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Maximal Domains of Preferences Preserving Strategy-Proofness for Generalized Median Voter Schemes*

by

Salvador Barberà†, Jordi Massó‡, and Alejandro Neme§

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Abstract

We characterize the maximal sets of preferences under which generalized median voter schemes are strategy-proof. Those domains are defined by a qualified version of single-peakedness, which depends on the distribution of power among agents implied by each generalized median voter scheme.

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†Departament d'Economia i d'Història Econòmica and CODE. Universitat Autònoma de Barcelona. 08193, Bellaterra (Barcelona), Spain.

‡Departament d'Economia i d'Història Econòmica and CODE. Universitat Autònoma de Barcelona. 08193, Bellaterra (Barcelona), Spain.

§Instituto de Matemática Aplicada. Universidad Nacional de San Luis and CONICET. Ejército de los Andes 950. 5700, San Luis, Argentina.

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Proofs should be sent to: Jordi Massó.

1. Introduction

This paper investigates the connections between single-peakedness and strategy-proofness. Whether or not nontrivial strategy-proof social choice functions exist depends on the environment where we want them to operate. When alternatives can be represented as points in a rectangular grid, and preferences are single-peaked, generalized median voter schemes are strategy-proof. Outside those situations, nontrivial strategy-proof social choice functions may still exist, but they are harder to find: domain restrictions become less natural.

Single-peakedness of the agent's preferences is often assumed in the existing literature and it is certainly a useful requirement toward the existence of nontrivial strategy-proof social choice functions. In some environments it is sufficient to guarantee it, in others it needs to be combined with additional restrictions. But it is always there. This leads us to investigate, in the present paper, the extent to which some form of single-peakedness might be necessary for strategy-proofness, as well as sufficient.

Our answer is partial, because it only refers to generalized median voter schemes, but it is precise. We start from any such scheme F, and we characterize the maximal set of preferences under which F is strategy-proof. It turns out that the condition characterizing this maximal domain is a qualified version of single-peakedness. Previous results in the same vein include Barberà, Sonnenschein, and Zhou [4], Serizawa [10], and Barberà, Massó, and Serizawa [3]. Our results improve upon these previous results in several directions. We allow for all types of generalized median voter schemes, by not ruling out the existence of vetoers or dummies. We also cover restricted domains under which the range of generalized median voter schemes might not be a cartesian product.

We have chosen to keep this introduction short, leaving further motivational remarks and examples for Section 2, which contains the definitions and a statement of our result. This is proven in Section 3. Section 4 concludes.

2. Definitions, Notation and the Theorem

Agents are the elements of a finite set $N = \{1, 2, ..., n\}$. We assume that n is at least 2.

Alternatives are K-tuples of integers numbers. For integers $a, b \in Z$, with $a < b$, we will denote the *integer interval* $[a, b] = \{a, a+1, ..., b\}$. A K-dimensional *box* B is a cartesian product of K integer intervals:

$$B = \prod_{k=1}^{K} B_k,$$

where $B_k = [a_k, b_k]$ and $a_k < b_k$. A *subbox* of B is any box A contained in B. We endow B with the L_1-norm. That is, for every $\alpha \in B$,

$$\|\alpha\| = \sum_{k=1}^{K} \mid \alpha_k \mid .$$

Given $\alpha, \beta \in B$, the *minimal box containing* α *and* β is defined by

$$MB(\alpha, \beta) = \{\gamma \in B \mid \|\alpha - \beta\| = \|\alpha - \gamma\| + \|\gamma - \beta\|\}.$$

Preferences are binary relations on alternatives (or subsets of alternatives). Let $\mathcal{U}$ be the set of complete, transitive and asymmetric preferences on B. *Preference profiles* are n-tuples of preferences on B, $\mathbf{P} \in \mathcal{U}^n$. Preference profiles $\mathbf{P} = (P_1, ..., P_n)$ are also represented by (P_i, P_{-i}) when we want to stress the role of i's preference. For $P \in \mathcal{U}$ and $A \subseteq B$, we denote the alternative in A most preferred by P as $\tau^A(P)$, and we call it *the top of* P *on* A. Therefore, $\tau^B(P)$ is the unconstrained top of P.

A *social choice function* on $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \subseteq \mathcal{U}^n$ is a function $F : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \to B$.

The range of a social choice function $F : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \to B$, is denoted by R_F. That is,

$$R_F = \left\{\alpha \in B \mid \exists\ \mathbf{P} = (P_1, ..., P_n) \in \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \text{ such that } F(\mathbf{P}) = \alpha\right\}.$$

Social choice functions require each agent to report some preference. A social choice function is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully. Formally,

Definition 1. *A social choice function* $F : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \to B$ *is manipulable on* $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ *if there exist* $\mathbf{P} = (P_1, ..., P_n) \in \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$, $i \in N$ *and* $P_i' \in \widetilde{\mathcal{P}}_i$ *such that* $F(P_i', P_{-i})\, P_i F(\mathbf{P})$. *A social choice function is strategy-proof on* $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ *if it is not manipulable on* $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$.

Definition 2. *Let* $F : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \rightarrow B$ *be a social choice function and let* $P_i \in \widetilde{\mathcal{P}}_i$. *The set of options left open to the other agents by* i *declaring* P_i *is defined as follows:*

$$o(P_i) = \left\{ \alpha \in R_F \mid \text{there exists } P_{-i} \in \widetilde{\mathcal{P}}_{-i} \text{ such that } F(P_i, P_{-i}) = \alpha \right\}.$$

We shall consider different restrictions on preferences, all of them related to single-peakedness. The first one is a natural extension of this classical condition and has been already used in the literature; see for instance Barberà, Gul and Stacchetti [1], Serizawa [10], and Barberà, Massó, and Neme [2]. The second one refers to any set O of K-dimensional alternatives. The third one involves three sets O, A, and D. We present these definitions in sequence for the benefit of the reader, since the first one is very natural while the others are a bit harder to interpret. Formally, though, we need only one of them (Definition 5).

Our first condition is a natural extension of single-peakedness and it coincides with the classical version when $K = 1$. It says that whenever alternative β is closer than γ to the best alternative $\tau^B(P)$ (lies on the minimal path from $\tau^B(P)$ to γ, in the sense of the L_1-norm) then $\beta P \gamma$.

Definition 3. *A preference* $P \in \mathcal{U}$ *is single-peaked if* $\beta P \gamma$ *for all* $\beta, \gamma \in B$ $(\beta \neq \gamma)$ *such that* $\beta \in MB(\tau^B(P), \gamma)$.

Preferences satisfying Definition 3 are characterized by the following two properties. The first one is goodwise single-peakedness: those preferences, restricted to sets of alternatives differing only on one component, are single-peaked. The second is peak-separability: the best alternative for those preferences on such one-dimensional sets are the projection of the global best on the set.[1]

Our next definition involves a subset $O \subseteq B$ and imposes conditions only on elements of this set. Therefore, it is weaker than Definition 3 and it coincides with single-peakedness whenever $O = B$.

Definition 4. *A preference* $P \in \mathcal{U}$ *is single-peaked on* $O \subseteq B$ *if* $\beta P \gamma$ *for all* $\beta, \gamma \in O$ $(\beta \neq \gamma)$ *such that* $\beta \in MB(\tau^B(P), \gamma)$.

