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A QUASI-MALMQUIST PRODUCTIVITY INDEX

E. Grifell-Tatjé
Departament d'Economia de l'Empresa
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
SPAIN
egrifell@volcano.uab.es

C. A. K. Lovell
Department of Economics
University of Georgia
Athens, GA 30602, USA
knox@rigel.econ.uga.edu

J. T. Pastor
Departamento de Estadistica e Investigacion Operativa
Universidad de Alicante
03071 Alicante, SPAIN
jtpastor@ua.es

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Abstract

The Malmquist productivity index is based on distance functions, which are reciprocals of radial Debreu-Farrell efficiency measures, and which have a number of desirable properties. Linear programming techniques are frequently employed to calculate the efficiency measures. However these techniques can leave slacks, which constitute a non-radial form of inefficiency which is not incorporated into the analysis. Thus a radial efficiency measure overstates true efficiency, the reciprocal distance function understates the distance to the relevant efficient subset, and the Malmquist productivity index is adversely affected, although in an analytically indeterminate direction. This has led us to consider a new definition of "one-sided" efficiency, and to develop a new nonradial efficiency measure which incorporates all slacks on the selected side. Replacing conventional radial efficiency measures with our new non-radial efficiency measures generates what we call a quasi-Malmquist productivity index. We illustrate our quasi-Malmquist productivity index with an application to productivity change in Spanish banking.

Keywords: distance functions, Malmquist productivity index, linear programming slacks

JEL codes: C43, O47

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A QUASI-MALMQUIST PRODUCTIVITY INDEX*

1. Introduction

The Malmquist (1953) productivity index has enjoyed widespread popularity in recent years. This popularity can be attributed to several factors. Caves, Christensen and Diewert (CCD) (1982) showed that, under fairly restrictive assumptions, the geometric mean of two adjacent-period Malmquist productivity indexes is equal to the product of a scale index and a Törnqvist (1936) productivity index, which is a superlative index, being exact for a flexible translog representation of production technology.[1] The Malmquist productivity index has the advantage that it can be constructed from quantity data only, but it requires knowledge of the underlying technology; the Törnqvist productivity index has the advantage of not requiring knowledge of the structure of the underlying technology, but it requires both price and quantity data, and restrictive behavioral assumptions as well, in its construction. CCD argued that the empirical usefulness of the Malmquist productivity index was limited by the need to estimate the parameters of the underlying technology, but that one could construct a Malmquist productivity index indirectly, by constructing a Törnqvist productivity index and exploiting the relationship between the two indexes.

It is possible to reverse the logic employed by CCD, and to argue that in some situations one might prefer to construct a Malmquist productivity index directly. Färe, Grosskopf, Lindgren and Roos (FGLR) (1992, 1994) provided the foundation for this reversal by showing how to use linear programming techniques to calculate a Malmquist productivity index. The Malmquist productivity index is based on distance functions, output distance functions for an output-oriented index and input distance functions for an input-oriented index. The FGLR approach is an adaptation of the Charnes et al. (1978), Banker et al. (1984) oriented DEA models designed to construct nonparametric Debreu (1951) - Farrell (1957) radial efficiency measures. Distance functions are reciprocals of radial efficiency measures, and so the FGLR technique can be used to construct a nonparametric Malmquist productivity index. However there is a difficulty with the use of oriented DEA models in this context, since they frequently leave slacks in the solutions to problems. When slacks are present, radial efficiency measures overstate true efficiencies. This is a problem in and of itself, but it also creates an additional problem: when slacks are present, the Malmquist productivity index is affected in an unknown way.

One is thus faced with a tradeoff. Radial efficiency measures, being reciprocals of distance functions, have desirable properties, as does the Malmquist productivity index constructed from them.[2] However when oriented DEA problems are formulated to calculate the radial efficiency measures, slacks are typically ignored, thereby adversely affecting both the efficiency measures and the resulting Malmquist productivity index. Stated differently, the Malmquist productivity index is based on Debreu-Farrell measures of radial efficiency, and radial efficiency is necessary, but not sufficient, for satisfaction of Koopmans' (1951; 60) definition of overall efficiency (the absence of slacks). Thus, as frequently happens, empirical measures frequently fall short of the standards established by theoretical definitions. It is, however, possible to calculate nonradial efficiency measures which do incorporate slacks appearing in solutions to oriented DEA problems, and which do correspond to Koopmans' definition of efficiency. These nonradial efficiency measures collapse to conventional radial efficiency measures in the absence of slacks (Cooper and Pastor (1996)), and they possess a desirable property we call **inclusion** (the incorporation of nonradial as well as radial slacks). Unfortunately they sacrifice a desirable property of radial efficiency measures, **monotonicity**. Since we view monotonicity as an essential property of any efficiency measure, in this paper we develop a new nonradial efficiency measure having the two desirable properties of (semi-strict) monotonicity and (weak) inclusion. This measure is derived from an additive DEA model rather than from an oriented DEA model, and it is unlikely to collapse to the radial efficiency measure. As we formulate it, it incorporates only output slacks, although it could be extended to incorporate input slacks as well. We choose to incorporate only output slacks because productivity measurement involves maximum outputs obtainable from given inputs, or minimum inputs required to obtain given outputs. Thus we feel justified in ignoring input (output) slacks when measuring productivity with an output (input) orientation.

