac{M_k}{|z - e^{ia}|^{n-k}} + M_n + |J|M_{n+1}\right).$$

We shall need in the following a modification $\tilde\rho(t)$ of $\rho(t)$ on T, such that $\tilde\rho \sim \rho$ and $\tilde\rho$ is C^∞ in $T\backslash E$. For instance,

$$\tilde\rho(t) = \frac{(t-a_n)(b_n-t)}{b_n-a_n}, \qquad t \in (a_n,b_n)$$

is such a function.

We now give a first sufficient condition. Though it will be later generalized, we prefer to present it as a preliminary step in the proof of the main result of this section (see first point in the proof of Theorem 2.6).

In the statement that follows, recall that $J_x = \{e^{it}: |e^{it}-x| \le \rho(x)/4\}$.

2.3 Theorem. *Suppose that $\rho^{-\alpha} \in L^1(T)$ and that the following holds:*

$$(6)\qquad |J_x| \int_{T\backslash J_x} \frac{\rho^{-\alpha}(e^{it})}{|e^{it}-x|^2}\,dt \le \text{const}\, J_x(\rho^{-\alpha}), \qquad x \in T\backslash E.$$

Then E is a peak set for Lip α $(0 < \alpha < 1)$.

Proof. We define

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,\tilde\rho^{-\alpha}(e^{it})\,dt, \qquad z \in D.$$

Re g is the Poisson integral of $\tilde\rho^{-\alpha}$. First we shall prove

$$(7)\qquad \operatorname{Re} g(z) \ge \text{const}\,\rho(z)^{-\alpha}, \qquad z \in D.$$

Take I related to z; for $e^{it} \in I$, $P_z(e^{it}) \sim |I|^{-1}$ and

$$\rho(e^{it}) \le \rho(z) + |z - e^{it}| \le \rho(z) + \text{const}(1-|z|) \le \text{const}\,\rho(z).$$

Hence,

$$\operatorname{Re} g(z) = \int_T P_z(e^{it})\tilde\rho^{-\alpha}(e^{it})\,dt \ge \int_I P_z(e^{it})\tilde\rho^{-\alpha}(e^{it})\,dt \sim I(\tilde\rho^{-\alpha})$$
$$\ge \text{const}\,\rho(z)^{-\alpha}.$$

The proof of the theorem will be finished if we show that $f = g^{-1}$ is in Lip α. By the Hardy-Littlewood theorem [2, page 74], this is equivalent to $|f'(z)| = 0(1 - |z|)^{\alpha-1}$, i.e.,

$$\frac{|g'(z)|}{|g(z)|^2} \leq \text{const}(1 - |z|)^{\alpha-1}. \tag{8}$$

(In fact, it is shown in [2] that this is equivalent to $f \in A$, the disc algebra, and $f(e^{it}) \in \text{Lip}(\alpha,T)$. But it is an old result that this is in turn equivalent to $f \in$ Lip α.)

If $\rho(z) \leq 4(1 - |z|)$, (8) follows from (5) and (7).

For $\rho(z) \geq 4(1 - |z|)$ we shall obtain the estimate

$$|g'(z)| \leq \text{const}\, \rho(z)^{-\alpha-1} \tag{9}$$

from which, using (7), (8) follows again. Now,

$$g'(z) = \frac{1}{\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it} - z)^2}\, \tilde{\rho}^{-\alpha}(e^{it})\, dt.$$

Let $x = z/|z|$ and break this integral into two parts corresponding to J_x and $T\backslash J_x$. We claim the following:

$$\rho(e^{it}) \sim \rho(z) \qquad \text{for} \quad e^{it} \in J_x. \tag{10}$$

$$|J_x| \sim \rho(z) \tag{11}$$

$$|e^{it} - z| \sim |e^{it} - x| \qquad \text{for} \quad e^{it} \notin J_x. \tag{12}$$

To prove (10), just observe that $\rho(e^{it}) \sim \rho(x)$ for $e^{it} \in J_x$ and that $\rho(x) \sim \rho(z)$, because $|z - x| = 1 - |z| \leq \rho(z)/4$. (11) is trivial. Finally, (12) follows from the triangle inequality and from the fact that $|z - x| = 1 - |z|$ can be absorbed in $|e^{it} - z|$ and in $|e^{it} - x|$ if $e^{it} \notin J_x$.