Finally, the third definition refers to three different sets O, A $(O \subseteq A)$, and D. Preferences will be restricted on the two sets $O \cap D$ and $A \cap D$, but not in

[1] Barberà, Gul, and Stacchetti [1] introduced first this concept and called it multidimensional single-peaked. Serizawa [10] calls those preferences cross-shaped.

the same way. They will be required to be single-peaked on $O \cap D$, although the reference point will be the top of P on A, $\tau^A(P)$, instead of the unconstrained top, $\tau^B(P)$. In addition, they will also be required to respect some milder restriction on $(A \cap D) \setminus (O \cap D)$. In applications, O will be the set of options left open by an agent, under a given social choice function, D will be a subset of alternatives where the agent is not a dummy and A will be the range of the social choice function.

Definition 5. *Consider $O \subseteq A \subseteq B$ and $D \subseteq B$. A preference $P \in \mathcal{U}$ is single-peaked on O relative to A and D if for every $\gamma \in A \cap D$ and every $\beta \in O \cap D \cap MB\left(\tau^A(P), \gamma\right)$ such that $\beta \neq \gamma$ we have that $\beta P \gamma$.*

We will say that a preference P is single-peaked on O relative to A, whenever $D = A$ and P satisfies Definition 5.

Single-peakedness and single-peakedness on O relative to A and D are related concepts but they define sets of preferences which are not necessarily subsets of each other. To see that there are single-peaked preferences which do not satisfy Definition 5, consider the case where $B = \{0,1\} \times \{0,1\}$, $O = A = D = \{(0,1), (1,0), (0,0)\}$, and the single-peaked preference $(1,1)\,P\,(0,1)\,P\,(1,0)\,P\,(0,0)$. Notice that P is not single-peaked on O relative to A since $\beta = (0,0)$ is not preferred to $\gamma = (1,0)$, $\beta \in O \cap D \cap MB\left(\tau^A(P), \gamma\right)$, and $\gamma \in A \cap D$. Obviously, a single-peaked preference on O relative to A and D may not be single-peaked because the ordering between some pairs $\beta, \gamma \in B \backslash D$ is free while it is not for a preference satisfying Definition 3.

Next, we define generalized median voter schemes. This class of social choice functions are interesting multidimensional extensions of the basic idea of median voting. Additionally, several papers have shown that, in this and similar settings,[2] they are strategy-proof rules under single-peakedness.

Definition 6. *A left (right)-coalition system on $B_k = [a_k, b_k]$ is a correspondence $\mathcal{W}_k$ that assigns to every $\alpha_k \in B_k$ a nonempty collection of nonempty coalitions $\mathcal{W}_k(\alpha_k)$ satisfying the following conditions:*
(1) If $W \in \mathcal{W}_k(\alpha_k)$ and $W \subset W'$, then $W' \in \mathcal{W}_k(\alpha_k)$.
(2) If $\beta_k > (<)\alpha_k$ and $W \in \mathcal{W}_k(\alpha_k)$, then $W \in \mathcal{W}_k(\beta_k)$.
(3) $\mathcal{W}_k(b_k) = 2^N \backslash \emptyset$ ($\mathcal{W}_k(a_k) = 2^N \backslash \emptyset$).

[2] See, for example, Moulin [8], Border and Jordan [6], Barberà, Sonnenschein, and Zhou [4], Barberà, Gul, and Stacchetti [1], Peters, van der Stel, and Storken [9], Barberà, Massó, and Neme [2], and Barberà, Massó, and Serizawa [3].

A family $\mathcal{L}$ of left-coalition systems on B is a collection $\{\mathcal{L}_k\}_{k=1}^{K}$ where each $\mathcal{L}_k$ is a left-coalition system on B_k. Similarly, a family $\mathcal{R}$ of right-coalition systems on B is a collection $\{\mathcal{R}_k\}_{k=1}^{K}$ where each $\mathcal{R}_k$ is a right-coalition system on B_k. Moreover, given a left (right)-coalition system $\mathcal{W}_k$ on B_k we say that $W \in \mathcal{W}_k(\alpha_k)$ is a *minimal* left (right) coalition if for every $i \in W$, $W \setminus \{i\} \notin \mathcal{W}_k(\alpha_k)$. Given $\mathcal{L}_k$ ($\mathcal{R}_k$) denote by $\mathcal{L}_k^m$ ($\mathcal{R}_k^m$) the corresponding sets of minimal left (right) coalitions.

For a preference profile $\mathbf{P} = (P_1, ..., P_n) \in \mathcal{U}^n$, $A \subseteq B$, and $\beta_k \in B_k$, denote by $\tau^A(\mathbf{P}) = \left(\tau^A(P_1), ..., \tau^A(P_n)\right)$ the vector of tops on A, and define the *coalition to the left (right) of* β_k at $\tau^A(\mathbf{P})$ by

$$\begin{aligned} l\left(\tau^A(\mathbf{P}), \beta_k\right) &= \left\{i \in N \mid \tau_k^A(P_i) \leq \beta_k\right\} \\ (r\left(\tau^A(\mathbf{P}), \beta_k\right) &= \left\{i \in N \mid \tau_k^A(P_i) \geq \beta_k\right\}). \end{aligned}$$

Definition 7. *Let $\{\mathcal{L}_k\}_{k=1}^{K}$ ($\{\mathcal{R}_k\}_{k=1}^{K}$) be a family of left (right)-coalition systems on B and let $A \subseteq B$. The social choice function $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \rightarrow A$ is called a generalized median voter scheme defined by $\mathcal{L}$ ($\mathcal{R}$) if it can be defined as follows: for every $\mathbf{P} \in \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ and every $k = 1, ..., K$*

$$\begin{aligned} &F_k^A(\mathbf{P}) = \beta_k \Leftrightarrow l\left(\tau^A(\mathbf{P}), \beta_k\right) \in \mathcal{L}_k(\beta_k) \text{ and } l\left(\tau^A(\mathbf{P}), \beta_k - 1\right) \notin \mathcal{L}_k(\beta_k - 1) \\ &(F_k^A(\mathbf{P}) = \beta_k \Leftrightarrow r\left(\tau^A(\mathbf{P}), \beta_k\right) \in \mathcal{R}_k(\beta_k) \text{ and } r\left(\tau^A(\mathbf{P}), \beta_k + 1\right) \notin \mathcal{R}_k(\beta_k + 1)). \end{aligned}$$

Notice that our definition is relative to the set A, since two preference profiles $\mathbf{P}$ and $\mathbf{P}'$ with the same top $\tau^B(\mathbf{P}) = \tau^B(\mathbf{P}')$ outside of A may lead to different choices under F^A, if their tops on A are not the same. However, this will not happen if A itself is box-shaped. Also notice that a generalized median voter scheme F^A respects unanimity on A and therefore the range of F^A contains A. Hence, when we write $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \rightarrow A$ we implicitly understand that the range of F^A is the set A.[3]

Before proceeding, it is useful to understand the relationship between right and left coalition systems, $\mathcal{R}_k$ and $\mathcal{L}_k$ that produce the same outcome for all $\left(\tau_k^A(P_1), ..., \tau_k^A(P_n)\right)$. Given $\mathcal{R}_k$, define $\mathcal{L}_k^*$ as follows: for all $a_k \leq \alpha_k < b_k$,

$$\begin{aligned} \mathcal{L}_k^*(\alpha_k) &= \{S \subseteq N \mid S \cap S' \neq \emptyset \text{ for all } S' \in \mathcal{R}_k(\alpha_k + 1)\}, \text{ and} \\ \mathcal{L}_k^*(b_k) &= 2^N \setminus \emptyset. \end{aligned}$$

[3]See Barberà, Massó, and Neme [2] for an explicit discussion over the ontoness of such functions and a characterization of all such generalized median voter schemes.