The reciprocal of our nonradial efficiency measure is not a distance function, but it is "like" a distance function, and so we call it a **quasi-distance function**. Inserting these quasi-distance functions into the definition of a Malmquist productivity index generates what we call a **quasi-Malmquist productivity index**. The quasi-Malmquist productivity index inherits the desirable properties of semi-strict monotonicity and weak inclusion from its component quasi-distance functions.

Whether the compromise is worthwhile is ultimately an empirical issue, depending on the frequency with which slacks occur in the solutions to the oriented DEA problems. If slacks are pervasive (because many observations lie outside the efficient subset cone), we prefer our quasi-Malmquist productivity index; if slacks occur infrequently, we prefer the original Malmquist productivity index. Since calculation of our nonradial efficiency measure involves the use of an additive DEA model, the following procedure suggests itself: use an oriented DEA model to calculate radial efficiency measures, check the pervasiveness of slacks, and decide whether or not to calculate nonradial efficiency measures based on an additive DEA model. This decision determines whether to calculate a Malmquist productivity index or a quasi-Malmquist productivity index.

The first objective of this paper is to introduce our quasi-Malmquist productivity index, and to explore its properties. This we do in Section 2. The second objective is to provide an empirical comparison of Malmquist and quasi-Malmquist productivity indexes. This we do in Section 3. The comparison is based on an unbalanced panel of about 60 Spanish banks over the period 1986-1993. We find slacks in the oriented DEA model to be large and pervasive, which motivates us to calculate both a Malmquist productivity index and a quasi-Malmquist productivity index, and to compare the two. We find a systematic relationship between the two productivity indexes: in every adjacent pair of years the quasi-Malmquist productivity index shows slower productivity growth (or faster productivity decline) than the Malmquist productivity index does. However, despite the large and pervasive slacks appearing in the oriented DEA model, the magnitude of the annual differences between the two productivity indexes is small. Over the entire period, the Malmquist productivity index suggests a 2.8% per annum rate of productivity growth, while the quasi-Malmquist productivity index suggests a negligible 0.4% per annum rate of productivity decline. Section 4 concludes with some thoughts on the value of the quasi-Malmquist productivity index.

2. Malmquist and Quasi-Malmquist Productivity Indexes

Let $x^t = (x_1^t,...,x_N^t) \geq 0$ and $y^t = (y_1^t,...,y_M^t) \geq 0$ denote vectors of inputs and outputs in period t, t = 1,...,T.[3] Since we maintain an output orientation to the measurement of productivity change, we represent the structure of production technology with the output distance function

$$D_O^t(x^t,y^t) = \inf\{\theta: y^t/\theta \in P^t(x^t)\}, \qquad (1)$$

where $P^t(x^t)$ is the set of all output vectors obtainable from x^t with period t technology. Färe (1988; 29-34) lists a number of properties that $D_o^t(x^t,y^t)$ satisfies, including $0 < D_o^t(x^t,y^t) \le 1$, $D_o^t(x^t,\lambda y^t) = \lambda D_o^t(x^t,y^t)$, $\lambda > 0$, and

$$Do^t(x^t,y^t) = 1 \Leftrightarrow y^t \in \text{Isoq}P^t(x^t), \qquad (2)$$

where $\text{Isoq}P^t(x^t) = \{y^t: y^t \in P^t(x^t), \lambda y^t \notin P^t(x^t), \lambda > 1\} \supseteq \text{Eff}P^t(x^t) = \{y^t: y^t \in P^t(x^t), y'^t \ge y^t \Rightarrow y'^t \notin P^t(x^t)\}$. A radial output-oriented Debreu-Farrell efficiency measure is defined as

$$E_o^t(x^t,y^t) = [D_o^t(x^t,y^t)]^{-1}, \qquad (3)$$

and so an observation can have $E_o^t(x^t,y^t) = 1$ even though $y^t \notin \text{Eff}P^t(x^t)$. Moreover, an observation can have $E_o^t(x^t,y^t) = 1$ and $y^t \in \text{Eff}P^t(x^t)$, but $x^t \notin \text{Eff}L^t(y^t) = \{x^t: x^t \in L^t(y^t), x'^t \le x^t \Rightarrow x'^t \notin L^t(y^t)\}$, $L^t(y^t)$ being the set of input vectors capable of producing y^t with period t technology. Thus $E_o^t(x^t,y^t) = 1$ is necessary, but not sufficient in two respects, for overall efficiency in the sense of Koopmans.

