Lipschitz Approximation by Harmonic Functions and Some Applications to Spectral Synthesis

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ABSTRACT. For $0 < s \leq 1$, we characterize those compact sets X with the property that each function harmonic in $\overset{\circ}{X}$ and satisfying a little **o** Lipschitz condition of order s is the limit in the Lipschitz norm of order s of functions harmonic on neighbourhoods of X. As an application of the methods we give a spectral synthesis result in the space of locally integrable functions whose laplacian belongs to $B^p(\mathbb{R}^d)$, the containing Banach space of the Hardy space $H^p(\mathbb{R}^d)$.

0. Introduction. Let X be a compact subset of $\mathbb{R}^d$ and let $\Lambda^s(X)$, $0 < s < 1$, be the usual Banach space of Lipschitz functions of order s on X. That is, $f \in \Lambda^s(X)$ if and only if

$$\|f\|_s = \sup\{|f(x) - f(y)|\,|x-y|^{-s} : x,\, y \in X,\, x \neq y\} < \infty.$$

If $\|\ \ \|_\infty$ is the supremum norm on X, $\|f\|_\infty + \|f\|_s$ is a Banach space norm on $\Lambda^s(X)$. An important role in what follows will be played by the space $\lambda^s(X)$ which is the set of functions f in $\Lambda^s(X)$ satisfying $|f(x) - f(y)| = o(|x-y|^s)$ as $|x-y| \to 0$. Alternatively $\lambda^s(X)$ can be described as the closure in $\Lambda^s(X)$ of $C^\infty(\mathbb{R}^d)_{|X}$.

We are interested in the problem of understanding the space $H^s(X)$, the closure in $\Lambda^s(X)$ of $\{f_{|X} : f$ is harmonic on some neighbourhood of $X\}$. There are two obvious necessary conditions for $f \in H^s(X)$: f must be harmonic on $\overset{\circ}{X}$, and f must belong to $\lambda^s(X)$. If we set

$$h^s(X) = \lambda^s(X) \cap \{f : \Delta f = 0 \text{ on } \overset{\circ}{X}\}$$

we then have $H^s(X) \subset h^s(X)$ and it turns out that the above inclusion can be strict. Thus the problem arises of characterizing those X for which $H^s(X) =$

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$h^s(X)$. A complete solution to the above problem is provided by our main result. We denote by M^α and M^α_* the α-dimensional and lower α-dimensional Hausdorff contents respectively (see Section 1 for precise definitions).

Theorem 1. *Let $X \subset \mathbb{R}^d$ be compact. Then the following are equivalent.*

(1) $H^s(X) = h^s(X)$.

(2) $M_*^{d-2+s}(B \setminus \mathring{X}) \leq C\, M^{d-2+s}(B \setminus X)$ *for all open balls B and some positive constant C.*

(3) $\limsup_{r\to 0} \dfrac{M^{d-2+s}(B(x,r)\setminus X)}{r^{d-2+s}} > 0$, M_*^{d-2+s}*-a.e. on ∂X.*

For $1 < s < 2$ the equivalence between (1) and (2) was proved in [22], in which one can also find a more general result including O'Farrell's Theorem on Lipschitz analytic approximation [15] and a complete discussion of the case $s \geq 2$. A slight variant of the techniques used there takes care of the the Zygmund class case ($s = 1$, see Section 4 below). However, they are not powerful enough to cover the case $0 < s < 1$. The main contribution of this paper is to show how one can exploit the new ideas presented in [13] to prove Theorem 1. For $s = 1$ it is also natural to replace the Zygmund class by C^1. The problem one gets is not covered by our methods because C^1 is not invariant by singular integrals. A complete solution of the C^1 problem has been found very recently by Paramanov [18].

As in [13] our technique is a combination of constructive and duality arguments. The constructive part uses Vitushkin's localization and matching coefficients method, while the duality argument involves a differentiation theorem for Riesz potentials of B^p distributions and some geometric measure theory. The space B^p, $s = d(\frac{1}{p} - 1)$, is the dual of $\lambda_0^s(\mathbb{R}^d)$, the subspace of $\lambda^s(\mathbb{R}^d)$ consisting of functions vanishing at ∞. It can also be described either as a space of distributions with some specific atomic descomposition or as the containing Banach space for $H^p(\mathbb{R}^d)$, the usual Fefferman-Stein Hardy space (see Section 1 for more details about these facts). The equivalence of (2) and (3) is a geometric result independent of the approximation problem. However, we don't know if there exists a proof of (3) $\Longrightarrow$ (2) which uses only geometric measure theory. We will give a direct proof of (3) $\Longrightarrow$ (2) (without going through (1)) which involves the solution of a very particular approximation problem.

It is worthwhile mentioning that combining the machinery developed in [22] with ours one can prove a version of Theorem 1 which applies to homogeneous elliptic operators with constant coefficients. If the operator is L and its order is r one is then approximating solutions of the equation $Lf = 0$ in $\Lambda^s(X)$, $r-2 < s < r-1$, and the characterizing conditions involve $(d-r+s)$-dimensional Hausdorff contents. New ideas seem to be required to obtain a complete solution for the case $0 < s \leq r-2$.

As an application of the ideas in the proof of Theorem 1 we obtain some results in spectral synthesis in the spirit of the Hedberg-Wolff theorem for the Sobolev spaces W^p_k, $1 < p < \infty$, $0 < k \in \mathbb{Z}$, [9]. For $k = 1$ this theorem follows readily from the truncation properties of W^p_1 (see [2]), but for $k > 1$ it is a deep achievement, both from the technical and conceptual points of view.

Our spectral synthesis results arise when we consider the problem of extending the Hedberg-Wolff theorem to indexes p with $0 < p \leq 1$. As it is well known in harmonic analysis, the space L^p, $0 < p \leq 1$, must often be replaced by H^p to get positive results. Since our approach to spectral synthesis is based on duality arguments, unlike in the classical Sobolev space setting, H^p is not an adequate substitute for L^p, $0 < p < 1$. This is so because H^p is not a Banach space in this case, and thus its duality theory is not satisfactory. It turns out that the family of spaces B^p, $0 < p < 1$, enjoys good duality properties and still is a natural extension of the L^p spaces to indexes p less than one.

The Sobolev type space to be considered in this context would be obtained by requiring that all derivatives up to some fixed order belong to B^p, but it will be more convenient to work with the essentially equivalent potential spaces

$$I_\alpha B^p = \{f : \ f = I_\alpha * g, \ g \in B^p\},$$

$0 < \alpha \leq d$, where $I_\alpha(x) = |x|^{\alpha-d}$ ($= \log|x|$ if $\alpha = d$). A Banach space norm on $I_\alpha B^p$ is $\|f\| = \|g\|_{B^p}$, $f = I_\alpha * g$, $g \in B^p$. By the Fractional Integration Theorem, distributions in $I_\alpha B^p$ are L^q functions, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, with p in the range $\frac{d}{d+\alpha} < p < 1$. They are shown to satisfy the inequality

$$M^\beta\big(\{x : \ Mf(x) > \lambda\}\big) \leq C\lambda^{-1}\|f\|, \qquad f \in I_\alpha B^p,$$

where Mf is the Hardy-Littlewood maximal function and $\beta = d\big(\frac{1}{p} - \frac{\alpha}{d}\big)$. As a consequence, a function f in $I_\alpha B^p$ can be strictly defined M^β-almost everywhere, in the sense that

$$f(x) \equiv \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

exists for M^β-almost all $x \in \mathbb{R}^d$. When so defined f turns out to be M^β-quasicontinuous, which means that given any $\varepsilon > 0$ there is an open set G with $M^\beta(G) < \varepsilon$ such that the restriction of f to $\mathbb{R}^d \setminus G$ is continuous there.

The spectral synthesis theorem referred to above reads as follows.

Theorem 2. *Let $F \subset \mathbb{R}^d$ be closed and let $f \in I_\alpha B^p$, $\frac{d}{d+\alpha} < p < \min\left(1, \frac{d}{d+\alpha-2}\right)$, be M^β-quasicontinuous, $\beta = d\left(\frac{1}{p} - \frac{\alpha}{d}\right)$. Then the following are equivalent.*

(i) *$f = 0$ on F, M^β-almost everywhere.*
(ii) *There exists a sequence (φ_n), $\varphi_n \in C_0^\infty(F^c)$, such that $\varphi_n \to f$ in $I_\alpha B^p$.*

Some comments on the restriction on p in the above statement are in order. As we said before, the condition $\frac{d}{d+\alpha} < p$ tells us that distributions in $I_\alpha B^p$ are functions, and thus we can formulate the spectral synthesis problem in the classical way. The upper bound on p is due to the fact that, in the setting of Theorem 1, our technique for the approximation of Λ^s functions applies only when $r - 2 < s$, where r is the order of the operator under consideration.

A remarkable fact about Theorem 2 is that, unlike the previously known similar results [9], [13], no assumption is needed on the vanishing of ∇f on F even in the case $\alpha > 1$. It turns out that ∇f vanishes on F in the appropiate sense whenever (i) holds and $\alpha > 1$. To understand this phenomenon and the role of the condition $p < 1$ consider the case $\alpha = 2$. If $f \in I_2 B^p$ is good enough then $\{f = 0 \text{ and } \nabla f \neq 0\}$ is a set of dimension $d - 1$, but the gradient of f need be considered only on sets of dimension $d\left(\frac{1}{p} - \frac{1}{d}\right)$ (because $\nabla f \in I_1 B^p$), which is larger than $d - 1$.

We point out that nothing is known so far for the potential spaces $I_\alpha H^p$, $0 < p < 1$, except for $0 < \alpha < 1$ (one can then use truncation). For $p = 1$ see [13], [17].

In Section 1 we collect some background information and establish some notation. The constructive part of the proof of Theorem 1 can be found in Section 2, and its completion in Section 3. In Section 4 we briefly discuss Theorem 1 in the limiting case $s = 1$ ($s = 0$ was dealt with in [13]). Finally, Section 5 contains the spectral synthesis results.

1. Definitions and basic results.

1.1 Hausdorff content. A measure function is a non-decreasing function $h(t)$, $t \geq 0$, such that $\lim_{t \to 0} h(t) = 0$. If h is a measure function and $F \subset \mathbb{R}^d$ we set

$$M^h(F) = \inf \sum_j h(\delta_j),$$

where the infimum is taken over all countable coverings of F by cubes with sides of length δ_j and parallel to the coordinate axes. When $h(t) = t^\alpha$, $\alpha > 0$, $M^h(F) = M^\alpha(F)$ is called the α-dimensional Hausdorff content of F. The lower α-dimensional Hausdorff content of F is defined by

$$M_*^\alpha(F) = \sup M^h(F),$$

the supremum being taken over all measure functions h which satisfy $h(t) \leq t^\alpha$ and $\lim_{t\to 0} h(t)t^{-\alpha} = 0$.