Using the hypothesis and (10), (11), (12), we find that the contribution of $T\backslash J_x$ satisfies an estimate like (9). To evaluate the contribution of J_x we use Lemma 2.2, with J_x, $n = 1$, and $\psi(e^{it}) = \tilde{\rho}^{-\alpha}(e^{it})$ ($\tilde{\rho}$ is C^∞ outside E). An easy computation shows that a k-derivative of $\tilde{\rho}^{-\alpha}$ is of the same order as $\rho^{-\alpha-k}$. This and (10) imply, with the notations of Lemma 2.2, that $M_k \leq \text{const}\, \rho(z)^{-\alpha-k}$. Thus,

$$\left|\int_{J_x} \frac{e^{it}}{(e^{it} - z)^2}\, \tilde{\rho}^{-\alpha}(e^{it})\, dt\right|$$

$$\leq \text{const}\left(\text{const}\, \frac{\rho(z)^{-\alpha}}{|z - e^{ia}|} + \text{const}\, \rho(z)^{-\alpha-1} + \text{const}\, \rho(z)^{-\alpha-1}\right) \leq \text{const}\, \rho(z)^{-\alpha-1}.$$

Here e^{ia} is an endpoint of J_x and we have used (11) and the fact that $|z - e^{ia}| \sim |e^{ia} - x| = \rho(x)/4 \sim \rho(z)$, by (10), (12).

Thus the estimate (9) is completely proved. The proof is finished. □

2.4 Corollary. *If* $\rho^{-\alpha}$ *satisfies Muckenhoupt's condition* (B_2)*, E is a peak set for* Lip α.

2.5 Remark. The above are not global conditions in the sense that they depend on the repartition of the complementary intervals (a_n, b_n) and not just on their lengths. But the same method gives that $\rho^{-(1+\alpha)/2} \in L^1(T)$ is a (global) sufficient condition. Just observe that (7) is changed to

$$\operatorname{Re} g(z) \geq \text{const}\, \rho(z)^{-(1+\alpha)/2}, \qquad z \in D$$

and that

$$|g'(z)| \leq \text{const}\, \rho(z)^{-2}, \qquad z \in D$$

so that (8) is again obtained. Since $(1+\alpha)/2 < 2/(3-\alpha)$, this condition is already better than (1).

Since $\rho(e^{it}) \sim \rho(x)$ for $e^{it} \in J_x$ and $|J_x| \sim \rho(x)$, condition (6) is equivalent to

$$\rho(x) \int_{T \setminus J_x} \frac{\rho^{-\alpha}(e^{it})}{|e^{it} - x|^2}\, dt \leq \text{const}\, \rho^{-\alpha}(x), \qquad x \in T \setminus E. \tag{13}$$

Now we will see that the existence of a majorant w of $\rho^{-\alpha}$ satisfying this is already sufficient for E to be a peak set.

2.6 Theorem. *Suppose that there exists* $w \in L^1(T)$ *such that*

$$\rho^{-\alpha}(x) \leq \text{const}\, w(x), \qquad x \in T \setminus E \tag{14}$$

$$\rho(x) \int_{T \setminus J_x} \frac{w(e^{it})}{|e^{it} - x|^2}\, dt \leq \text{const}\, w(x), \qquad x \in T \setminus E. \tag{15}$$

Then E is a peak set for Lip α.

Proof. We distinguish five steps.

1. In what follows we show why the method of Theorem 2.3 does not apply directly and we indicate what will be the line of proof. Thus, the following considerations are not rigorous and just pretend to motivate in a natural manner the other points of this proof.

We consider again the proof of Theorem 2.3, with the same notations, but with g defined with w instead of $\rho^{-\alpha}$. As before, the important case is $\rho(z) \geq 4(1 - |z|)$, i.e., z is closer to T than to E. In this case, (15) implies that the contribution of $T \setminus J_x$ in $|g'(z)|$ is bounded by some constant times $\rho(x)^{-1} w(x)$. But for the contribution of J_x, we can just obtain

$$\left| \int_{J_x} \frac{e^{it}}{(e^{it} - z)^2}\, w(e^{it})\, dt \right| \leq \frac{1}{(1 - |z|)^2} \int_{J_x} w$$

which is not good, because of the presence of $1 - |z|$. Lemma 2.2 improves this estimate when one has some differentiability assumptions on w.

But observe that this difficulty would not appear if a function w_G is assumed to exist and satisfy a relation like (15) but on the boundary ∂G of a domain G such that $\rho(z,\partial G) \sim \rho(z,E)$ for $z \in D$, as for instance the domain $G = \{z, |z| - 1 < \varepsilon\rho(z)\}$ (this comes from [3, page 111]). With the same construction, the contribution of $\partial G \backslash J_x$ (now $J_x \subset \partial G$) would be bounded by $\rho(x)^{-1} w_G(x)$ whereas that of J_x would be changed to

$$\left|\int_{J_x} \frac{w_G(\zeta)}{(\zeta - z)^2}\, d\zeta\right| \leq \rho(z,\partial G)^{-2} \int_{J_x} w_G \sim \rho(z)^{-2} \int_{J_x} w_G \sim \rho(x)^{-2} \int_{J_x} w_G$$

in which the "bad" quantity $1 - |z|$ does not appear.

