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Weak approximation of the Wiener process from a Poisson process: the multidimensional parameter set case

by

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Abstract

We give an approximation in law of the d -parameter Wiener process by processes constructed from a Poisson process with parameter in \mathbb{R}^d . This approximation is an extension of previous results of Stroock (1982) and Bardina and Jolis.

Key words and phrases: d -parameter Wiener process, Poisson process, weak convergence.

1 Introduction

Let $\{N(t), t \in \mathbb{R}_+\}$ be a standard Poisson process. Stroock (1982) proved that the laws in $\mathcal{C}([0, T])$, the Banach space of continuous functions on $[0, T]$, of the processes

$$z_n := \{z_n(t) = n^{\frac{1}{2}} \int_0^t (-1)^{N(s)} ds, t \in [0, T]\},$$

converges weakly towards the Wiener measure.

The purpose of this paper is to prove this kind of result in \mathbb{R}^d : we prove a weak convergence of processes constructed from a Poisson process with parameter in \mathbb{R}^d to a d -parameter Wiener process. Let $\{N(x), x \in \mathbb{R}_+^d\}$ be a Poisson

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process in \mathbb{R}^d (see Definition 2.2). Then, we consider the processes

$$y_n(s) = n^{\frac{d}{2}} \int_0^{s_d} \cdots \int_0^{s_1} \left(\prod_{i=1}^d x_i \right)^{\frac{d-1}{2}} (-1)^{N(x_1 n^{\frac{1}{d}}, \dots, x_d n^{\frac{1}{d}})} dx_1 \cdots dx_d, \quad (1)$$

$s = (s_1, \dots, s_d) \in \prod_{i=1}^d [0, S_i]$. In Theorem 3.1 we prove that the laws of such processes converge weakly to the Wiener measure in $\mathcal{C}(\prod_{i=1}^d [0, S_i])$. Bardina and Jolis proved this result in the particular case $d = 2$.

The proof of Theorem 3.1 follows the usual methods in the study of weak convergence. We check that the sequence of laws is tight and identify the limit of any subsequence weakly convergent. The tightness follows by a standard way but the study of the limit law requires a characterization of the d -parameter Wiener process (see Theorem 3.4) well posed to our problem. This characterization is one of the keys of our proof.

Notice that in the processes y_n appears a factor $(\prod_{i=1}^d x_i)^{\frac{d-1}{2}}$ in the integrand. The study of the covariance function of our processes shows us that this is the needed factor to obtain the weak convergence (see Bardina and Jolis for a more detailed discussion in the two parameter case).

The paper is organized as follows. Section 2 is devoted to introduce the necessary notations, definitions and some technical results about the Poisson process. In Section 3 we state our main result and we give the proof of the two ingredients: the tightness and the identification of the limit law.

2 Preliminaries

Suppose $d \geq 2$ and consider $[0, S] = \prod_{i=1}^d [0, S_i] \subset \mathbb{R}^d$ with the usual partial order. Define $I_0 = \{s \in [0, S] : s_1 \times \cdots \times s_d = 0\}$.

Let (Ω, \mathcal{F}, Q) be a complete probability space and let $\{\mathcal{F}_s, s \in [0, S]\}$ be a family of sub- σ -fields of \mathcal{F} such that: $\mathcal{F}_{s^1} \subseteq \mathcal{F}_{s^2}$ for any $s^1 \leq s^2$.

Fix $t \in [0, S]$ we also define $\mathcal{F}_t^1 := \mathcal{F}_{(t_1, S_2, \dots, S_d)}$ (the σ -field generated by the 1-past of t) and $\mathcal{F}_t^T := \bigvee_{i=1}^d \mathcal{F}_{(S_1, \dots, S_{i-1}, t_i, S_{i+1}, \dots, S_d)}$ (the σ -field generated by all the past of t).

Given $s < t$ we denote by $\Delta_s X(t)$ the increment of the process X over the rectangle $(s, t] = \prod_{i=1}^d (s_i, t_i] \subset \mathbb{R}^d$.