One has $M_*^\alpha \leq M^\alpha$ but it can happen that $M_*^\alpha(F) = 0 < M^\alpha(F)$. For instance, if F is the segment $[0,1]$ in the plane, then $M_*^1(F) = 0$ but $M^1(F) = 1$. An old result of Sion and Sjerve [19] in geometric measure theory asserts that $M_*^\alpha(F) = 0$ if and only if F is a countable union of sets with finite α-dimensional Hausdorff measure. For a ball B, $M_*^\alpha(B) = M^\alpha(B)$ and for open sets U, $M_*^\alpha(U) \simeq M^\alpha(U)$.

1.2. In [1] it is shown that

$$M^{d-s}(\{x \in \mathbb{R}^d :\ Mf(x) > \lambda\}) \leq C\lambda^{-1}\|h\|_{H^1}, \tag{1.1}$$

where $f = I_s * h$, $0 < s \leq d$, $h \in H^1(\mathbb{R}^d)$, and M is the Hardy-Littlewood maximal operator. It follows readily that any $f \in I_s H^1$ has a (essentially unique) M^{d-s}-quasicontinuous representative.

1.3. Lipschitz spaces. Given $0 < s < 1$ the space $\Lambda^s(\mathbb{R}^d)$ consists of those functions f such that

$$\|f\|_s = \sup\{\omega_s(f,\delta)\ :\ \delta > 0\} < \infty,$$

where $\omega_s(f,\delta) = \sup\{|f(x) - f(y)|\ |x-y|^{-s} :\ |x-y| \leq \delta\}$.

Let $s > 1$ be non-integer. The space $\Lambda^s(\mathbb{R}^d)$ consists of those functions f in $C^{[s]}(\mathbb{R}^d)$ such that $\partial^\beta f \in \Lambda^{s-[s]}(\mathbb{R}^d)$ for $|\beta| = [s]$.

The space $\Lambda^1(\mathbb{R}^d)$, the Zygmund class, consists those continuous functions such that

$$\|f\|_1 = \sup\{\omega_1(f,\delta)\ :\ \delta > 0\} < \infty,$$

where $\omega_1(f,\delta) = \sup\{|f(x+h) + f(x-h) - 2f(x)|\ |h|^{-1} :\ h \in \mathbb{R}^d\}$.

If $s > 1$ is an integer, we define $\Lambda^s(\mathbb{R}^d)$ as the functions in $C^{s-1}(\mathbb{R}^d)$ such that $\partial^\beta f \in \Lambda^1(\mathbb{R}^d)$ for all $|\beta| = s-1$.

A function $f \in \lambda^s(\mathbb{R}^d)$ if and only if $f \in C^{s-r}(\mathbb{R}^d)$ and $\omega_r(\nabla^{s-r} f, \delta) \to 0$ as $\delta \to 0$, where $r = 1$ if $s \in \mathbb{N}$ and $r = s - [s]$ otherwise.

To each compact subset X of $\mathbb{R}^d$ one associates the restriction spaces

$$\Lambda^s(X) = \Lambda^s(\mathbb{R}^d)_{|X} \simeq \Lambda^s(\mathbb{R}^d)/I_s(X)$$
$$\lambda^s(X) = \lambda^s(\mathbb{R}^d)_{|X} \simeq \lambda^s(\mathbb{R}^d)/J_s(X),$$

where

$$I_s(X) = \{f \in \Lambda^s(\mathbb{R}^d) :\ f = 0 \text{ on } X\}$$
$$J_s(X) = \{f \in \lambda^s(\mathbb{R}^d) :\ f = 0 \text{ on } X\}.$$

We refer the reader to [11, III, 1 and 2] for intrinsic descriptions of the functions in $\Lambda^s(X)$, involving the Whitney Extension Theorem.

1.4 The containing Banach space of H^p. Let $H^p(\mathbb{R}^d)$ be the usual Fefferman-Stein Hardy space, $0 < p \leq 1$. One can characterize H^p as the space of distributions of the form

$$f = \sum_{j=0}^{\infty} \lambda_j a_j, \qquad \sum_j |\lambda_j|^p < \infty, \tag{1.2}$$

where the a_j are p-atoms. Recall that one says that a is a p-atom if:

(i) spt $a \subset B$ where B is a ball,
(ii) $\|a\|_\infty \leq |B|^{-1/p}$ and
(iii) $\int a(x)x^\beta\,dx = 0$ for $|\beta| \leq [d(\frac{1}{p} - 1)]$, where the brackets mean integer part.

The quantity

$$\|f\|_{H^p} = \inf(\sum_j |\lambda_j|^p)^{1/p},$$

the infimum being taken over all possible expressions (1.2), is a quasi-norm for $0 < p < 1$ and a norm only for $p = 1$.

When $0 < p < 1$ H^p is not a normed space. However, it is known that the dual of $H^p(\mathbb{R}^d)$ is $\Lambda^s(\mathbb{R}^d)$, $s = d(\frac{1}{p} - 1)$. Since Λ^s separates elements of H^p, we may regard H^p as being embedded in its bidual. We define B^p as the weak-$*$ closure of H^p in $(H^p)^{**}$. It is a consequence of the following general result that H^p and B^p have the same dual.

Lemma. *If X is a Banach space and $T : H^p \to X$ a continuous linear map, then T extends continuously from B^p to X.*

Proof. We have to show that $\|Tf\|_X \leq C\|f\|_{B^p}$ for all $f \in H^p$. We know that $\|Tf\|_X = \sup\{|\varphi(Tf)| : \|\varphi\|_{X^*} \leq 1\}$. Since $\psi = (\varphi \circ T) \in (H^p)^*$ we have

$$\begin{aligned}\|Tf\|_X &\leq \sup\{|\psi(f)| : \psi \in (H^p)^* \text{ and } \|\psi\| \leq \|T\|\} \\ &\leq \|T\|\ \|f\|_{B^p}.\end{aligned}$$

□

By the Fractional Integration Theorem (e.g. [12]) and the above lemma the Riesz potential of order s, I_s, maps B^p in H^1, when $s = d(\frac{1}{p} - 1)$. Therefore, inequality (1.1) tell us that functions in $I_\alpha B^p$ are $M^{(d/p)-\alpha} = M^{d-\alpha+s}$-quasicontinuous.

We also consider B^p_a (atomic B^p), the space of distributions of the form $f = \sum_j \lambda_j a_j$ where a_j are p-atoms and $\sum_j |\lambda_j| < \infty$. The norm of f in B^p_a is defined as the infimum of all sums $\sum_j |\lambda_j|$.

It is clear that $H^p \subset B^p_a \subset B^p$ continuously because $\|a\|_{B^p} \leq \|a\|_{H^p} \leq C$, where C is a constant not depending on a. Using the lemma and the fact that B^p_a is a Banach space we get $B^p_a = B^p$.

Let $\lambda^s_0(\mathbb{R}^d)$ be the closure in $\Lambda^s(\mathbb{R}^d)$ of $C^\infty_0(\mathbb{R}^d)$, the space of indefinitely differentiable functions with compact support. Using the arguments of [4, p.638-642] one can prove the following.

Theorem. *B^p is the dual of $\lambda^s_0(\mathbb{R}^d)$ $(s = d(\frac{1}{p} - 1))$. More precisely, given any continuous linear functional φ on $\lambda^s_0(\mathbb{R}^d)$ one can find $f \in B^p$ such that*

$$\varphi(v) = \langle f, v\rangle, \qquad v \in C^\infty_0(\mathbb{R}^d), \tag{1.3}$$

where the bracket means the action of the distribution f on the test function v. Moreover, $\|f\|_{B^p}$ is equivalent to the linear functional norm. Conversely, if $f \in B^p$ and φ is defined by (1.3) then φ extends to a linear continuous functional on $\lambda^s_0(\mathbb{R}^d)$.

We refer the reader to [5], [6] for an exhaustive information about H^p.

1.5. A family (E_j) of subsets of $\mathbb{R}^d$ is said to be almost disjoint with constant N whenever each $x \in \mathbb{R}^d$ belongs to at most N sets E_j.

The letter C will denote a constant, which may be different at each occurence and which is independent of the relevant variables under consideration.

2. Approximation by potentials of measures. Our first lemma is a generalization in $\mathbb{R}^d$ of [13, 1.1].

2.1 Lemma. *Let (B_j) be a finite family of open balls such that for some $\lambda > 1$, (λB_j) is almost disjoint with constant N. Let $h_j \in \Lambda^s(\mathbb{R}^d)$, $0 < s < 1$, be harmonic outside a compact subset of B_j and assume that*

$$h_j(x) = O(|x|^{-d}) \qquad \text{as } x \to \infty.$$

Then

$$\|\sum_j h_j\|_s \leq C \max_j \|h_j\|_s$$

for some positive constant $C = C(\lambda, d, N)$.

Proof. We start by remarking that the lemma is obviously true when each h_j is supported in λB_j, because the family (λB_j) is almost disjoint. We will now show how the general case can be reduced to this particular case. The key fact in the argument below is the invariance of Lipschitz spaces under Calderón-Zygmund operators.

The main idea is to write each h_j in the form

$$h_j = \mathcal{X}_j + b_j, \tag{2.1}$$

with spt $b_j \subset \lambda B_j$, $\|b_j\|_s \leq C\|h_j\|_s$ and

$$\mathcal{X}_j = \sum_{|\beta|=2} \partial^\beta E * H^j_\beta,$$

where E is the fundamental solution for the laplacian and spt $H^j_\beta \subset \lambda B_j$, $\|H^j_\beta\|_s \leq C\|h_j\|_s$. Since $\partial^\beta E$ is the kernel of a Calderón-Zygmund operator [20, II], we then have

$$\begin{aligned}
\Big\|\sum_j h_j\Big\|_s &\leq \sum_{|\beta|=2} \Big\|\partial^\beta E * \sum_j H^j_\beta\Big\|_s + \Big\|\sum_j b_j\Big\|_s \\
&\leq C \sum_{|\beta|=2} \Big\|\sum_j H^j_\beta\Big\|_s + \Big\|\sum_j b_j\Big\|_s \\
&\leq C \sum_{|\beta|=2} \max_j \|H^j_\beta\|_s + \max_j \|b_j\|_s \\
&\leq C \max \|h_j\|_s,
\end{aligned}$$

where in the next to the last inequality we applied the initial remark.

Let's now turn to the proof of (2.1). Fix j and set $h = h_j$ and $B = B_j$ to simplify notation. Assume that B is a ball centered at the origin with radius δ. The expansion of h at infinity is

$$h(x) = \sum_{|\alpha|\geq 2} C^\alpha \partial^\alpha E(x) \qquad \text{if } |x| \geq \delta,$$

where $C^\alpha = \big((-1)^{|\alpha|}/\alpha!\big)\langle \Delta h, y^\alpha\rangle$.