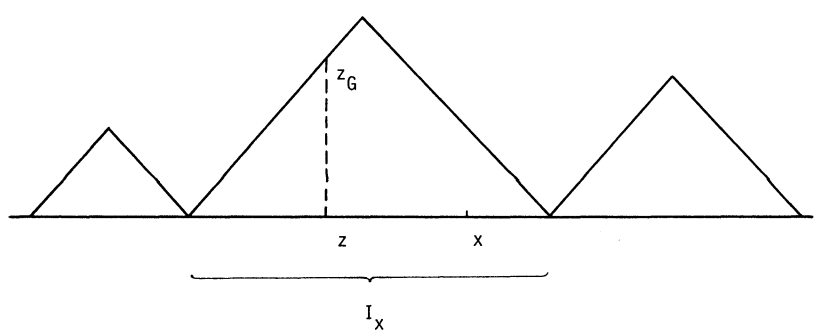
The problem is that the required domain G is just a Lipschitz domain so that we do not have at our disposal a good Poisson kernel theory for G to carry over the other points in the proof of Theorem 2.3. For instance, an estimate of the Poisson kernel $P_G(z,\zeta)$ for G of the type

$$(16) \qquad P_G(z,\zeta) \sim \frac{\rho(z,\partial G)}{|z - \zeta|^2}, \qquad z \in G,\ \zeta \in \partial G$$

is not known.

In spite of this, these considerations suggest that we substitute for w on T a function u on T which essentially is $P_G w_G$ restricted to T and then apply to u the procedure of Theorem 2.3. This is the central idea and the rest of the proof is devoted to showing that it works. Thus, given w, the next step of the proof will be to find a w_G and to "guess" u. In doing so, we shall proceed as if the Poisson kernel P_G would be given by (16).

2. The shape of the domain G is sketched in the following figure where we draw three complementary intervals of E and the part of ∂G over them (the same construction has been used in another context in [8]).



For $z \in T \backslash E$, we denote by z_G the point of ∂G over z. By the correspondence $z \to z_G$ the Lebesgue measures of T and ∂G map one into the other up to a constant. Since $\rho(x) \sim \rho(x_G)$ and $|z - x| \sim |z_G - x_G|$, this shows that the function w_G defined by

$$w_G(z_G) = w(z), \qquad z \in \partial G \backslash E$$

satisfies a relation like (15) on ∂G.

Now, for $x \in T\backslash E$ and $z_G \in \partial G\backslash E$,

$$P_G(x,z_G) \sim \frac{\rho(x,\partial G)}{|x - z_G|^2} \sim \frac{\rho(x)}{|x - z|^2 + \rho(z)^2}.$$

So, according to the planning of point 1, we should consider the function

$$u(x) \stackrel{\text{def}}{=} \rho(x) \int_T \frac{w(z)}{|x - z|^2 + \rho(z)^2}\, dz \sim \int_{\partial G} P_G(x,z_G)\, w_G(z_G)\, dz_G. \tag{17}$$

For further use, we estimate now the function u in terms of w. Note that $|x - z|^2 + \rho(z)^2 \sim \rho(x)^2$ for $z \in J_x$ (since $\rho(z) \sim \rho(x)$ and $|x - z| \leq \rho(x)/4$) and that $|x - z|^2 + \rho(z)^2 \sim |x - z|^2$ for $z \in T\backslash J_x$ (since $\rho(z) \leq \rho(x) + |x - z| \leq 5|x - z|$). Hence we get

$$u(x) \sim \frac{1}{\rho(x)} \int_{J_x} w + \rho(x) \int_{T\backslash J_x} \frac{w(z)}{|z - x|^2}\, dz, \qquad x \in T\backslash E. \tag{18}$$

Therefore, by (15),

$$u(x) \leq \text{const} \left(\frac{1}{\rho(x)} \int_{J_x} w + w(x) \right), \qquad x \in T\backslash E. \tag{19}$$

3. Now we will show that the relations (14) and (15) remain true for the function u:

$$\rho(x)^{-\alpha} \leq \text{const}\, u(x), \qquad x \in T\backslash E \tag{20}$$

$$\rho(x) \int_{T\backslash J_x} \frac{u(y)}{|y - x|^2}\, dy \leq \text{const}\, u(x), \qquad x \in T\backslash E. \tag{21}$$

To prove (20), note that $\rho(z) \leq |x - z|$ for $z \in T\backslash I_x$ and so, by (14) and just using the second term in (18),

$$u(x) \geq \rho(x) \int_{T\backslash I_x} |x - z|^{-\alpha-2}\, dz \geq \text{const}\, \rho(x)^{-\alpha}.$$