We can give now the definitions of the two processes involved in our paper.

Definition 2.1 *A d -parameter continuous process $W = \{W(s); s \in [0, S] \subset \mathbb{R}_+^d\}$ is called a d -parameter $\{\mathcal{F}_t\}$ -Wiener process if it is $\{\mathcal{F}_t\}$ -adapted, null on I_0 and for any $s < t$ the increment $\Delta_s W(t)$ is independent of \mathcal{F}_s^T and is normally distributed with zero mean and variance $\prod_{i=1}^d (t_i - s_i)$.*

If we do not specify the filtration, $\{\mathcal{F}_t\}$ will be the one generated by the process itself completed with the necessary null sets.

Definition 2.2 *An \mathcal{F}_s -Poisson process in \mathbb{R}^d with intensity μ is an adapted, càdlàg process $N_\mu = \{N_\mu(s); s \in \mathbb{R}_+^d\}$, null on I_0 and such that for all $s < t$ the increment $\Delta_s N_\mu(t)$ is independent of $\bigvee_{i=1}^d \mathcal{F}_{(\infty, \dots, \infty, s_i, \infty, \dots, \infty)}$ with a Poisson law of parameter $\mu \prod_{i=1}^d (t_i - s_i)$. Here, we are denoting $\mathcal{F}_{(\infty, \dots, \infty, s_i, \infty, \dots, \infty)} := \bigvee_{x_j > 0, j \neq i} \mathcal{F}_{(x_1, \dots, x_{i-1}, s_i, x_{i+1}, \dots, x_d)}$.*

If we do not specify the filtration, $\{\mathcal{F}_t\}$ will be the filtration generated by the process itself, completed with the null sets of $\mathcal{F}^{N_\mu} = \sigma\{N_\mu(x), x \in \mathbb{R}^d\}$. If we do not specify the intensity, μ will be 1.

We finish this section with a technical lemma. The proof, based in Definition 2.2 and the fact that for Z a random variable with a Poisson law with parameter λ we have $E((-1)^Z) = \exp(-2\lambda)$, is omitted. We need to introduce first some more notation.

Given $x^1 = (x_1^1, \dots, x_d^1)$, $x^2 = (x_1^2, \dots, x_d^2)$ in \mathbb{R}^d we define, for any $i \in \{1, \dots, d\}$, $x_i^{1 \wedge 2} = \min(x_i^1, x_i^2)$, $x_i^{1 \vee 2} = \max(x_i^1, x_i^2)$ (we can extend this definitions for more than two points). Moreover, for $j \geq i$ we define $x_j^{1 \vee 2(i)}$ as x_j^1 when $x_i^1 \geq x_i^2$ and as x_j^2 when $x_i^1 < x_i^2$. The differential dx^i will denote the differential $dx_1^i \cdots dx_d^i$ in \mathbb{R}^d , and for $\varepsilon \in \mathbb{R}$, $x^i + \varepsilon$ denotes the point $(x_1^i + \varepsilon, \dots, x_d^i + \varepsilon)$. Finally, when $J = \emptyset$ we consider the usual convention $\sum_{j \in J} x_j = 0$ and $\prod_{j \in J} x_j = 1$.

Lemma 2.3 *Let $\{N_\mu(x), x \in \mathbb{R}^d\}$ be a Poisson process with intensity μ in \mathbb{R}^d , and for $j \in \{1, 2, 3, 4\}$, $x^j \in \mathbb{R}^d$. Then:*