Consider λ_1 and λ_2 such that $1 < \lambda_1 < \lambda_2 < \lambda$. Let $\psi \in C_0^\infty(\lambda_1 B)$ such that $\psi = 1$ on B and $|\partial^\alpha \psi| \leq C(\lambda_1\delta)^{-|\alpha|}$, $0 \leq |\alpha| \leq 2$. Then

$$\begin{aligned}
|\alpha! C^\alpha| &= |\langle \Delta h, \psi y^\alpha\rangle| = |\langle \Delta(h - h(0)), \psi y^\alpha\rangle| \\
&\leq \int_{\lambda_1 B} |h(y) - h(0)|\, |\Delta(\psi(y) y^\alpha)|\, dy,
\end{aligned}$$

and since

$$\max_{|y|\leq 1} |y^\alpha| = \left(\frac{\alpha_1^{\alpha_1}\cdots\alpha_d^{\alpha_d}}{|\alpha|^{|\alpha|}}\right)^{1/2},$$

where we have set $\alpha_i^{\alpha_i} = 1$ if $\alpha_i = 0$, one obtains

$$|\alpha! C^\alpha| \leq C\|h\|_s |\alpha|^2 (\lambda_1\delta)^{d+|\alpha|-2+s} \left(\frac{\alpha_1^{\alpha_1}\cdots\alpha_d^{\alpha_d}}{|\alpha|^{|\alpha|}}\right)^{1/2}, \tag{2.2}$$

where C is a constant that depends only on d.

Take $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\varphi = 0$ on $\lambda_2 B$, $\varphi = 1$ on $(\lambda B)^c$ and $|\partial^\alpha\varphi| \leq C(\lambda\delta)^{-|\alpha|}$, $0 \leq |\alpha| \leq 3$.

(2.3) Given $\alpha = (\alpha_1, ..., \alpha_d)$, $|\alpha| \geq 2$, choose any two multiindexes $\gamma = \gamma(\alpha)$ and $\beta = \beta(\alpha)$ such that $\alpha = \gamma + \beta$ and $|\beta| = 2$.

It is clear that

$$\varphi(x)\partial^\gamma E(x) = \int E(x-y)\Delta\big(\varphi(y)\partial^\gamma E(y)\big)\, dy,$$

and so

$$\partial^\gamma E(x) = \int E(x-y)\Delta\big(\varphi(y)\partial^\gamma E(y)\big)\, dy \qquad \text{on } (\lambda B)^c.$$

Differentiating this identity, we obtain

$$\partial^\alpha E(x) = \partial^\beta\partial^\gamma E(x) = \int \partial^\beta E(x-y)\Delta\big(\varphi(y)\partial^\gamma E(y)\big)\, dy, \ x \in (\lambda B)^c,$$

and consequently on $(\lambda B)^c$,

$$\begin{aligned} C^\alpha\partial^\alpha E &= \partial^\beta E * C^\alpha \Delta(\varphi\partial^\gamma E) \\ &= \partial^\beta E * C^\alpha G^\gamma, \end{aligned}$$

where the last identity is a definition of G^γ.

We want an estimate for $\|G^\gamma\|_s$. Since

$$\Delta(\varphi\partial^\gamma E) = \Delta\varphi\partial^\gamma E + 2\sum_{i=1}^d \partial^i\varphi\partial^i\partial^\gamma E$$

we get

$$\begin{aligned}\|\nabla G^\gamma\|_\infty &\le C(\delta^{-3}\max_{|y|=\lambda_2\delta}|\partial^\gamma E| + \delta^{-2}\max_{|y|=\lambda_2\delta}|\nabla\partial^\gamma E| + +\delta^{-1}\max_{|y|=\lambda_2\delta}|\nabla^2\partial^\gamma E|)\\ &= C\,(\lambda_2\delta)^{-d-|\gamma|-1}(A_\gamma + B_\gamma + D_\gamma),\end{aligned}$$

where

$$\begin{aligned}A_\gamma &= \sup\{|\partial^\gamma E(x)| : |x| = 1\}\\ B_\gamma &= \sup\{|\nabla\partial^\gamma E(x)| : |x| = 1\}\\ D_\gamma &= \sup\{|\nabla^2\partial^\gamma E(x)| : |x| = 1\}.\end{aligned}$$

To control the s norm of G^γ it is enough to estimate first differences for couples of points $x, y \in \lambda B$, because the support of G^γ is in λB. For such x and y we have

$$\begin{aligned}(2.4)\qquad |G^\gamma(x) - G^\gamma(y)|\,|x-y|^{-s} &\le C\|\nabla G^\gamma\|_\infty \delta^{1-s}\\ &\le C(\lambda_2\delta)^{-d-|\gamma|-1}\delta^{1-s}(A_\gamma + B_\gamma + D_\gamma).\end{aligned}$$

Now we need upper bounds for A_γ, B_γ and D_γ. The fundamental solution of the laplacian E is a real analytic function on $\mathbb{R}^d \setminus \{0\}$. Fix $y \in \mathbb{R}^d$, $|y| = 1$, so that $E(x-y) = \sum_{|\alpha|\ge 0} a_\alpha (x-y)^\alpha$ for $|x-y| < 1$, where $a_\alpha = (\alpha!)^{-1}\partial^\alpha E(y)$, and the convergence of the above series is uniform on $|x-y| < k^{-1}$, for any fixed $k > 1$. Given a multiindex γ consider the polydisc $\{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_i - y_i| < k^{-1}\sqrt{\gamma_i|\gamma|^{-1}}\}$. Since $E(z-y) = \sum a_\alpha (z-y)^\alpha$ is holomorphic in this polydisc, Cauchy inequalities give the estimate

$$|\partial^\gamma E(y)| \le C(k) k^{|\gamma|}\gamma!\left(\frac{|\gamma|^{|\gamma|}}{\gamma_1^{\gamma_1}\cdots\gamma_d^{\gamma_d}}\right)^{1/2}, \qquad |y| = 1,$$

where $C(k)$ is a constant depending only on k. Therefore

$$(2.5)\qquad \begin{aligned}A_\gamma &\le C(k)k^{|\gamma|}\gamma!\left(\frac{|\gamma|^{|\gamma|}}{\gamma_1^{\gamma_1}\cdots\gamma_d^{\gamma_d}}\right)^{1/2}\\ B_\gamma &\le C(k)k^{|\gamma|}\gamma!(|\gamma|+1)\left(\frac{(|\gamma|+1)^{|\gamma|+1}}{\gamma_1^{\gamma_1}\cdots\gamma_d^{\gamma_d}}\right)^{1/2}\\ D_\gamma &\le C(k)k^{|\gamma|}\gamma!(|\gamma|+1)^2\left(\frac{(|\gamma|+2)^{|\gamma|+2}}{\gamma_1^{\gamma_1}\cdots\gamma_d^{\gamma_d}}\right)^{1/2}.\end{aligned}$$

Combining (2.2), (2.4) and (2.5), we get

$$\|C^\alpha G^\gamma\|_s \le C(k)\left(\frac{\lambda_1 k}{\lambda_2}\right)^{d-2+|\alpha|}|\alpha|^{1/2}\|h\|_s. \tag{2.6}$$

On the other hand, given $|\beta| = 2$ we define

$$I(\beta) = \{\alpha : \alpha = \beta + \gamma \quad \text{in the representation (2.3)}\}.$$

Set

$$H_\beta = \sum_{\alpha\in I(\beta)} C^\alpha G^\beta \quad \text{and} \quad \mathcal{X} = \sum_{|\beta|=2} \partial^\beta E * H_\beta.$$

Hence spt $H_\beta \subset \lambda B$ and

$$\begin{aligned}\|H_\beta\|_s &\le \sum_{|\alpha|\ge 2}\|C^\alpha G^\gamma\|_s \le C(k)\|h\|_s \sum_{|\alpha|\ge 2}|\alpha|^{1/2}\left(\frac{\lambda_1 k}{\lambda_2}\right)^{d-2+|\alpha|}\\ &\le C(k)\|h\|_s\sum_{m=2}^{\infty} m^{1/2}\left(\frac{\lambda_1 k}{\lambda_2}\right)^m\binom{m+d-1}{m}\\ &\le C\|h\|_s,\end{aligned}$$

where the second inequality follows from (2.6) and in the last we choosed $k > 1$ such that $\lambda_1 k < \lambda_2$.

Finally, set $b = h - \mathcal{X}$. Then $\|b\|_s \le C\|h\|_s$ and spt $b \subset \lambda B$, because $\mathcal{X}$ was defined so that $h = \mathcal{X}$ on $(\lambda B)^c$. □

The next lemma is a variant of 2.1, in which we require less decay of h_j at the infinity but we have a packing condition on the family of balls B_j.

2.2 Lemma. *Let $\omega(t)$, $t > 0$, a non-decreasing function satisfying $\omega(2t) \le C\omega(t)$, $t > 0$. Let (B_j) be a finite family of open balls of radii δ_j satisfying*

(i) *(λB_j) is an almost disjoint family for some $\lambda > 1$,*
(ii) *For any ball B of radius δ,*

$$\sum_{B_j\subset B}\delta_j^{d-1+s}\omega(\delta_j) \le C\delta^{d-1+s}\omega(\delta).$$

Let $h_j \in \Lambda^s(\mathbb{R}^d)$ be harmonic outside a compact subset of B_j, $\|h_j\|_s \le \omega(\delta_j)$ and $h_j(x) = O(|x|^{1-d})$ as $x \to \infty$.

Then

$$\|\sum_j h_j\|_s \le C\omega(L),$$

L being the diameter of $\cup B_j$.

Proof. We will perform a reduction to 2.1. Write the expansion of h_j at ∞

$$h_j(x) = \sum_{i=1}^{d} C_j^i \partial^i E(x) + O(|x|^{-d}),$$

and notice that by (2.2)

$$|C_j^i| \leq C\delta_j^{d-1+s} \|h_j\|_s \leq C\delta_j^{d-1+s}\omega(\delta_j).$$

Set

$$\mu_j^i = C_j^i |\frac{1}{2}B_j|^{-1}\chi_{(1/2)B_j}$$

and

$$P_j = \sum_{i=1}^{d} \mu_j^i * \partial^i E,$$

so that

$$h_j - P_j = O(|x|^{-d}).$$

Since

$$\|\sum_j h_j\|_s \leq \|\sum_j h_j - P_j\|_s + \|\sum_j P_j\|_s$$

by 2.1 it is clearly enough to show that

$$\|P_j\|_s \leq C\|h_j\|_s \tag{2.7}$$

$$\|\sum_j P_j\|_s \leq C\omega(L). \tag{2.8}$$

Now, (2.7) is a consequence of the well known inequality (e.g.[3, p.91])

$$\|\mu * \partial^i E\|_s \leq C \sup\{|\mu|(B(x,r))r^{-d+1-s} : x \in \mathbb{R}^d, r > 0\}, \tag{2.9}$$

μ being any locally finite measure, and (2.8) follows also from (2.9) provided we ascertain that for each ball $B(x,r)$

$$\sum_j |\mu_j^i|(B(x,r)) \leq \omega(L)r^{d-1+s}, \qquad i = 1,..,d.$$

This can be done as follows.

$$\begin{aligned}\sum |\mu_j^i|(B(x,r)) &= \sum_{r\leq\delta_j} + \sum_{r>\delta_j} \\ &\leq C\omega(L)\sum_{r\leq\delta_j} r^{s-1}|B(x,r)\cap B_j| + C\sum_{B_j\subset B(x,3r)} \delta_j^{d-1+s}\omega(\delta_j) \\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

Clearly $\mathrm{I} \le C\omega(L)r^{d-1+s}$, because the B_j are almost disjoint. If $r \le L$ we estimate II by $Cr^{d-1+s}\omega(r) \le Cr^{d-1+s}\omega(L)$. Otherwise, given any $z \in \bigcup B_j$, II can be estimated by

$$\sum_j \delta_j^{d-1+s}\omega(\delta_j) \le \sum_{B_j \subset B(z,L)} \delta_j^{d-1+s}\omega(\delta_j) \le CL^{d-1+s}\omega(L)$$

$$\le Cr^{d-1+s}\omega(L). \qquad \square$$

2.3 Covering Lemma. *Let $h(t) = t^{d-1+s}\omega(t)$ be a measure function with ω non-decreasing and satisfying $\omega(2t) \le C\omega(t)$. Then for any compact set $K \subset \mathbb{R}^d$ there exists a finite family of balls B_j of radii δ_j with the following properties.*

(a) $K \subset \cup B_j$.
(b) $(2B_j)$ *is an almost disjoint family with constant independent of* K.
(c) $\sum_j h(\delta_j) \le CM^h(K)$, *where* C *is independent of* K.
(d) *For each ball* B *of radius* δ

$$\sum_{B_j \subset B} h(\delta_j) \le Ch(\delta),$$

where C *is independent of* K.