Remark 1. *It is easy to see that $\mathcal{R}_k$ and $\mathcal{L}_k$ will select the same outcome for all $\left(\tau_k^A(P_1), ..., \tau_k^A(P_n)\right)$ if and only if $\mathcal{L}_k = \mathcal{L}_k^*$.*

Our definition of generalized median voter schemes induces some distribution of power among agents. Some agents may never be able to influence the outcome at all: they are dummies. Some agents may always dictate the outcome to be in a specific subset: they are decisive. Some agents may avoid some outcomes, if they want: they are vetoers. These possibilities are some times global, but they can also be defined in a local sense: power may depend on the alternative under consideration and also on each of the dimensions defining this alternative. The definition below makes all these notions precise. To do so, let $\mathcal{L} = \{\mathcal{L}_k\}_{k=1}^K$ $\left(\mathcal{L}^m = \{\mathcal{L}_k^m\}_{k=1}^K\right)$ be a (minimal) left-coalition system defined on B, $A \subseteq B$, and let $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^K$ $\left(\mathcal{R}^m = \{\mathcal{R}_k^m\}_{k=1}^K\right)$ and $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \to A$ be its associated (minimal) right-coalition system and generalized median voter scheme, respectively.

Definition 8. *We say that agent i is left (right) dummy at $\beta_k \in B_k$ if $i \notin \underset{S\in\mathcal{L}_k^m(\beta_k)}{\cup} S$ $\left(i \notin \underset{S\in\mathcal{R}_k^m(\beta_k)}{\cup} S\right)$.*
We say that agent i is left (right) vetoer at $\beta_k \in B_k$ if $i \in \underset{S\in\mathcal{L}_k(\beta_k)}{\cap} S$ $\left(i \in \underset{S\in\mathcal{R}_k(\beta_k)}{\cap} S\right)$.
We say that agent i is left (right) decisive at $\beta_k \in B_k$ if $\{i\} \in \mathcal{L}_k(\beta_k)$ $(\{i\} \in \mathcal{R}_k(\beta_k))$.

Remark 2. *The following relationships result from Remark 1.*
(1) Assume that $a_k \leq \beta_k < b_k$: (1.a) if agent i is left dummy at β_k then agent i is right dummy at $\beta_k + 1$, and (1.b) if agent i is left vetoer at β_k then agent i is right decisive at $\beta_k + 1$.
(2) Assume that $a_k < \beta_k \leq b_k$: (2.a) if agent i is right dummy at β_k then agent i is left dummy at $\beta_k - 1$, and (2.b) if agent i is right vetoer at β_k then agent i is left decisive at $\beta_k - 1$.

The definition of a decisive agent follows Serizawa [10]. Notice that its power is weaker than what the name may suggest. If i is left decisive at β_k, then he can guarantee that the outcome will not be strictly above β_k. In other words, i can veto all values strictly above β_k.

Definition 9. *We say that F^A is a generalized median voter scheme without dummies if for all $k = 1, ..., K$ the set of left (right) dummies at β_k is empty for all $\beta_k \in [a_k, b_k]$.*

Our next definition requires the domain of the social choice function to be sufficiently large: this avoids cases where strategy-proofness might be trivially obtained because agents' preferences are almost fixed.

Definition 10. *We say that a domain* $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ *is rich on* $A \subseteq B$ *if for all* $i \in N$ *and* $\alpha \in A$ *there exists* $P_i \in \widetilde{\mathcal{P}}_i$ *such that* $\tau^A(P_i) = \alpha$.

The richness condition simply requires that there should be, for each alternative, at least one admissible preference ranking this alternative as best. This is a standard assumption (see, for instance, Barberà, Sonnenschein, and Zhou [4] and Serizawa [10]). Notice that if a domain of preferences is rich, its supersets are also rich.

As a starting point, we remind the reader the following result.

Theorem 1. *(Serizawa (1995)) Let* $F^B : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \to B$ *be a strategy-proof generalized median voter scheme without dummies with rich domain on* B. *Let* $i \in N$. *For any* $k = 1, ..., K$, *let* d_k *be the maximal level such that* i *is right decisive in* $\mathcal{R}_k$, *and let* v_k *be the minimal level such that he is a right vetoer in* $\mathcal{R}_k$. *Then any* $P_i \in \widetilde{\mathcal{P}}_i$ *is single-peaked on*

$$S(P_i) = \left\{\alpha \in B \mid \forall k = 1, ..., K, \min\left\{d_k, \tau_k^B(P_i)\right\} \leq \alpha_k \leq \max\left\{v_k, \tau_k^B(P_i)\right\}\right\}.$$

Our results improve upon this one in several directions. In order to motivate our contributions, let us first rephrase the essential intuition behind Theorem 1. The set $S(P_i)$ is almost the option set $o(P_i)$, i.e. the set of alternatives that, given that i votes P_i, may be the final outcome, depending on the votes of others. Precisely,[4]

$$o(P_i) = \left\{\alpha \in B \mid \forall k = 1, ..., K, \min\left\{d_k, \tau_k^B(P_i)\right\} \leq \alpha_k \leq \max\left\{v_k - 1, \tau_k^B(P_i)\right\}\right\}.$$

Then, Theorem 1 requires that i's preferences are single-peaked on $S(P_i)$. This statement is equivalent to requiring that (a) P_i is single-peaked on $o(P_i)$, and (b) v_k is worse than any point different than v_k in $MB\left(\tau_k^B(P_i), v_k\right)$ if $v_k \neq \tau_k^B(P_i)$.[5] (This rewording may seem artificial, but wait). In fact, single-peakedness on $o(P_i)$ is necessary. But, because agent i, by changing his preference from P_i to P_i', can change these options, and shift the outcome, a further requirement is also

[4] See Lemma 1 in the Appendix.

[5] This heuristic argument is done assuming implicitly that $K = 1$.

necessary: other points which might be attained by declaring preferences other than P_i must be worse than some points in the option set. Serizawa's condition requires this for the point v_k only (and does so implicitly). If we want to get a condition which is not only necessary but also sufficient for strategy-proofness we must require it explicitly and for a (generally) larger set of alternatives.

To be more specific, consider Example 1, which shows that the set of single-peaked preferences on $S(P_i)$ is still too large in the sense that with those preferences generalized median voter schemes may be manipulable.

Example 1. Consider a one-dimensional problem $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and agents 1 and 2. Define the generalized median voter scheme F^B as follows: $\mathcal{R}^m(\alpha_4) = \{1, 2\}$, and $\mathcal{R}^m(\alpha_3) = \mathcal{R}^m(\alpha_2) = \mathcal{R}^m(\alpha_1) = \{\{1\}, \{2\}\}$. Notice that F^B does not have a right-dummy agent and, by Remark 2, it does not have a left-dummy agent. Consider the preference P_2 of agent 2 such that $\alpha_4 P_2 \alpha_1 P_2 \alpha_2 P_2 \alpha_3$. Since α_3 is the maximal level such that agent 2 is right decisive and α_4 is the minimum level such that agent 2 is right vetoer, we have that $S(P_2) = [\min\{\alpha_3, \alpha_4\}, \max\{\alpha_4, \alpha_4\}] = [\alpha_3, \alpha_4]$. Since $\alpha_4 P_2 \alpha_3$ we have that P_2 is single-peaked on $S(P_2)$. However, to see that agent 2 can manipulate F^B let P_1 be any single-peaked preference for agent 1 with the property that $\tau^B(P_1) = \alpha_1$ and $P_2' = P_1$. Then, $F^B(P_1, P_2') = \alpha_1 P_2 \alpha_3 = F^B(P_1, P_2)$.

