H^p-theory for Generalized M-harmonic Functions in the Unit Ball

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ABSTRACT. In this paper we study the space of functions in the unit ball in $\mathbf{C}^n$ annihilated by the differential operators $\Delta_{\alpha,\beta}$, α, $\beta \in \mathbf{C}$, given by

$$\Delta_{\alpha,\beta} = (1-|z|^2)\left\{\sum_{i,j}(\delta_{i,j} - z_i\bar{z}_j)D_i\overline{D}_j + \alpha R + \beta\overline{R} - \alpha\beta\right\}.$$

We obtain growth estimates and several equivalent characterizations of those such functions having boundary values in $H^p(\mathbf{S}^n)$, in terms of maximal and area functions.

1. Statement of the problems and results

1.1. Let $\mathbf{B}^n$ denote the unit ball in $\mathbf{C}^n$, $\mathbf{S}^n$ its boundary. In [**Ge**] Geller introduced a family of differential operators,

$$\Delta_{\alpha,\beta} = (1-|z|^2)\left\{\sum_{i,j}(\delta_{i,j} - z_i\bar{z}_j)D_i\bar{D}_j + \alpha R + \beta\bar{R} - \alpha\beta\right\}$$

where $D_i = \partial/\partial z_i$, and R is the radial derivative given by $R = \sum_i z_i D_i$. If $\alpha = \beta = 0$, $\Delta_{0,0}$ is the invariant laplacian or *Bergman laplacian*. It can be shown that $\Delta_{\alpha,\alpha}$ is the laplacian with respect to the weighted Bergman metric, with weight $(1-|z|^2)^\alpha$ (see Section 2). The functions annihilated by $\Delta_{0,0}$ are called *invariantly harmonic* or *M-harmonic* (see [**Ru**, Chapter 4] for general properties of these functions). We will call (α,β)-*harmonic* the functions u such that $\Delta_{\alpha,\beta}u=0$.

With $\Delta_{\alpha,\beta}$ there is associated a kernel

$$P_{\alpha,\beta}(z,\zeta)=c_{\alpha,\beta}\frac{(1-|z|^2)^{n+\alpha+\beta}}{(1-z\bar\zeta)^{n+\alpha}(1-\zeta\bar z)^{n+\beta}},\qquad z\in\mathbf{B}^n,\ \zeta\in\mathbf{S}^n,$$

where

$$c_{\alpha,\beta}=\frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n)\Gamma(n+\alpha+\beta)}.$$

If $\operatorname{Re}(n+\alpha+\beta)>0$, and neither $n+\alpha$ nor $n+\beta$ is zero or a negative integer, $P_{\alpha,\beta}$ is an approximation of the identity, and if f is continuous on $\mathbf{S}^n$, the function $P_{\alpha,\beta}[f]$ defined on $\mathbf{B}^n$ by

$$P_{\alpha,\beta}[f](z)=\int_{\mathbf{S}^n}P_{\alpha,\beta}(z,\zeta)\,f(\zeta)\,d\sigma(\zeta),$$

solves the Dirichlet problem for $\Delta_{\alpha,\beta}$ with boundary values equal to f (see Section 2.1).

The operators $\Delta_{\alpha,\beta}$ appear in a natural way when we consider certain derivatives of M-harmonic functions. It is proved in [**Ge**], (see also [**ACa**]), that $\Delta_{\alpha,\beta}u=0$ implies $\Delta_{\alpha,\beta-1}(Ru-\beta u)=0$ (in particular, radial derivatives of M-harmonic functions are no longer M-harmonic). The operators $\Delta_{\alpha,\beta}$ also appear when computing the Laplace-Beltrami operator on forms.

In [**Ge**], Geller studied the space of functions in the Siegel upper half-plane harmonic with respect to the corresponding invariant laplacian. He obtained several characterizations of functions in such a space with boundary values in the Hardy space $H^p(\mathbf{H}^n)$, $p\geq 1$, and some partial results for $p<1$, where $\mathbf{H}^n$ is the Heisenberg group.

In this paper we will deal with analogous questions in the context of the unit ball and for the laplacians $\Delta_{\alpha,\beta}$ for α,β satisfying $\operatorname{Re}(n+\alpha+\beta)>0$, $n+\alpha$, $n+\beta$ not zero or negative integer.

We will deal with the following expressions, defined for a smooth function u in $\mathbf{B}^n$:

(a) The *radial maximal function*

$$u^+(\zeta)=\sup\{|f(r\zeta)|\,;\,0\leq r<1\}.$$

(b) The *admissible maximal function*

$$M_\delta[u](\zeta)=M[u](\zeta)=\sup\{|f(z)|\,;\,z\in\mathcal{A}_\delta(\zeta)\}.$$

(c) The *admissible area function*

$$S[u](\zeta)=\left\{\int_{\mathcal{A}_\delta(\zeta)}\left\|\nabla_{\mathbf{B}}u(z)\right\|_{\mathbf{B}}^2\,d\lambda(z)\right\}^{1/2}.$$

Here, in (b) and (c), $\mathcal{A}_\delta(\zeta)$ is the admissible approach region given by

$$\mathcal{A}_\delta(\zeta) = \{z \in \mathbf{B}^n\,;\, |1 - z\bar{\zeta}| < \delta(1 - |z|^2)\},$$

$$d\lambda(z) = \frac{1}{(1-|z|^2)^{n+1}}\, dV(z),$$

dV denoting Lebesgue measure, and $\big\|\nabla_{\mathbf{B}} u\big\|_{\mathbf{B}}$ is the Bergman length of the Bergman gradient, given in coordinates by

$$\|\nabla_{\mathbf{B}} u\|_{\mathbf{B}}^2 = (1-|z|^2)\Big\{\sum_{i=1}^n |D_i u|^2 - \Big|\sum_{i=1}^n z_i D_i u\Big|^2 + \sum_{i=1}^n |\bar{D}_i u|^2 - \Big|\sum_{i=1}^n \bar{z}_i \bar{D}_i u\Big|^2\Big\}.$$

The aim of this paper is to prove the (expected) result that for an (α,β)-harmonic function u, if one of the functions u^+, $M_\delta[u]$, $S[u]$ belongs to $L^p(\mathbf{S}^n)$, $0 < p < +\infty$, so do the other two, and that this fact is equivalent to $u = P_{\alpha,\beta}[f]$, where f is in the *atomic Hardy space* $H^p(\mathbf{S}^n)$, as defined in [**GaLa**], which equals $L^p(\mathbf{S}^n)$ for $p > 1$.

There are certain serious technical difficulties in adapting the proofs of [**GaLa**] or [**Ge**] (modelled after Fefferman-Stein fundamental paper [**FeSt**] for the euclidean case) to the present situation. The same comment applies to Uchiyama's papers on H^p-spaces. Our setting falls within the situation considered in [**U**], general homogeneous spaces, but unfortunately the main result there on maximal characterization of H^p-spaces does not apply in our case. There is also a related paper of Arai [**A**] in which he obtains similar results to ours for certain real coercive operators with respect to the Bergman metric of a general strictly pseudoconvex domain. It can be proved that $\Delta_{\alpha,\beta}$ is indeed coercive in this sense exactly when $\operatorname{Re}(n+\alpha+\beta) > 0$, but only $\Delta_{0,0}$ is covered by the results of Arai.

Instead we combine explicit formulae with results from the theory of *tent spaces*, as developped in [**CoMeSt**]; strictly speaking, we use the analogue of this theory for the ball or a general homogeneous space. We also use a version of the $T(1)$-theorem due to Christ and Journé [**ChJo**]. The proof is done in Section 4. Other ingredients of the proof are the existence of developments of (α,β)-harmonic functions in terms of hypergeometric functions, mean-value estimates (Section 2) and a couple of Green formulas for the laplacians $\Delta_{\alpha,\beta}$ (Section 3).

1.2. The following notations and facts will be used. First, there is a “radial-tangential” expression for $\Delta_{\alpha,\beta}$ obtained in [**Ge**] that will be often used:

$$\Delta_{\alpha,\beta} = (1-|z|^2)\left\{\frac{1}{|z|^2}\left((1-|z|^2)R\bar{R} - \mathcal{L}_0 + \frac{n-1}{2}(R+\bar{R})\right) + \alpha R + \beta\bar{R} - \alpha\beta\right\}, \tag{1.1}$$

where

$$L_{ij} = \bar{z}_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial z_i}.$$

and $\mathcal{L}_0 = -\frac{1}{2}(\sum_{i<j} \bar{L}_{ij}L_{ij} + L_{ij}\bar{L}_{ij})$.

We denote by $(u,v)_{\mathbf{B}}$ the inner product

$$(u,v)_{\mathbf{B}}(z) = (1-|z|^2)\Big(\sum_{i,j}(\delta_{ij} - z_i\bar{z}_j)u_i\bar{v}_j\Big).$$

Note that $\|\nabla_{\mathbf{B}}u\|^2_{\mathbf{B}} = (Du, Du)_{\mathbf{B}} + (D\bar{u}, D\bar{u})_{\mathbf{B}}$. A computation shows that

$$\big\|\nabla_{\mathbf{B}}u\big\|^2_{\mathbf{B}} = (1-|z|^2)\frac{1}{|z|^2}\left\{(1-|z|^2)(|Ru|^2 + |\bar{R}u|^2) + \sum_{i<j}|L_{ij}u|^2 + \sum_{i<j}|\bar{L}_{ij}u|^2\right\},$$

which exhibits the usual non-isotropic behaviour of this gradient.

A number of hypergeometric functions will appear throughout. We use the classical notation $F(a,b,c;x)$ to denote

$$F(a,b,c;x) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b+k)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+k)}\frac{x^k}{k!},$$

$c \neq 0, -1, -2, \ldots$. We refer to [**Er**] for the theory of these functions.

Finally $B(z,\delta)$ will denote, for $z \in \mathbf{S}^n$, the non-isotropic ball in $\mathbf{S}^n$ given by $\{\zeta \in \mathbf{S}^n\,;\, |1-z\bar{\zeta}| < \delta\}$, and $\hat{B}(z,\delta) = \{w \in \mathbf{B}^n\,;\, |1-z\bar{w}| < \delta\}$ is the admissible tent over $B(z,\delta)$. Here and throughout the paper we use $\zeta\bar{\eta}$ to denote the usual hermitian inner product. The notation $r\mathbf{B}^n$ will stand for the closed ball of radious r.

2. Some preliminary results

2.1. Our first step will be to prove that any function u such that $\Delta_{\alpha,\beta}u = 0$ has a series expansion in homogeneous polynomials. We denote by $H(p,q)$ the space of homogeneous harmonic polynomials of bidegree (p,q).

Theorem 2.1 *Let α, $\beta \in \mathbf{C}$, and let u be a C^2 function in $\mathbf{B}^n$ satisfying $\Delta_{\alpha,\beta}u = 0$. Then*

$$u(r\zeta) = \sum_{p,q} F_{p,q}(r^2)u_{p,q}(r\zeta),$$

where $F_{p,q}(x)$ is the hypergeometric function given by

$$F_{p,q}(x) = F(p-\beta\ ,\ q-\alpha;\ p+q+n;\ x),$$

$u_{p,q} \in H(p,q)$, and where the series converges uniformly and absolutely on compact sets in $\mathbf{B}^n$.

Proof. For each $0 < r < 1$ the L^2-decomposition in harmonic polynomials of $u(r\zeta)$ (see [**Ru**, page 256]) gives that

$$u(r\zeta) = \sum_{p,q} \int_{\mathbf{S}^n} K_{pq}(\zeta\overline{\eta})\, u(r\eta)\, d\sigma(\eta),$$

where $K_{pq}(\zeta,\eta) = K_{pq}(\zeta\overline{\eta})$ is the orthogonal projection of $L^2(\mathbf{S}^n)$ onto $H(p,q)$.