Remark. If $M^h(K) = 0$ (c) should be replaced by $\sum_j h(\delta_j) \le \varepsilon$, where $\varepsilon > 0$ has been fixed in advance.

Proof. We start by proving that there exists a finite family of dyadic cubes satisfying (a), (b), (c) and (d).

It is easy to find a finite family (D_j) of dyadic cubes (with disjoint interiors) of side length r_j such that

(i) $K \subset \bigcup_j D_j$.
(ii) $\sum_j h(r_j) \le CM^h(K)$.
(iii) $\sum_{D_j \subset D} h(r_j) \le Ch(r)$, for each cube D of side length r.

To construct the family (D_j) we consider a family (P_j) of dyadic cubes of side length σ_j satisfying $K \subset \bigcup_j Q_j$ and $\sum_j h(\sigma_j) \le CM^h(K)$. This family can be easily modified so that in addition one has

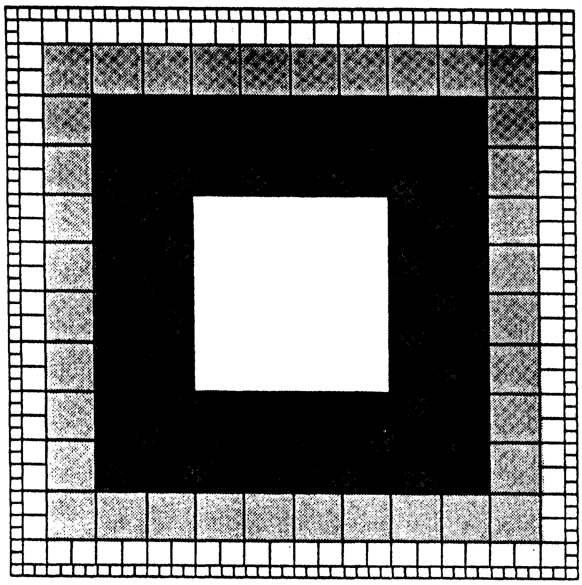
$$\sum_{P_j \subset P} h(\sigma_j) \le h(\sigma), \tag{2.10}$$

for each dyadic cube P of the side length σ. In fact, if for such a P (2.10) fails then we remove the P_j contained in P from our family and we put P in it.

Our goal now is to modify the family (D_j) so that the double cubes are almost disjoint and properties (i)-(iii) are preserved.

Given a dyadic cube D of side length r we let $\mathcal{Q}_1$ stand for the family of cubes consisting of D and of those dyadic cubes of side length $r/2$ touching D but not contained in D. For $n \geq 2$ the family $\mathcal{Q}_n$ is defined inductively as follows. A cube is in $\mathcal{Q}_n$ if either it is in $\mathcal{Q}_{n-1}$ or it is a dyadic cube of side length $r2^{-n}$ touching some cube in $\mathcal{Q}_{n-1}$ but not contained in any cube in $\mathcal{Q}_{n-1}$.

We define the halo of a dyadic cube D as $halo(D) = \bigcup_{n=1}^{\infty} \mathcal{Q}_n$.



FIGURE

Let $\mathcal{F}$ be the maximal cubes in the family $\bigcup_j halo(D_j)$, and let $\mathcal{H} = \{Q \in \mathcal{F} : Q \cap K \neq \emptyset\}$. Observe that the family $\mathcal{H}$ is finite because it contains at most as many cubes as the original family (D_j).

We are going to see that the family $\mathcal{H}$ satisfies properties (a), (b), (c) and (d).

Property (a) is clearly satisfied.

To show that (b) holds just remark that, because of maximality, a cube $Q \in \mathcal{H}$ of side length r can only touch cubes of $\mathcal{H}$ of side lengths $\frac{r}{2}$, r or $2r$.

Let's turn to (c). Given a dyadic cube D of side length r let $N_n(D)$ be the number of cubes of side length $r2^{-n}$ which belong to *halo*(D). A simple computation shows that $N_n(D) \le 3^{d-1}d^2 2^{n(d-1)}$. We then have

$$\begin{aligned}(2.11)\qquad \sum_{Q\in halo(D)} h(\text{side length of } Q) &\le C\sum_{n=0}^{\infty}\omega(r2^{-n})(r2^{-n})^{d-1+s}2^{n(d-1)}\\ &\le C\omega(r)r^{d-1+s}\\ &= Ch(\text{side length of } D).\end{aligned}$$

Let's mention in passing that (2.11) holds because $s > 0$, and that if $s = 0$ the left hand side may be infinity.

Set $\mathcal{H} = \{Q_i\}$ and let δ_i be the side length of the cube Q_i. Then by (2.11) and property (ii) of (D_j) we have

$$\sum_i h(\delta_i) \le \sum_j \sum_{Q_i\in halo(D_j)} h(\delta_i) \le C\sum_j h(r_j) \le CM^h(K).$$

It is clearly enough to prove (d) with the test ball B replaced by a dyadic cube. Let's then fix a dyadic cube Q of side length δ. We define the index sets

$$I = \{i : Q_i \subset Q \text{ and } Q_i \in halo(D_j) \text{ for some } D_j \subset Q\}.$$
$$J = \{i : Q_i \subset Q \text{ and } Q_i \notin halo(D_j) \text{ for all } D_j \subset Q\}.$$

Therefore

$$\sum_{Q_i\subset Q} h(\delta_i) = \sum_{i\in I} h(\delta_i) + \sum_{i\in J} h(\delta_i)$$

and by (2.11) and property (iii) of (D_j) it follows that

$$\sum_{i\in I} h(\delta_i) \le \sum_{D_j\subset Q}\ \sum_{Q_i\in halo(D_j)} h(\delta_i) \le C\sum_{D_j\subset Q} h(r_j) \le Ch(\delta).$$

Let $\mathcal{F}(Q) = \{R \in \mathcal{F} : R\cap\partial Q \ne \emptyset \text{ and } R \subset Q\}$. By construction, for each Q_i, $i \in J$, there is a cube $R \in \mathcal{F}(Q)$ such that $Q_i \in halo(R)$. Let r be the side length of R. Because of (2.11)

$$\sum_{i\in J} h(\delta_i) \le C\sum_{R\in\mathcal{F}(Q)} h(r),$$

and now, since the (disjoint) family $\mathcal{F}(Q)$ lies on the faces of Q we get

$$\sum_{i\in J} h(\delta_i) \le Ch(\delta),$$

which completes the proof of (d).

Finally, the lemma follows replacing the dyadic cubes Q_i by their respective circumscribed balls. □

Given a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ one associates to it the Vitushkin localization operator (see [22, p. 168])

$$V_\varphi f = \varphi\Delta f * E,$$

where E is the fundamental solution of the laplacian and f is any distribution in $\mathbb{R}^d$. Next lemma will allow us to use the Vitushkin's localization technique.

2.4 Lemma. *Let $\varphi \in C_0^\infty(B)$, B being a ball of center a and radius δ. Then for any function $f \in \Lambda^s(\mathbb{R}^d)$, $0 < s < 1$, we have*

$$\omega_s(V_\varphi f, r) \le C(\varphi)\omega_s\big(f, \min(\delta, r)\big),$$

where $C(\varphi) = C \sum_{|\alpha|\le 3} \delta^{|\alpha|}\|\partial^\alpha \varphi\|_\infty$.

In particular, $\|V_\varphi f\|_s \le C(\varphi)\omega_s(f,\delta)$.

Proof. Without loss of generality we can assume a to be the origin. Put $F(x) = f(x) - f(0)$. Thus

$$V_\varphi f = \varphi\Delta F * E = \varphi F - 2(\nabla\varphi\nabla f) * E - (F\Delta\varphi) * E.$$

Since $\sum_{i=1}^d \partial^i(F\partial^i\varphi) = F\Delta\varphi + \nabla f\nabla\varphi$, we get an useful expression for $V_\varphi f$, namely

$$\begin{aligned} V_\varphi f &= \varphi F + (F\Delta\varphi) * E - 2\sum_{i=1}^d F\partial^i\varphi * \partial^i E \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

We are going to estimate ω_s for the three terms separately. Set $|x-y| = r$.

For the first term we can suppose that x, $y \in \overline{B}$ and so $r \le 2\delta$. If $|y| \le r$ then obviously $|F(y)| \le \omega_s(f,r)r^s$ and if $|y| > r$ then $|F(y)| = |f(y) - f(0)| \le 2|y|r^{-1}\omega_s(f,r)r^s$. Thus

$$|(\varphi F)(x) - (\varphi F)(y)| \le \|\varphi\|_\infty\ |f(x) - f(y)| + |F(y)|\ \|\nabla\varphi\|_\infty\ |x-y|,$$

and so

$$\omega_s(I, r) \le C(\varphi)\omega_s(f, \min(r,\delta)).$$

On the other hand, since $\|F\Delta\varphi * E\|_\infty \le C(\varphi)\omega_s(f,\delta)\delta^s$ we have $\omega_s(\mathrm{II}, r) \le C(\varphi)\omega_s(f,\delta)$ if $r \ge \delta$. When $r < \delta$

$$\begin{aligned} &|(F\Delta\varphi * E)(x) - (F\Delta\varphi * E)(y)| \\ &\le \int_{B(x,2\delta)} \frac{|(F\Delta\varphi)(x-z) - (F\Delta\varphi)(y-z)|}{|z|^{d-2}}\, dz. \end{aligned}$$

If one distinguishes as above the cases $|x-z| \le r$ and $|x-z| > r$ one gets

$$\begin{aligned}
&|(F\Delta\varphi)(x-z)-(F\Delta\varphi)(y-z)| \\
&\quad \le |\Delta\varphi(y-z)|\;|F(x-z)-F(y-z)| + |F(x-z)|\;|\Delta\varphi(x-z)-\Delta\varphi(y-z)| \\
&\quad \le C(\varphi)\omega_s(f,r)r^s\delta^{-2},
\end{aligned}$$

and therefore

$$\omega_s(\mathrm{II},r) \le C(\varphi)\omega_s(f,\min(r,\delta)).$$

One proves the analagous estimate for III in a similar way. □

Given a compact set X of $\mathbb{R}^d$, let $P^s(X)$ be the linear span in $\Lambda^s(X)$ of the functions of the type $\mu * E$ and $\nu * \partial^i E$, where μ and ν are Borel measures with compact support disjoint from $\overset{\circ}{X}$ satisfying

$$|\mu|(B(x,r)) \le \varepsilon_1(r)r^{d-2+s} \qquad x \in \mathbb{R}^d,\ r>0,$$

$$|\nu|(B(x,r)) \le \varepsilon_2(r)r^{d-1+s} \qquad x \in \mathbb{R}^d,\ r>0,$$

for some $\varepsilon_i(r) \to 0$ as $r \to 0$.