In view of this, we proceed as follows. We provide necessary and sufficient conditions for strategy-proofness of generalized median voter schemes for the general case where the range is not necessarily equal to B. Before that, in order to allow for better comparison with Serizawa's result and to proceed more smoothly, we state an intermediate result which maintains the non-dummy condition and highlights one of the directions of our extension. Since it will become a Corollary of Theorem 3 (proven in Section 3), we state it without proof.

Theorem 2. *Let $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \to A$ be a generalized median voter scheme with rich domain on A without dummies. Then, F^A is strategy-proof on $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ if and only if for every $i \in N$ and every $P_i \in \widetilde{\mathcal{P}}_i$, P_i is single-peaked on $o(P_i)$ relative to A.*

Example 2 below illustrates that the non-dummy condition in Theorems 1 and 2 is very restrictive because many generalized median voter schemes do not satisfy it. It is obvious that any maximality result should exclude agents which are dummies at all points, but there is a wide gap between the trivial case where an agent is dummy everywhere and those where he might be dummy locally, especially in a multidimensional setting.

Example 2. Consider a one-dimensional problem $B = [\alpha_1, \alpha_{10}]$ with ten alternatives $\alpha_1 < ... < \alpha_{10}$, and agents i and j $(i \neq j)$ in a set $N = \{1, ..., n\}$ where $n \geq 3$. Define the generalized median voter scheme F^B as follows: $\mathcal{L}^m(\alpha_1) = N \backslash \{i\}$, $\mathcal{L}^m(\alpha_2) = ... = \mathcal{L}^m(\alpha_9) = \{N \backslash \{i\}, N \backslash \{j\}\}$, and $\mathcal{L}^m(\alpha_{10}) = 2^N \backslash \emptyset$. Notice that although agent i is only left dummy at α_1, F^B does not satisfy the non-dummy condition, and therefore we can not apply Theorems 1 and 2.

Consider a generalized median voter scheme $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \rightarrow A$ defined by $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^K$ (and $\mathcal{L} = \{\mathcal{L}_k\}_{k=1}^K$) and let $i \in N$ and $k \in K$ be given. Consider the set of points $\{x_k^1, ..., x_k^T\}$ where $a_k < x_k^1 < ... < x_k^T \leq b_k$ and agent i is right dummy at x_k^t for all $1 \leq t \leq T$. Denote by $\mathcal{D}_k(i) = \{\mathcal{D}_k^0(i), ..., \mathcal{D}_k^T(i)\}$ the partition of $[a_k, b_k]$ where $\mathcal{D}_k^0(i) = [a_k, x_k^1 - 1]$, $\mathcal{D}_k^t(i) = [x_k^t, x_k^{t+1} - 1]$ for all $1 \leq t < T$, and $\mathcal{D}_k^T(i) = [x_k^T, b_k]$.

Definition 11. *Let $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \rightarrow A$ be a generalized median voter scheme defined by $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^K$ (and $\mathcal{L} = \{\mathcal{L}_k\}_{k=1}^K$) and let $i \in N$. The partition $\mathcal{D}(i) = \prod_{k=1}^{K} \mathcal{D}_k(i)$ of B is called the non-dummy partition of i.*

Theorem 3. *Let $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \rightarrow A$ be a generalized median voter scheme with rich domain on A. Then, F^A is strategy-proof on $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ if and only if for every $i \in N$, every $P_i \in \widetilde{\mathcal{P}}_i$, and every $D \in \mathcal{D}(i)$, P_i is single-peaked on $o(P_i)$ relative to A and D.*

Example 3 below illustrates some of the main concepts used in the definition of single-peaked preferences on $o(P_i)$ relative to A and D.

Example 3. Consider the case with two coordinates where $B = \{\alpha_1^1, ..., \alpha_1^7\} \times \{\alpha_2^1, ..., \alpha_2^5\}$ and the set of agents is $N = \{1, 2, 3, 4, 5\}$. Consider the generalized median voter scheme F^B defined by the following family of right-coalition systems

$\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2\}$:

$$\begin{aligned}
\mathcal{R}_1^m\left(\alpha_1^7\right) &= \{1,2,3,4,5\}, \\
\mathcal{R}_1^m\left(\alpha_1^6\right) &= \{2,3,4,5\}, \\
\mathcal{R}_1^m\left(\alpha_1^5\right) &= \mathcal{R}_1^m\left(\alpha_1^4\right) = \{\{2,3,4,5\},\{1,2,3,4\}\}, \\
\mathcal{R}_1^m\left(\alpha_1^3\right) &= \{\{1,2,3\},\{2,3,4\},\{3,4,5\}\}, \\
\mathcal{R}_1^m\left(\alpha_1^2\right) &= \{\{2,3\},\{3,4,5\}\}, \\
\mathcal{R}_1^m\left(\alpha_1^1\right) &= 2^N\backslash\emptyset, \\
\mathcal{R}_2^m\left(\alpha_2^5\right) &= \{1,2,3,4\}, \\
\mathcal{R}_2^m\left(\alpha_2^4\right) &= \{\{2,3,4\},\{3,4,5\}\}, \\
\mathcal{R}_2^m\left(\alpha_2^3\right) &= \mathcal{R}_2^m\left(\alpha_2^2\right) = \{\{1,2\},\{3,4\}\}, \text{ and} \\
\mathcal{R}_2^m\left(\alpha_2^1\right) &= 2^N\backslash\emptyset.
\end{aligned}$$

Notice that agent 1 is right dummy at α_1^2, α_1^6, and α_2^4 but agent 2 is never right dummy. Therefore, the non-dummy partition of agent 1 is $\mathcal{D}(1) = \mathcal{D}_1(1) \times \mathcal{D}_2(1)$, where $\mathcal{D}_1(1) = \{\{\alpha_1^1\},\{\alpha_1^2,\alpha_1^3,\alpha_1^4,\alpha_1^5\},\{\alpha_1^6,\alpha_1^7\}\}$ and $\mathcal{D}_2(1) = \{\{\alpha_2^1,\alpha_2^2,\alpha_2^3\},\{\alpha_2^4,\alpha_2^5\}\}$, while the non-dummy partition of agent 2 is the box B itself since $\mathcal{D}_1(2) = \{\{\alpha_1^1,...,\alpha_1^7\}\}$ and $\mathcal{D}_2(2) = \{\{\alpha_2^1,...,\alpha_2^5\}\}$. Consider any set of preferences $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_5$ for which F^B has rich domain on B. Notice that since $\mathcal{D}(2) = \{B\}$, any single-peaked preference P_2 on $o(P_2)$ is indeed single-peaked on $o(P_2)$ relative to B and D. Consider any preference $P_1 \in \widetilde{\mathcal{P}}_1$ such that $\tau^B(P_1) = (\alpha_1^3, \alpha_2^2)$. Notice that $o(P_1) = \{\alpha_1^1,\alpha_1^2,\alpha_1^3,\alpha_1^4,\alpha_1^5,\alpha_1^6\} \times \{\alpha_2^1,\alpha_2^2,\alpha_2^3,\alpha_2^4\}$. If F^B is strategy-proof we must have, for instance that $(\alpha_1^6,\alpha_2^2)\, P_1\, (\alpha_1^7,\alpha_2^1)$ and $(\alpha_1^4,\alpha_2^3)\, P_1\, (\alpha_1^5,\alpha_2^3)$ but we could have either $(\alpha_1^5,\alpha_2^5)\, P_1\, (\alpha_1^4,\alpha_2^3)$ or $(\alpha_1^4,\alpha_2^3)\, P_1\, (\alpha_1^5,\alpha_2^5)$ since (α_1^5,α_2^5) and (α_1^4,α_2^3) belong to different elements of the non-dummy partition of agent 1.