An output-oriented period t Malmquist productivity index is given by

$$M_o^t(x^t,y^t,x^{t+1},y^{t+1})=D_o^t(x^{t+1},y^{t+1})/D_o^t(x^t,y^t), \qquad (4)$$

where

$$D_o^t(x^{t+1},y^{t+1}) = \inf\{\theta: y^{t+1}/\theta \in P^t(x^{t+1})\}. \qquad (5)$$

$M_o^t(x^t,y^t,x^{t+1},y^{t+1})$ is greater than, equal to or less than unity according as productivity increases, remains unchanged or declines between periods t and t+1, from the perspective of period t technology. The geometric mean of $M_o^t(x^t,y^t,x^{t+1},y^{t+1})$ and $M_o^{t+1}(x^t,y^t,x^{t+1},y^{t+1})$, which is defined analogously, provides the Malmquist-type productivity index which CCD showed to be equal to the product of a Törnqvist productivity index and a Törnqvist scale index.

FGLR showed how to provide a nonparametric characterization of the structure of period t technology. They also showed how to calculate the efficiency measure $E_o^t(x^t,y^t)$ for all observations indexed by $i = 1,\ldots,I^t$ in a data set.[4] Both objectives are achieved by solving the following output-oriented DEA problem for each observation

$$[D_o^t(x^{it},y^{it})]^{-1} = \max \ \phi \qquad (6)$$

$$\text{subject to} \quad \phi y^{it} \leq \Sigma_j \lambda^{jt} y^{jt}$$

$$\Sigma_j \lambda^{jt} x^{jt} \leq x^{it}$$

$$\lambda^{jt} \geq 0, \qquad j = 1,\ldots,I^t$$

$$\Sigma_j \lambda^{jt} = 1.$$

From equation (3), $\phi = E_o^t(x^t,y^t)$. The structure of period t technology is characterized by the dual variables associated with program (6). An output distance function for each observation is provided by the reciprocals of the solutions to program (6). $D_o^t(x^{t+1},y^{t+1})$ is calculated for each producer by substituting (x^{it+1},y^{it+1}) for (x^{it},y^{it}), and retaining (y^{jt},x^{jt}), in program (6). Solving these $2 \bullet I^t$ programs generates all the information required to calculate $M_o^t(x^t,y^t,x^{t+1},y^{t+1})$ for each producer.

The Malmquist productivity index is the ratio of two output distance functions. When an output-oriented DEA model is used to calculate the output distance functions, they become reciprocals of the solutions to the linear programs. However slacks can occur in the output constraints and the input constraints of the linear programs, and they represent a nonradial component of overall inefficiency, but they are not incorporated into the Malmquist productivity index. We now propose an alternative approach which does incorporate slacks into a quasi-Malmquist productivity index.

As we suggested in Section 1, we adopt a balanced position between Koopmans' definition of technical efficiency and the Debreu-Farrell measure of technical efficiency. This position suggests the following DEA-based, output-oriented definition of technical efficiency: An observation is classified as being **output-efficient** if it is technically efficient in the sense of Debreu and Farrell, and if in addition it has all its output slacks at level zero. It should be clear that output-efficient observations belong to $EffP^t(x^t)$ because they have no output slacks, although they may not necessarily be Koopmans-efficient because they may have input slacks. Consequently the set of output-efficient observations is nested between the set of Koopmans-efficient observations and the set of radially efficient observations in the sense of Debreu and Farrell.

A natural approach is to base a nonradial efficiency measure on the oriented DEA program (6), but to incorporate output slacks as well as output-oriented radial inefficiency into an inclusive output-oriented efficiency measure. Total (radial plus nonradial) output slacks are obtained from (6) as

$$S_m^t = (\phi - 1)y_m^t + r_m^t, \quad m = 1,\ldots,M. \qquad (7)$$

where: $r_m^t = \Sigma_j \lambda^{jt} y_m^{jt} - \phi y_m^t$.

If nonradial slack (r_m^t) in output m is zero, total slack reduces to radial inefficiency, while if nonradial slack in output m is positive, total slack exceeds radial inefficiency. An output-oriented nonradial technical efficiency measure can now be defined as

$$\Omega_o^t =: [1 + (M)^{-1} \Sigma_m(S_m^t/y_m^t)] . \qquad (8)$$

Inserting (7) into (8), it is clear that $\Omega_o^t(x^t,y^t) = E_o^t(x^t,y^t)$ in the absence of output slacks in the oriented DEA problem (6). It is appealing that $\Omega_o^t(x^t,y^t)$ and $E_o^t(x^t,y^t)$ are derived from the same oriented DEA problem, although they are based on different projections. The radial efficiency measure $E_o^t(x^t,y^t)$ is based on a radial projection which belongs to the output isoquant, while the nonradial efficiency measure $\Omega_o^t(x^t,y^t)$ is based on a nonradial projection which belongs to the output efficient subset. Unfortunately, Ω_o^t is not monotonic (in inputs and in outputs). In fact, let us assume that we can establish that unit 1 is less efficient than unit 2 (e.g., by direct comparison of inputs and outputs). Then we cannot conclude that the new efficiency score (Ω_o^t) of unit 1 is less or equal than the new efficiency score of unit 2.[5] This motivates our search for an alternative DEA model that guarantees that the formal expression of Ω_o^t is a monotonic measure.