The potentials $E * \mu$ and $\partial^i E * \nu$ belong to $\lambda^s(\mathbb{R}^d)$ because of the required growth conditions on μ and ν. Consequently $P^s(X)$ is a subspace of $h^s(X)$.

Theorem 2.5. *For each compact set $X \subset \mathbb{R}^d$, $P^s(X)$ is dense in $h^s(X)$.*

Proof. Let $f \in h^s(X)$. Applying the Whitney Extension Theorem for Lipschitz functions (see [20, p. 175]) and using a cut-off function identically equal to 1 on a neighbourhood of X, we can assume without loss of generality that $f \in \lambda^s(\mathbb{R}^d)$ and has compact support.

Fix $\delta > 0$ and let (B_k, φ_k, f_k) a δ-Vitushkin scheme for the approximation of f. This means (see [22, p. 168]) that the following holds.

(1) (B_k) is an almost disjoint family of open balls of radii δ covering $\mathbb{R}^d$.
(2) $\varphi_k \in C_0^\infty(B_k)$, $\sum \varphi_k \equiv 1$ on $\mathbb{R}^d$ and $|\partial^\alpha \varphi_k| \le C\delta^{-|\alpha|}$, $0 \le |\alpha| \le 3$.
(3) $f_k = \varphi_k \Delta f * E$, and so $f = \sum_k f_k$.

Notice that $f_k \equiv 0$ whenever $B_k \cap \operatorname{spt} f = \emptyset$, and thus only finitely many f_k do not vanish identically.

Fix a point $x_k \in B_k$ and expand f_k at ∞

$$f_k(x) = a_k E(x-x_k) + \sum_{i=1}^d b_k^i \partial^i E(x-x_k) + O(|x|^{-d}).$$

We now show that we have the estimates

$$|a_k| \le CM^{h_1}(B_k \setminus \overset{\circ}{X}) \tag{2.12}$$

$$|b_k^i| \le C\delta M^{h_1}(B_k \setminus \overset{\circ}{X}), \qquad i = 1,..,d, \tag{2.13}$$

where $h_1(t) = t^{d-2+s}\omega_s(t)$ and $\omega_s(t) = \omega_s(f, \min(t,\delta))$.

Consider a family of open balls D_j, with centers c_j and radii δ_j, which covers $B_k \setminus \overset{\circ}{X}$. Let $\phi_j \in C_0^\infty(2D_j)$ with $|\partial^\alpha \phi_j| \le C\delta_j^{-|\alpha|}$, $0 \le |\alpha| \le 2$, and $\sum \phi_j \equiv 1$ on $\bigcup D_j$. Since $\Delta f_k = \varphi_k \Delta f$ has support contained in $B_k \setminus \overset{\circ}{X}$,

$$\begin{aligned}|a_k| &\le |\langle \Delta f_k, 1\rangle| = \Big|\langle \Delta f_k, \sum_j \phi_j\rangle\Big| \le \sum_j |\langle \Delta(f_k - f_k(c_j)),\ \phi_j\rangle| \\ &\le \sum_j \int_{2D_j} |f_k(x) - f_k(c_j)|\ |\Delta\phi_j(x)|\, dx \\ &\le C\sum_j \delta_j^{d-2+s}\omega_s(f_k, \delta_j) \le C\sum_j h_1(\delta_j).\end{aligned}$$

Hence (2.12) follows, and similarly one obtains (2.13).

We would like to have a better estimate for b_k^i, namely

$$|b_k^i| \le CM^{h_2}(B_k \setminus \overset{\circ}{X}) \qquad \text{where } h_2(t) = th_1(t),$$

but unfortunately it can be shown that the above inequality is not always satisfied. To get around this difficulty we will again localize each f_k to exploit the basic estimate (2.13) at a lower level, forcing in this way M^{h_2} to enter the scene.

We know that f_k is harmonic outside a compact $K_k \subset B_k \setminus \overset{\circ}{X}$. Take a covering of K_k by open balls B_{kj}, with radii δ_{kj}, which satisfies the properties of the Covering Lemma 2.3 with $h = h_2$.

By a lemma of Harvey and Polking [8, 3.1, p. 43] we can construct functions $\psi_j \in C_0^\infty\big(\frac{3}{2}B_{kj}\big)$ such that $\sum \psi_j \equiv 1$ on $\bigcup_j B_{kj}$ and $|\partial^\alpha \psi_j| \le C\delta_{kj}^{-|\alpha|}$, $0 \le |\alpha| \le 3$. Let's mention that the family $\big(\frac{3}{2}B_{kj}\big)$ also satisfies the conclusions of Lemma 2.3.

Let $f_{kj} = \psi_j \Delta f_k * E$, so that $f_k = \sum_j f_{kj}$.

Consider a positive Borel measure μ_{kj} with spt $\mu_{kj} \subset \big(\frac{3}{2}B_{kj}\big) \setminus \overset{\circ}{X}$ such that

(a) $\mu_{kj}(B(x,r)) \le h_1(r)$, $x \in \mathbb{R}^d$, $r > 0$,

(b) $\|\mu_{kj}\| \ge CM^{h_1}\big(\frac{3}{2}B_{kj} \setminus \overset{\circ}{X}\big)$.

It is clear that the function

$$h_{kj} = a_{kj} \frac{\mu_{kj}}{\|\mu_{kj}\|} * E,$$

where a_{kj} is the first coefficient of the expansion at ∞ of f_{kj}, satisfies $f_{kj} - h_{kj} = O(|x|^{1-d})$ and moreover, using (2.9),

$$\|h_{kj}\|_s \leq \frac{|a_{kj}|}{\|\mu_{kj}\|} \|\mu_{kj} * E\|_s \leq C\omega_s(f,\delta_{kj}).$$

Let $H_k = \sum_j h_{kj}$, so that $f_k - H_k = O(|x|^{1-d})$ as $x \to \infty$. Applying Lemmas 2.2 and 2.4

$$\|H_k - f_k\|_s = \|\sum_j f_{kj} - h_{kj}\|_s \leq C\omega_s(f,\delta),$$

and so $\|H_k\|_s \leq \|H_k - f_k\|_s + \|f_k\|_s \leq C\omega_s(f,\delta)$.

Consider the expansions

$$f_k(x) - H_k(x) = \sum_{i=1}^{d} b_k^i \partial^i E(x - x_k) + O(|x|^{-d})$$

$$f_{kj}(x) - h_{kj}(x) = \sum_{i=1}^{d} b_{kj}^i \partial^i E(x - x_{kj}) + O(|x|^{-d}),$$

where $x_{kj} \in B_{kj}$. Using (2.13) for b_{kj}^i

$$|b_k^i| \leq \sum_j |b_{kj}^i| \leq C \sum_j h_2(\delta_{kj}) \leq CM^{h_2}(K_k),$$

where the last inequality follows from the properties of the family $\left(\frac{3}{2}B_{kj}\right)$.

We can now match the coefficients b_k^i by means of appropiate functions. Let ν_k a positive Borel measure with support contained in K_k such that

(a) $\nu_k(B(x,r)) \leq h_2(r) \qquad x \in \mathbb{R}^d,\ r > 0,$
(b) $\|\nu_k\| \geq CM^{h_2}(K_k)$.

We define

$$P_k = H_k + \sum_{i=1}^{d} (b_k^i \frac{\nu_k}{\|\nu_k\|} * \partial^i E),$$

which belongs to $P^s(X)$ and satisfies $\|P_k\|_s \leq C\omega_s(f,\delta)$ and $f_k - P_k = O(|x|^{-d})$. An application of Lemma 2.1 gives

$$\Big\|f - \sum_k P_k\Big\|_s = \Big\|\sum_k (f_k - P_k)\Big\|_s \leq C \max_k \|f_k - P_k\|_s$$

$$\leq C\omega_s(f,\delta) \to 0 \qquad \text{as } \delta \to 0,$$

and this completes the proof of the theorem. □

3. Proof of Theorem 1. We start by the most difficult step in the proof of Theorem 1, that is, that (3) is sufficient for the approximation.

Proof of (3) $\Longrightarrow$ (1). The argument at the beginning of the proof of the Theorem 2.5 shows that

$$\lambda^s(X) = \lambda_0^s(\mathbb{R}^d)/K_s(X),$$

where $K_s(X) = \{f \in \lambda_0^s(\mathbb{R}^d) : f = 0 \text{ on } X\}$.

Since $(\lambda_0^s(\mathbb{R}^d))^* = B^p$, $s = d(\frac{1}{p} - 1)$, we have

$$(\lambda^s(X))^* = \{b \in B^p : \text{spt } b \subset X\}.$$

Let $b \in (\lambda^s(X))^*$ and assume that b annihillates $H^s(X)$. We must show that b annihillates $h^s(X)$, and in view of 2.5 it is in fact enough to ascertain that b annihillates functions of the form $\mu * E$ and $\nu * \partial^i E$, where μ and ν are Borel measures satisfying

$$|\mu|(B(x,r)) \leq \varepsilon_1(r) r^{d-2+s} \quad \text{and spt } \mu \subset (\mathring{X})^c, \tag{3.1}$$

$$|\nu|(B(x,r)) \leq \varepsilon_2(r) r^{d-1+s} \quad \text{and spt } \nu \subset (\mathring{X})^c. \tag{3.2}$$

Since b annihillates $H^s(X)$ we have $b * E = 0$ on X^c. We will show that

$$b * E = 0, \qquad M_*^{d-2+s} - \text{a.e. on } (\mathring{X})^c \tag{3.3}$$

and

$$\nabla(b * E) = 0, \qquad M_*^{d-1+s} - \text{a.e. on } (\mathring{X})^c. \tag{3.4}$$

Then, from (3.3) we get

$$\langle b, \mu * E\rangle = \langle b * E, \mu\rangle = \int b * E \, d\mu = 0,$$

because, by the growth condition (3.1), $\mu \ll M_*^{d-2+s}$ (that is, $\mu(A) = 0$ whenever $M_*^{d-2+s}(A) = 0$).