Next, let $\zeta \in \mathbf{S}^n$ and p, $q \in \mathbf{Z}^+$ be fixed, and for $\lambda \in \mathbf{D}$, let

$$f_\zeta(\lambda) = \int_{\mathbf{S}^n} K_{pq}(\zeta\overline{\eta})\, u(\lambda\eta)\, d\sigma(\eta).$$

Since $\Delta_{\alpha,\beta} u = 0$, the "radial-tangential" expression of $\Delta_{\alpha,\beta}$ (see (1.2)) gives that

$$\begin{aligned} (2.1) \qquad & (1-|\lambda|^2)|\lambda|^2 \frac{\partial^2 f_\zeta}{\partial\lambda\partial\overline{\lambda}} + \frac{(n-1)}{2}\left(\lambda\frac{\partial f_\zeta}{\partial\lambda} + \overline{\lambda}\frac{\partial f_\zeta}{\partial\overline{\lambda}}\right) \\ & \qquad + |\lambda|^2\left(\alpha\lambda\frac{\partial f_\zeta}{\partial\lambda} + \beta\overline{\lambda}\frac{\partial f_\zeta}{\partial\overline{\lambda}}\right) - |\lambda|^2\alpha\beta f_\zeta \\ = & \int_{\mathbf{S}^n} K_{pq}(\zeta\overline{\eta})\, \mathcal{L}_0 u(\lambda\eta)\, d\sigma(\eta) \\ = & \int_{\mathbf{S}^n} \mathcal{L}_0 K_{pq}(\zeta\overline{\eta})\, u(\lambda\eta)\, d\sigma(\eta) \\ = & \ c_{pq} \int_{\mathbf{S}^n} K_{pq}(\zeta\eta)\, u(\lambda\eta)\, d\sigma(\eta) = c_{pq} f_\zeta(\lambda), \end{aligned}$$

where in the second equality we have used that $\mathcal{L}_0$ is a self-adjoint operator and in the previous to the last identity, that $\mathcal{L}_0 H(p,q) = c_{pq} H(p,q)$, with $c_{pq} = pq + (n-1)(p+q)/2$ (see [**ABr**, page 138]).

Since for $0 < r < 1$, $0 \le \theta \le 2\pi$, $f_\zeta(re^{i\theta}) = e^{i(p-q)\theta} f_\zeta(r)$, the function $f_\zeta(\lambda)/(\lambda^p\overline{\lambda}^q)$ is radial. Hence there exists $g(x)$ defined on $0 < x < 1$ such that $f_\zeta(\lambda) = \lambda^p\overline{\lambda}^q g(|\lambda|^2)$. Expressing in terms of g and its derivatives, and writing $x = |\lambda|^2$, we obtain

$$\begin{aligned} & (1-x)x^2 g''(x) + x\left[(1-x)(p+q+1) + (n-1) + (\alpha+\beta)x\right] g'(x) + \\ & \qquad \left(pq + \frac{n-1}{2}(p+q) - c_{pq} - x(pq - \alpha p - \beta q + \alpha\beta)\right) g(x) = 0. \end{aligned}$$

Now, inserting the definition of c_{pq} in the equation above, we deduce that g satisfies the hypergeometric equation

$$(1-x)xg''(x) + ((p+q+n) - x(p+q+1-\alpha-\beta))\, g'(x) - (p-\beta)(q-\alpha)g(x) = 0. \tag{2.2}$$

As a consequence of Frobenius' Theorem, every solution of this equation is a linear combination of two functions $g_1(x)$, $g_2(x)$, whose behaviour at $x=0$ is respectively like 1 and $x^{1-p-q-n}$ (when $n=1$ and $p=q=0, x^{1-p-q-n}$ must be replaced by $\ln x$). Since clearly $g(x)$ is bounded near zero while g_2 is not, we conclude that g is a multiple of $g_1(x)$, which is known to be the hypergeometric function $F(p{-}\beta\,, q{-}\alpha\,, p+q+n;\ x)$. Hence,

$$g(x) = C_{pq}(\zeta)\, F(p{-}\beta\,, q{-}\alpha;\ p+q+n;\ x),$$

for some constant $C_{pq}(\zeta)$. Therefore,

$$f_\zeta(\lambda) = C_{pq}(\zeta)\lambda^p\bar{\lambda}^q F(p{-}\beta\,, q{-}\alpha;\ p+q+n;\ |\lambda|^2).$$

This last expression, together with the definition of f_ζ gives that for each fixed $0<r<1$, the function $G(\zeta) = f_\zeta(r)$ is in $H(p,q)$, and consequently, that there exists $u_{pq} \in H(p,q)$ so that $C_{pq}(\zeta) = u_{pq}(\zeta)$. Thus

$$u(z) = \sum_{p,q} F(p{-}\beta\,, q{-}\alpha;\ p+q+n;\ |z|^2)\, u_{pq}(z).$$

Since u is regular, each term in the above expansion satisfies an adequate estimate on compact sets of $\mathbf{B}^n$ that assures the absolute and uniform convergence of the series (see [**St**, Apendix C]). □

Let us look in detail at the solvability of the Dirichlet problem for $\Delta_{\alpha,\beta}$

$$\Delta_{\alpha,\beta} u = 0,\ \ u \in \mathcal{C}(\bar{\mathbf{B}}^n),\ \ u = \varphi \ \text{ on } \mathbf{S}^n,\ \ \varphi \in \mathcal{C}(\mathbf{S}^n).$$

Theorem 2.2 ([**Ge**]) *The Dirichlet problem has a solution for all* $\varphi \in \mathcal{C}(\mathbf{S}^n)$ *if and only if* $\mathrm{Re}\,(n+\alpha+\beta) > 0$ *and* $n+\alpha \notin \mathbf{Z}_-$, $n+\beta \notin \mathbf{Z}_-$. *In this case the solution is unique and is given by*

$$u(z) = \int_{\mathbf{S}^n} \varphi(\zeta) P_{\alpha,\beta}(z,\zeta)\, d\sigma(\zeta) = P_{\alpha,\beta}[\varphi](z) \tag{2.3}$$

with

$$P_{\alpha,\beta}(z,\zeta) = c_{\alpha,\beta} \frac{(1-|z|^2)^{n+\alpha+\beta}}{(1-z\bar{\zeta})^{n+\alpha}(1-\bar{z}\zeta)^{n+\beta}},\quad c_{\alpha,\beta} = \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!\Gamma(n+\alpha+\beta)}$$

or, alternatively, by

$$u(z)=\sum_{p,q} f_{pq}(r^2)r^{p+q}\varphi_{pq}(\zeta),\ z=r\zeta, \tag{2.4}$$

if $\varphi=\sum_{p,q}\varphi_{pq}$ *is the expansion of* φ*, with*

$$f_{pq}(x)=\frac{F_{pq}(x)}{F_{pq}(1)}.$$

Proof. Suppose that the Dirichlet problem has a solution for all $\varphi\in\mathcal{C}(\mathbf{S}^n)$. Take $\varphi_{pq}\in H(p,q)$, $\varphi_{pq}\not\equiv 0$ and let u be a solution of the Dirichlet problem. By Theorem 1,

$$u(r\zeta)=\sum_{p'q'} F_{p'q'}(r^2)u_{p'q'}(r\zeta),$$

and hence

$$\int_{\mathbf{S}^n} u(r\zeta)\overline{\varphi_{pq}(\zeta)}\,d\sigma(\zeta)=F(p{-}\beta\ ,\ q{-}\alpha\ ,\ p+q+n;\ r^2)\langle u_{pq},\varphi_{pq}\rangle.$$

Since the left-hand side has limit $\|\varphi_{pq}\|_2^2$, it follows that

$$\lim_{r\to 1} F(p{-}\beta\ ,\ q{-}\alpha, p+q+n; r^2)$$

exists and is not zero. From [**Er**] we know that if $\mathrm{Re}\,(c-a-b)\le 0$, the hypergeometric function $F(a,b,c,x)$ has a limit at 1 only if a or b is a non-positive integer. Taking p,q large enough it follows that we must have

$$\mathrm{Re}\,(p+q+n-(p{-}\beta)-(q{-}\alpha))=\mathrm{Re}\,(n+\alpha+\beta)>0.$$

In this case, the limit above is

$$\frac{\Gamma(p+q+n)\Gamma(n+\alpha+\beta)}{\Gamma(n+q+\alpha)\Gamma(n+p+\beta)},$$

and this is non-zero for all p, q if and only if $n+\alpha$, $n+\beta$ are not zero or negative integers.

The proof of Theorem 2.1 shows that

$$F_{pq}(r^2)r^{p+q}u_{pq}(\zeta)=\int_{\mathbf{S}^n} K_{pq}(\zeta\bar{\eta})u(r\eta)\,d\sigma(\eta),$$

and letting $r\to 1$ we see that $F_{pq}(1)u_{pq}=\varphi_{pq}$ which shows unicity and establishes formula (2.4). To show formula (2.3) one can argue as follows. By direct computation, one first shows that $P_{\alpha,\beta}[\varphi]$ is (α,β)-harmonic. It is also clear that

$P_{\alpha,\beta}[1]$ is radial, hence with the notations above it is a multiple of $F_{00}(|z|^2)$. Comparing the values at 0 we conclude that $P_{\alpha,\beta}[1](z) = c_{\alpha,\beta}F_{00}(|z|^2)$. The above choice of $c_{\alpha,\beta}$ makes $P_{\alpha,\beta}$ an approximation of the identity as $|z| \to 1$, and therefore $P_{\alpha,\beta}[\varphi]$ is the solution of the Dirichlet problem (see 3.3 for another direct argument). □

A slight variant of the argument above shows that if $\operatorname{Re}(n+\alpha+\beta) \leq 0$, then, except for special values of α, β, no "reasonable" function is annihilated by $\Delta_{\alpha,\beta}$.

Proposition 2.3 *Suppose* $\operatorname{Re}(n+\alpha+\beta) \leq 0$ *and neither* α *nor* β *is a non-negative integer; then* $u \equiv 0$ *is the only* $u \in \mathcal{C}^2(\mathbf{B})$ *such that*

(i) $\Delta_{\alpha,\beta}u = 0$, *and*
(ii) $\sup_{0<r<1} \int_{\mathbf{S}^n} |u(r\zeta)|\, d\sigma(\zeta) < +\infty$.

Proof. As before

$$u(r\zeta) = \sum_{pq} f_{pq}(r^2) r^{p+q} u_{pq}(\zeta).$$

Multiplying by $\bar{u}_{pq}$ and integrating, we get

$$f_{pq}(r^2)r^{p+q}\|u_{pq}\|_2^2 = \left| \int_{\mathbf{S}^n} u(r\zeta)\overline{u_{pq}(\zeta)}\, d\sigma(\zeta) \right| \leq \int_{\mathbf{S}^n} |u(r\zeta)|\, d\sigma(\zeta)\ \|u_{pq}\|_\infty.$$

Under our assumptions on α, β,

$$f_{pq}(r^2) \to \infty \text{ as } r \to 1,$$

and hence $\|u_{pq}\|_2 = 0$ for all p, q. □

We point out without giving the full details that if α or β is a non-negative integer, then there are always bounded ($\not\equiv 0$) solutions to $\Delta_{\alpha,\beta}u = 0$. For example if $\alpha = 0$, then any holomorphic function u is a solution to $\Delta_{0\beta}u = 0$.

We do not know whether the proposition still holds under the assumption

$$\sup_{0<r<1} \int_{\mathbf{S}^n} |u(r\zeta)|^p\, d\sigma(\zeta) < +\infty$$

for some $p < 1$.

When $\operatorname{Re}(n+\alpha+\beta) > 0$ and either $n+\alpha$ or $n+\beta$ is zero or a negative integer, the proof of Theorem 2.2 shows that for the Dirichlet problems to have a solution it is necessary that the boundary data φ have zero components in certain of the $H(p,q)$.

Proposition 2.4 *Let u be (α,β)-harmonic, and assume* $\mathrm{Re}\,(n+\alpha+\beta)>0$. *Then:*

(i) $u=P_{\alpha,\beta}[f]$ *for some* $f\in L^p(\mathbf{S}^n)$, $1<p<+\infty$ *if and only if*

$$\sup_{0<r<1}\int_{\mathbf{S}^n}|u(r\zeta)|^p\,d\sigma(\zeta)<+\infty. \tag{2.5}$$

In this case, u has admissible limit f a.e. and $M[u]\in L^p(\mathbf{S}^n)$.

(ii) $u=P_{\alpha,\beta}[\mu]$ *for some measure μ if and only if*

$$\sup_{0<r<1}\int_{\mathbf{S}^n}|u(r\zeta)|\,d\sigma(\zeta)<+\infty. \tag{2.6}$$

In this case, u has admissible limit $d\mu/d\sigma$ a.e. Moreover, if $M[u]\in L^1(\mathbf{S}^n)$, then $d\mu$ is absolutely continuous.