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Consider the following output-oriented weighted additive model (Pastor (1994), Lovell, Pastor and Turner (1995), Cooper and Pastor (1996)):

$$\max \quad \Sigma_m(s_m^t/y_m^t) \tag{9}$$

$$\text{subject to} \quad \Sigma_j\lambda^{jt}y_m^{jt} - s_m^t = y_m^t \quad j = 1,\ldots,I^t,\ m = 1,\ldots,M$$

$$\Sigma_j\lambda^{jt}x_n^{jt} - s_n^t = x_n^t \quad j = 1,\ldots,I^t,\ n = 1,\ldots,N$$

$$s_m^t \geq 0 \quad m = 1,\ldots,M$$

$$s_n^t \geq 0 \quad n = 1,\ldots,N$$

$$\lambda^{jt} \geq 0 \quad j = 1,\ldots,I^t$$

$$\Sigma_j\lambda^{jt} = 1.$$

Based on this weighted additive model, we define our nonradial technical efficiency measure as

$$\Gamma_o^t(x^t,y^t) =: [1 + (M)^{-1}\Sigma_m(s_m^t/y_m^t)]. \tag{10}$$

The expression for $\Gamma_o^t(x^t,y^t)$ has the same form as the expression for $\Omega_o^t(x^t,y^t)$, but there is a significant difference between the two. $\Gamma_o^t(x^t,y^t)$ is based on total output slacks s_m^t obtained from the weighted additive DEA model (9), whereas $\Omega_o^t(x^t,y^t)$ is based on total (= radial plus nonradial) output slacks S_m^t obtained from the oriented DEA model (6). The implication of this difference is that, while $\Omega_o^t(x^t,y^t)$ is **not** monotonic (in outputs and in inputs), $\Gamma_o^t(x^t,y^t)$ is monotonic (in outputs and in inputs) as we demonstrate below.[6]

The DEA program (9) is an output-oriented weighted additive model which projects each observation onto $\text{EffP}^t(x^t)$, where output slacks are all at level zero. Thus an observation is classified as being output-efficient if the optimal solution to program (9) is zero, and output-inefficient otherwise.[7] The efficiency measure (10) has the following properties:

Γ1 $\Gamma_o^t(x^t,y^t) \geq 1$;

Γ2 $\Gamma_o^t(x^t,y^t) = 1 \Leftrightarrow y^t \in \mathrm{Eff}P^t(x^t)$;

Γ3 $\Gamma_o^t(x^t,y^t)$ is strictly monotonic in y^t and weakly monotonic in x^t;

Γ4 $\Gamma_o^t(x^t,y^t)$ is invariant to changes in units of measurement;

Γ5 $\Gamma_o^t(x^t,y^t) \geq E_o^t(x^t,y^t)$.

Proofs of these properties can be found in the Appendix.

Russell (1988) notes four desirable properties of an efficiency measure: inclusion (which he calls indication, meaning that the measure equals one if, and only if, the observation is Koopmans efficient), strict monotonicity, units invariance (which he calls commensurability) and homogeneity. The radial Debreu-Farrell measure satisfies two of the four properties, commensurability and homogeneity, but for reasons mentioned above it does not satisfy the inclusion property and it satisfies only a weak monotonicity property. Our new efficiency measure satisfies a weak inclusion property (Γ2), a stronger monotonicity property (Γ3) and the commensurability property (Γ4), although it does not satisfy a homogeneity property. Finally, because our new measure satisfies a weak inclusion property and the radial measure does not, there is a natural ordering between the two measures (Γ5). Thus there is a tradeoff between the two measures.[8]

An output-oriented quasi-distance function $Q_o^t(x^t,y^t)$ can now defined as

$$Q_o^t(x^t,y^t) = [\Gamma_o^t(x^t,y^t)]^{-1}. \qquad (11)$$

It is straightforward to list the properties satisfied by the quasi-distance function:

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Q1 $0 < Q_o^t(x^t,y^t) \leq 1$;

Q2 $Q_o^t(x^t,y^t) = 1 \Leftrightarrow y^t \in EffP^t(x^t)$;

Q3 $Q_o^t(x^t,y^t)$ is strictly monotonic in y^t and weakly monotonic in x^t;

Q4 $Q_o^t(x^t,y^t)$ is invariant with respect to units of measurement;

Q5 $Q_o^t(x^t,y^t) \leq D_o^t(x^t,y^t)$.

Typically the last inequality is a strict inequality. Only in the unusual case in which the radial DEA model (6) and our new weighted additive DEA model (9) generate the same projection with no output slacks does $D_o^t(x^t,y^t) = Q_o^t(x^t,y^t)$.