$$\begin{aligned}
(i) \quad & E \left[(-1)^{\sum_{j=1}^2 N_\mu(x^j)} \right] = \exp \left[-2\mu \left(\prod_{i=1}^d x_i^1 + \prod_{i=1}^d x_i^2 - 2 \prod_{i=1}^d x_i^{1 \wedge 2} \right) \right], \\
(ii) \quad & E \left[(-1)^{\sum_{j=1}^4 N_\mu(x^j)} \right] = \exp \left[-2\mu \left[\sum_{j=1}^4 \left(\prod_{i=1}^d x_i^j \right) - 2 \sum_{j \neq k} \left(\prod_{i=1}^d x_i^{j \wedge k} \right) \right. \right. \\
& \quad \left. \left. + 4 \sum_{j \neq k \neq l} \left(\prod_{i=1}^d x_i^{j \wedge k \wedge l} \right) - 8 \left(\prod_{i=1}^d x_i^{1 \wedge 2 \wedge 3 \wedge 4} \right) \right] \right] \\
(iii) \quad & \prod_{i=1}^d x_i^1 + \prod_{i=1}^d x_i^2 - 2 \prod_{i=1}^d x_i^{1 \wedge 2} = \sum_{i=1}^d \left(\prod_{k=1}^{i-1} x_k^{1 \wedge 2} \right) |x_i^1 - x_i^2| \left(\prod_{k=i+1}^d x_k^{1 \vee 2(i)} \right), \\
(iv) \quad & \prod_{i=1}^d x_i^1 + \prod_{i=1}^d x_i^2 - 2 \prod_{i=1}^d x_i^{1 \wedge 2} \leq \prod_{i=1}^d x_i^{1 \vee 2} - \prod_{i=1}^d x_i^{1 \wedge 2}.
\end{aligned}$$

3 Weak convergence

Our main result reads as follows.

Theorem 3.1 *Let P_n be the image law of the process y_n , defined in (1), in the Banach space $\mathcal{C}([0, S], \mathbb{R})$ of continuous functions on $[0, S]$. Then $\{P_n\}_n$ converges weakly, as n tends to infinity, towards the law of a d -parameter Wiener process on $\mathcal{C}([0, S], \mathbb{R})$.*

Proof: It follows from the tightness proved in Subsection 3.1 and the identification of the limit law given in Subsection 3.2. ■

In order to simplify the notation, we denote by $N_n(x)$ the random variable $N(x_1 n^{\frac{1}{d}}, \dots, x_d n^{\frac{1}{d}})$. Then, $\{N_n(x), x \in \mathbb{R}_+^d\}$ is a Poisson process in \mathbb{R}^d with intensity n . Moreover, we can write

$$y_n(s) = n^{\frac{d}{2}} \int_0^{s_d} \cdots \int_0^{s_1} \left(\prod_{i=1}^d x_i \right)^{\frac{d-1}{2}} (-1)^{N_n(x)} dx.$$

3.1 Tightness

In this section we prove that $\{P_n\}_n$ is tight. Using a criterium given by Bickel and Wichura (1971), it suffices to prove the following lemma.

Lemma 3.2 *There exists a positive constant K such that $E[(\Delta_s y_n(t))^4] \leq K \prod_{i=1}^d (t_i - s_i)^2$, for any $0 \leq s \leq t \leq S$, $n \geq 1$.*

Before the proof we need a technical lemma.

Lemma 3.3 *Let $X = \{X(s), s \in [0, S] \subset \mathbb{R}^d\}$ be a continuous process. Assume that there exists $K > 0$ such that*

$$E[(\Delta_s X(t))^4] \leq K \prod_{i=1}^d (t_i - s_i)^2, \quad (2)$$

for any $0 < s < t < 2s$. Then there exists $K_1 > 0$ such that X enjoys (2) for any $0 \leq s \leq t$, with K_1 instead of K .