Proof. If $u=P_{\alpha,\beta}[f]$, obviously by Hölder's inequality

$$\begin{aligned} |u(z)|^p &\le \int_{\mathbf{S}^n}|P_{\alpha,\beta}(z,\zeta)||f(\zeta)^p\,d\sigma(\zeta) \\ &= |c_{\alpha,\beta}|\int_{\mathbf{S}^n}\frac{(1-|z|^2)^{n+\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta}}{|1-z\bar{\zeta}|^{2n+\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta}}|f(\zeta)|^p\,d\sigma(\zeta), \end{aligned} \tag{2.7}$$

and then (2.5) follows from [**Ru**, Proposition 1.4.10]. In the other direction, the fact that the L^p-norms are uniformly bounded gives that there exists $\varphi\in L^p(\mathbf{S}^n)$ and a sequence $r_m\to 1$ such that $u(r_m\zeta)\to\varphi(\zeta)$, as $m\to+\infty$ weakly in $L^p(\mathbf{S}^n)$. In particular, for each $z\in\mathbf{B}^n$ fixed, by Theorem 2.1,

$$\begin{aligned} P_{\alpha,\beta}[\varphi](z) &= \lim_{m\to+\infty}\int_{\mathbf{S}^n}P_{\alpha,\beta}(z,\zeta)u(r_m\zeta)\,d\sigma(\zeta) \\ &= \lim_{m\to+\infty}\sum_{pq}f_{pq}(r_m^2)r_m^{p+q}\int_{\mathbf{S}^n}P_{\alpha,\beta}(z,\zeta)u_{pq}(\zeta)\,d\sigma(\zeta) \\ &= \lim_{m\to+\infty}\sum_{pq}f_{pq}(r_m^2)r_m^{p+q}F_{pq}(|z|^2)u_{pq}(z) \\ &= \sum_{p,q}f_{pq}(|z|^2)u_{pq}(z)=u(z). \end{aligned}$$

From the explicit formula for $P_{\alpha,\beta}$ one easily obtains, as in the classical case, that $M[u]$ is dominated by the Hardy-Littlewood maximal function of f. This implies that $M[u]\in L^p(\mathbf{S}^n)$, and the existence of admissible limits is proved in the standard way.

The first part of (ii) is proved similarly. If $M[u]\in L^1(\mathbf{S}^n)$, then the convergence of u_r is dominated, hence its weak limit $d\mu$ is absolutely continuous. □

2.2. In this section we study sub-mean value properties of the functions annihilated by $\Delta_{\alpha,\beta}$ with no restrictions on α, β. We begin with some similar to those found by Geller. For each $z \in \mathbf{B}^n$ we denote $E(z)$ the Bergman ball of radious $\frac{1}{2}$ centered at z, i.e. $E(z) = \varphi_z(\frac{1}{2}\mathbf{B}^n)$, where φ_z is the automorphism of the ball that maps z to 0, and such that $\varphi_z^2 = \mathrm{Id}$ (see [**Ru**, pg 297]).

Lemma 2.5 *There exists $C = C(\alpha, \beta)$ such that if $\Delta_{\alpha,\beta} u = 0$, then*

$$|u(z)| \leq \frac{C}{(1-|z|)^{n+1}} \int_{E(z)} |u(\omega)|\, dV(\omega).$$

Proof. Using Theorem 2.1 (see also [**Ge**, Theorem 1.1]), for each r, $0 < r < 1$

$$F_{00}(r^2)u(0) = \int_{\mathbf{S}^n} u(r\zeta)\, d\sigma(\zeta).$$

Hence, if $\psi(x)$ is a $\mathcal{C}^\infty$ function supported in $[0, \frac{1}{4}]$ and conveniently normalized,

$$u(0) = \int_{\mathbf{B}^n} u(\zeta)\psi(|\zeta|^2)\, d\lambda(\zeta).$$

It is proved in [**ACa**] that for each $z \in \mathbf{B}^n$, the function $h_z^{\alpha,\beta} \cdot (u \circ \varphi_z)$, where $h_z^{\alpha,\beta}(\zeta) = (1 - z\bar{\zeta})^\alpha (1 - \bar{z}\zeta)^\beta$, is also annihilated by $\Delta_{\alpha,\beta}$. Applying the above to it, we get

$$u(z) = \int_{\mathbf{B}^n} h_z^{\alpha,\beta}(\varphi_z(\omega)) u(\omega) \psi(|\varphi_z(\omega)|^2)\, d\lambda(\omega).$$

Given that

$$|\varphi_z(\omega)|^2 = 1 - \frac{(1-|z|^2)(1-|\omega|^2)}{|1 - z\bar{\omega}|^2}$$

and the fact that $|1 - z\bar{\omega}| \simeq (1 - |z|) \simeq (1 - |\omega|)$ when $|\varphi_z(\omega)| \leq \frac{1}{2}$, the relation is easily seen to hold. □

The following geometrical lemma will be needed:

Lemma 2.6 *Let $0 < r < \rho < 1$, $\omega \in \mathbf{B}^n$ with $|\omega| = r$ and let $\delta = \rho - r$. Then*

$$\varphi_\omega(\delta \mathbf{B}^n) \subset \rho \mathbf{B}^n.$$

Proof. It is enough to show that for any $\zeta \in \mathbf{B}^n$, $|\zeta| \leq \delta$, and $z = \varphi_w(\zeta)$, then $1 - \rho^2 \leq 1 - |z|^2$. But (see [**Ru**, Thm. 2.2.2.])

$$1 - |z|^2 = 1 - |\varphi_w(\zeta)|^2 = \frac{(1-|w|^2)(1-|\zeta|^2)}{|1 - w\bar{\zeta}|^2} \geq \frac{(1-r^2)(1-\delta^2)}{(1+r\delta)^2}.$$

And it is easy to check that if $0 < \delta = \rho - r$, the above is bounded from below by $1 - \rho^2$. □

The next lemma in the classical case is well known and due to Hardy-Littlewood (see also [**FeSt**]).

Lemma 2.7 *Let $0 < p < +\infty$. There exists $C = C(\alpha, \beta, p, n) > 0$, so that if $\Delta_{\alpha,\beta} u = 0$, then*

$$|u(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1}} \int_{E(z)} |u(w)|^p \, dV(w).$$

Proof. Since the case $p = 1$ is just Lemma 2.5, and $p > 1$ follows from this one by Hölder's inequality, we just need to deal with the case $0 < p < 1$. It is then enough to show that there exists $C > 0$ so that

$$|u(0)|^p \leq C \int_{\frac{1}{2}\mathbf{B}^n} |u(w)|^p \, dV(w),$$

since the general situation follows applying it to $h_z^{\alpha,\beta} \cdot (u \circ \varphi_z)$. This will be deduced once we show the following statement:

There exists $C > 0$ so that for all u satisfying $\Delta_{\alpha,\beta} u = 0$ and $\int_{(1/2)\mathbf{B}^n} |u(w)|^p \, dV(w) \leq 1$, we have

$$|u(0)| \leq C.$$

Let u be a function such that $\Delta_{\alpha,\beta} u = 0$ and $\int_{(1/2)\mathbf{B}^n} |u(w)|^p \, dV(w) \leq 1$. Observe that we also may assume that $|u(0)| > 1$, otherwise there is nothing to prove. Arguing similarly to the proof of Lemma 2.5, we deduce that for each $0 < \delta < \frac{1}{2}$,

$$|u(0)| \leq \frac{C}{\delta^{2n}} \int_{\delta \mathbf{B}^n} |u(w)| \, dV(w).$$

Next, for $z \in \mathbf{B}^n$ we apply the estimate above to $h_z^{\alpha,\beta} \cdot (u \circ \varphi_z)$, and we get

$$\begin{aligned} |u(z)| &\leq \frac{C}{\delta^{2n}} \int_{\delta \mathbf{B}^n} |h_z^{\alpha,\beta}(w)| \, |u(\varphi_z(w))| \, dV(w) \\ &\leq \frac{C}{\delta^{2n}} \int_{\delta \mathbf{B}^n} |u(\varphi_z(w))| \, d\lambda(w) \\ &= \frac{C}{\delta^{2n}} \int_{\varphi_z(\delta \mathbf{B}^n)} |u(w)| \, d\lambda(w). \end{aligned}$$

Now suppose $0 < r < \rho < \frac{1}{2}$ and let $z = r\zeta$. The estimate above with $\delta = \rho - r$ and Lemma 2.6 give

$$|u(r\zeta)| \leq \frac{C}{(\rho - r)^{2n}} \int_{\rho \mathbf{B}^n} |u(w)|\, d\lambda(w) \leq \frac{C}{(\rho - r)^{2n}} \int_{(1/2)\mathbf{B}^n} |u(w)|\, dV(w),$$

since $\rho < \frac{1}{2}$.

Let $m(r) = \sup_{|z| \leq r} |u(z)|$. Then we have that

$$|u(z)| \leq \frac{Cm(\rho)^{1-p}}{(\rho - r)^{2n}} \int_{(1/2)\mathbf{B}^n} |u(w)|^p\, dV(w) = \frac{Cm(\rho)^{1-p}}{(\rho - r)^{2n}},$$

where we have used that the L^p-norm is bounded by 1. Hence,

$$m(r) \leq Cm(\rho)^{1-p}(\rho - r)^{-2n},$$

provided $0 < r < \rho < \frac{1}{2}$. Taking logarithms and integrating, we obtain

$$\int_{1/4}^{1/2} \log m(r)\, \frac{dr}{r} \leq C + (1 - p) \int_{1/4}^{1/2} \log m(\rho)\, \frac{dr}{r} - 2n \int_{1/4}^{1/2} \log(\rho - r)\, \frac{dr}{r}$$

Next, choosing $\rho = 2^{a-1} r^a$, where $0 < a < 1$, we deduce

$$\int_{1/4}^{1/2} \log m(r)\, \frac{dr}{r} \leq C + \frac{(1 - \rho)}{a} \int_{1/4}^{1/2} \log m(r) \frac{dr}{r},$$

and, if a also satisfies $1 - (1 - p)/a > 0$, that

$$\int_{1/4}^{1/2} \log m(r)\, \frac{dr}{r} \leq C.$$

Finally $m(0) = |u(0)| \leq m(\frac{1}{4}) \leq C$. □

As a consequence of Lemma 2.7, we obtain two results on the growth of the functions u annihilated by $\Delta_{\alpha,\beta}$.

Lemma 2.8 *Let α, $\beta \in \mathbf{C}$, $0 < p < +\infty$ and u (α, β)-harmonic. Suppose*

$$\sup_{0<r<1} \int_{\mathbf{S}^n} |u(r\zeta)|^p\, d\sigma(\zeta) < +\infty.$$

Then

$$|u(z)| \leq \frac{C}{(1 - |z|)^{n/p}}.$$

Proof. For $z \in \mathbf{B}^n$, $E(z) \subset \{w\,;\, \varepsilon(1-|z|^2) \leq 1-|w|^2 \leq \frac{1}{\varepsilon}(1-|z|)^2\}$. Applying Lemma 2.7, we deduce

$$\begin{aligned} |u(z)|^p &\leq \frac{C}{(1-|z|)^{n+1}} \int_{E(z)} |u(w)|^p \, dV(w) \\ &\leq \frac{C}{(1-|z|^2)^{n+1}} \int_{\{r\,;\,\varepsilon(1-|z|^2)1-r^2 \leq 1/\varepsilon(1-|z|)^2\}} \int_{\mathbf{S}^n} |u(r\zeta)|^p \, d\sigma(\zeta) \, r^{2n-1} \, dr \\ &\leq \frac{C}{(1-|z|)^n}. \end{aligned}$$

□

Lemma 2.9 *Let α, $\beta \in \mathbf{C}$, $0 < p < +\infty$ and u (α,β)-harmonic. Suppose that $S[u] \in L^p(\mathbf{S}^n)$. Then*

$$|u(z)| \leq \frac{C\|S[u]\|_p}{(1-|z|)^{n/p}} + |u(0)|.$$

Proof. It is easy to check that if $\Delta_{\alpha,\beta} u = 0$, then for each $1 \leq i \leq n$,

$$\Delta_{\alpha,\beta-1} \frac{\partial u}{\partial z_i} = \Delta_{\alpha-1,\beta} \frac{\bar{\partial} u}{\partial \bar{z}_i} = 0.$$

Applying Lemma 2.7 to each partial derivative, we get

$$\begin{aligned} |\nabla u(z)|^2 &\leq \frac{C}{(1-|z|)^{n+1}} \int_{E(z)} |\nabla u(w)|^2 \, dV(w) \\ &\leq \frac{C}{(1-|z|)^2)^2} \int_{E(z)} (1-|w|^2)^{1-n} |\nabla u(w)|^2 \, dV(w) \\ &\leq \frac{C}{(1-|z|^2)^2} S[u]^2(\zeta), \end{aligned}$$

for all $\zeta \in X(z) = \{\zeta \in \mathbf{S}^n\,;\, E(z) \subset D(\zeta)\}$. Note that in the last inequality we use that $(1-|\omega|^2)^2|\nabla u|^2 \leq \|\nabla_{\mathbf{B}} u\|_2^2$.