By (19), (21) will follow from

$$\rho(x) \int_{T\backslash J_x} \frac{dy}{|y - x|^2} \frac{1}{\rho(y)} \int_{J_y} w(z)\, dz \leq \text{const}\, u(x) \tag{22}$$

and

$$\rho(x) \int_{T\backslash J_x} \frac{w(y)}{|y - x|^2}\, dy \leq \text{const}\, u(x), \qquad x \in T\backslash E.$$

This last relation is implied by (18). In (22), $z \in J_y$ implies $|y - z| \leq \rho(z)/3$, $\rho(y) \sim \rho(x)$ and so, by Fubini's theorem, the left member of (22) is less than some constant times

$$\rho(x) \int_T \frac{w(z)}{\rho(z)} \left(\int_A \frac{dy}{|y-x|^2} \right) dz$$

where the set A is defined by the inequalities $|y - x| \geq \rho(x)/4$ and $|y - z| \leq \rho(z)/3$. If $\rho(z) \leq 2|z - x|$, $|y - z| \leq (2/3)|z - x|$ implies $|z - x| \sim |x - y|$ and so the interior integral $\sim \rho(z)|z - x|^{-2}$. If $\rho(z) \geq 2|z - x|$, one has $\rho(x) \sim \rho(z)$ and then the interior integral is bounded by some constant times $\rho(z)^{-1}$. So in any case,

$$\frac{1}{\rho(z)} \int_A \frac{dy}{|y-x|^2} \leq \text{const} \frac{1}{|z-x|^2 + \rho(z)^2}.$$

Hence the left member of (22) is bounded by

$$\text{const}\, \rho(x) \int_T \frac{w(z)}{|z-x|^2 + \rho(z)^2} dz = \text{const}\, u(x).$$

This proves (22) and hence (21).

4. Now we will see what is the gain after the change of w by u. By the formula (17) defining $u(x)$ we see that u is infinitely differentiable in $T \backslash E$. We are going to show that for $y = e^{it} \in J_x$ one has

$$u(y) \sim u(x) \tag{23}$$

$$\left| \frac{du(e^{it})}{dt} \right| \leq \text{const} \frac{u(x)}{\rho(x)} \tag{24}$$

$$\left| \frac{d^2u(e^{it})}{dt^2} \right| \leq \text{const} \frac{u(x)}{\rho(x)^2}. \tag{25}$$

To prove (23) it is enough to show that $|x - z|^2 + \rho(z)^2 \sim |y - z|^2 + \rho(z)^2$ for all z, $y \in J_x$, because $\rho(y) \sim \rho(x)$. If $|z - x| \leq \rho(x)/2$, then $\rho(z) \sim \rho(x)$ and $|z - y| \leq 3\rho(x)/4$ and so both quantities are $\sim\rho(x)^2$. If $|z - x| \geq \rho(x)/2$, then $|z - y| \geq \rho(x)/4$ and $\rho(z) \leq 3|z - x|$, which give that both terms are $\sim|z - x|^2$.

From the definition (17), we obtain ($y = e^{it}$)

$$\left| \frac{du(e^{it})}{dt} \right| \leq \text{const} \left(\int_T \frac{w(z)}{|y-z|^2 + \rho(z)^2} dz + \rho(y) \int_T \frac{w(z)|y-z|}{(|y-z|^2 + \rho(z)^2)^2} dz \right).$$

Taking into account that $\rho(y) \sim \rho(x)$ for $y \in J_x$ and (23), to get (24) it is enough to bound this by $u(y)\rho(y)^{-1}$. This is immediate for the first term on the right. To obtain the same estimate for the second one it is clearly sufficient to show

$$\frac{|z-y|}{|z-y|^2 + \rho(z)^2} \leq \frac{\text{const}}{\rho(y)}.$$

To see this, we distinguish two cases: if $|z - y| \geq \rho(y)/2$, we use that the left member is bounded by $|z - y|^{-1}$. In case $|z - y| \leq \rho(y)/2$, we use the bound

$\rho(y)\rho(z)^{-2}$ and the fact that $\rho(y) \sim \rho(z)$. Hence (24) is proved.

The proof of (25) is similar and is left to the reader.

5. At this point we are in position to finish the proof of the theorem by applying the argument from the proof of Theorem 2.3. We just sketch it. Put

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, u(e^{it})\,dt, \qquad z \in D.$$

The estimate (7) is obtained as before, by (20), and so it is enough to show the estimate (8). The case $\rho(z) \leq 4(1-|z|)$ is proved as in Theorem 2.3 using (5) in Lemma 2.1. In case $\rho(z) \geq 4(1-|z|)$, and using the same notations, we will prove

$$|g'(z)| \leq \text{const}\, \frac{u(x)}{\rho(x)} \tag{26}$$

$$\operatorname{Re} g(z) \geq \text{const}\, u(x) \tag{27}$$

which, with (20), give $|g'(z)|/|g(z)|^2 \leq \text{const}\ \rho(x)^{\alpha-1}$. Then $\rho(x) \sim \rho(z) \geq 1 - |z|$ shows that (8) holds.