Proof: Changing K by \bar{K} , it is easy to check that X satisfies (2) for any $0 < s < t < 8s$. Now, we can give the sketch of the proof for the case $s = 0$. By using the inequality $(a + b)^4 \leq 8a^4 + 8b^4$ and the continuity of X , we get

$$\begin{aligned} E[(\Delta_0 X(t))^4] &\leq \sum_{j_1, \dots, j_d=1}^{\infty} 8^{j_1 + \dots + j_d} E[(\Delta_{(\frac{t_1}{8^{j_1}}, \dots, \frac{t_d}{8^{j_d}})} X(\frac{t_1}{8^{j_1-1}}, \dots, \frac{t_d}{8^{j_d-1}}))^4] \\ &\leq \bar{K} \sum_{j_1, \dots, j_d=1}^{\infty} \prod_{i=1}^d 8^{j_i} (\frac{7t_i}{8^{j_i}})^2 \leq K_1 \prod_{i=1}^d t_i^2. \end{aligned}$$

The proof of the general case follows the same method. ■

Proof of Lemma 3.2: Using Lemma 3.3 we can assume $0 < s < t < 2s$. We have

$$E[(\Delta_s y_n(t))^4] = n^{2d} \int_{\prod_{i=1}^d [s_i, t_i]^4} (\prod_{i=1, j=1}^{d, 4} x_i^j)^{\frac{d-1}{2}} E[(-1)^{\sum_{j=1}^4 N_n(x^j)}] \prod_{j=1}^4 dx^j.$$

Notice that

$$E[(-1)^{\sum_{j=1}^4 N_n(x^j)}] \leq \prod_{i=1}^d |E[(-1)^{\sum_{j=1}^4 \Delta_{(0, \dots, 0, s_i, 0, \dots, 0)} N_n(s_1, \dots, s_{i-1}, x_i^j, s_{i+1}, \dots, s_d)}]|. \quad (3)$$

Indeed, for $x \in \prod_{i=1}^d [s_i, t_i]$ we have $[0, x] = A_x \cup B_x$, with

$$A_x := \cup_{i=1}^d [(0, \dots, 0, s_i, 0, \dots, 0), (s_1, \dots, s_{i-1}, x_i, s_{i+1}, \dots, s_d)] \subset G,$$

where $G := \cup_{i=1}^d [(0, \cdot, 0, s_i, 0, \cdot, 0), (s_1, \cdot, s_{i-1}, t_i, s_{i+1}, \cdot, s_d)]$. Moreover $B_x \subset G^c$. Fixed $x^j, j \in \{1, 2, 3, 4\}$, then $\cup_{j=1}^4 A_{x^j}$ and $\cup_{j=1}^4 B_{x^j}$ are disjoint. So, the corresponding increments will be independent and we get (3) easily.

Assume now $x_i^1 \leq x_i^2 \leq x_i^3 \leq x_i^4$ for all $i \in \{1, \dots, d\}$. Then (3) is equal to

$$\begin{aligned} & \prod_{i=1}^d \exp \left[-2n[(x_i^4 - x_i^3) + (x_i^2 - x_i^1)] \prod_{k \neq i} s_k \right] \\ & \leq \prod_{i=1}^d \exp \left[-2^{2-d} n[(x_i^2 - x_i^1) \prod_{k \neq i} x_k^1 + (x_i^4 - x_i^3) \prod_{k \neq i} x_k^3] \right], \end{aligned}$$

since $x_i^j \leq t_i < 2s_i$ for all $i \in \{1, \dots, d\}, j \in \{1, 2, 3, 4\}$. Notice that (3) is symmetric in the variables $x_i^1, x_i^2, x_i^3, x_i^4$ for each i . So, we get

$$\begin{aligned} E \left[(\Delta_s y_n(t))^4 \right] & \leq (4!)^d 2^{d-1} \left(n^d \int_{\prod_{i=1}^d [s_i, t_i]^2} \left(\prod_{i=1}^d x_i^1 \right)^{d-1} \left(\prod_{i=1}^d \mathbf{1}_{\{x_i^1 \leq x_i^2\}} \right) \right. \\ & \quad \times \left. \exp \left[-2^{2-d} n \sum_{i=1}^d [(x_i^2 - x_i^1) \prod_{k \neq i} x_k^1] \prod_{j=1}^2 dx^j \right]^2 \right), \end{aligned}$$

since $x_i^2 \leq t_i < 2s_i \leq 2x_i^1$ for all $i \in \{1, \dots, d\}$. Integrating now with respect to $x_i^2, i \in \{1, \dots, d\}$, we can bound it by $K(\int_{\prod_{i=1}^d [s_i, t_i]} dx^1)^2$ and the proof is completed. \blacksquare