Before proving the main result of the paper we illustrate, in Example 4 below, that the class of preferences identified in Theorem 3 may be very large, indeed.

Example 4. Consider the case with two coordinates where $B = \{\alpha_1^1,\alpha_1^2,\alpha_1^3,\alpha_1^4,\alpha_1^5\} \times \{\alpha_2^1,\alpha_2^2,\alpha_2^3,\alpha_2^4,\alpha_2^5\}$ and the set of agents is $N = \{1,2\}$. Let F^B be the generalized median voter scheme where each agent is a dictator in one of the coordinates; that is, F^B is defined by the following family of right-coalition systems $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2\}$:

$$\begin{aligned}
\mathcal{R}_1^m\left(\alpha_1^2\right) &= ... = \mathcal{R}_1^m\left(\alpha_1^5\right) = \{1\}, \\
\mathcal{R}_2^m\left(\alpha_2^2\right) &= ... = \mathcal{R}_2^m\left(\alpha_2^5\right) = \{2\}, \text{ and} \\
\mathcal{R}_1^m\left(\alpha_1^1\right) &= \mathcal{R}_2^m\left(\alpha_2^1\right) = \{\{1\},\{2\}\}.
\end{aligned}$$

Following Le Breton and Sen [7] we say that a preference $P_1 \in \mathcal{U}$ is top unconditional for agent 1 if given $\tau^B(P_1) = (\alpha_1, \alpha_2)$ we have that $(\alpha_1, \alpha_2') P_1 (\alpha_1', \alpha_2')$ for all $\alpha_1' \neq \alpha_1$ and all $\alpha_2' \in B_2$. Similarly, we say that a preference $P_2 \in \mathcal{U}$ is top unconditional for agent 2 if given $\tau^B(P_2) = (\alpha_1, \alpha_2)$ we have that $(\alpha_1', \alpha_2) P_2 (\alpha_1', \alpha_2')$ for all $\alpha_2' \neq \alpha_2$ and all $\alpha_1' \in B_1$. Denote by $\mathcal{TU}_i$ the set of top unconditional preferences for agent i.

Le Breton and Sen [7] show, in a more general set up, that the maximal domain of preferences under which this coordinatewise dictator F^B is strategy-proof is precisely $\mathcal{TU}_1 \times \mathcal{TU}_2$. We will see that, even though preferences on $\mathcal{TU}_1$ are far from being single peaked, the set of top unconditional preferences for agent 1 coincides with the class of preferences identified in Theorem 3.[6] Given F^B, and since agent 1 is a dummy at every $\alpha_2 \neq \alpha_2^1$ we have that $\mathcal{D}_2(1) = \{\{\alpha_2^1\}, ..., \{\alpha_2^5\}\}$. Moreover, since agent 1 is never a dummy at any $\alpha_1 \in B_1$, we have that $\mathcal{D}_1(1) = \{\alpha_1^1, ..., \alpha_1^5\}$. Therefore, a generic element D in $\mathcal{D}(1)$ can be written as $\{\alpha_1^1, ..., \alpha_1^5\} \times \{\alpha_2'\}$, an horizontal integer segment. Given $P_1 \in \mathcal{TU}_1$ and its associated top element $\tau^B(P_1) = (\alpha_1, \alpha_2)$, we have that $o(P_1) = \{(\alpha_{1,}\alpha_2^1), ..., (\alpha_1, \alpha_2^5)\}$, a vertical integer segment, because agent 1 is a dictator in the first coordinate and a dummy in the second one. But Definition 5 just says that $(\alpha_1, \alpha_2') P_1 (\alpha_1', \alpha_2')$ for all $\alpha_1' \neq \alpha_1$ and all $\alpha_2' \in B_2$ which is the top unconditional condition for agent 1.

3. Proof of Theorem 3

Let $A \subseteq B$ be a subset of alternatives and let $F^A : \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n \to A$ be a generalized median voter scheme defined by $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^K$ (and $\mathcal{L} = \{\mathcal{L}_k\}_{k=1}^K$) with rich domain on A. Since the set A will be kept fixed throughout the proof we will omit its use as a superscript; that is, in this section, F and τ should be understood as F^A and τ^A. Let $i \in N$, $k = 1, ..., K$ and P_i be a preference ordering in $\widetilde{\mathcal{P}}_i$. Define:

$$v_k^i = \begin{cases} \max\left\{\beta_k \in B_k \mid i \in \underset{S \in \mathcal{L}_k(\beta_k)}{\cap} S\right\} & \text{if the set is nonempty} \\ a_k - 1 & \text{otherwise} \end{cases}, \tag{3.1}$$

$$d_k^i = \min\left\{\beta_k \in B_k \mid \{i\} \in \mathcal{L}_k(\beta_k)\right\}, \tag{3.2}$$

[6] We omit the argument for agent 2 since it is identical after interchanging the role of the coordinates.

$$a_k(P_i) = \min\left\{v_k^i + 1, \tau_k(P_i)\right\}, \tag{3.3}$$

$$b_k(P_i) = \max\left\{d_k^i, \tau_k(P_i)\right\}, \text{and} \tag{3.4}$$

$$B(P_i) = A \cap \left[\prod_{k=1}^{K} [a_k(P_i), b_k(P_i)]\right].^7 \tag{3.5}$$

Lemma 1 below describes, for any given generalized median voter scheme, the exact shape of the set of options left open by an agent i to the other agents.

Lemma 1. $B(P_i) = o(P_i)$.

Proof. Let $\beta \in B(P_i)$. For every $j \neq i$, consider any $P_j \in \widetilde{\mathcal{P}}_j$ with the property that $\tau(P_j) = \beta$. We will show that $F(P_i, P_{-i}) = \beta$. Let $k = 1, ..., K$ be arbitrary and define the set $S = \{j \in N \mid \tau_k(P_j) \leq \beta_k\}$; by construction, $N \setminus \{i\} \subseteq S$. First, suppose that $i \notin S$. Then, $v_k^i + 1 \leq \beta_k$, since $a_k(P_i) \leq \beta_k < \tau_k(P_i)$ implies that $a_k(P_i) = v_k^i + 1$. Therefore, $S \in \mathcal{L}_k(\beta_k)$. Moreover, $\{j \in N \mid \tau_k(P_j) \leq \beta_k - 1\} = \emptyset$, which implies that $F_k(P_i, P_{-i}) = \beta_k$. Second, assume that $i \in S$; that is $S = N$, which implies that $S \in \mathcal{L}_k(\beta_k)$. By construction, the set $\overline{S} = \{j \in N \mid \tau_k(P_j) \leq \beta_k - 1\}$ is either empty or is equal to the set $\{i\}$. Suppose $\overline{S} = \{i\}$, then $\beta_k \leq d_k^i$ since $\tau_k(P_i) < \beta_k$ and $b_k(P_i) = \max\{d_k^i, \tau_k(P_i)\}$ imply that $b_k(P_i) = d_k^i$. From the hypothesis that $\beta_k \leq b_k(P_i)$ it follows that $\beta_k - 1 < d_k^i$. Therefore $\{i\} \notin \mathcal{L}_k(\beta_k - 1)$ which implies that $F_k(P_i, P_{-i}) = \beta_k$. Since $k \in K$ was arbitrary, we have that $\beta \in o(P_i)$.