To evaluate $|g'(z)|$, we again break the integral defining it into two parts, corresponding to J_x and $T\backslash J_x$. It follows from (12) and (21) that the contribution of $T\backslash J_x$ satisfies an estimate like (26). In the contribution of J_x we use Lemma 2.2, (11), (23), (24) and (25) to obtain the same result.

Now it remains to prove (27) for $\rho(z) \geq 4(1-|z|)$. If I is related to z, we have $I \subset J_x$ and so, by (23),

$$\operatorname{Re} g(z) \geq \int_I P_z(y)u(y)\,dy \sim I(u) \sim u(x).$$

Thus, the proof of Theorem 2.6 is completely finished. □

2.7 Theorem. *If* $\rho^{-\alpha} \in L^{1+\delta}(T)$ *for some* $\delta > 0$, *i.e., if*

$$|E| = 0, \qquad \sum_n \varepsilon_\eta^{1-\alpha-\delta} < +\infty, \qquad \delta > 0,$$

then E is a peak set for Lip α.

Proof. We recall first the well-known inequality

$$|I| \int_{T\backslash I} \frac{w(e^{it})}{|e^{it}-x_I|^2}\,dt \leq \text{const}\, w^*(x_I), \qquad I \subset T,\ w \in L^1(T).$$

This is proved by the usual doubling technique, estimating the contribution of $I_k\backslash I_{k-1}$ and adding on k, where $I_k = 2^k I$. Now, since $|J_x| \sim \rho(x)$ in (15), we see that the existence of a majorant w of $\rho^{-\alpha}$ satisfying Muckenhoupt's condition (A_1) is a sufficient condition. But for a given measurable function u on T, such majorant exists if and only if $u \in L^{1+\delta}(T)$ for some $\delta > 0$ [5, Theorem 9.1]. The theorem follows. □

2.8 Remark. The proof of Theorem 2.6 is also valid if (15) is replaced by

$$\rho(x)\int_{T\setminus J_x}\frac{w(e^{it})}{|e^{it}-x|^2}\,dt \le \text{const}\, J_x(w), \qquad x\in T\setminus E.$$

This leads to the following sufficient condition: there exists a majorant of $\rho^{-\alpha}$ which satisfies condition (B_2). We do not know of any characterization of the functions having such majorant. An answer to this question would surely improve the result of Theorem 2.7 (the author thanks Carlos Kenig for this observation).

III. Necessary conditions. In the first result of this section we use an argument similar to that of the important Lemma 4.3 in [7], to see that a function $f\in \text{Lip}\,\alpha$ vanishing on E satisfies a sort of equilibrium condition. We can suppose $\|f\|\le 1$ and define, for $x\in T\setminus E$,

$$a_f(x)=\int_{T\setminus I_x}\frac{-\log|f(e^{it})|}{|e^{it}-x|^2}\,dt.$$

We also consider the analogous function defined by log ρ (we can assume without loss of generality that $\rho\le 1$)

$$b(x)=\int_{T\setminus I_x}\frac{-\log \rho(e^{it})}{|e^{it}-x|^2}\,dt.$$

Since $|f|\le \rho^\alpha$ and $\log|f|\in L^1(T)$, we see that

$$\text{const}\, b(x)\le a_f(x)\le \text{const}\,\rho(x)^{-2}, \qquad x\in T\setminus E. \tag{28}$$

From the fact that $\rho(e^{it})\le |e^{it}-x|$ for $e^{it}\in T\setminus I_x$ it follows that

$$b(x)\ge \text{const}\,\frac{|\log\rho(x)|}{\rho(x)}. \tag{29}$$

We can assume as well that $a_f(x)$ and $b(x)$ are ≥ 1.