An output-oriented period t quasi-Malmquist productivity index can now be defined in terms of quasi-distance functions as

$$QM_o^t(x^t,y^t,x^{t+1},y^{t+1}) = Q_o^t(x^{t+1},y^{t+1})/Q_o^t(x^t,y^t). \qquad (12)$$

Although $Q_o^t(x^t,y^t) \leq D_o^t(x^t,y^t)$, we have that $QM_o^t(x^t,y^t,x^{t+1},y^{t+1})$ can be greater than, equal to or less than $M_o^t(x^t,y^t,x^{t+1},y^{t+1})$. Since no analytical ordering can be established between $QM_o^t(x^t,y^t,x^{t+1},y^{t+1})$ and $M_o^t(x^t,y^t,x^{t+1},y^{t+1})$, we now turn to an empirical investigation of the relationship between the two productivity indexes.[9]

3. An Empirical Comparison of the Performance of the Two Productivity Indexes

The data we use to conduct our empirical comparison of the Malmquist and quasi-Malmquist productivity indexes comprise an unbalanced panel describing the operations of about two-thirds of all Spanish commercial banks during the period 1986-1993. Although the population of commercial banks increased from 97 to 110 during the period, data limitations restricted our sample size to between 61 and 67 commercial banks per year. Nonetheless, the commercial banks included in our sample accounted for 93% of aggregate commercial bank assets in 1993.

We follow the value added approach to bank productivity measurement, in which banks incur costs in the provision of deposit and loan services to their customers. We specify three outputs and two inputs. The outputs are the value of loan accounts, the value of savings accounts, and the value of checking accounts. All three account types include private, public and non-resident accounts, and all are deflated to 1986 values using the consumer price index. The two inputs are the number of employees, and the sum of non-labor operating expense, direct expenditure on buildings, and amortization expense. The second input is also deflated to 1986 values.

Table 1 shows annual arithmetic means, maximum values and minimum values for each variable, and the number of banks in the sample. Three features of the data stand out. Banks have changed their product mix during the period, with mean loan account size having grown more rapidly than mean deposit account sizes of either type. Banks have also substituted out of labor, and actually shed labor during the second half of the period. The third feature of the data is the enormous dispersion in bank size. The maxima and minina are not outliers; several banks have fewer than 100 employees, and several banks have several thousand employees.

Table 2 provides a comparison of the radial efficiency measure $E_0^t(x^t,y^t) = [D_o^t(x^t,y^t)]^{-1}$ and the nonradial efficiency measure $\Gamma_0^t(x^t,y^t) = [Q_o^t(x^t,y^t)]^{-1}$. Both measures trend upward through time, suggesting that dispersion in performance is increasing. However in every year the geometric mean of the nonradial efficiency measure exceeds the geometric mean of the radial efficiency measure, and the overall geometric mean is 27% higher. The nonradial measure suggests that banks are capable of a 55% overall improvement in their service provision, while the radial measure suggests that banks are capable of only a 22% increase in their service provision. This difference provides a dramatic empirical illustration of the magnitude of the difference between the Debreu-Farrell radial measure of efficiency and the Koopmans weakly inclusive definition of efficiency (on the output side only) when linear programming techniques are used to calculate the two efficiency measures.

The severity of the slack problem is only observable after solving a radial model (in our case, the BCC model.) This is the key for switching from a radial to an additive model, i.e., for considering Quasi-Malmquist instead of Malmquist indices. The (nonradial) slacks of our example, r-sub m-super t, are summarized in Table 3; there we illustrate the *magnitude* of the problem by listing the arithmetic means of slacks, expressed as a per cent of actual values, for each output in each year, and we illustrate the *frequency* of the problem by listing the percent of observations exhibiting positive slack in at least one output dimension in each year. Slacks associated with the three outputs, average 9.2% of actual values. These slacks are *nonradial* slacks obtained from the radial model (6). As for frequency, slacks occur in at least one output dimension in over 74.5% of the observations. The majority of inefficient observations lie outside the efficient subset cone, and some of them are far removed from it. These two aspects of the slack problem lead one to expect that the quasi-Malmquist productivity index may differ from the Malmquist productivity index.

We now conduct a comparison of the quasi-Malmquist and Malmquist productivity indexes to determine whether the slack problem carries over from the efficiency measurement exercise to the productivity measurement exercise. The results summarized in Table 4 suggest that it does. Wether the differences are relevant or not requires an ex-post analysis, after computing both the Malmquist and the Quasi-Malmquist indices. Although there is no analytical ordering between $QM_o^t(x^t,y^t,x^{t+1},y^{t+1})$ and $M_o^t(x^t,y^t,x^{t+1},y^{t+1})$, the empirical relationship summarized in Table 4 is strong. The two indexes exhibit the same trend through time, although at different levels. In every pair of adjacent years the geometric mean (over all banks) of the quasi-Malmquist productivity index is smaller than the geometric mean of the Malmquist productivity index, although over the entire period it is only 3 percentage points lower. Moreover, the two indexes paint a qualitatively different picture of the productivity performance of commercial banks during the period. The Malmquist productivity index suggests that productivity *growth* has proceeded at an average annual rate of 2.8%, while the quasi-Malmquist productivity index suggests that productivity *decline* has occurred at an average annual rate of 0.4%. The difference does not appear to be due to a few large outlying values of the Malmquist productivity index or to a few small outlying values of the quasi-Malmquist productivity index. Generally speaking, a large value of the Malmquist productivity index coincides with a large value of the quasi-Malmquist productivity index, and a small value of the quasi-Malmquist productivity index coincides with a small value of the Malmquist productivity index. The difference appears to be due to the fact that most banks have smaller values of the quasi-Malmquist productivity index in each pair of adjacent years.