Thus if $\zeta \in X(z)$, then

$$\big((1-|z|^2)|\nabla u(z)|\big)^p \leq CS[u](\zeta)^p.$$

Since $\sigma(X(z)) \simeq (1-|z|)^n$, integrating the estimate above, we have

$$(1-|z|^2)^{p+n}|\nabla u(z)|^p \leq C \int_{X(z)} S[u](\zeta)^p \, d\sigma(\zeta) \leq C\|S[u]\|_p^p.$$

In particular, for $0 < r < 1$ and $\zeta \in \mathbf{S}^n$,

$$\left|\frac{\partial u}{\partial r}(r\zeta)\right| \leq C\|S[u]\|_p (1-r)^{n/p+1},$$

which implies that

$$|u(r\zeta)| \leq \frac{C\|S[u]\|_p}{(1-r)^{n/p}} + |u(0)|. \qquad \square$$

2.3. Now we will show that when $p < 1$, and the L^p-norms are uniformly bounded, then u has also boundary values in the sense of distributions. We will need the following technical lemma:

Lemma 2.10 *Let $F \in \mathcal{C}^2([\frac{1}{2},1])$, and $h \in \mathcal{C}^1([\frac{1}{2},1])$ satisfying* $\operatorname{Re} h(1) > -1$. *Suppose that*

$$(1-x)F''(x) + h(x)F'(x) = \mathrm{O}(1-x)^{-A},$$

as $x \to 1$. Then:

(i) *If $A > 1$, $F(x) = \mathrm{O}(1-x)^{-A+1}$.*
(ii) *If $0 < A < 1$, then $\lim_{x\to 1} F(x)$ exists and is finite.*

Proof. If we define

$$\mu(x) = \exp \int_{1/2}^{x} \frac{h(t)}{1-t}\, dt,$$

then the equation gives that

$$(\mu F')'(x) = \mathrm{O}\left((1-x)^{-A-1}\mu(x)\right).$$

It is also easy to show that $\mu(x)$ behaves exactly as $(1-x)^{-h(1)}$. Indeed,

$$\begin{aligned}(1-x)^{h(1)}\mu(x) &= \exp\left(h(1)\log(1-x) + \int_{1/2}^{x}\frac{h(t)}{1-t}\,dt\right) \\ &\quad \times \exp\left(\int_{1/2}^{x}\frac{h(t)-h(1)}{1-t}\,dt + h(1)\log\frac{1}{2}\right),\end{aligned}$$

and since $h \in \mathcal{C}^1([\frac{1}{2},1])$, the limit as $x \to 1$ of the integral in the right hand side exists, and we obtain the desired conclussion.

Integrating and using the estimate above , we deduce that

$$|\mu(x)\, F'(x)| = \mathrm{O}\left(\int_{1/2}^{x}(1-t)^{-A-1-\operatorname{Re}h(1)}\,dt\right).$$

From this estimate the conclusion of the lemma easily follows. $\square$

Proposition 2.11 *Assume u is (α,β)-harmonic,* $\mathrm{Re}\,(n+\alpha+\beta)>0$, *and that for some $p<1$*

$$\sup_{0<r<1}\int_{\mathbf{S}^n}|u(r\zeta)|^p\,d\sigma(\zeta)<+\infty.$$

Then there exists a distribution ϕ satisfying:

(i) $\lim_{r\to 1}u(r\zeta)=\phi$, *in the sense of distributions,*
(ii) $u=P_{\alpha,\beta}[\phi]$.

Proof. Let $N=(R+\bar{R})/2$, $T=i\,(\bar{R}-R)/2$, then T is tangential, $TN=NT$ and $R=N+i\,T$, $\bar{R}=N-i\,T$. The expression in radial-tangential derivatives of the laplacian $\Delta_{\alpha,\beta}$ gives that the operator $(1/(1-|z|^2))\Delta_{\alpha,\beta}$ takes the form

$$\frac{1}{|z|^2}(1-|z|^2)\,N^2+(\alpha+\beta+n-1)\,N+\frac{(1-|z|^2}{|z|^2}\,T^2 - \mathcal{L}_0+i\,(\alpha-\beta)\,T-\alpha\beta. \tag{2.8}$$

Now suppose that $\Delta_{\alpha,\beta}u=0$ and $\varphi\in\mathcal{C}^\infty(\mathbf{S}^n)$. Define

$$F(r)=\int_{\mathbf{S}^n}u(r\zeta)\,\varphi(\zeta)\,d\sigma(\zeta).$$

Formula (2.8) together with the fact that $Nu(r\zeta)=(\frac{1}{2}r(\partial/\partial r))u(r\zeta)$, gives that

$$\frac{1}{r^2}(1-r^2)\left(\frac{1}{2}r\frac{\partial}{\partial r}\right)^2F(r)+(\alpha+\beta+n-1)\frac{1}{2}rF'(r)+\alpha\beta F(x) = \int_{\mathbf{S}^n}(Xu)(r\zeta)\,\varphi(\zeta)\,d\sigma(\zeta),$$

where

$$X=-\left\{\frac{1-r^2}{r^2}T^2-\mathcal{L}_0+i(\alpha-\beta)T\right\}$$

is a tangential derivative. Thus, writing $\psi=X^*\varphi\in\mathcal{C}^\infty(\mathbf{S}^n)$, with X^* the adjoint operator, we have

$$\frac{1}{r^2}(1-r^2)\left(rF'(x)+r^2F''(x)\right)+2\,(\alpha+\beta+n-1)\,rF'(x)+\alpha\beta F(x) = \int_{\mathbf{S}^n}u(r\zeta)\,\psi(\zeta)\,d\sigma(\zeta).$$

Iterating the process above, and if

$$L=(1-r^2)\frac{d^2}{dr^2}+\left(\frac{1-r^2}{r}+2(\alpha+\beta+n-1)r\right)\frac{d}{dr}+\alpha\beta,$$

we deduce that for each $k = 1, 2, \ldots$, there exists $\varphi_k \in \mathcal{C}^\infty(\mathbf{S}^n)$ so that

$$\left(L^k F\right)(r) = \int_{\mathbf{S}^n} u(r\zeta)\varphi_k(\zeta)\, d\sigma(\zeta).$$

Since

$$\sup_{0<r<1} \int_{\mathbf{S}^n} |u(r\zeta)|^p\, d\sigma(\zeta) < +\infty.$$

Lemma 2.8 gives that there exists $A > 0$ so that $u(r\zeta) = \mathrm{O}(1-r)^{-A}$. Then we have

$$L^k F(r) = \mathrm{O}(1-r)^{-A},$$

and applying Lemma 2.10 that

$$L^{k-1} F(r) = \mathrm{O}(1-r)^{-A+1}.$$

Iterating the process we deduce that $\lim_{r\to 1} F(r)$ exists.

Part (ii) follows similarly. □

3. Green's Formula for $\Delta_{\alpha,\beta}$

3.1. The first result concerns a Green's formula for our laplacians.

Theorem 3.12 *Let $u,\ v \in \mathcal{C}^2(\mathbf{B}^n)$, $r < 1$. Then*

$$\text{(3.1)}\quad \frac{\pi r^{2n-2}}{(n-1)!}(1-r^2)^{-\alpha-\beta-n+1} \int_{\mathbf{S}^n} \left(u\, Rv - v\, \bar{R}u\right)(r\zeta)\, d\sigma(\zeta)$$
$$= \int_{r\mathbf{B}^n} \left\{\Delta_{\beta,\alpha} v(z) u(z) - \Delta_{\alpha,\beta} u(z) v(z)\right\} (1-|z|^2)^{-\alpha-\beta}\, d\lambda(z).$$

Proof. We first deal with the case $\alpha = 0$. The proof will be based in the following formula (1^{st} Green identity)

$$\text{(3.2)}\quad \frac{\pi r^{2n-2}}{(n-1)!}(1-r)^{-\beta-n+1} \int_{r\mathbf{S}^n} v\, \bar{R}u\, d\sigma(\zeta)$$
$$= \int_{r\mathbf{B}^n} (Dv, D\bar{u})_{\mathbf{B}}\, (1-|z|^2)^{-\beta}\, d\lambda(z) + \int_{r\mathbf{B}^n} v(z)\Delta_{0,\beta} u(z)(1-|z|^2)^{-\beta}\, d\lambda(z).$$

Note that this says that $\Delta_{0,\beta}$ is essentially $\partial\partial^*$, where the adjoint ∂^* is taken with respect the Bergman metric weighted by the factor $(1-|z|^2)^{-\beta}$. To simplify notations we use

$$\rho = (1-|z|^2),\ \ \omega_i = d\bar{z}_i \bigwedge_{j\neq i} d\bar{z}_j \wedge dz_j,\ \ \omega = \bigwedge_{j=1}^{n} d\bar{z}_j \wedge dz_j = (2i)^n\, dV(z).$$

We apply Stokes' Theorem to the form $v\rho^{-\beta-n}\sum_{i=1}^n \bar D_i u\,\omega_i$:

$$\begin{aligned}(1-r^2)^{-\beta-n}&\int_{r\mathbf{S}^n} v\sum_{i=1}^n \bar D_i u\,\omega_i\\ = \quad -&\int_{r\mathbf{B}^n}\rho^{-\beta-n}\sum_{i=1}^n \bar D_i u D_i v\,\omega - \int_{r\mathbf{B}^n}\rho^{-\beta-n} v\sum_{i=1}^n D_i\bar D_i u\,\omega\\ &\qquad - (\beta+n)\int_{r\mathbf{B}^n}\rho^{-\beta-n-1} v\bar R u\,\omega.\end{aligned}$$

Next we apply Stokes' theorem to the form $v\bar R u\rho^{-\beta-n}\sum_{i=1}^n z_i\omega_i$:

$$\begin{aligned}(1-r^2)^{-\beta-n}&\int_{r\mathbf{S}^n} v\,\bar R u\sum_{i=1}^n z_i\omega_i\\ = \quad -&\int_{r\mathbf{B}^n}\rho^{-\beta-n}\bar R u\,R v\,\omega - n\int_{r\mathbf{B}^n}\rho^{-\beta-n} v\bar R u\,\omega\\ &- (\beta+n)\int_{r\mathbf{B}^n}\rho^{-\beta-n-1} v\bar R u|z|^2\,\omega - \int_{r\mathbf{B}^n}\rho^{-\beta-n} v\sum_{i,j}\bar z_i z_j\bar D_i D_j u\,\omega.\end{aligned}$$

Subtracting, we get

$$\begin{aligned}&\int_{r\mathbf{B}^n}(Dv, D\bar u)_{\mathbf{B}}\,\rho^{-\beta-n-1}\,\omega + \int_{r\mathbf{B}^n}\rho^{-\beta-n-1}\Delta_{0,\beta} u\,v\,\omega\\ = \quad &(1-r^2)^{-\beta-n}\int_{r\mathbf{S}^n} v\bar R u\sum_{i=1}^n z_i\omega_i - v\sum_{i=1}^n \bar D_i u\,\omega_i.\end{aligned}$$

If $z = r\zeta$, a computation shows that

$$\omega_i(z) = -r^{2n-1}\frac{(2\pi i)^n}{(n-1)!}\bar\zeta_i\,d\sigma.$$

Then

$$\sum_{i=1}^n z_i\omega_i = -r^{2n}\bar R u\frac{(2\pi i)^n}{(n-1)!}\,d\sigma,$$

and we obtain (3.2).