3.1 Theorem. *Suppose $f\in \text{Lip}\,\alpha$ vanishes on E (and so E is a Carleson set, i.e., $\log\rho\in L^1(T)$). Then, for $x\in T\setminus E$*

$$|f(x)|\le \text{const}\left(\frac{\log a_f(x)}{a_{f(x)}}\right)^\alpha. \tag{30}$$

In particular,

$$|f(x)|\le \text{const}\left(\frac{\log b(x)}{b(x)}\right)^\alpha \le \text{const}\left(\frac{|\log\rho(x)|}{b(x)}\right)^\alpha. \tag{31}$$

Proof. We put $z=rx$, $r<1$ and write

$$|f(x)|\le |f(z)|+(1-r)^\alpha.$$

Now, choosing $1 - r < \rho(x)$, we have $|e^{it} - z| \leq 2|e^{it} - x|$ for $e^{it} \in T \setminus I_x$, and so taking into account that $\log |f| \leq 0$

$$\log |f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{|e^{it} - z|^2} \log |f(e^{it})|$$
$$\leq \frac{1 - r}{8\pi} \int_{T \setminus I_x} \frac{\log |f(e^{it})|}{|e^{it} - x|^2}\, dt = -\frac{1 - r}{8\pi} a_f(x).$$

So we can write

$$|f(x)| \leq \inf_{0 < 1 - r < \rho(x)} \left\{ \exp\left(-\frac{1 - r}{8\pi} a_f(x) \right) + (1 - r)^{\alpha} \right\}. \tag{32}$$

If $\alpha |\log \rho(x)|/\rho(x) < a_f(x)/8\pi$, since $|\log t|/t$ is decreasing in $[0,1]$, there exists $t(x)$, $0 \leq t(x) \leq \rho(x)$, such that $\alpha |\log t(x)|/t(x) = a_f(x)/8\pi$, and $t(x) \sim (\log a_f(x))/a_f(x)$. Then, choosing $1 - r = t(x)$ in (32) we obtain (30).

In case $\alpha |\log \rho(x)|/\rho(x) \geq a_f(x)/8\pi$, one has $\rho(x) \leq \text{const} \log a_f(x)/a_f(x)$ and then (30) follows from the trivial estimate $|f| \leq \rho^{\alpha}$. (31) follows from (30) and (28). □

3.2 Theorem. *Suppose that E is a peak set for* Lip α. *Then* $(b(x)/|\log \rho(x)|)^{-\alpha} \in$ weak $L^1(T)$. *In particular,* $\rho^{-\alpha} \in$ weak $L^1(T)$.

Proof. We just observe that the boundary values of a holomorphic function in D with positive real parts are in weak $L^1(T)$ (by Kolmogorov's theorem), apply this to $1/f$ and use (31), (29). □

3.3 Remark. Of course, $\rho^{-\alpha} \in$ weak $L^1(T)$ can be obtained without Theorem 3.1, by using the trivial estimate $|f| \leq \text{const}\, \rho^{\alpha}$. We think that the assertion concerning $b(x)/|\log \rho(x)|$ could be used to disprove the conjecture (see the last section). On the other hand, from $\rho^{-\alpha} \in$ weak $L^1(T)$ one can obtain different necessary conditions, by consideration of the functions $\psi(\rho^{-\alpha})$ where $\psi : (1,\infty) \to (0,\infty)$ is such that $s^{-1}\psi'(s)$ is integrable (so that $\psi(\rho^{-\alpha}) \in L^1(T)$). The necessary condition (2) corresponds to $\psi(s) = s(\log s)^{-\delta}$.

IV. Strong peak sets, Gevrey classes and interpolation sets. In [4] it is proved that a closed set $E \subset T$ is an interpolation set for the analytic Gevrey class (see Section I) if and only if

$$I(\rho^{-\alpha}) \leq \text{const}\, |I|^{-\alpha} \qquad \text{for every} \quad I \subset T. \tag{33}$$

We note that this condition implies $\rho^{-\alpha} \in (A_2)$. For if $I \cap E \neq \emptyset$, then $I(\rho^{\alpha}) \leq |I|^{\alpha}$ and so $I(\rho^{-\alpha}) I(\rho^{\alpha}) \leq$ const. In the case $I \cap E = \emptyset$, $I(\rho^{\alpha}) \sim (\sup_I \rho)^{\alpha}$, $I(\rho^{-\alpha}) \sim (\sup_I \rho)^{-\alpha}$ and condition (A_2) follows again (this is essentially the fact that x^{α} is an A_2-weight). In fact, (33) is equivalent to $\rho^{\alpha} \in (A_2)$, $I(\rho^{\alpha}) \leq \text{const}\, |I|^{\alpha}$, and implies $\rho^{-\alpha} \in (A_1)$ [5, Proposition 9.1.1].

Therefore, by Corollary 2.4, E is a peak set for Lip α. We shall see, in fact, that they are characterized as peak sets of a special type:

We say that E is a *strong peak set* for Lip α if there exists $f \in$ Lip α such that $f = 0$ on E and

$$\text{(34)} \qquad \operatorname{Re} f(z) \geq \text{const } \rho(z)^{\alpha}, \qquad z \in \bar{D} \backslash E.$$

This amounts to saying, due to the estimate $|f(z)| \leq \text{const } \rho(z)^{\alpha}$, that f takes values in a sector of opening $<\pi$ and that $\operatorname{Re} f(z)$, $|f(z)|$ and $\rho(z)^{\alpha}$ are of the same order.