4. Summary and Conclusions

FGLR showed how to use linear programming techniques to construct the distance functions on which the Malmquist productivity index is based. This has led to a spate of recent empirical applications of the Malmquist productivity index. However in all but one of the applications of which we are aware (the exception being Thompson et al. (1996)), the slacks which inevitably occur in the solutions to the linear programming problems have been omitted from the analysis. Although slacks do not constitute a source of inefficiency in the radial Debreu-Farrell efficiency measures, they do constitute a source of inefficiency in the Koopmans definition of efficiency. It is of interest, then, to determine the impact on a Malmquist productivity index of incorporating slacks into the component efficiency measures.

We have followed a linear programming approach quite different from that employed by FGLR, who implemented a radial DEA model which ignores slacks. We implemented a weighted additive DEA model, and we incorporated output slacks into our nonradial efficiency measure. We called the reciprocal of our nonradial efficiency measure a quasi-distance function, and we explored its properties. We found the quasi-distance function to lack the homogeneity property satisfied by a distance function, in return for which it gains the desirable properties of weak inclusion and semi-strict monotonicity not satisfied by a distance function. We then built a quasi-Malmquist productivity index from the quasi-distance functions, and despite the differences between the component functions, we found no analytical ordering between the Malmquist and quasi-Malmquist productivity indexes.

We then used Spanish commercial bank data to conduct an empirical comparison of the nonradial and radial efficiency measures, and of the quasi-Malmquist and Malmquist productivity indexes. We found large and systematic differences between the two efficiency measures, with the nonradial efficiency measure being empirically smaller than the radial efficiency measure, as it must be analytically. Although there is no analytical ordering between the quasi-Malmquist productivity index and the Malmquist productivity index, the quasi-Malmquist productivity index is empirically smaller, on average, in every pair of adjacent years.

It is well known that neglecting slacks when linear programming techniques are used to calculate efficiency measures leads to a general overstatement of efficiency. What has not been known is the impact of neglecting slacks when calculating Malmquist productivity indexes. We have shown in this paper that, at least for our Spanish commercial bank data, the impact is not very large. It appears that large differences in annual efficiency scores largely wash out in adjacent-year comparisons, and generate much smaller differences in annual productivity indexes. Consequently, the ex-post analysis reveals that the DEA-based Malmquist productivity indices are accurate enough and do not need to be substituted by the Quasi-Malmquist productivity indices. Nevertheless, in the presence of large and pervasive nonradial slacks we do not know in advance how similar the results will be. If we do not need to consider any decomposition of the productivity indices the best strategy

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is to compute directly the Quasi-Malmquist indices. Otherwise, we recommend to compute the Malmquist indices and to check if they are accurate enough. If this is the case, we prefer the Malmquist over the Quasi-Malmquist indices.

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Footnotes

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1. Under somewhat different conditions Färe and Grosskopf (1992) and Balk (1993) have related the Malmquist productivity index to the Fisher (1922) ideal index, which also uses prices in its construction.

2. Althin (1995; Ch. 4) has a nice discussion of the properties of the Malmquist productivity index.

3. We use the following vector inequality notation: $z' \geq z \Leftrightarrow z'_i \geq z_i$ but $z' \neq z$.

4. FGLR (1992) imposed nonincreasing returns to scale on technology by replacing $\Sigma_j\lambda^{jt} = 1$ with $\Sigma_j\lambda^{jt} \leq 1$in (6), and FGLR (1994) imposed constant returns to scale on technology by deleting $\Sigma_j\lambda^{jt} = 1$ from (6). These assumptions are unnecessary for an analysis of Malmquist productivity indexes; their value is empirical rather than theoretical, since, together with other assumptions typically maintained, they guarantee existence of solutions to the linear program (6) for all producers.

5. In order to show that Ω_o^t is not even monotonic (in inputs and in outputs) let us consider the next counterexample, based on the experimental results of Pastor (1995). First, and in order to show that $\Omega_o^t(x^t,y^t)$ is not monotonic in inputs let us consider four different units with one input and two outputs. Let us define U1 as (5;38,31), U2 as (4;37,39), U3 as (5;30,24) and U4 as (4;30,24). If we compare U3 and U4 it is quite obvious that U3 is less efficient than U4 (U3 consumes more input than U4 and reaches the same output levels). In fact, if we perform a DEA analysis resorting to the BCC output-oriented model, we get that U1 and U2 are efficient units, while U3 and U4 are inefficient units, with efficiency scores $\phi_3 = 1.267$ and $\phi_4 = 1.233$. Moreover, the total slacks obtained for U3 are (0;8,7), and the total slacks obtained for U4 are (0;7,15). Easy computations lead us to the next results: Ω_o^t (U3) = 1.279 and Ω_o^t (U4) = 1.424. These results suggest that U4 is less efficient than U3, which is a contradiction. Secondly, let us show that Ω_o^t is not monotonic in outputs. Now we consider four units with one input and three outputs. Let us define U1 as (1; 70, 29.5, 22), U2 as (1; 37.5, 30, 12.5),