Similarly (3.4) implies that

$$\langle b, \nu * \partial^i E\rangle = \langle \partial^i (b * E), \nu\rangle = 0,$$

because $\nu \ll M_*^{d-1+s}$, which follows from (3.2).

To prove (3.3) we need to introduce a new notion. We say that a function f, which is defined M^α-almost everywhere, is M^α-pseudocontinuous at a point a if for all $\varepsilon > 0$

$$\lim_{r \to 0} \frac{M^\alpha(\{x : |f(x) - f(a)| > \varepsilon\} \cap B(a,r))}{r^\alpha} = 0.$$

Also we set $\int |f(x)| \, dM^\alpha(x) = \int_0^\infty M^\alpha\{x : |f(x)| > \lambda\} \, d\lambda$.

Lemma 3.1. *Let f be a M^{d-t}-quasicontinuous function in I_tH^1, $0<t<d$. Then f is M^{d-t}-pseudocontinuous, M^{d-t}-almost everywhere.*

Proof. We define

$$Sf(x)=\sup_{r>0}\frac{1}{r^{d-t}}\int_{B(x,r)}Mf(y)\,dM^{d-t}(y),$$

where Mf is the Hardy-Littlewood maximal function. We start by showing the weak type estimate

$$M^{d-t}(\{x:Sf(x)>\lambda\})\le C\lambda^{-1}\|f\|_{I_tH^1}. \tag{3.5}$$

The set $\{x: Sf(x)>\lambda\}$ is contained in an union of balls $B(z,r)$ satisfying $r^{d-t}<\lambda^{-1}\int_{B(z,r)}M(\nabla f)(x)\,dM^{d-t}$ and by a well-known covering lemma (see [20, p. 9]) we can select a disjoint sequence of balls $B(z_i,r_i)$ such that each ball $B(z,r)$ is contained in $\bigcup B(z_i,5r_i)$.

By Melnikov's Covering Lemma [16, p. 72] there exits a subfamily $B(z_j, 5r_j)$ satisfying

$$M^{d-t}\big(\bigcup_i B(z_i,5r_i)\big)\le C\sum_j r_j^{d-t}$$

and the packing condition $\sum_{B_j\subset B}r_j^{d-t}\le C\delta^{d-t}$, for all balls B of radius δ. Then

$$\begin{aligned}M^{d-t}\big(\{a\in\mathbb{R}^d:Sf(a)>\lambda\}\big)&\le C\sum_j r_j^{d-t}\\&\le\frac{C}{\lambda}\sum_j\int_{B(z_j,r_j)}M(\nabla f)(x)\,dM^{d-t}\\&\le\frac{C}{\lambda}\sum_j\int_{\mathbb{R}^d}M(\nabla f)(x)\,d\mu_j,\end{aligned}$$

where each μ_j is a positive Borel measure with compact support in $B(z_j,r_j)$ satisfying $\mu_j(B(x,\delta))\le\delta^{d-t}$ for all balls B of radius δ [1, p. 118].

Set $\mu=\sum_j\mu_j$. Given any ball $B(x,\delta)$ the number of balls in the family $B(z_j,r_j)$ which intersect $B(x,\delta)$ and have radius greater than δ is less than a constant $C(d)$, because this family is disjoint. Therefore

$$\begin{aligned}\mu(B(x,\delta))&=\sum_{r_j\le\delta}\mu_j(B(x,\delta))+\sum_{r_j>\delta}\mu_j(B(x,\delta))\\&\le\sum_{B(z_j,r_j)\subset B(x,3\delta)}r_j^{d-t}+C\delta^{d-t}\le C\delta^{d-t},\end{aligned}$$

where the last inequality follows from the packing condition.

We finally obtain

$$\begin{aligned} M^{d-t}\left(\{x \in \mathbb{R}^d : Sf(a) > \lambda\}\right) &\leq \frac{C}{\lambda}\int_{\mathbb{R}^d} M(\nabla f)(x)\,d\mu \\ &\leq \frac{C}{\lambda}\|g\|_{H^1}, \end{aligned}$$

where the last inequality is proved in [1, sec. 5].

Now, from (3.5) and the fact that f is strictly defined M^{d-t} almost everywhere by

$$f(x) \equiv \lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy,$$

we deduce that, M^{d-t}-almost all $a \in \mathbb{R}^d$,

$$\lim_{r\to 0} r^{t-d}\int_{B(a,r)} |f(x) - f(a)|\,dM^{d-t}(x) = 0. \tag{3.6}$$

Since for all positive ε we have

$$\begin{aligned} &\frac{M^{d-t}(\{x : |f(x) - f(a)| > \varepsilon\} \cap B(a,r))}{r^{d-t}}\varepsilon \\ &\qquad \leq \frac{1}{r^{d-t}}\int_{B(a,r)} |f(x) - f(a)|\,dM^{d-t}(x), \end{aligned}$$

the lemma follows from (3.6). □

We can now proceed to prove (3.3). Set $f = E * b \in I_2B^p \subset I_{2-s}H^1$. Recall that $f = 0$ on X^c. Let A be the set of points $a \in \partial X$ such that $f(a)$ is strictly defined, f is M^{d-2+s}-pseudocontinuous at a and $f(a) \neq 0$. Then if $a \in A$

$$\begin{aligned} &\limsup_{r\to 0}\frac{M^{d-2+s}(B(a,r)\setminus X)}{r^{d-2+s}} \\ &\leq \limsup_{r\to 0}\frac{M^{d-2+s}(\{x : |f(x) - f(a)| > |f(a)|/2\} \cap B(a,r))}{r^{d-2+s}} = 0, \end{aligned}$$

because f is M^{d-2+s}-pseudocontinuous at a. Condition (4) in Theorem 1 implies that $M_*^{d-2+s}(A) = 0$, and so (3.3) follows.

The proof of (3.4) requires a differentiability result for functions in the potential space I_tH^1, $1 < t \leq d$.

Lemma 3.2. *Let $f \in I_t H^1$ with $1 < t \le d$. Then M^{d+1-t}-almost all $a \in \mathbb{R}^d$ satisfies*

$$\lim_{r\to 0} \frac{1}{r^{d+1-t}} \int_{B(a,r)} \frac{|f(x)-f(a)-\nabla f(a)\cdot(x-a)|}{|x-a|}\, dM^{d+1-t}(x) = 0. \tag{3.7}$$

Remark. When $t = d$ one can get a better result, namely, ordinary differentiability M^1-a.e. (see [5]).

Proof. We set $f = I_t * g$, where $g \in H^1$. To prove the lemma it is enough to show that the maximal operator

$$Tf(a) = \sup_{r>0} \frac{1}{r^{d+1-t}} \int_{B(a,r)} \frac{|f(x)-f(a)-\nabla f(a)(x-a)|}{|x-a|}\, dM^{d+1-t}(x) \tag{3.8}$$

satisfies the weak type estimate

$$M^{d+1-t}\left(\{a \in \mathbb{R}^d : Tf(a) > \lambda\}\right) \le C\lambda^{-1}\|g\|_{H^1}, \tag{3.9}$$

because smooth functions obviously satisfy (3.7) and $C^\infty \cap \mathrm{I}_t H^1$ is dense in $\mathrm{I}_t H^1$.

To estimate the integrand in (3.8) we assume first that $f \in C^\infty$. In this case if $x \ne a$

$$\begin{aligned} |f(x)-f(a)-\nabla f(a)(x-a)|\; |x-a|^{-1} &\le |f(x)-f(a)|\; |x-a|^{-1} + |\nabla f(a)| \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Clearly, $\mathrm{II} \le M(\nabla f)(a)$. To estimate I we put $\delta = |x-a|$, $B = B(x,\delta)$ and $f_B = \frac{1}{|B|}\int_B f(y)\,dy$. We have

$$|f(x)-f(a)| \le |f(x)-f_B| + |f_B - f(a)|$$

and

$$\begin{aligned} |f(x)-f_B| &\le \frac{1}{|B|}\int_B |f(x)-f(y)|\,dy \\ &= \frac{1}{|B|}\int_{|\xi|=1}\int_0^\delta |f(x)-f(x+t\xi)|\,t^{d-1}\,dt\,d\sigma(\xi) \\ &\le \frac{1}{|B|}\int_{|\xi|=1}\int_0^\delta\int_0^t |\nabla f(x+u\xi)|\,du\; t^{d-1}\,dt\,d\sigma(\xi) \\ &\le C\int_{|\xi|=1}\int_0^\delta |\nabla f(x+u\xi)|\,du\,d\sigma(\xi) \\ &= C\int_{|z|\le\delta} |\nabla f(x+z)|\;|z|^{1-d}\,dz \le C\delta M(\nabla f)(x), \end{aligned}$$

where the last step follows from [20, III, 2.2].

Also similarly

$$\begin{aligned}|f_B - f(a)| &\le \frac{1}{|B|}\int_B |f(y) - f(a)|\,dy \\ &\le \frac{1}{|B|}\int_{B(a,2\delta)} |f(y) - f(a)|\,dy \\ &\le C\delta M(\nabla f)(a).\end{aligned}$$

Therefore, $\mathrm{I} \le C(M(\nabla f)(a) + M(\nabla f)(x))$.

Consequently when $f \in C^\infty(\mathbb{R}^d)$ and $x \ne a$ we have

$$\begin{aligned}|f(x) - f(a) - \nabla f(a)(x-a)|\;|x-a|^{-1}& \\ \le C\,(M(\nabla f)(x) + M(\nabla f(a))).&\end{aligned} \tag{3.10}$$

Using inequality (1.1) and a standard argument we can prove that given $f \in \mathrm{I}_t H^1$ there exists a sequence (f_j) in $C^\infty \cap \mathrm{I}_t H^1$ such that $f_j(x) \to f(x)$, $\nabla f_j(x) \to \nabla f(x)$ and $M(\nabla f_j)(x) \to M(\nabla f)(x)$, M^{d+1-t}-a.e.. Then for M^{d+1-t} almost all $a \in \mathbb{R}^d$, (3.10) holds for M^{d+1-t} almost all $x \in \mathbb{R}^d$, and so

$$Tf(a) \le C(M(\nabla f)(a) + Sf(a)) \qquad M^{d+1-t}\text{-a.e.}, \tag{3.11}$$

where

$$Sf(a) = \sup_{r>0} \frac{1}{r^{d+1-t}} \int_{B(a,r)} M(\nabla f)(x)\,dM^{d+1-t}(x).$$

Since $\partial^i f \in \mathrm{I}_{t-1}H^1$, $i = 1,\dots,d$, inequality (1.1) gives

$$M^{d+1-t}\left(\{a \in \mathbb{R}^d : M(\nabla f)(a) > \lambda\}\right) \le C\lambda^{-1}\|g\|_{H^1}. \tag{3.12}$$

On the other hand, as in the proof of Lemma 3.1 one can prove

$$M^{d+1-t}\left(\{a \in \mathbb{R}^d : Sf(a) > \lambda\}\right) \le C\lambda^{-1}\|g\|_{H^1}. \tag{3.13}$$

Finally (3.9) follows from (3.11)-(3.13). □

Lemma 3.3. *Let $f \in \mathrm{I}_t H^1$, $1 \le t < 2$ and $D = \{x \in \mathbb{R}^d : f(x) = 0 \quad \text{and}\ \nabla f(x) \ne 0\}$. Then $M^{d+1-t}(D) = 0$.*

Proof. Given $a \in D$ we can assume $\partial^i f(a) = 0$, $i = 2,..,d$, and $\partial^1 f(a) = 1$. Therefore, by the above lemma, M^{d+1-t} almost all $a \in D$ we have

$$\begin{aligned} 0 &= \lim_{r\to 0} \frac{1}{r^{d+1-t}} \int_{B(a,r)\cap D} \frac{|\partial^1 f(a)(x_1 - a_1)|}{|x-a|} dM^{d+1-t}(x) \\ &\geq \lim_{r\to 0} \frac{\xi M^{d+1-t}(B(a,r)\cap D \cap S(a,\xi))}{r^{d+1-t}} \end{aligned}$$

where $S(a,\xi)$ is the cone $\{x \in \mathbb{R}^d : \xi|x-a| \leq |x_1 - a_1|\}$, $0 < \xi < 1$.