4.1 Theorem. *A closed set E is a strong peak set for* Lip α *if and only if (33) holds, i.e., if and only if it is an interpolation set for* G_{α}.

Proof. We begin with the necessity. Suppose that E is a strong peak set and let f be as before. By the Helson-Szego theorem (or also directly), every holomorphic function in D which takes values in a sector of opening $<\pi$ is an outer function with boundary values in (A_2). By the Poisson formula and condition (B_2), and if I is related to z, one has

$$|f(z)| \leq \text{const } I(|f|).$$

But since the same applies to f^{-1} and $|f| \in A_2$, we have that $|f(z)| \sim I(|f|)$ and $|f(z)|^{-1} \sim I(|f|^{-1})$. Hence,

$$I(\rho^{-\alpha}) \sim I(|f|^{-1}) \sim |f(z)|^{-1} \sim \rho(z)^{-\alpha} \leq (1 - |z|)^{-\alpha} = |I|^{-\alpha}$$

and (33) is proved.

To prove the sufficiency, suppose (33) holds, and consider the functions g and f constructed in Theorem 2.3. Since by (7) we already know that $\operatorname{Re} g(z) \geq \text{const } \rho(z)^{-\alpha}$; proving (34) is equivalent to proving

$$\text{(35)} \qquad |g(z)| \leq \text{const } \rho(z)^{-\alpha}, \qquad z \in D.$$

The function g differs from the Cauchy integral of $\tilde{\rho}^{-\alpha}$ only by constants, and hence we can assume in this paragraph that

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\tilde{\rho}^{-\alpha}(e^{it}) e^{it}}{e^{it} - z} \, dt.$$

Consider first the case $\rho(z) \leq 4(1 - |z|)$ and take I related to z. Then $|e^{it} - z| \sim 1 - |z| = |I|$ for $e^{it} \in I$ and $|e^{it} - z| \sim |e^{it} - x_I|$ for $e^{it} \notin I$. So

$$|g(z)| \leq \text{const} \left(I(\tilde{\rho}^{-\alpha}) + \int_{T \backslash I} \frac{\rho^{-\alpha}(e^{it})}{|e^{it} - x_I|} \, dt \right).$$

Now we apply the usual doubling technique: let $I_k = 2^k I$ for $k = 0, \ldots, N-1$ where N is the smallest integer such that $2^N |I| \geq 2\pi$. Let $I_N = T$. We have, since $|e^{it} - x_I| \geq \text{const } 2^k |I|$ outside I_{k-1}, that the last integral is less than some constant times

$$I(\tilde{\rho}^{-\alpha}) + \sum_{k=1}^{N} 2^{-k}|I| \int_{I_k} \tilde{\rho}^{-\alpha} = \sum_{k=0}^{N} I_k(\tilde{\rho}^{-\alpha}).$$

By (33), this is dominated by

$$\sum 2^{-k\alpha}|I|^{-\alpha} = \text{const}(1 - |z|)^{-\alpha} \leq \text{const}\, \rho(z)^{-\alpha},$$

which is (35).

In case $\rho(z) \geq 4(1 - |z|)$, we consider x and J_x as in the proof of Theorem 2.3 and apply the same technique as before, but this time with J_x instead of I. The contribution of $T\backslash J_x$ is evaluated exactly as before and we obtain a bound of type const $|J_x|^{-\alpha}$. By (10) and (11), this is of type (35). Now it remains to estimate the contribution of J_x

$$\left| \int_{J_x} \frac{\tilde{\rho}^{-\alpha}(e^{it})e^{it}}{(e^{it} - z)} dt \right|.$$

To do this we use Lemma 2.2. With the notation there, $M_0 \sim \rho(z)^{-\alpha}$, $M_1 \sim \rho(z)^{-\alpha-1}$ and $|J_x| \sim \rho(z)$. Thus (35) is again obtained. □

One more relation between these two classes of functions is furnished by the following observation. Conditions (14) and (15) imply

$$\int_{T\backslash I_x} \frac{w(e^{it})}{|e^{it} - x|^2} dt \leq \text{const}\, w(x)^{1+1/\alpha}, \qquad x \in T\backslash E. \tag{36}$$

This is called condition (α) in [5] and the existence of a majorant w of $\rho^{-\alpha}$ satisfying it turns out to be a necessary and sufficient condition for E to be a set of nonuniqueness for G_α, i.e. there exists $f \in G_\alpha$ vanishing on E together with all its derivatives. So, all the known peak sets for Lip α are nonuniqueness sets for G_α. Similarly, a computation shows that condition (6) in Theorem 2.3 is equivalent (only for $\rho^{-\alpha}$) to

$$\int_{T\backslash I_x} \frac{\rho^{-\alpha}(e^{it})}{|e^{it} - x|^2} dt \leq \text{const}\, \rho(x)^{-\alpha-1}, \qquad x \in T\backslash E$$

(that is, the contribution of $I_x\backslash J_x$ always satisfies this). The above condition is called condition $(\mathrm{PS})_\alpha$ in [5] and turns out to be necessary and sufficient for the existence of an $f \in G_\alpha$ such that $-\log|f(x)| \sim \rho(x)^{-\alpha}$.