Applying (3.2) to the pair $(\bar v, \bar u)$ and $\Delta_{0,\bar\beta}$, instead of (u, v) and $\Delta_{0,\beta}$, and conjugating, we get the result for $\alpha = 0$. The general case will follow once we prove that

$$\int_{r\mathbf{B}^n}(v\Delta_{\alpha,\beta}u - u\Delta_{\beta,\alpha}v)\,\rho^{-\alpha-\beta}\,d\lambda = \int_{r\mathbf{B}^n}(v\Delta_{0,\alpha+\beta}u - u\Delta_{\alpha+\beta,0}v)\,\rho^{-\alpha-\beta}\,d\lambda.$$

Indeed, the difference of these expressions equals

$$\alpha \int_{r\mathbf{B}^n} \left(R(uv) - \bar{R}(uv)\right) \rho^{-\alpha-\beta}\, d\lambda,$$

which is obviously zero because $R - \bar{R}$ is tangential. □

3.2. From now on we will suppose that $\mathrm{Re}\,(n+\alpha+\beta) > 0$, $n+\alpha \notin \mathbf{Z}_-$, $n+\beta \notin \mathbf{Z}_-$. Next we introduce, following [**Gr**], the fundamental solution for $\Delta_{\alpha,\beta}$, i.e., the function $G_{\alpha,\beta}(z)$ playing the role of the Green's function in the classical potential theory. We look for a radial function $G_{\alpha,\beta}(z) = g_{\alpha,\beta}(|z|^2)$, which is (α,β)-harmonic away from zero. This means that $g_{\alpha,\beta}$ must satisfy Equation (2.2) for $p = q = 0$:

$$x(1-x)h''(x) + [n - (-\alpha-\beta+1)x]h'(x) - \alpha\beta h(x) = 0, \quad 0 < x < 1. \tag{3.3}$$

Of course, one solution of this equation is $F_{00}(x)$, the hypergeometric function with parameters $-\alpha$, $-\beta$, n. Now [**Er**, pg. 105-106], a second independent solution is

$$h(x) = (1-x)^{n+\alpha+\beta} F(n+\beta\,,\, n+\alpha\,,\, n+\alpha+\beta+1;\; 1-x).$$

It is well known that, if $n > 1$,

$$\begin{aligned} & F(n+\beta\,,\, n+\alpha\,,\, n+\alpha+\beta+1;\; x) \\ = \;& (1-x)^{1-n} F(\beta+1\,,\, \alpha+1\,,\, n+\alpha+\beta+1;\; x). \end{aligned}$$

This shows that $F(n+\beta\,,\, n+\alpha\,,\, n+\alpha+\beta+1;\; x)$ is equivalent to

$$(1-x)^{1-n} \frac{\Gamma(n+\alpha+\beta+1)\Gamma(n-1)}{\Gamma(n+\alpha)\Gamma(n+\beta)},$$

when $x \to 1$; if $n = 1$,

$$F(1+\beta\,,\, 1+\alpha\,,\, \alpha+\beta+2;\; x) = \left(\log \frac{1}{1-x}\right) f_2(x),$$

with

$$f_2(1) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

Let $g_{\alpha,\beta} = d_{\alpha,\beta} h$, with a constant $d_{\alpha,\beta}$ to be determined, i.e.,

$$G_{\alpha,\beta}(z) = d_{\alpha,\beta}(1-|z|^2)^{n+\alpha+\beta} F(n+\beta\,,\, n+\alpha\,,\, n+\alpha+\beta+1;\; 1-|z|^2).$$

By well-known properties of hypergeometric functions, it is easy to see that

$$RG_{\alpha,\beta}(z) = -d_{\alpha,\beta}(n+\alpha+\beta)|z|^2(1-|z|^2)^{n+\alpha+\beta-1}F(n{+}\beta\,,\,n{+}\alpha\,,\,n{+}\alpha{+}\beta;\;1-|z|^2).$$

Since $F(n{+}\beta\,,\,n{+}\alpha\,,\,n{+}\alpha{+}\beta;\;x) = (1-x)^{-n}F(\alpha\,,\,\beta\,,\,n{+}\alpha{+}\beta;\;x)$, and

$$F(\alpha\,,\,\beta\,,\,n{+}\alpha{+}\beta;\;1) = \frac{\Gamma(n)\Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha)\Gamma(n+\beta)},$$

$RG_{\alpha,\beta}(z)$ behaves like

$$-d_{\alpha,\beta}(n+\alpha+\beta)|z|^{2-2n}\frac{\Gamma(n)\Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha)\Gamma(n+\beta)} \qquad \text{as } |z|\to 0.$$

Let $u\in\mathcal{C}^2(\bar{\mathbf{B}}^n)$. Obviously Theorem 3.12 implies, for $0<\varepsilon<r<1$,

$$\begin{aligned}
&\frac{\pi^n r^{2n-2}}{(n-1)!}(1-r^2)^{-\alpha-\beta-n+1}\int_{\mathbf{S}^n}\left\{u(r\zeta)RG_{\alpha,\beta}(r\zeta)-G_{\alpha,\beta}(r\zeta)\bar{R}u(r\zeta)\right\}\,d\sigma(\zeta)\\
&\qquad-\frac{\pi^n\varepsilon^{2n-2}}{(n-1)!}(1-\varepsilon^2)^{-\alpha-\beta-n+1}\\
&\qquad\qquad\int_{\mathbf{S}^n}\left\{u(\varepsilon\zeta)RG_{\alpha,\beta}(\varepsilon\zeta)-G_{\alpha,\beta}(\varepsilon\zeta)\bar{R}u(\varepsilon\zeta)\right\}\,d\sigma(\zeta)\\
=\;&-\int_{r\mathbf{B}^n\setminus\varepsilon\mathbf{B}^n}\Delta_{\alpha,\beta}u(z)G_{\alpha,\beta}(z)(1-|z|^2)^{-\alpha-\beta}\,d\lambda(z).
\end{aligned}$$

Choosing

$$d_{\alpha,\beta} = -\frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\pi^n\Gamma(n+\alpha+\beta+1)},$$

the limit as $\varepsilon\to 0$ of the second term is exactly $-u(0)$ and the limit as $r\to 1$ of the first is

$$c_{\alpha,\beta}\int_{\mathbf{S}^n}u(\zeta)\,d\sigma(\zeta),$$

(recall that in Theorem 2.2 we have seen that

$$c_{\alpha,\beta} = \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n)\Gamma(n+\alpha+\beta)}\;).$$

Thus, the following holds for $u \in \mathcal{C}^2(\bar{\mathbf{B}}^n)$:

$$(3.4) \qquad u(0) = c_{\alpha,\beta} \int_{\mathbf{S}^n} u(\zeta)\, d\sigma(\zeta) + \int_{\mathbf{B}^n} \Delta_{\alpha,\beta} u(z) G_{\alpha,\beta}(z)(1-|z|^2)^{-\alpha-\beta}\, d\lambda(z).$$

When this is applied to $h_z^{\alpha,\beta} \cdot (u \circ \varphi_z)$ the following Riesz-type decomposition formula is obtained for $u \in \mathcal{C}^2(\bar{\mathbf{B}}^n)$ after some computation

$$(3.5) \quad u(z) = \int_{\mathbf{S}^n} u(\zeta) P_{\alpha,\beta}(z,\zeta)\, d\sigma(\zeta) + \int_{\mathbf{B}^n} \Delta_{\alpha,\beta} u(\omega) G_{\alpha,\beta}(z,\omega)(1-\bar{z}\omega)^\alpha (1-z\bar{\omega})^\beta (1-|\omega|^2)^{-\alpha-\beta}\, d\lambda(\omega),$$

with $G_{\alpha,\beta}(z,\omega) = g_{\alpha,\beta}(|\varphi_z(\omega)|^2) = G_{\alpha,\beta}(\varphi_z(\omega))$. This amounts to

$$(\Delta_{\alpha,\beta})_w (G_{\alpha,\beta}(z,\omega)(1-\bar{z}w)^\alpha (1-z\bar{w})^\beta) = \delta_z,$$
$$P_{\alpha,\beta}(z,\zeta) = \lim_{r\to 1} (1-r)^{1-n-\alpha-\beta} R_\zeta G_{\alpha,\beta}(z,\zeta).$$

In case $\alpha = \beta = 0$, this last formula says that $P_{0,0}$ is the Bergman normal derivative of $G_{0,0}$. This is another way of finding $P_{\alpha,\beta}$.

Strictly speaking, formula (3.5) has only been obtained for $u \in \mathcal{C}^2(\bar{\mathbf{B}}^n)$, but it can be seen to hold under more general conditions. For instance, it holds if $u \in \mathcal{C}^2(\mathbf{B}^n) \cap \mathcal{C}(\bar{\mathbf{B}}^n)$ and

$$\int_{\mathbf{B}^n} |\Delta_{\alpha,\beta} u(\omega)| \frac{dV(\omega)}{1-|\omega|} < +\infty.$$

This can be seen as follows: Fix $r < 1$ and apply the same argument as before with $v(z) = G_{\alpha,\beta}(z) - g_{\alpha,\beta}(r^2)$. After letting $\varepsilon \to 0$, one obtains

$$\begin{aligned} u(0) = {} & c_{\alpha,\beta} r^{2n} F(n+\beta\,,\, n+\alpha\,,\, n+\alpha+\beta;\; 1-r^2) \int_{\mathbf{S}^n} u(r\zeta)\, d\sigma(\zeta) \\ & + \int_{r\mathbf{B}^n} \Delta_{\alpha,\beta} u(z) \left\{ G_{\alpha,\beta}(z) - g_{\alpha,\beta}(r^2) \right\} (1-|z|^2)^{-\alpha-\beta} \Big\}\, d\lambda(z) \\ & - \alpha\beta g_{\alpha,\beta}(r^2) \int_{r\mathbf{B}^n} u(z)(1-|z|^2)^{1-\alpha-\beta}\, d\lambda(z). \end{aligned}$$

By dominated convergence one gets (3.4), hence (3.5), making $r \to 1$.

In case $\alpha\beta = 0$, there is a simple explicit integral formula for $g_{\alpha,\beta}$. Namely, solving (3.3) leads to

$$g_{\alpha,\beta}(x) = -\frac{(n-1)!}{\pi^n} \int_x^1 \frac{(1-t)^{n+\beta-1}}{t^n}\, dt, \;\; \alpha = 0.$$

3.3. We now choose

$$v(z) = (1-|z|^2)^{n+\alpha+\beta}$$

in Theorem 12. Then $Rv(z) = -(n+\alpha+\beta)|z|^2(1-|z|^2)^{n+\alpha+\beta-1}$, and a computation shows that

$$\Delta_{\alpha,\beta} v(z) = -(n+\alpha)(n+\beta)(1-|z|^2)^{n+\alpha+\beta+1}.$$

(Note in passing that this gives an example related to Theorem 2.2: if $n+\alpha$ or $n+\beta$ is zero then v is a non-zero solution to $\Delta_{\alpha,\beta} u = 0$, which is zero on the boundary.)

Fix r, $0 < r < 1$. Applying (3.1) above with v replaced by $v - v(r)$, we obtain

$$\begin{aligned} &\frac{(n+\alpha+\beta)}{(n-1)!}\pi^n r^{2n} \int_{\mathbf{S}^n} u(r\zeta)\, d\sigma(\zeta) \\ = &\int_{r\mathbf{B}^n} \left\{ (n+\alpha)(n+\beta) + \alpha\beta \frac{(1-r^2)^{n+\alpha+\beta}}{(1-|z|^2)^{n+\alpha+\beta}} \right\} u(z)\, dV(z) \\ &\qquad + \int_{r\mathbf{B}^n} \Delta_{\alpha,\beta} u(z) \left\{ 1 - \left(\frac{1-r^2}{1-|z|^2} \right)^{n+\alpha+\beta} \right\} \frac{dV(z)}{1-|z|^2} \end{aligned}$$

From this it follows that, if $u \in \mathcal{C}(\bar{\mathbf{B}}^n)$ and

$$\int_{\mathbf{B}^n} |\Delta_{\alpha,\beta} u(z)| \frac{dV(z)}{1-|z|^2} < +\infty,$$

say, then

$$\begin{aligned} (3.6)\quad \frac{(n+\alpha+\beta)}{(n-1)!}\pi^n \int_{\mathbf{S}^n} u(\zeta)\, d\sigma(\zeta) \quad &= \quad (n+\alpha)(n+\beta) \int_{\mathbf{B}^n} u(z)\, dV(z) \\ &\qquad + \int_{\mathbf{B}^n} \Delta_{\alpha,\beta} u(z) \frac{dV(z)}{1-|z|^2}. \end{aligned}$$

4. Characterizations of H^p spaces

4.1. In this paragraph we will prove a Fefferman-Stein type characterization of the H^p spaces on $\mathbf{S}^n$ in terms of their (α,β)-harmonic extensions. Here, of course, we understand by H^p the atomic H^p-space of Garnett and Latter ([**GaLa**]) for $p \leq 1$ and $L^p(\mathbf{S}^n)$ for $1 < p < +\infty$. We recall that α, β are always assumed to satisfy $\mathrm{Re}(n+\alpha+\beta) > 0$ and $n+\alpha$, $n+\beta \notin \mathbf{Z}_-$.