U3 as (1; 40, 24, 20) and U4 as (1; 30, 24, 10). Here U4 is less efficient than U3. If we perform a DEA analysis resorting to the BCC output-oriented model we get that U1 and U2 are efficient units and that the value of the efficiency score associated with U3 is 1.10, while the value associated with U4 is 1.25. Surprisingly enough, if we consider the total (radial plus nonradial) slacks associated with U3 and with U4, which are given by (0; 30, 5.5, 2) and (0; 7.5, 6, 2.5) respectively, we get Ω_o^t (U3) = 1.340 and Ω_o^t (U4) = 1.25, which constitutes again a contradiction.

6. It is possible to incorporate weighted input slacks into equation (10). In this event our new efficiency measure would satisfy strict inclusion (of output slacks and input slacks). However in this output-oriented productivity measurement context we choose not to do so.

7. Although we assume that $y^t \geq 0$, (10) requires that $y^t > 0$. In our empirical illustration, $y^t > 0$.

8. Our nonradial technical efficiency measure is similar to the oriented nonradial technical efficiency measure proposed by Zieschang (1984), and to the "slack-adjusted" measures proposed by Torgersen et al. (1996). However Torgersen et al. did not weight slacks by observed values, as we do, and so they are unable to obtain a single efficiency measure. Their output-oriented efficiency measures are output-specific, and cannot be aggregated across noncommensurate outputs. Other nonradial efficiency measures have recently been proposed by Aida et al. (1996), Briec (1996), Frei and Harker (1996), González and Alvarez (1996), and Pastor (1995), and a nonradial Malmquist productivity index has been proposed by Thompson et al. (1996). A review of nonradial DEA-based efficiency measures can be found in Cooper and Pastor (1996).

9. Like the Malmquist productivity index, the quasi-Malmquist productivity index decomposes into indexes of technical change and efficiency change, and this two-way decomposition can be extended if desired. However because quasi-distance functions do not satisfy a homogeneity property, the interpretation of the components of the quasi-Malmquist productivity index is complicated.

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Appendix

(Γ1) $\Gamma_o{}^t(x^t,y^t) \geq 1$

This property is a direct consequence of the definition of $\Gamma_o{}^t(x^t,y^t)$ and of the fact that all the output values as well as all the output slacks are non-negative numbers.

(Γ2) $\Gamma_o{}^t(x^t,y^t) = 1 \Leftrightarrow y^t \in EffP^t(x^t)$

This is also a direct consequence of the definition of $\Gamma_o{}^t(x^t,y^t)$ and of the definition of an output-efficient observation. In fact, $\Gamma_o{}^t(x^t,y^t)=1$ if, and only if, all output slacks in the solution to model (9) are zero, and this is precisely the definition of an output-efficient observation.

(Γ3) *$\Gamma_o{}^t(x^t,y^t)$ is strictly monotonic in y^t and weakly monotonic in x^t.*

Let us start with the weak input monotonicity. For the sake of clarity we do not consider any time period in our notation. Let us denote by (x^1,y^1) and (x^2,y^2) two observations which differ in only one input, for instance the first input. We can write $(x^1,y^1) = (x_1,x_2,...x_N, y_1,...,y_M)$, $(x^2,y^2) =(x_1 -k,x_2,...x_N, y_1,...,y_M)$, where k is a positive constant. If both observations have the same radial projection, it is obvious that both have the same optimal output slacks and, consequently, $\Gamma_o{}^t(x^1,y^1)= \Gamma_o{}^t(x^2,y^2)$. If both observations have different radial projections, let us consider the following path from (x^1,y^1) to the projection of (x^2,y^2): make a displacement of k units on the X_1 axis and then follow the optimal path of (x^2,y^2) to its projection. If we denote by $s_n{}^2$ the optimal input slacks, n=1,..,N, and by $s_m{}^2$ the optimal output slacks, m = 1,...,M, associated with the second unit, then the value of the slacks associated to the first unit and to the path described above are: $s_1{}^{12} = s_1{}^2+k$, $s_n{}^{12} = s_n{}^2$, n=2,..,N, $s_m{}^{12} = s_m{}^2$, m=1,...,M. Since (9) is a maximization model it is clear that $\sum_m(s_m{}^1/y_m) \geq \sum_m(s_m{}^{12}/y_m) = \sum_m(s_m{}^2/y_m)$, and therefore $\Gamma_o{}^t(x^1,y^1) \geq \Gamma_o{}^t(x^2,y^2)$. Strict monotonicity in outputs is proved similarly and can be found in the Appendix of Lovell, Pastor and Turner (1995).