On the other hand, using the density theorem for the Hausdorff content (see [6, (2.10.19)(2)]), M^{d+1-t} almost all $a \in D$

$$\begin{aligned} 0 < c &\leq \limsup_{r\to 0} \frac{M^{d+1-t}(B(a,r)\cap D)}{r^{d+1-t}} \\ &\leq \limsup_{r\to 0} \frac{M^{d+1-t}(B(a,r)\cap D\cap S(a,\xi))}{r^{d+1-t}} \\ &\quad + \limsup_{r\to 0} \frac{M^{d+1-t}(B(a,r)\cap D\cap T(a,\xi))}{r^{d+1-t}} \end{aligned}$$

where $T(a,\xi) = \mathbb{R}^d \setminus S(a,\xi)$.

Then for all $0 < \xi < 1$ one has

$$\begin{aligned} 0 < c &\leq \limsup_{r\to 0} \frac{M^{d+1-t}(B(a,r)\cap D\cap T(a,\xi))}{r^{d+1-t}} \\ &\leq \limsup_{r\to 0} \frac{M^{d+1-t}(B(a,r)\cap T(a,\xi))}{r^{d+1-t}} = C(\xi), \end{aligned}$$

and it is easy to see that $C(\xi) \leq C(\tan\xi)^{2-t} \to 0$ as $\xi \to 0$. Consequently $M^{d+1-t}(D) = 0$. $\square$

It is now clear that from (3.3) and the above lemma we even get a stronger result than (3.4), namely,

$$\nabla(b * E) = 0, \qquad M^{d-1+s}\text{-a.e. on } (\overset{\circ}{X})^c.$$

Proof of (1) $\Longrightarrow$ (2). For a compact set $K \subset \mathbb{R}^d$ and $0 < s < 1$ define

$$\begin{aligned} \gamma_s(K) &= \sup|\langle T, 1\rangle| \\ \alpha_s(K) &= \sup|\langle T, 1\rangle|, \end{aligned}$$

where the first supremum is over those distributions T with compact support contained in K, such that $\|T * E\|_s \leq 1$. In addition we require in the second that $\omega_s(T * E, \delta) \to 0$ as $\delta \to 0$.

For an arbitrary set $F \subset \mathbb{R}^d$ we set

$$\gamma_s(F) = \sup \gamma_s(K) \qquad \text{and} \qquad \alpha_s(F) = \sup \alpha_s(K),$$

both supremums being over the compact subsets of F.

Next lemma is analogous to a result of Melnikov on removable sets for Lipschitz holomorphic functions [14].

Lemma 3.4. *There exists a constant $C \geq 1$ such that*

$$C^{-1}M^{d-2+s}(F) \leq \gamma_s(F) \leq CM^{d-2+s}(F),$$
$$C^{-1}M_*^{d-2+s}(F) \leq \alpha_s(F) \leq CM_*^{d-2+s}(F),$$

for any σ-compact $F \subset \mathbb{R}^d$.

Proof. See [22, p. 178]. □

Now, the argument for the proof of (1)$\Longrightarrow$(2) in Theorem 1 is similar to that presented in [22, p. 185]. We refer the reader there for more details.

Relationships between conditions (2) and (3). We begin by showing a density result for the lower Hausdorff content, analogous to the density property for the Hausdorff content.

Lemma 3.5. *For all subset F of $\mathbb{R}^d$ and $0 < \alpha < d$ one has*

$$\limsup_{r \to 0} \frac{M_*^\alpha(B(x,r) \cap F)}{r^\alpha} \geq C \qquad M_*^\alpha\text{-a.e. } x \in F,$$

where C is a constant which only depends on α and d.

Proof. Let h be a measure function such that $h(t) \leq t^\alpha$ and $\lim_{t\to 0} h(t)/t^\alpha = 0$. We write $h(t) = \varepsilon(t)t^\alpha$, where $0 < \varepsilon(t) \leq 1$ and $\varepsilon(t) \to 0$ as $t \to 0$. Given $0 < \eta < 1$, if $B(x,r) \cap F \subset \bigcup_j B(x_j, r_j)$ with $r_j \leq r$ we have

$$M^h(B(x,r) \cap F) \leq \sum h(r_j) \leq \varepsilon^{1-\eta}(r) \sum \varepsilon^\eta(r_j) r_j^\alpha$$

and consequently

$$M^h(B(x,r) \cap F) \leq \varepsilon^{1-\eta}(r) M_*^\alpha(B(x,r) \cap F),$$

which gives, by letting $\eta \to 0$,

$$M^h(B(x,r) \cap F) \leq \varepsilon(r) M_*^\alpha(B(x,r) \cap F).$$

Then, by the density theorem for the Hausdorff content

$$\limsup_{r\to 0} \frac{M_*^\alpha(B(x,r) \cap F)}{r^\alpha} \geq \limsup_{r \to 0} \frac{M^h(B(x,r) \cap F)}{h(r)} \geq C, \quad M^h\text{-a.e. } x \in F,$$

where C is independent of h and F. This proves the lemma. □

In view of 3.5 it becomes now clear that (2) implies (3).

We are going to see how condition (3) gives us condition (2), without going through (1), solving a particular approximation problem. Let B be a ball. To show (2) is sufficient to see that $M_*^{d-2+s}(B \cap \partial X) \leq CM^{d-2+s}(B \setminus X)$.

We choose a measure function $h(t) = \varepsilon(t)t^{d-2+s}$, with $0 < \varepsilon(t) \leq 1$, $\varepsilon(t) \to 0$ as $t \to 0$, and a positive Borel measure μ supported on $B \cap \partial X$ and satisfying $\mu(B(x,r)) \leq h(r)$, $\|\mu\| \geq CM_*^{d-2+s}(B \cap \partial X)$.

If $p = d/(d+s)$, B^p is the dual space of λ_0^s. Consider $T \in B^p$, such that T annhillates functions of λ_0^s with laplacian supported on $B \setminus X$. Then, $E * T = 0$ on $B \setminus X$ and from condition (3) we deduce as before that $E * T = 0$ on $B \cap \partial X$, M_*^{d-2+s}-a.e.

Then $\langle T, E * \mu\rangle = \langle E * T, \mu\rangle = 0$ (because $\mu \ll M_*^{d-2+s}$). Therefore there exists a sequence $\{\varphi_j\}$ in C_0^∞ satisfying

(a) $\varphi_j \to E * \mu$ weak-$*$ in Λ^s,
(b) spt $\Delta\varphi_j \subset B \setminus X$.

Since

$$\langle \Delta\varphi_j, 1\rangle \to \langle \mu, 1\rangle \quad \text{and} \quad \|\Delta\varphi_j\| \leq CM^{d-2+s}(B \setminus X),$$

we obtain $\|\mu\| \leq CM^{d-2+s}(B \setminus X)$, which gives the desired inequality.

4. The case $s = 1$: the Zygmund class. Theorem 2.5 and Theorem 1 are also true for $s = 1$. In fact much simpler proofs can be given in this case. We now present a brief account of them.

Given a compact set $X \subset \mathbb{R}^d$ let $P^1(X)$ be the linear span in $\Lambda^1(X)$ of the functions of the type $\mu * E$ where μ is a Borel measure with compact support disjoint of $\mathring{X}$ and $|\mu|(B(x,r) \leq \varepsilon(r)r^{d-1}$, $\varepsilon(r) \to 0$ as $r \to 0$. It then follows that $P^1(X) \subset h^1(X)$.

4.1 Theorem. *For each compact $X \subset \mathbb{R}^d$, $P^1(X)$ is dense in $h^1(X)$.*

The proof, like in Theorem 2.5, is based on the Vitushkin's localization and matching coefficients technique. But now as in the case $\lambda^s(X)$, $1 < s < 2$, (see [22]), we only have to match the first coefficient, because of the following result.

Lemma 4.2. *Let (B_j) be a finite almost disjoint family of open balls of radii $\delta > 0$. Let $f_j \in \Lambda^1(\mathbb{R}^d)$ be harmonic outside a compact set of B_j and $f_j(x) = O(|x|^{1-d})$. Then*

$$\|\sum_j f_j\|_1 \leq C \max_j \|f_j\|_1.$$

Sketch of the proof. We reduce the problem to the case $f_j = O(|x|^{-d})$ and then we proceed as in [22, p. 179]. The reduction argument is based on the fact that if μ is a Borel measure satisfying $|\mu|(B(x,\delta) \leq Cr^d$, that is, $\mu = f(x)dx$ with $\|f\|_\infty \leq C$, then $\|\mathrm{I}_1 * \mu\|_1 \leq C$. □

It is now clear how to get (3) $\Longrightarrow$ (1) in Theorem 1 for $s = 1$ using the above lemma. To prove (1) $\Longrightarrow$ (2) we just need to follow the argument in [22, p. 185] using the version of the Whitney Extension Theorem for the Zygmund class given in [10] or [11].

5. Spectral synthesis. Since $\mathrm{I}_\alpha B^p \hookrightarrow \mathrm{I}_{\alpha-s}H^1$ (i) implies (ii) in Theorem 2 follows from a standard argument from inequality (1.1).

To show (ii) implies (i) in Theorem 2 we will first present a reduction argument to the case $\alpha = 2$ and then we will deal with this particular case.

5.1 Reduction to the case α=2. By [21, p. 553] $\mathrm{I}_\alpha(\Lambda^s(\mathbb{R}^d)) = \Lambda^{s+\alpha}(\mathbb{R}^d)$, $\alpha > 0$, $s > 0$, and so the dual map $\mathrm{I}_\alpha^* = \mathrm{I}_\alpha$ satisfies

$$\mathrm{I}_\alpha B^p = B^q$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ and $0 < p < 1$.

Then

$$\mathrm{I}_\alpha B^p = \mathrm{I}_2 B^r \qquad \text{where } r = \frac{dp}{d-p(\alpha-2)}.$$

Now $\frac{d}{d+\alpha} < r < 1$ follows from $\frac{d}{d+\alpha} < p < \min(1, \frac{d}{d+\alpha-2})$ and thus the reduction is performed.