Theorem 4.13 *Let u be (α,β)-harmonic in $\mathbf{B}^n$. The following are equivalent for $p \geq 1$:*

(i) *There exists* $f \in H^p(\mathbf{S}^n)$ *such that* $u = P_{\alpha,\beta}[f]$.
(ii) *The admissible maximal function* $M[u] \in L^p(\mathbf{S}^n)$.
(iii) *The radial maximal function* $u^+ \in L^p(\mathbf{S}^n)$.
(iv) *The area function* $S[u] \in L^p(\mathbf{S}^n)$.

First note that the equivalence of (i), (iii) and (iii) for $p > 1$ follows inmediately from Proposition 2.4.

The equivalence of (ii) and (iii) for $p = 1$ is very much alike the corresponding real variable result in [**FeSt**]. Indeed, with Lemma 2.7 we can follow the same argument there to show that $M[u]^{1/2}$ is pointwise dominated by the (non-isotropic) Hardy-Littlewood maximal function of $(u^+)^{1/2}$.

4.2. We start proving the equivalence between (i) and (ii) for $p = 1$. In case $\alpha = \beta = 0$ this is proved in [**GaLa**]. We have not been able to carry over their proof for general α, β. In fact we do not know for which Poisson-type kernels this maximal characterization of $H^p(\mathbf{S}^n)$ holds. There is a result of Uchiyama [**U**] in this direction, which holds in spaces of homogeneous type. However, the kernels $P_{\alpha,\beta}$ do not satisfy all assumptions of Uchiyama's theorem. Instead we will use the theory of tent spaces. Assume first that $M[u] \in L^1(\mathbf{S}^n)$, we know from Proposition 2.4 that $u = P_{\alpha,\beta}[f]$ for some $f \in L^1(\mathbf{S}^n)$. We must show that $f \in H^1(\mathbf{S}^n)$. First we assume that f is smooth and we will prove the a priori estimate

$$\|f\|_{H^1} \leq C\|M[u]\|_1.$$

The idea is to show that $\|M[P_{0,0}f]\|_1 \leq C\|M[u]\|_1$ and apply the Garnett and Latter result. For this we consider the Riesz decomposition of u (formula (3.5))

$$u(z) = P_{0,0}[f](z) + \int_{\mathbf{B}^n} \Delta_{0,0}u(\omega)G_{0,0}(z,\omega)\,d\lambda(\omega).$$

For this to make sense we must discuss the convergence of the last integral. Since $\Delta_{\alpha,\beta}u = 0$, we have that

$$\Delta_{0,0}u(\omega) = (1 - |\omega|^2)\left\{\alpha\beta u(\omega) - \alpha R u(\omega) - \beta \bar{R} u(\omega)\right\},$$

and therefore we must have

$$\int_{\mathbf{B}^n} |Ru(\omega)| dV(\omega) < +\infty,$$

and similarly for $\bar{R}$. This is acomplished if f is sufficiently smooth, as shown by next lemma.

Lemma 4.14 *If* f *is of class* $\mathcal{C}^1$ *on* $\mathbf{S}^n$, *then* $u = P_{\alpha,\beta}[f]$ *satisfies*

$$|Ru(z)|,\ |\bar{R}u(z)| = \mathrm{O}(1 - |z|)^{\varepsilon - 1},$$

with $\varepsilon = \min(\frac{1}{2}, \mathrm{Re}\,(n + \alpha + \beta))$.

Proof. Write

$$Ru(z) = \int_{\mathbf{S}^n} R_z P_{\alpha,\beta}(z,\zeta)\,\{f(\zeta) - f(\eta)\}\,d\sigma(\zeta) + f(\eta)R_z \int_{\mathbf{S}^n} P_{\alpha\beta}(z,\zeta)\,d\sigma(\zeta), \quad z = r\eta.$$

The first term is bounded by

$$\int_{\mathbf{S}^n} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^{n+1/2}} = \mathrm{O}(1 - |z|)^{-1/2}.$$

The other integral is $P_{\alpha,\beta}[1] = c_{\alpha,\beta}F(-\alpha\ ,\ -\beta\ ,\ n;\ |z|^2)$ and its radial derivative is $c_{\alpha,\beta}F(1-\alpha\ ,\ 1-\beta\ ,\ n+1\ ,\ |z|^2)$, which has growth $(1 - |z|^2)^{\mathrm{Re}\,(n+\alpha+\beta)-1}$ if $\mathrm{Re}\,(n+\alpha+\beta) < 1$, $\log(1-|z|^2)$ if $\mathrm{Re}\,(n+\alpha+\beta) = 1$ or bounded if $\mathrm{Re}\,(n+\alpha+\beta) > 1$. □

The lemma implies the integrability of Ru, $\bar{R}u$, and hence

$$u(z) - P_{0,0}[f](z) = \int_{\mathbf{B}^n} \left\{\alpha\beta u(\omega) - \alpha Ru(\omega) - \beta\bar{R}u(\omega)\right\} G_{0,0}(z,\omega)(1 - |\omega|^2)^{-n} dV(\omega).$$

Integrating by parts in $r\mathbf{B}^n$, using

$$P_{0,0}(z,\zeta) = \lim_{r\to 1}(1 - r)^{1-n} R_\zeta G_{0,0}(z,\zeta) = -n \lim_{r\to 1} \frac{G_{0,0}(z,\zeta)}{(1 - r)^n}$$

as pointed out in paragraph 3.2, and letting $r \to 1$, we get

$$\begin{aligned} u(z) &= (1 + \frac{\alpha+\beta}{n})P_{0,0}f(z) + \int_{\mathbf{B}^n} \alpha\beta u(\zeta)G_{0,0}(z,\zeta)(1 - |\zeta|^2)^{-n}\,dV(\zeta) \\ &\quad + \alpha \int_{\mathbf{B}^n} u(\zeta)R_\zeta \left\{G_{0,0}(z,\zeta)(1 - |\zeta|^2)^{-n}\right\} dV(\zeta) \\ &\quad + \beta \int_{\mathbf{B}^n} u(\zeta)\bar{R}_\zeta \left\{G_{0,0}(z,\zeta)(1 - |\zeta|^2)^{-n}\right\} dV(\zeta). \end{aligned}$$

We will see that each of these three integrals defines operators that preserve the space of functions with admissible maximal function in $L^1(\mathbf{S}^n)$ (the tent space T_1^∞ in the terminology of [**CoMeSt**]). We only deal with the second one, the third beeing essentially the same and the first easier. The kernel is, with $c_n = -(n-1)!/\pi^n$:

$$\begin{aligned} \text{(4.1)}\quad K(z,\zeta) &= n|\zeta|^2 G_{0,0}(z,\zeta)(1 - |\zeta|^2)^{n-1} \\ &\quad - (1 - |\zeta|^2)^{-n}\frac{(1 - |\varphi_z(\zeta)|^2)^{-n-1}}{|\varphi_z(\zeta)|^{2n}} c_n R_\zeta |\varphi_z(\zeta)|^2. \end{aligned}$$

Clearly $g_{0,0}(x) = c_n(1/n)(1-x)^n + \mathrm{O}(1-x)^{n+1}$. Also, from

$$|\varphi_z(\zeta)|^2 = 1 - \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-z\bar{\zeta}|^2},$$

we see that

$$R_\zeta|\varphi_z(\zeta)|^2 = \frac{1-|z|^2}{(1-\bar{\zeta}z)}\frac{\zeta(\bar{\zeta}-\bar{z})}{(1-\zeta\bar{z})^2}.$$

Thus,

$$\begin{aligned}
& K(z,\zeta) \\
= \;& c_n(1-|\varphi_z(\zeta)|^2)^{n-1}(1-|\zeta|^2)^{-n-1}\Big\{|\zeta|^2(1-|\varphi_z(\zeta)|^2) - (1-|\zeta|^2) \\
& \quad \frac{(1-|z|^2)}{|\varphi_z(\zeta)|^{2n}}\frac{\zeta(\bar{\zeta}-\bar{z})}{|1-\zeta\bar{z}|^2}\frac{1}{(1-\zeta\bar{z})}\Big\} + \mathrm{O}(1-|\varphi_z(\zeta)|^2)^{n+1}(1-|\zeta|^2)^{-n-1} \\
= \;& c_n(1-|\varphi_z(\zeta)|^2)^{n}(1-|\zeta|^2)^{-n-1}\left\{|\zeta|^2 - \frac{1}{|\varphi_z(\zeta)|^{2n}}\frac{\zeta(\bar{\zeta}-\bar{z})}{1-\zeta\bar{z}}\right\} \\
& \quad + \mathrm{O}\left(\frac{(1-|z|^2)^{n+1}}{|1-z\bar{\zeta}|^{2n+2}}\right) \\
= \;& c_n\frac{(1-|z|^2)^n}{|1-z\bar{\zeta}|^{2n}(1-|\zeta|^2)}\frac{\mathrm{O}(1-|\zeta|^2) + (1-\zeta\bar{z})\mathrm{O}(1-|\varphi_z(\zeta)|^2)}{|\varphi_z(\zeta)|^{2n}(1-\zeta\bar{z})} \\
& \quad + \mathrm{O}\left(\frac{(1-|z|^2)^{n+1}}{|1-z\bar{\zeta}|^{2n+2}}\right).
\end{aligned}$$

This is

$$\mathrm{O}\left(\frac{(1-|z|^2)^n}{|1-z\bar{\zeta}|^{2n+1}}\right)$$

if $|\varphi_z(\zeta)|$ is bounded below.

We call $\mathbf{K}_i$, $i=1,2$, the operators

$$\begin{aligned}
\mathbf{K}_1u(z) &= \int_{|\varphi_z(\zeta)|\le\rho} K(z,\zeta)u(\zeta)\,dV(\zeta), \\
\mathbf{K}_2u(z) &= \int_{\mathbf{B}^n}\frac{(1-|z|^2)^n}{|1-z\bar{\zeta}|^{2n+1}}u(\zeta)\,dV(\zeta).
\end{aligned}$$

We must show that both preserve T_1^∞. We start with $\mathbf{K}_2$. Equivalently, we must show that there exists $C>0$, so that for any tent-atom a,

$$\int_{\mathbf{S}^n}\sup_{z\in\mathcal{A}(\eta)}(1-|z|^2)^n\int_{\mathbf{B}^n}\frac{|a(\zeta)|}{|1-z\bar{\zeta}|^{2n+1}}\,dV(\zeta)\,d\sigma(\eta)\le C.$$

Here a tent-atom is a function in $\mathbf{B}^n$ supported in an admissible tent $\hat{B}$ over a non isotropic ball $B(\omega_0,\delta)$, such that $\|a\|_\infty \le 1/\delta^n$.

Thus, let a be any such atom, and denote by $\widetilde{B} = B(\omega_0, K\delta)$, where $K>0$ is a constant to be fixed. Let $z\in\mathcal{A}(\eta)$, with $\eta\in\mathbf{S}^n$. We will compute the L^1 integral by estimating the integrals over $\widetilde{B}$ and the complementary $\widetilde{B}^c$.

Assume first that $\eta\in\widetilde{B}$. Using [**Ru**, Proposition 1.4.10], and the size condition on a, we obtain that

$$\int_{\hat{B}} \frac{|a(\zeta)|}{|1-z\bar{\zeta}|^{2n+1}} dV(\zeta) \le \frac{1}{\delta^n}\int_{\mathbf{B}^n}\frac{dV(\zeta)}{|1-z\bar{\zeta}|^{2n+1}} \simeq \frac{1}{\delta^n(1-|z|^2)^n}, \tag{4.2}$$

and

$$\sup_{z\in\mathcal{A}(\eta)}(1-|z|^2)^n\int_{\hat{B}}\frac{|a(\zeta)|}{|1-z\bar{\zeta}|^{2n+1}}dV(\zeta)\le\frac{C}{\delta^n}.$$

Integrating over $\widetilde{B}$, we get the desired estimate.