Let $\alpha \in o(P_i)$. That is, there exists $P_{-i} \in \widetilde{\mathcal{P}}_{-i}$ such that $F(P_i, P_{-i}) = \alpha$. Define $\mathbf{P} = (P_i, P_{-i})$. Let $k = 1, ..., K$ be arbitrary. Notice that if $\tau_k(P_i) = \alpha_k$ the result follows immediately by (3.3) and (3.4). Assume first that $\tau_k(P_i) < \alpha_k$. It implies that $a_k(P_i) < \alpha_k$. Define the set $S = \{j \in N \mid \tau_k(P_j) \leq \alpha_k - 1\}$. Since $F(\mathbf{P}) = \alpha$ we know that $S \notin \mathcal{L}_k(\alpha_k - 1)$. However, since $i \in S$ we have that $\alpha_k - 1 < d_k^i$ implying that $\alpha_k \leq b_k(P_i)$ since $\tau_k(P_i) < \alpha_k \leq d_k^i$ and (3.4) hold. Assume now that $\tau_k(P_i) > \alpha_k$. It implies that $\alpha_k < b_k(P_i)$. Define the set $S = \{j \in N \mid \tau_k(P_j) \leq \alpha_k\}$ which belongs to $\mathcal{L}_k(\alpha_k)$ since $F_k(\mathbf{P}) = \alpha_k$. Notice that $i \notin S$ which means that $v_k^i + 1 \leq \alpha_k$. Therefore, (3.3) and $v_k^i + 1 \leq \alpha_k < \tau_k(P_i)$

[7] Notice that a_k and b_k were already defined as the extreme values B_k. The values $a_k(P_i)$ and $b_k(P_i)$ are defined here. We keep a parallel notation, since $[a_k(P_i), b_k(P_i)]$ will again stand for intervals defined by their extremes.

imply that $a_k(P_i) = v_k^i + 1$. Hence, $a_k(P_i) \le \alpha_k$. Since $k \in K$ was arbitrary, we have that $\alpha \in B(P_i)$. ■

3.1. Necessity

Let F be strategy-proof on $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$. Consider $i \in N$, $P_i \in \widetilde{\mathcal{P}}_i$, and $D \in \mathcal{D}(i)$. Let $\gamma \in A \cap D$ and $\beta \in B(P_i) \cap D \cap MB(\tau(P_i), \gamma)$ be such that $\beta \neq \gamma$.

Let $K_1 = \{k \in K \mid \gamma_k < \beta_k \le \tau_k(P_i)\}$ and $K_2 = \{k \in K \mid \tau_k(P_i) \le \beta_k < \gamma_k\}$. Notice that $K_1 \cup K_2 \neq \emptyset$ since $\beta \neq \gamma$ and $\beta \in MB(\tau(P_i), \gamma)$.

Lemma 2. *If $B(P_i) \cap D \cap MB(\beta, \gamma) = \{\beta, \gamma\}$ then $\beta P_i \gamma$.*

Proof. The proof is based on the choice of a profile such that, when i declares his top $\tau(P_i)$ on A, then β obtains, but i could change the outcome to γ by voting for γ. To find such profile, we will divide the proof into two different cases.

Case 1: Assume that $K_1 \neq \emptyset$. That is, there exists $\tilde{k}$ such that $\gamma_{\tilde{k}} < \beta_{\tilde{k}} \le \tau_{\tilde{k}}(P_i)$ and i is not a left dummy at $\gamma_{\tilde{k}}$ because $\gamma_{\tilde{k}}, \beta_{\tilde{k}} \in \left[x_{\tilde{k}}^t, x_{\tilde{k}}^{t+1} - 1\right]$ for some $0 \le t \le T_{\tilde{k}}$ and $\gamma_{\tilde{k}} < \beta_{\tilde{k}}$. Let $S \subseteq N$ be such that $i \in S \in \mathcal{L}_{\tilde{k}}^m(\gamma_{\tilde{k}})$, and consider P_{-i} where for every $j \in N \setminus \{i\}$, $P_j \in \widetilde{\mathcal{P}}_j$ is such that

$$\tau(P_j) = \begin{cases} \gamma & \text{if } j \in S \setminus \{i\} \\ \beta & \text{if } j \in N \setminus S \end{cases},$$

which exist since $\beta, \gamma \in A$ and F has rich domain on A.

First, for every $k \notin K_1 \cup K_2$ we have that $N \setminus \{i\} \subset \{j \in N \mid \tau_k(P_j) \le \beta_k = \gamma_k\}$. Using the fact that $\beta, \gamma \in B(P_i)$ we will show that $F_k(P_i, P_{-i}) = \gamma_k = \beta_k$. To see it, first assume that $F_k(P_i, P_{-i}) < \gamma_k = \beta_k$, which would imply that $\{i\} \in \mathcal{L}_k(F_k(P_i, P_{-i}))$ and $\tau_k(P_i) \le \gamma_k = \beta_k$; but from these two conditions we could conclude that $b_k(P_i) = \max\{d_k^i, \tau_k(P_i)\} < \gamma_k = \beta_k$ contradicting the hypothesis that $\beta, \gamma \in B(P_i)$. Assume now that $F_k(P_i, P_{-i}) > \gamma_k = \beta_k$, which would imply that $\tau_k(P_i) > \gamma_k = \beta_k$. Therefore, $N \setminus \{i\} \notin \mathcal{L}_k(\gamma_k)$ implying that $i \in \bigcap_{S \in \mathcal{L}_k(\gamma_k)} S$, which would mean that $v_k^i + 1 > \gamma_k = \beta_k$. Therefore, we would have that $\gamma_k = \beta_k < \min\{v_k^i + 1, \tau_k(P_i)\} = a_k(P_i)$, contradicting the hypothesis that $\beta, \gamma \in B(P_i)$. Hence,

$$F_k(P_i, P_{-i}) = \beta_k \text{ for all } k \notin K_1 \cup K_2. \tag{3.6}$$

Second, for every $k \in K_1$ the set $\{j \in N \mid \tau_k(P_j) \leq \beta_k\}$ contains the set $N \setminus \{i\}$. Since $\beta \in B(P_i)$ by hypothesis, $N \setminus \{i\} \in \mathcal{L}_k(\beta_k)$ and therefore $F_k(P_i, P_{-i}) \leq \beta_k$. Moreover, $\gamma_k \leq F_k(P_i, P_{-i})$ because $\tau_k(P_j) \geq \gamma_k$ for every $j \in N$. Hence,

$$\gamma_k \leq F_k(P_i, P_{-i}) \leq \beta_k \text{ for all } k \in K_1. \tag{3.7}$$

The set $\{j \in N \mid \tau_{\tilde{k}}(P_j) \leq \gamma_{\tilde{k}}\}$ is equal to $S \setminus \{i\}$. Since $S \setminus \{i\} \notin \mathcal{L}_{\tilde{k}}(\gamma_{\tilde{k}})$ we must have that

$$\gamma_{\tilde{k}} < F_{\tilde{k}}(P_i, P_{-i}). \tag{3.8}$$

Third, for every $k \in K_2$ the set $\{j \in N \mid \tau_k(P_j) < \beta_k\}$ is either empty, in which case $F_k(P_i, P_{-i}) \geq \beta_k$, or else it is equal to the set $\{i\}$. But since $\beta \in B(P_i)$ implies that $\{i\} \notin \mathcal{L}_k(\beta_k - 1)$ we must have that $F_k(P_i, P_{-i}) \geq \beta_k$. Hence,

$$F_k(P_i, P_{-i}) \geq \beta_k \text{ for all } k \in K_2. \tag{3.9}$$

It is straightforward to see that from (3.6), (3.7), (3.8), (3.9), and the hypothesis of Lemma 2 it follows that $F(P_i, P_{-i}) = \beta$.