To finish this section, we wish to comment on the interpolation sets for Lip α, that is, those sets $E \subset T$ such that for every $\varphi \in \mathrm{Lip}(\alpha,E)$ there exists $f \in$ Lip α such that $f = \varphi$ on E. These sets are characterized by the so-called condition (K) (see [3])

$$\text{``}\forall I \subset T,\ \sup_{x\in I} \rho(x) \geq \text{const}|I|\text{''}. \tag{K}$$

In particular, they do not depend on α and they are the interpolation sets for the classes A^p as well, with p an integer [1]. Also, it is known (see [1], [3]) that (K) holds if and only if (33) holds for some α. Hence every strong peak set for Lip α is an interpolation set for Lip α and every interpolation set for Lip α is a strong peak set for some Lip β, with a possibly different β.

In spite of this, it is not difficult to produce examples showing that there is no relation between the class of peak sets and that of interpolation sets. For instance, the set $E = \{\exp i/n, n \geq 1\} \cup \{1\}$ is a peak set for Lip α if $\alpha < 1/2$ (this follows from Corollary 2.7 because $\varepsilon_n \sim 1/n^2$), and does not satisfy (K), and so it is not interpolating. In the other direction, the Cantor perfect set (adjusted to T) satisfies (K) and a computation shows that it is not a peak set for Lip α if $\alpha > 1 - (\log 2/\log 3)$ (using that $\sum \varepsilon_n^{1-\alpha'} < \infty$ for $\alpha' < \alpha$ is a necessary condition).

V. Conclusions and questions. In this concluding section we comment on the difficulties of reaching a complete characterization of peak sets and we pose three questions.

There seems to exist some reason for thinking that conjecture (3) is not true in the sense that it is a *global* condition, not depending on the *disposition* of the complementary intervals of E. That some more subtle condition should exist is suggested by the relationship with exceptional sets for the Gevrey classes, characterized by nonglobal conditions. Incidentally, concerning nonuniqueness sets for G_α, it was also conjectured that condition (3) is necessary and sufficient. It was in the excellent paper [5] where the necessity of (36) was first shown (as well as its sufficiency). Also in this direction, conditions like (6) or (15) appear to be indispensable for the techniques developed here, and so we think that any attempt to work with global conditions should use other constructions.

The important point, of course, is to find some necessary condition of a nonglobal character. The possibility of this seems to be related to the following question: we know that if $u \in L^\infty_{\mathbf{R}}$ and $\|u\|_\infty < \pi/2$, then $\exp \tilde{u}$ is a weight in (A_2). If $\|u\|_\infty \leq \pi/2$, when can it be said that $\exp \tilde{u}$ still satisfies a kind of equilibrium condition, or is it even integrable? (In general it is clearly not true.) In our opinion, the answer to this question should determine whether conjecture (3) is appropriate or not.

In conclusion we mention four questions:

1. Concerning the necessary condition found in Theorem 3.2, it is easy to see that for every $\delta > 0$

$$\text{const}\, \rho(x)^{-\alpha} \leq \left(\frac{b(x)}{|\log \rho(x)|}\right)^{-\alpha} \leq \text{const}\, \frac{\rho(x)^{-\alpha-\delta}}{|\log \rho(x)|^{\alpha}}.$$

Here we see that, of course, $\rho^{-\alpha} \in L^{1+\delta}(T)$ implies $(b/|\log \rho|)^{-\alpha} \in L^1(T)$ but not that $\rho^{-\alpha} \in L^1(T)$. Is it possible to produce an example so that $\rho^{-\alpha} \in L^1(T)$ but $(b/|\log \rho|)^{-\alpha}$ is not in weak $L^1(T)$? This would prove that the conjecture is false.

2. Does there exist a direct relation for the good understanding of the equality between strong peak sets for Lip α and interpolation sets for G_α?

3. As was said before, every strong peak set is a peak interpolation set. Is the converse true?

4. After this paper was written, S. V. Hruscev brought to my attention that in the Zapicki Nauchnykh Seminarov LOMI, **81** 249-251, E. M. Dyn'kin proves in fact using results from [5] that every nonuniqueness set for G_α is a peak set for Lip α. It seems reasonable to conjecture that the converse is also true.

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