Now, if $\eta\in\widetilde{B}^c$, and $z\in\mathcal{A}(\eta)$, it is easy to see, choosing $K>0$ big enough, that $|1-\bar{\zeta}z|\simeq(1-|z|^2)+|1-\omega_0\bar{\eta}|$. Thus,

$$\begin{aligned}\int_{\hat{B}}\frac{|a(\zeta)|}{|1-z\bar{\zeta}|^{2n+1}}\,dV(\zeta) &\le C\frac{1}{\delta^n}\frac{(1-|z|^2)^n}{(1-|z|^2)^{2n+1}+|1-\omega_0\bar{\eta}|^{2n+1}}\int_{\hat{B}}dV(\zeta)\\ &\le C\delta\frac{(1-|z|^2)^n}{(1-|z|^2)^{2n+1}+|1-\omega_0\bar{\eta}|^{2n+1}}.\end{aligned}$$

But

$$\sup_z\frac{(1-|z|^2)^n}{(1-|z|^2)^{2n+1}+|1-\omega_0\bar{\eta}|^{2n+1}}\preceq\frac{1}{|1-\omega_0\bar{\eta}|^{n+1}},$$

and integrating over $\widetilde{B}^c$, we obtain

$$\int_{\widetilde{B}^c}\frac{\delta}{|1-\omega_0\bar{\eta}|^{n+1}}\,d\sigma(\eta)\le C.$$

So we are left with $\mathbf{K}_1$. We will estimate separately both integrals that appear. Assume that $|\varphi_z(\zeta)|<\rho$. It is then inmediate that, provided ρ is small enough,

$$|1-z\bar{\zeta}|\simeq 1-|z|^2\simeq 1-|\zeta|^2,$$

and that (widening the aperture of the admissible region if necessary), if $z\in\mathcal{A}(\eta)$, then $\zeta\in\mathcal{A}(\eta)$. The first term is then bounded by

$$\begin{aligned}M[u](\eta)\int_{\{\zeta;\ |\varphi_z(\zeta)|<\frac12\}}G_{0,0}(z,\zeta)d\lambda(\zeta) &\preceq M[u](\eta)\int_{\{|\zeta|<\frac12\}}G_{0,0}(\zeta)d\lambda(\zeta)\\ &\preceq M[u](\eta).\end{aligned}$$

The remaining term is estimated by

$$M[u](\eta)\int_{|\varphi_z(\zeta)|<\rho}(1-|\zeta|^2)^{-n}\frac{(1-|\varphi_z(\zeta)|^2)^{n-1}}{|\varphi_z(\zeta)|^{2n}}\frac{1-|z|^2}{|1-z\bar{\zeta}|^3}|\zeta(\bar{\zeta}-\bar{z})|\,dV(\zeta).$$

Write $\zeta - z = \lambda\zeta + v$ with $v\bar{\zeta} = 0$. Then

$$|1 - z\bar{\zeta}|^2 - (1 - |z|^2)(1 - |\zeta|^2) = |\lambda|^2|\zeta|^2 + (1 - |\zeta|^2)|v|^2.$$

Hence

$$|\zeta(\bar{\zeta} - \bar{z})| = |\lambda|\ |\zeta|^2 \leq \left[|1 - z\bar{\zeta}|^2 - (1 - |z|^2)(1 - |\zeta|^2)\right]^{1/2} = |\varphi_z(\zeta)|\ |1 - z\bar{\zeta}|.$$

With all this, the integral above can be estimated by

$$\int_{|\varphi_z(\zeta)|<\rho} \frac{(1 - |\varphi_z(\zeta)|^2)^{n-1}}{|\varphi_z(\zeta)|^{2n-1}}\, d\lambda(\zeta) = \int_{|\zeta|<\rho} \frac{(1 - |\zeta|^2)^{n-1}}{|\zeta|^{2n-1}}\, d\lambda(\zeta) < +\infty.$$

This ends the proof of the a priori inequality

$$\|f\|_{H^1} \leq C\|M[u]\|_1, \ \ u = P_{\alpha\beta}[f]$$

when f is smooth. Assume now that $f \in L^1(\mathbf{S}^n)$ and that $u = P_{\alpha,\beta}[f]$ has $M[u]$ integrable. Define

$$f_\varepsilon(\zeta) = \int_{\mathbf{S}^n} f(\eta) h_\varepsilon(\bar{\eta}\zeta)\, d\sigma(\eta),$$

where $h_\varepsilon(\lambda)$ are functions of one variable chosen so that $h_\varepsilon(\bar{\eta}\zeta)\, d\sigma(\eta) \to \delta_\zeta$, i.e. h_ε are positive smooth functions supported in $|\lambda - 1| \leq \varepsilon$ and such that

$$\int_{|\lambda|<1} (1 - |\lambda|^2)^{n-2} h_\varepsilon(\lambda)\, dV(\lambda) = 1.$$

Then, f_ε is smooth and converges to f in $L^1(\mathbf{S}^n)$; let $u_\varepsilon = P_{\alpha,\beta}[f_\varepsilon]$. Using [**ACo**, Corollary 2.2] a computation gives

$$u_\varepsilon(r\eta) = \int_{\mathbf{S}^n} u(r\zeta) h_\varepsilon(\bar{\eta}\zeta)\, d\sigma(\zeta).$$

Therefore

$$u_\varepsilon^+(\eta) \leq \int_{\mathbf{S}^n} u^+(\zeta) h_\varepsilon(\bar{\eta}\zeta)\, d\sigma(\zeta),$$

which trivially implies

$$\|u_\varepsilon^+\|_1 \leq \|u^+\|_1.$$

From the equivalence between (ii) and (iii) we conclude that

$$\|M[u_\varepsilon]\|_1 \leq C\|M[u]\|_1$$

and, by the a priori inequality, that $\|f_\varepsilon\|_{H^1} \leq C\|M[u]\|_1$. Since H^1 is the dual of VMO, every bounded sequence in H^1 has a subsequence with a weak-star limit in H^1. But f_ε converges to f in $L^1(\mathbf{S}^n)$, hence $f \in H^1$.

This proves that (ii) implies (i). The reverse implication can be obtained just interchanging the roles of $P_{0,0}$ and $P_{\alpha,\beta}$ or, alternatively, checking directly that $P_{\alpha,\beta}$ sends an H^1-atom to a function in T_1^∞.

4.3. To prove that (i) implies (iv) we will use the following theorem, which is the ball version of a result of M. Christ and J. L. Journé ([**ChJo**]). We recall that a *Carleson measure* on $\mathbf{B}^n$ is a positive measure μ on $\mathbf{B}^n$ satisfying that $\mu(\hat{B}(\zeta,\delta)) \le C\sigma(B(\zeta,\delta))$ for any $\zeta \in \mathbf{S}^n$, $\delta > 0$. Here $\hat{B}(\zeta,\delta)$ is the admissible tent over $B(\zeta,\delta)$.

Theorem 4.15 *Let $K(z,\zeta)$, $z \in \mathbf{B}^n$, $\zeta \in \mathbf{S}^n$ be a kernel satisfying for some $\varepsilon > 0$, $c > 0$, $C > 0$*

(i) $|K(z,\zeta)| \le C\dfrac{(1-|z|^2)^\varepsilon}{|1-z\bar{\zeta}|^{n+\varepsilon}}$,

(ii) $|K(z,\zeta) - K(z,\eta)| \le C\dfrac{(1-|z|^2)^\varepsilon}{|1-z\bar{\zeta}|^{n+\varepsilon}}\left(\dfrac{|1-\zeta\bar{\eta}|}{|1-z\bar{\zeta}|}\right)^\varepsilon$,

whenever $|1-\eta\bar{\zeta}| \le c|1-z\bar{\zeta}|$. Let $\mathbf{K}$ be the operator

$$\mathbf{K}f(z) = \int_{\mathbf{S}^n} f(\zeta)K(z,\zeta)\,d\sigma(\zeta).$$

Then

$$\int_{\mathbf{B}^n} |\mathbf{K}\varphi(z)|^2 \frac{dV(z)}{1-|z|^2} \le M\|\varphi\|_2^2, \forall\varphi \in L^2(\mathbf{S}^n)$$

if and only if $\left(|\mathbf{K}1(z)|^2/(1-|z|^2)\right)dV(z)$ is a Carleson measure.

Proof. Let us see how the theorem above gives the implication (i) $\Rightarrow$ (iv) in case $p = 2$. By definition, if $u = P_{\alpha,\beta}[f]$,

$$\|S[u]\|_2^2 = \int_{\mathbf{S}^n}\int_{D(\zeta)} |\nabla_{\mathbf{B}} u(z)|^2\,d\lambda(z)\,d\sigma(\zeta) \simeq \int_{\mathbf{B}^n} |\mathbf{K}f(z)|^2\frac{dV(z)}{1-|z|^2},$$

where $\mathbf{K}f(z) = \nabla_{\mathbf{B}}u(z)$ (we look at $\mathbf{K}$ as a vector-valued operator). The corresponding kernel is

$$\begin{aligned} K(z,\zeta) &= \nabla_{\mathbf{B}} P_{\alpha,\beta}(z,\zeta) \\ &= \frac{1}{|z|}\Big((1-|z|^2)R_z P_{\alpha,\beta}(z,\zeta)\,,\,(1-|z|^2)\bar{R}_z P_{\alpha,\beta}(z,\zeta)\,, \\ &\qquad (1-|z|^2)^{1/2}T_{ij}P_{\alpha,\beta}(z,\zeta)\,,\,(1-|z|^2)^{1/2}\bar{T}_{ij}P_{\alpha,\beta}(z,\zeta)\Big). \end{aligned}$$

Properties (i) and (ii) above are routinely checked for K. Indeed, using

$$|\nabla_{\mathbf{B}}(1-z\bar{\zeta})| \preceq |1-z\bar{\zeta}|,$$

it is inmediate that (i) holds with $\varepsilon = \mathrm{Re}\,(n+\alpha+\beta)$. For (ii), one proceeds in the standard way (see for instance [**Ru**, pg 92]) using the mean-value theorem, the fact that $|1-z\bar{\zeta}|^{1/2}$ satisfies a triangle inequality and the inequality

$$|z(\bar{\zeta}-\bar{\eta})| \preceq |1-\zeta\bar{\eta}|^{1/2}|1-\zeta\bar{z}|^{1/2};\ |1-\zeta\bar{\eta}| \le c|1-\zeta\bar{z}|.$$

If $|1-\zeta\bar{\eta}| \le c|1-\zeta\bar{z}|$, the final result is,

$$|K(z,\zeta)-K(z,\eta)| \le \mathrm{O}\frac{(1-|z|^2)^\delta}{|1-z\bar{\zeta}|^{n+\delta}}\left(\frac{|1-\zeta\bar{\eta}|}{|1-z\bar{\zeta}|}\right)^{1/2} \quad \text{with } \delta = \mathrm{Re}(n+\alpha+\beta).$$

Therefore (i) and (ii) hold with $\varepsilon = \min(\delta\ ,\ \frac{1}{2})$.

We now must check that the measure $(|\mathbf{K}1(z)|^2/(1-|z|^2))\,dV(z)$ is a Carleson measure. Recall from Theorem 2.2 that

$$\mathbf{K}1(z) = c_{\alpha,\beta}\nabla_{\mathbf{B}}F(-\alpha\ ,-\beta\ ,n;\ |z|^2),$$

and hence, as it is radial, it will be enough to show that

$$\int_0^1 (1-r^2)|F'(-\alpha\ ,-\beta\ ,n;\ r^2)|^2\,dr < +\infty.$$

But $F'(-\alpha\ ,-\beta\ ,n;\ r^2) = ((\alpha,\beta)/n)F(1-\alpha\ ,1-\beta\ ,1+n;\ r^2)$. If $\mathrm{Re}\,(n+\alpha+\beta) > 1$ this has a limit at 1. If $\mathrm{Re}\,(n+\alpha+\beta) = 1$ this growths at most like $\log(1-r)$ and finally, if $0 < \mathrm{Re}\,(n+\alpha+\beta) < 1$ this growths like $(1-r)^{\mathrm{Re}\,(n+\alpha+\beta)-1}$. In all cases we have the desired result. □

Next we show the implication (i) ⇒ (iv) for $p \ge 1$. It is enough to prove that if a is an atom, and $u = P_{\alpha,\beta}[a]$, then

$$\int_{\mathbf{S}^n} S[u](\zeta)\,d\sigma(\zeta) \le C,$$

for some absolute constant C. This is done in a standard way: in fact it only depends on the properties (i) and (ii) of the kernel K and the L^2-estimate already proved. Namely, if $K(z,\zeta)$ satisfies them, and

$$S[f](\zeta) = \left\{\int_{D(\zeta)} |\mathbf{K}f(z)|^2 d\lambda(z)\right\}^{1/2}, \quad \zeta \in \mathbf{S}^n,$$

then S is bounded from $H^p(\mathbf{S}^n)$ to $L^p(\mathbf{S}^n)$, $1 \le p < +\infty$ and $\big(|\mathbf{K}f(z)|^2/(1-|z|)\big)\,dV(z)$ is a Carleson measure for all $f \in \mathrm{BMO}$. This is surely known but we have not found a reference. First, for $p = 1$ it is enough to prove that if a is an atom, say supported in $B(\zeta_0,\delta) = \{\zeta \in \mathbf{S}^n\ ;\ |1-\zeta\bar{\zeta}_0| < \delta\}$, then

$$\int_{\mathbf{S}^n} S[a](\zeta)\,d\sigma(\zeta) \le C \tag{4.3}$$

for some absolute constant. This is seen as follows: the contributions of $B(\zeta_0, k\delta)$ to this integral is estimated using Schwarz's inequality and the L^2-estimate already proved. For points $\zeta \notin B(\zeta_0, k\delta)$ far from $B(\zeta_0, \delta)$, one uses the cancellation of the atom and property (ii) to obtain the pointwise bound

$$S[a](\zeta) \leq C\delta^{\varepsilon}|1-\zeta\bar{\zeta}_0|^{-n-\varepsilon};\ |1-\zeta\bar{\zeta}_0| \geq k\delta,$$

which finishes the proof of (4.3).