(Γ4) *$\Gamma_o{}^t(x^t,y^t)$ is invariant to changes in units of measurement.*

The definition of the objective function in (9) as a sum of ratios shows that if a change in the scale of any output is performed, its effect on the correponding slack (numerator) cancels with the effect on the output value (denominator). On the other hand, $\Gamma_o{}^t(x^t,y^t)$ does not depend directly on input slacks or input values and so, since the feasible region does not change, it is unaffected by them.

(Γ5) $\Gamma_o{}^t(x^t,y^t) \geq E_o{}^t(x^t,y^t)$.

For the sake of brevity let us write ϕ^t for $E_o{}^t(x^t,y^t)$. ϕ^t is the Banker et al. (1984) output-oriented radial efficiency measure. Let us denote by $r_m{}^t$ the (non-radial) slack obtained for output *m* by means of the same model. Let us further denote by $S_m{}^t$ the total slack, that is, the sum of the non-radial slack r_m^t and the radial slack, given by $(\phi^t - 1)\cdot y_m^t$. Since model (9) maximizes the sum of output slacks, it is clear that $\sum_m(s_m^t/y_m^t) \geq \sum_m(S_m^t/y_m^t)$. Consequently

$$\Gamma_o^t(x^t,y^t) = [\ 1 + (M^{-1})\sum_m(s_m^t/y_m^t)\ ] \ \geq\ [\ 1 + (M^{-1})\sum_m(S_m^t/y_m^t)\ ] \geq$$

$$[\ 1 + (M^{-1})\sum_m(r_m^t + (\phi - 1)\cdot y_m^t)/y_m^t)\ ] = [\ 1 + (\phi - 1) + (M^{-1})\sum_m(r_m^t/y_m^t)\ ] =$$

$$= [\ \phi + (M^{-1})\sum_m(r_m^t/y_m^t)\ ] \geq \phi.$$

A further interesting consequence of the first inequality is that if an observation is output-efficient (s_m^t = 0 for all *m*), then S_m^t = 0 for all *m*, and, consequently, ϕ = 1, i.e., the observation is weakly efficient.

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Table 1
Spànish Commercial Bank Data, 1986 - 1993

	1986	1987	1988	1989	1990	1991	1992	1993
OUTPUTS								
Loan Accounts (million of 1986 ptas.)								
Mean	166	175	197	204	209	241	229	205
Max	1.323	1.267	2.078	2.063	1.997	2.943	2.866	2.853
Min	1.0	1.4	1.2	1.5	1.4	1.5	1.5	1.1
Savings Accounts (millions of 1986 ptas.)								
Mean	143	154	160	166	157	158	163	166
Max	1.107	1.200	1.371	1.407	1.350	1.901	2.057	2.199
Min	1.6	2.4	2.2	0.5	1.1	1.1	1.5	1
Checking Accounts (millions of 1986 ptas.)								
Mean	98	100	113	116	136	144	123	107
Max	863	906	1.377	1.312	1.476	1.669	1.441	1.426
Min	0.6	0.6	1.2	0.2	0.7	1.0	0.4	1.1
INPUTS								
Employees (numbers)								
Mean	2.364	2.318	2.364	2.303	2.266	2.329	2.192	1.960
Max	18.399	18.051	24.761	23.400	21.974	30.175	28.084	26.231
Min	29	36	41	15	24	29	29	28
Operating Expense (millions of 1986 ptas.)								
Mean	3.5	3.7	4.2	4.2	4.6	5.2	5.1	4.9
Max	25.5	26.6	45.7	49.1	49.2	50.8	51.8	51.9
Min	0.05	0.06	0.06	0.06	0.04	0.08	0.09	0.04
Number of Commercial Banks								
	64	65	64	66	67	65	67	61

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Table 2
Nonradial and Radial Efficiency Measures, 1986-1993

	1986	1987	1988	1989	1990	1991	1992	1993	Geometric Mean
$\Gamma_0^t(x^t,y^t)$	1.359	1.439	1.443	1.407	1.923	1.653	1.592	1.653	1.547
$[D_0^t(x^t,y^t)]^{1}$	1.190	1.183	1.195	1.206	1.245	1.215	1.230	1.305	1.221

Table 3
Non-Radial Slacks as a Percent of Observed Output Values. Radial Model, 1986-1993

	1986	1987	1988	1989	1990	1991	1992	1993
Output								
Loans	5.29	8.77	7.40	8.15	2.11	4.33	6.25	14.87
Savings	1.11	7.76	17.97	10.81	41.78	7.88	5.68	8.57
Checking	1.41	3.36	5.00	4.74	12.40	2.95	15.87	15.91
Frequency (%)								
	67.71	68.72	71.88	70.71	79.10	79.19	76.26	82.41

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Table 4
Quasi-Malmquist and Malmquist Productivity Indexes, 1986/87 - 1992/93

	1986/87	**1987/88**	**1988/89**	**1989/90**	**1990/91**	**1991/92**	**1992/93**	**Geometric Mean**
Quasi-Malmquist	1.034	1.071	1.073	0.928	0.959	0.941	0.977	0.996
Malmquist	1.084	1.110	1.074	0.995	1.005	0.958	0.980	1.028