Consider any $\bar{P}_i \in \widetilde{\mathcal{P}}_i$ with the property that $\tau(\bar{P}_i) = \gamma$, which exists since $\gamma \in A$ and F has rich domain on A. Now, $F(\bar{P}_i, P_{-i}) \in MB(\beta, \gamma)$ because for every $j \in N$ we have that $\tau(P_j) \in \{\beta, \gamma\}$. Consider again the coordinate $\tilde{k} \in K_1$ and the set $S = \{j \in N \mid \tau_{\tilde{k}}(P_j) \leq \gamma_{\tilde{k}}\}$, which belongs to $\mathcal{L}^m_{\tilde{k}}(\gamma_{\tilde{k}})$. Therefore, $F_{\tilde{k}}(\bar{P}_i, P_{-i}) = \gamma_{\tilde{k}}$, which implies, by the hypothesis of Lemma 2, that $F(\bar{P}_i, P_{-i}) = \gamma$. Since F is strategy-proof on $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ we must have that $\beta P_i \gamma$.

Case 2: Assume that $K_1 = \emptyset$ and $K_2 \neq \emptyset$. That is, there exists $\tilde{k}$ such that $\gamma_{\tilde{k}} > \beta_{\tilde{k}} \geq \tau_{\tilde{k}}(P_i)$. Notice that i is not a right dummy at $\gamma_{\tilde{k}}$ because $\gamma_{\tilde{k}}, \beta_{\tilde{k}} \in \left[x^t_{\tilde{k}}, x^{t+1}_{\tilde{k}} - 1\right]$ for some $0 \leq t \leq T_{\tilde{k}}$ and $\gamma_{\tilde{k}} > \beta_{\tilde{k}}$. Let $S \subseteq N$ be such that $i \in S \in \mathcal{R}^m_{\tilde{k}}(\gamma_{\tilde{k}})$, and consider P_{-i} where for every $j \in N \setminus \{i\}$, $P_j \in \widetilde{\mathcal{P}}_j$ is such that

$$\tau(P_j) = \begin{cases} \gamma & \text{if } j \in S \setminus \{i\} \\ \beta & \text{if } j \in N \setminus S \end{cases},$$

which exist since $\beta, \gamma \in A$ and F has rich domain on A.

First, for every $k \notin K_2$ we have that $F_k(P_i, P_{-i}) = \gamma_k = \beta_k$ since $N \setminus \{i\} \subset \{j \in N \mid \tau_k(P_j) \geq \beta_k = \gamma_k\}$ and i is neither a right-decisive nor a right-vetoer agent at $\beta_k = \gamma_k$. Therefore,

$$F_k(P_i, P_{-i}) = \beta_k \text{ for all } k \notin K_2. \tag{3.10}$$

Second, for every $k \in K_2$ the set $\{j \in N \mid \tau_k(P_j) \geq \beta_k\}$ contains the set $N \backslash \{i\}$. Since i is not a right-vetoer agent at β_k (remember that $\beta \in B(P_i)$), we have that $N \backslash \{i\} \in \mathcal{R}_k(\beta_k)$. Therefore,

$$F_k(P_i, P_{-i}) \geq \beta_k \text{ for all } k \in K_2. \tag{3.11}$$

Moreover, the set $\{j \in N \mid \tau_{\tilde{k}}(P_j) \geq \gamma_{\tilde{k}}\}$ is equal to $S \backslash \{i\}$. Since $S \backslash \{i\} \notin \mathcal{R}_{\tilde{k}}(\gamma_{\tilde{k}})$ we must have that

$$\gamma_{\tilde{k}} > F_{\tilde{k}}(P_i, P_{-i}). \tag{3.12}$$

It is straightforward to see that from (3.10), (3.11), (3.12) and the hypothesis of Lemma 2 it follows that $F(P_i, P_{-i}) = \beta$.

Consider any $\bar{P}_i \in \widetilde{\mathcal{P}}_i$ with the property that $\tau(\bar{P}_i) = \gamma$, which exists since $\gamma \in A$ and F has rich domain on A. Now, $F(\bar{P}_i, P_{-i}) \in MB(\beta, \gamma)$ because for every $j \in N$ we have that $\tau(P_j) \in \{\beta, \gamma\}$. Consider again the coordinate $\tilde{k} \in K_2$ and the set $S = \{j \in N \mid \tau_{\tilde{k}}(P_j) \geq \gamma_{\tilde{k}}\}$, which belongs to $\mathcal{R}^m_{\tilde{k}}(\gamma_{\tilde{k}})$. Therefore, $F_{\tilde{k}}(\bar{P}_i, P_{-i}) = \gamma_{\tilde{k}}$, which implies, by the hypothesis of Lemma 2, that $F(\bar{P}_i, P_{-i}) = \gamma$. Since F is strategy-proof on $\widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ we must have that $\beta P_i \gamma$. ■

Lemma 3. *If $B(P_i) \cap D \cap MB(\beta, \gamma) \supsetneq \{\beta, \gamma\}$ then $\beta P_i \gamma$.*

Proof. Given γ and β, there will exist $\alpha_1 = \beta$, α_2,..., α_{h-1}, $\alpha_h = \gamma$ such that, for each j, $B(P_i) \cap D \cap MB(\alpha_j, \alpha_{j+1}) = \{\alpha_j, \alpha_{j+1}\}$. Specifically, we can choose such α_j's by letting α_{j+1} be one of the closest elements (in the L_1-norm) to α_j in $B(P_i) \cap D \cap MB(\alpha_j, \gamma)$. Now, to prove Lemma 3, apply successively Lemma 2 and the transitivity of the preference ordering P_i. ■

Lemma 4. *If $\gamma \notin B(P_i)$ then $\beta P_i \gamma$.*

Proof. For each $j \in N$ consider any $\bar{P}_j \in \widetilde{\mathcal{P}}_j$ such that $\tau(\bar{P}_j) = \gamma$, which exists since $\gamma \in A$ and F has rich domain on A. Obviously, $F(\bar{P}_1, ..., \bar{P}_n) = \gamma$. The proof will consists of two steps.

Step 1: We want to show that $a_k(P_i) \leq F_k(P_i, \bar{P}_{-i}) \leq b_k(P_i)$ for all $k = 1, ..., K$. But this is immediate, because by definition of option set $F(P_i, \bar{P}_{-i}) \in$

$o(P_i)$, and by Lemma 1, we have that $a_k(P_i) \leq F_k\left(P_i, \bar{P}_{-i}\right) \leq b_k(P_i)$ for all $k = 1, ..., K$.

Step 2: We want to show that for all $k = 1, ..., K$:

$$\text{(1) if } \beta_k \leq \gamma_k \text{ then } \beta_k \leq F_k\left(P_i, \bar{P}_{-i}\right) \leq \gamma_k,$$
$$\text{(2) if } \gamma_k < \beta_k\ (\leq \tau_k(P_i)) \text{ then } \gamma_k \leq F_k\left(P_i, \bar{P}_{-i}\right) \leq \beta_k.$$