In a completely analogous way to the Fefferman-Stein proof, it is easily proved that

$$\frac{|\mathbf{K}f(z)|^2}{1-|z|^2}dV(z)$$

is a Carleson measure for all $f \in \mathrm{BMO}$.

For $1 < p < +\infty$ the result follows from interpolation, using the theory of tent-spaces of [**CoMeSt**], or, more precisely, its non-isotropic version on the ball. With the notations of [**CoMeSt**], the statement $S[f] \in L^p(\mathbf{S}^n)$ means that $\mathbf{K}f \in T_2^p$ and to say that $\big(|\mathbf{K}f(z)|^2/(1-|z|^2)\big)\, dV(z)$ is a Carleson measure means that $\mathbf{K}f \in T_2^\infty$. Since we have seen that $\mathbf{K}$ is bounded from $L^\infty(\mathbf{S}^n)$ to T_2^∞, it follows from the interpolation theorems in [**CoMeSt**] that $\mathbf{K}$ is bounded from $L^p(\mathbf{S}^n)$ to T_2^p. □

4.4. We prove now that (iv) implies (i). We assume without loss of generality that $u(0) = 0$. Let assume first that $u = P_{\alpha,\beta}[f]$ for $f \in \mathcal{C}(\mathbf{S}^n)$. We will prove that

$$\|f\|_p \leq C\|S[u]\|_p,\ \ \|f\|_{H^1} \leq C\|S[u]\|_1.$$

We assume first that $p > 1$. Obviously

$$\|f\|_p = \sup\left\{\left|\int_{\mathbf{S}^n} f(\zeta)g(\zeta)\, d\sigma(\zeta)\right|;\ g \in \mathcal{C}(\mathbf{S}^n),\ \|g\|_q \leq 1\right\}$$

with q the conjugate exponent of p. Let $v = P_{\alpha,\beta}[g]$; we will apply formula (3.6) to uv. A computation shows that

$$\Delta_{\alpha,\beta}(uv) = \alpha\beta(1-|z|^2)uv + (\nabla_{\mathbf{B}}u, \nabla_{\mathbf{B}}v)_{\mathbf{B}}.$$

We obtain

$$\begin{aligned} &\frac{(n+\alpha+\beta)}{(n-1)!}\pi^n \int_{\mathbf{S}^n} u(\zeta)v(\zeta)\, d\sigma(\zeta) \\ =\ &[(n+\alpha)(n+\beta)+\alpha\beta]\int_{\mathbf{B}^n} u(z)v(z)\, dV(z) + \int_{\mathbf{B}^n} (\nabla_{\mathbf{B}}u\, ,\, \nabla_{\mathbf{B}}v)_{\mathbf{B}}\frac{dV(z)}{1-|z|^2}. \end{aligned}$$

(The discussion that follows will show that this last integral is convergent.)

The advantage of using formula (3.6) instead of the Riesz decomposition is that with this last one there would appear a term

$$\int_{\mathbf{B}^n} u(z)v(z)G_{\alpha,\beta}(z)(1-|z|^2)^{-\alpha-\beta-n}\, dV(z).$$

Even though $G_{\alpha,\beta}(z)(1-|z|^2)^{-\alpha-\beta-n}$ behaves like 1 near $|z|=1$, some arguments that follow do not allow us to put the absolute value inside the integral.

Since on compact sets u, v are uniformly estimated by $S[u]$, $S[v]$ respectively, by Lemma 9 it follows that

$$\begin{aligned}\left|\int_{\mathbf{S}^n} f(\zeta)g(\zeta)\, d\sigma(\zeta)\right| \quad &\leq \quad C_\varepsilon \|S[u]\|_p \ \|S[v]\|_q + \left|\int_{1-\varepsilon\leq|z|<1} u(z)v(z)\, dV(z)\right| \\ &\qquad + \int_{\mathbf{B}^n} \|\nabla_{\mathbf{B}} u\|_{\mathbf{B}} \ \|\nabla_{\mathbf{B}} v\|_{\mathbf{B}} \frac{dV(z)}{1-|z|^2}.\end{aligned}$$

Writing the second integral in polar coordinates, we bound it using Proposition 2.4 by

$$\int_{1-\varepsilon}^1 \left\{\int_{\mathbf{S}^n} |u(r\zeta)||v(r\zeta)|\, d\sigma(\zeta)\right\} dr \leq \int_{1-\varepsilon}^1 \|u_r\|_p \ \|v_r\|_q dr \leq C\varepsilon\|u\|_p \ \|v\|_q.$$

For the third integral we use (5.1) of [**CoMeSt**] to estimate it by

$$\int_{\mathbf{S}^n} S[u](\zeta)S[v](\zeta)\, d\sigma(\zeta) \leq \|S[u]\|_p \ \|S[v]\|_q.$$

As we already know that $\|S[v]\|_q \preceq C\|g\|_q$ we obtain

$$\|f\|_p \leq C_\varepsilon \|S[u]\|_p + C\varepsilon\|f\|_p,$$

which gives the result.

For $p=1$, we must use the duality $H^1 - \text{VMO}$:

$$\|f\|_{H^1} = \sup\left\{\left|\int_{\mathbf{S}^n} f(\zeta)g(\zeta)\, d\sigma(\zeta)\right|;\ g\in\mathcal{C}(\mathbf{S}^n),\ \|g\|_* \leq 1\right\}.$$

Two modifications are needed in the previous argument: first

$$\left|\int_{\mathbf{S}^n} u(r\zeta)v(r\zeta)\, d\sigma(\zeta)\right| \leq \|u_r\|_{H^1} \ \|v_r\|_*.$$

At this point the following lemma is needed.

Lemma 4.16 *If* $u = P_{\alpha,\beta}[f]$, *and* $u_r(\zeta) = u(r\zeta)$,

$$\|u_r\|_{H^1} \leq C\|f\|_{H^1},\ \ \|u_r\|_* \leq C\|f\|_*.$$

Proof. By the equivalence between (i) and (iii)

$$\|u_r\|_{H^1} \simeq \|P_{\alpha,\beta}[u_r]^+\|_1.$$

But clearly by Theorem 2.2,

$$|P_{\alpha,\beta}[u_r](s\zeta)| = |P_{\alpha\beta}[u_s](r\zeta)| \le \int_{\mathbf{S}^n} \frac{(1-r^2)^{n+\operatorname{Re}\alpha+\operatorname{Re}\beta}}{|1-r\zeta\bar{\omega}|^{2n+\operatorname{Re}\alpha+\operatorname{Re}\beta}} u^+(\omega)\, d\sigma(\omega),$$

and then

$$\|u_r\|_{H^1} \preceq \|u^+\|_1 \simeq \|f\|_{H^1}.$$

The other inequality follows by duality, using that, if $v = P_{\alpha,\beta}[g]$

$$\int_{\mathbf{S}^n} u_r(\zeta) g(\zeta)\, d\sigma(\zeta) = \int_{\mathbf{S}^n} f(\zeta) v_r(\zeta)\, d\sigma(\zeta).$$

□

With this lemma we see that

$$\left| \int_{1-\varepsilon\le|z|\le 1} u(z)v(z)\, dV(z) \right| \le C\varepsilon \|f\|_{H^1}.$$

(Here is where the advantage of formula (3.6) plays a role.)

The second modification is to replace (5.1) of [**CoMeSt**] by (4.1) of the same paper:

$$\int_{\mathbf{B}^n} \|\nabla_{\mathbf{B}} u\|_{\mathbf{B}}\ \|\nabla_{\mathbf{B}} v\|_{\mathbf{B}} \frac{dV(z)}{1-|z|^2} \le \int_{\mathbf{S}^n} S[u](\zeta) C[v](\zeta)\, d\sigma(\zeta),$$

where

$$C[v](\zeta) = \sup_{\zeta\in B} \left(\frac{1}{\delta(B)} \int_{\hat{B}} \frac{|\nabla_{\mathbf{B}} v(z)|^2}{1-|z|^2} dV(z) \right)^{1/2}.$$

In the previous paragraph we saw that $C[v]$ is bounded whenever $g \in \mathrm{BMO}$, and thus we arrive in the same way at

$$\|f\|_{H^1} \le C_\varepsilon \|S[u]\|_1 + C\varepsilon \|f\|_{H^1}.$$

To finish, it remains to remove the extra assumption on u. Assume $S[u] \in L^p$; by Lemma 2.9, $u = P_{\alpha,\beta}[f]$ for some distribution f. Define the same regularization as before

$$f_\varepsilon(\zeta) = \int_{\mathbf{S}^n} f(\omega) h_\varepsilon(\bar{\omega}\zeta)\, d\sigma(\zeta).$$

Then

$$u_\varepsilon(r\eta) = \int_{\mathbf{S}^n} u(r\zeta) h_\varepsilon(\bar{\eta}\zeta)\, d\sigma(\zeta)$$

and

$$(\nabla_{\mathbf{B}} u_\varepsilon)(r\eta) = \int_{\mathbf{S}^n} \nabla_{\mathbf{B}} u(r\zeta) h_\varepsilon(\bar{\eta}\zeta)\, d\sigma(\zeta).$$

Let us see now that this implies

$$S[u_\varepsilon](\omega) \leq \int_{\mathbf{S}^n} S[u](\zeta) h_\varepsilon(\bar{\omega}\zeta)\, d\sigma(\zeta). \tag{4.4}$$

For $z = r\eta \in \mathcal{A}(\omega)$ close to ω, we may choose in a smooth way a unitary map U_η such that $U_\eta^* \eta = \omega$ so that

$$(\nabla_{\mathbf{B}} u_\epsilon)(r\eta) = \int_{\mathbf{S}^n} \nabla_{\mathbf{B}} u(rU_\eta\zeta) h_\varepsilon(\bar{\omega}\zeta)\, d\sigma(\zeta),$$

and, consequently,

$$\begin{aligned} &\left(\int_{\mathcal{A}(\omega)} |\nabla_{\mathbf{B}} u_\varepsilon|^2(z)\, d\lambda(z) \right)^{1/2} \\ \leq\ & \int_{\mathbf{S}^n} \left(\int_{\mathcal{A}(\omega)} |\nabla_{\mathbf{B}} u(rU_\eta\zeta)|^2\, d\lambda(z) \right)^{1/2} h_\varepsilon(\bar{\omega}\zeta)\, d\sigma(\zeta). \end{aligned}$$

In the last inner integral, the change of variables $U_\eta\zeta = \tau$ turns it into

$$\int_{\mathcal{A}(\zeta)} |\nabla_{\mathbf{B}} u(r\tau)|^{1/2}\, d\lambda(r\tau),$$

(with possibly a larger opening for the admissible region $\mathcal{A}(\zeta)$). This proves 4.4.

Hence $\|S[u_\varepsilon]\|_p \leq C\|S[u]\|_p$ and, by what has already been proved, we have $\|f_\varepsilon\|_p \leq C\|S[u]\|_p$, $\|f_\varepsilon\|_{H^1} \leq C\|S[u]\|_1$ respectively. Since L^p and H^1 are dual spaces, f_ε has a weak-limit in the same space. But f_ε tends to f as distributions and, therefore, f is in L^p, H^1 respectively. □

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