5.2 The case α=2. We employ a duality argument. We observe that the mapping

$$\begin{aligned} \mathrm{I}_2 B^p(\mathbb{R}^d) &\to B^p(\mathbb{R}^d) \\ f &\longmapsto \Delta f \end{aligned}$$

is an onto isomorphism. Hence the dual of $\mathrm{I}_2 B^p(\mathbb{R}^d)$ is $\Lambda^s(\mathbb{R}^d)$, $s = d(\frac{1}{p} - 1)$, and the action of a $b \in \Lambda^s(\mathbb{R}^d)$ on a $f \in \mathrm{I}_2 B^p$ is given by

$$b(f) = \langle b, \Delta f \rangle,$$

where $\langle , \rangle$ is the Λ^s-B^p duality.

Let $b \in \Lambda^s(\mathbb{R}^d)$ such that

$$b(\varphi) = 0 \qquad \text{for all } \varphi \in C_0^\infty(F^c). \tag{5.1}$$

We must see that $b(f) = 0$, f being the function in the statement of Theorem 2. Obviously (5.1) is equivalent to the harmonicity of b on F^c. The idea of the proof consists in applying the Vitushkin's scheme to approximate b in the weak-$*$

topology of $\Lambda^s(\mathbb{R}^d)$ by functions β for which we know that $\beta(f)$ is either zero or small.

We may assume that b is harmonic outside a compact set K of F (see [13, p. 314] for the argument).

Fix $\delta > 0$ and let (B_j, φ_j, b_j) a δ-Vitushkin scheme for the approximation of b. Then $b_j = \varphi_j \Delta b * \mathrm{I}_2$ and so $b = \sum b_j$, where the sum is over those j such that B_j intersects K. But if $B_j \subset \overset{\circ}{K}$ then $b_j(f) = \langle b_j, \Delta f\rangle = \langle \Delta b_j, f\rangle = 0$, because spt $\Delta b_j \subset B_j \subset \overset{\circ}{K}$ and $f = 0$ on K, M^{d-2+s}-a.e. Therefore

$$b(f) = \Big(\sum_{j\in J} b_j\Big)(f) \qquad \text{where } J = \{j : B_j \cap \partial K \neq \emptyset\}.$$

We will distinguish three cases:

(1) $\quad \dfrac{d}{d+1} < p < 1$, corresponding to $0 < s < 1$.

(2) $\quad \dfrac{d}{d+2} < p < \dfrac{d}{d+1}$, corresponding to $1 < s < 2$.

(3) $\quad p = \dfrac{d}{d+1}$, corresponding to $s = 1$.

First case. As in the proof of Theorem 2.5 we proceed to localize again each b_j, $j \in J$, this time using the Covering Lemma 2.3 with $h(t) = t^{d-1+s}$.

We get functions

$$P_j = \sum_k \mu_{jk} * \mathrm{I}_2 + \sum_{i=1}^{d} \nu_j^i * \partial^i \mathrm{I}_2,$$

where μ_{jk} and ν_j^i are Borel measures with compact support contained in K, satisfying

$$|\mu_{jk}|(B(x,r)) \leq C r^{d-2+s} \qquad x \in \mathbb{R}^d,\ r > 0,$$

$$|\nu_j^i|(B(x,r)) \leq C r^{d-1+s} \qquad x \in \mathbb{R}^d,\ r > 0.$$

The functions P_j satisfy also

$$b_j = P_j + O(|x|^{-d}) \qquad \text{as } x \to \infty$$

and

$$\|P_j\|_s \leq C\|b_j\|_s \leq C\|b\|_s.$$

Write

$$\sum_{j\in J} b_j = \sum_{j\in J} P_j + \sum_{j\in J}(b_j - P_j)$$
$$\equiv B_\delta + D_\delta.$$

Then

$$b(f) = B_\delta(f) + D_\delta(f), \qquad \text{for all } \delta > 0$$

Now

$$(\mu * \mathrm{I}_2)(f) = \langle \mu * \mathrm{I}_2, \Delta f\rangle = \langle \mu, f\rangle = 0,$$

because $f = 0$ on K M^{d-2+s}-a.e., $\operatorname{spt}\mu \subset K$ and μ vanishes on sets of zero M^{d-2+s}.

Also

$$(\nu * \partial^i \mathrm{I}_2)(f) = \langle \nu * \partial^i \mathrm{I}_2, \Delta f\rangle = \langle \nu, \partial^i f\rangle = 0,$$

since Lemma 3.2 implies that $\nabla f = 0$ on K M^{d-1+s}-a.e., spt $\nu \subset K$ and ν vanishes on sets of zero M^{d-1+s}.

Therefore

$$B_\delta(f) = 0.$$

By Lemma 2.1 $\|D_\delta\|_s \le C\|b\|_s$, so there is a sequence $\delta_n \to 0$ and $D \in \Lambda^s(\mathbb{R}^d)$ such that $D_{\delta_n} \to D$ weak-$*$ in $\Lambda^s(\mathbb{R}^d)$. Thus $b(f) = b(D)$. We claim now that

$$D_\delta \to 0 \qquad \text{as } \delta \to 0,$$

uniformly on compact subsets of $(\partial K)^c$. Thus D vanishes on $(\partial K)^c$ and by continuity D vanishes on the whole of $\mathbb{R}^d$. Hence $b(f) = 0$.

To prove the claim consider a compact $H \subset (\partial K)^c$ and let l be the distance from H to ∂K. Fix $x \in H$ and define for $n = 0, 1, 2, \ldots$

$$A_n = \{y : n\delta \le |x - y| < (n+1)\delta\} \quad \text{and} \quad J_n = \{j \in J : a_j \in A_n\}.$$

Since the (B_j) is an almost disjoint family of balls, $\sharp J_n \le C n^{d-1}$. Taking $\delta < \ell/3$, $|x - a_j| > \ell - \delta > 2\delta$, we can apply for each $j \in J$ the decay lemma (see [22, p. 163]) and so

$$|b_j(x) - P_j(x)| \le C\delta^{d+s}|x - a_j|^{-d}\|b\|_s, \text{ when } |x - a_j| > 2\delta.$$

Therefore

$$|D_\delta(x)| \le \sum_{j\in J} |b_j(x) - P_j(x)| = \sum_{n=N_x}^{M_x} \sum_{j\in J_n} |b_j(x) - P_j(x)|,$$

where

$$N_x = \left[\frac{\operatorname{dist}(x,K)}{\delta}\right], \quad \text{and} \quad M_x = \left[\frac{\operatorname{dist}(x,K)+\operatorname{diam}(K)}{\delta}\right]+1.$$

Next

$$|D_\delta(x)| \le C\sum_{N_x}^{M_x} n^{d-1}\frac{\delta^{d+s}}{(n\delta)^d}\|b\|_s \le C\delta^s\|b\|_s \log\left(\frac{M_x}{N_x}\right)$$

$$\le C\delta^s\|b\|_s\log\left(2\left(1+\frac{\operatorname{diam}(K)}{\ell}\right)\right) \to 0 \qquad \text{as } \delta\to 0,$$

and the claim is proved.

SECOND CASE. Taking into account [22, p. 184] we can construct a function

$$P_j = \mu_j * \mathrm{I}_2,$$

where μ_j is a Borel measure with compact support contained in K such that $|\mu_j|(B(x,r)) \le Cr^{d-2+s}$, $x\in\mathbb{R}^d$, $r>0$, satisfying

$$b_j = P_j + O(|x|^{1-d}) \qquad \text{as } x\to\infty$$

and

$$\|P_j\|_s \le C\|b_j\|_s \le C\|b\|_s.$$

Set

$$\sum_{j\in J} b_j = \sum_{j\in J} P_j + \sum_{j\in J}(b_j - P_j) \equiv B_\delta + D_\delta,$$

so that, as before,

$$b(f) = D_\delta(f).$$

Again it is enough to show that $D_\delta\to 0$ uniformly on compact subsets of $(\partial K)^c$.

Given a compact $H\subset(\partial K)^c$ and $x\in H$, by the decay lemma we have

$$|D_\delta(x)| \le \sum_{n=N_x}^{M_x} |b_j - P_j|(x) \le C\delta^s\|b\|_s\frac{\operatorname{diam}(K)}{\delta} \to 0 \qquad \text{as } \delta\to 0$$

because $s>1$, and this ends the proof of the second case.

THIRD CASE. As in Theorem 4.1, setting, for each $j \in J$,

$$P_j = \mu_j * \mathrm{I}_2,$$

where μ_j is some Borel measure with compact support contained in K such that $|\mu_j|(B(x,r)) \leq Cr^{d-1}$, $x \in \mathbb{R}^d$, $r > 0$, we get

$$b_j - P_j = O(|x|^{1-d}) \qquad \text{as } x \to \infty,$$

and

$$\|P_j\|_1 \leq C\|b_j\|_1 \leq C\|b\|_1.$$

This time we need a zero of order d at ∞, so we set

$$(b_j - P_j)(x) = \sum_{i=1}^{d} C_j^i \partial^i \mathrm{I}_2(x - a_j) + O(|x|^{-d}) \qquad \text{as } x \to \infty,$$

a_j being the center of B_j. It is easy to see that $|C_j^i| \leq C\delta^d \|b_j - P_j\|_1 \leq C\delta^d\|b\|_1$. Let $\psi_j \in C_0^\infty(B_j)$, $\int \psi_j = 1$ and $\|\psi_j\|_\infty \leq C\delta^{-d}$. Define

$$H_j = \sum_{i=1}^{d} C_j^i(\partial^i \mathrm{I}_2 * \psi_j).$$

Thus

$$b_j = P_j + H_j + O(|x|^{-d}) \qquad \text{as } x \to \infty$$

and by (2.9)

$$\|H_j\|_1 \leq C\|b\|_1.$$

Set

$$\begin{aligned}\sum_{j\in J} b_j &= \sum_{j \in J}(P_j + H_j) + \sum_{j\in J}(b_j - P_j - H_j) \\ &\equiv B_\delta + D_\delta.\end{aligned}$$

Then

$$b(f) = B_\delta(f) + D_\delta(f).$$

It is clear that $\langle P_j * \mathrm{I}_2, \Delta f\rangle = \langle \mu_j, f\rangle = 0$. On the other hand, using that (B_j) is an almost disjoint family and $\|C_j^i\psi_j\|_\infty \leq C\|b\|_1$ we have

$$|\langle \sum_{j\in J} H_j, \Delta f\rangle| \leq C\|b\|_1 \int_{\cup B_j \setminus K} |\nabla f| \to 0 \qquad \text{as } \delta \to 0,$$

because $|\nabla f| \in L^1(\mathbb{R}^d)$ and $\bigcup B_j$ decreases to K.

Hence $B_\delta(f) \to 0$ as $\delta \to 0$.

As in the preceeding cases we can also show that $D_\delta(f) \to 0$ as $\delta \to 0$, thus completing the proof of Theorem 2. □

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