Define $\hat{\mathbf{P}} = \left(P_i, \bar{P}_{-i}\right)$. To show (1) assume that $\beta_k \leq \gamma_k$ and notice that the set $\left\{j \in N \mid \tau_k\left(\hat{P}j\right) \leq \gamma_k\right\}$ contains $N\backslash\{i\}$. Therefore, $F_k\left(\hat{\mathbf{P}}\right) \leq \gamma_k$, because $\beta \in B(P_i)$. If $\beta_k = a_k(P_i)$ then $\beta_k \leq F_k\left(\hat{\mathbf{P}}\right)$. Assume that $a_k(P_i) < \beta_k$. Since $\beta \in B(P_i)$ we know that $\beta_k \in [a_k(P_i), b_k(P_i)]$, which implies that $\beta_k \leq d_k^i$ because $\tau_k(P_i) \leq \beta_k$. Therefore, $\{i\} \notin \mathcal{L}_k(\beta_k - 1)$. Moreover, since $\left\{j \in N \mid \beta_k > \tau_k\left(\hat{P}_j\right)\right\} \subseteq \{i\}$ we must have that $\beta_k \leq F_k\left(\hat{\mathbf{P}}\right)$. To show (2), assume that $\gamma_k < \beta_k$ and notice that the set $\left\{j \in N \mid \gamma_k \leq \tau_k\left(\hat{P}_j\right)\right\}$ is equal to N. This implies that $\gamma_k \leq F_k\left(\hat{\mathbf{P}}\right)$. Since $\beta \in B(P_i)$ we know that $a_k(P_i) \leq \beta_k$, but because $\tau_k(P_i) \geq \beta_k$, we must have that $\beta_k > v_k^i$, implying that there exists $S \in \mathcal{L}_k^m(\beta_k)$ such that $i \notin S$. Since $\left\{j \in N \mid \beta_k \geq \tau_k\left(\hat{P}_j\right)\right\} \supseteq N \setminus \{i\} \supseteq S$ it follows that $F_k\left(\hat{\mathbf{P}}\right) \leq \beta_k$.

From Steps 1 and 2 we have established that $\gamma \neq F\left(P_i, \bar{P}_{-i}\right) \in B(P_i) \cap MB(\beta, \gamma)$. Since $\{\beta, \gamma\} \subset D$, from (1) and (2) we have that $F\left(P_i, \bar{P}_{-i}\right) \in D$. Moreover, since F is strategy-proof and $F\left(\bar{\mathbf{P}}\right) = \gamma$ we must have $F\left(P_i, \bar{P}_{-i}\right) P_i \gamma$. Define $F\left(P_i, \bar{P}_{-i}\right) = \gamma'$. Notice that $\beta \in MB(\tau(P_i), \gamma')$ and $\gamma' \in B(P_i) \cap D$, implying that the hypothesis of either Lemma 2 or Lemma 3 is satisfied. Therefore, we can deduce that $\beta P_i \gamma'$ and by transitivity of P_i we can conclude that $\beta P_i \gamma$. ∎

3.2. Sufficiency

Assume that F is not strategy-proof. Then, there exist $i \in N$, $\mathbf{P} = (P_1, ..., P_n) \in \widetilde{\mathcal{P}}_1 \times ... \times \widetilde{\mathcal{P}}_n$ and $P_i' \in \widetilde{\mathcal{P}}_i$ such that

$$F(P_i', P_{-i})\, P_i F(\mathbf{P}). \tag{3.13}$$

Denote by $\bar{\mathbf{P}}$ the profile (P_i', P_{-i}) and let $\gamma = F\left(\bar{\mathbf{P}}\right)$ and $\beta = F(\mathbf{P})$. We want to show that there exists $D \in \mathcal{D}(i)$ such that $\beta \in B(P_i) \cap D \cap MB(\tau(P_i), \gamma)$

and $\gamma \in A \cap D$. First, notice that $\beta = F(P_i, P_{-i})$ implies that $\beta \in o(P_i)$, and therefore, by Lemma 1, we have that $\beta \in B(P_i)$.

Lemma 5. $\beta \in MB(\tau(P_i), \gamma)$.

Proof. To show it, assume first that $\beta_k < \tau_k(P_i)$. We will show that $\gamma_k \leq \beta_k$. Since $F_k(\mathbf{P}) = \beta_k$ we have that $S = \{j \in N \mid \tau_k(P_j) \leq \beta_k\} \in \mathcal{L}_k(\beta_k)$ and because $i \notin S$ we have that $S \subseteq \left\{j \in N \mid \tau_k\left(\bar{P}_j\right) \leq \beta_k\right\} \in \mathcal{L}_k(\beta_k)$ by condition (1) in the definition of a left-coalition system. Then, clearly $F_k\left(\bar{\mathbf{P}}\right) \leq \beta_k$ which is the desired result because $\gamma_k = F_k\left(\bar{\mathbf{P}}\right)$. Assume that $\beta_k > \tau_k(P_i)$. We will show that $\beta_k \leq \gamma_k$. Since $F_k(\mathbf{P}) = \beta_k$, the set $S = \{j \in N \mid \tau_k(P_j) \leq \beta_k - 1\} \notin \mathcal{L}_k(\beta_k - 1)$ and because $i \in S$ we have that $\left\{j \in N \mid \tau_k\left(\bar{P}_j\right) \leq \beta_k - 1\right\} \subseteq S \notin \mathcal{L}_k(\beta_k - 1)$ implying that $\gamma_k = F_k\left(\bar{\mathbf{P}}\right) \geq \beta_k$. Finally, if $\beta_k = \tau_k(P_i)$ we do not have to prove anything since the minimal box condition for dimension k is irrelevant; that is, γ_k could be both higher or smaller than $\beta_k = \tau_k(P_i)$. ■

Lemma 6. *There exists $D \in \mathcal{D}(i)$ such that $\{\gamma, \beta\} \subset D$.*

Proof. We have to show that:

(1) If $\gamma_k < \beta_k$ then i is not left dummy at ξ for every $\gamma_k \leq \xi < \beta_k$, and
(2) If $\beta_k < \gamma_k$ then i is not right dummy at ξ for every $\beta_k < \xi \leq \gamma_k$.

We will show only (1), since the argument to show (2) is the symmetric one using right instead of left coalitions. Assume $\gamma_k < \beta_k (\leq \tau_k(P_i))$. The inequality $\beta_k \leq \tau_k(P_i)$ follows from Lemma 5. By condition (3.13) the coalition $S = \{j \in N \mid \tau_k(P_j) \leq \gamma_k\} = \left\{j \in N \setminus \{i\} \mid \tau_k\left(\bar{P}_j\right) \leq \gamma_k\right\}$ is not a member of $\mathcal{L}_k(\gamma_k)$ since $\gamma_k < \beta_k = F_k(\mathbf{P})$. However, $\bar{S} = \left\{j \in N \mid \tau_k\left(\bar{P}_j\right) \leq \gamma_k\right\} \in \mathcal{L}_k(\gamma_k)$ since $F_k\left(\bar{\mathbf{P}}\right) = \gamma_k$ implying that $i \in \bar{S}$ and $\bar{S} \setminus \{i\} = S \notin \mathcal{L}_k(\gamma_k)$ which in turn implies that there exists $T \subseteq \bar{S}$ such that $i \in T$ and $T \in \mathcal{L}_k^m(\gamma_k)$ which means that i is not left dummy at γ_k. Let $\gamma_k < \xi < \beta_k$ be arbitrary. By definition of left coalition system $\bar{S} \in \mathcal{L}_k(\xi)$ and by condition (3.13) $\bar{S} \setminus \{i\} \subseteq \{j \in N \mid \tau_k(P_j) \leq \xi\} \notin \mathcal{L}_k(\xi)$. Therefore, there exists $T \subseteq \bar{S}$ such that $i \in T$ and $T \in \mathcal{L}_k^m(\xi)$. Hence i is not a left-dummy agent at ξ, which shows (1). ■

4. Conclusion and Final Remark

We have characterized the maximal domains of preferences under which generalized median voter schemes are strategy-proof. The extent of these domains depends on the distribution of power among agents which is implied by each generalized median voter scheme. It is still an open question whether some form of single-peakedness is necessary for a domain of preferences to admit some strategy-proof social choice function (not necessarily a generalized median voter scheme). An interesting partial answer is provided in Berga and Serizawa [5].

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