3.2 Identification of the limit law

Let $\{P_{n_i}\}_i$ be a subsequence of $\{P_n\}_n$ (that we will also denote by $\{P_n\}_n$) weakly convergent to some probability P . We have to check that the canonical process $\{X(s)[y] := y(s)\}$ is a d -parameter Wiener process under the probability P .

We will need the following characterization of the d -parameter Wiener process. An important point of this characterization is that its quadratic variation part does not need conditioning on all the past but only on the 1-past. On the other hand, if we only consider the case in which the filtration is the natural one, we can avoid to study increments whose left points are in I_0 . For the sake of completeness, we will give the proof of this result, although some ideas involved in it were also used by Tudor (1980) in the two parameter case.

Theorem 3.4 *Let $X = \{X(s); s \in [0, S] \subset \mathbb{R}_+^d\}$ be a continuous process null on the axes and let $\{\mathcal{F}_t\}$ be its natural filtration. Then the following statements are equivalent:*

- (i) X is a d -parameter Wiener process.
- (ii) For all $0 < s \leq t$, $E(\Delta_s X(t) | \mathcal{F}_s^T) = 0$ and $E((\Delta_s X(t))^2 | \mathcal{F}_s^1) = \prod_{i=1}^d (t_i - s_i)$.

Proof: Obviously (i) implies (ii). Let us check now that (ii) yields (i). Fixed $0 < s < t$, consider the process $Y(u) := \Delta_s X(u, t_2, \dots, t_d)$, $u \in [s_1, S_1] \subset \mathbb{R}_+$, and the σ -fields $\mathcal{G}_u := \mathcal{F}_{(u, s_2, \dots, s_d)}^1$, $u \in [s_1, S_1]$. Clearly $\{Y(u), \mathcal{G}_u, u \in [s_1, S_1]\}$ is a martingale. Indeed, $Y(u)$ is \mathcal{G}_u -adapted and

$$\begin{aligned} E(Y(v) - Y(u) | \mathcal{G}_u) &= E(\Delta_{(u, s_2, \dots, s_d)} X(v, t_2, \dots, t_d) | \mathcal{G}_u) \\ &= E(E(\Delta_{(u, s_2, \dots, s_d)} X(v, t_2, \dots, t_d) | \mathcal{F}_{(u, s_2, \dots, s_d)}^T) | \mathcal{G}_u) = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} E((Y(v) - Y(u))^2 | \mathcal{G}_u) &= E((\Delta_{(u, s_2, \dots, s_d)} X(v, t_2, \dots, t_d))^2 | \mathcal{G}_u) \\ &= (v - u) \prod_{i=2}^d (t_i - s_i). \end{aligned}$$

Then, Paul Lévy's theorem gives us that $\{Y(u), \mathcal{G}_u, u \in [s_1, S_1]\}$ is a one parameter Wiener process with variance $(u - s_1) \prod_{i=2}^d (t_i - s_i)$.

So, the increments $\Delta_s X(t) = Y(t_1) - Y(s_1)$ are normally distributed with zero mean and variance $\prod_{i=1}^d (t_i - s_i)$.

Let now $s \in I_0$ and consider $\Delta_{s+\varepsilon} X(t)$ with $\varepsilon > 0$. They are centered gaussian random variables. Taking the limit when ε tends to 0, we get that $\Delta_s X(t)$ is also a centered gaussian random variable with variance $\prod_{i=1}^d (t_i - s_i)$.

Finally, since $E(\Delta_s X(t) | \mathcal{F}_s^T) = 0$ for $0 < s < t$, all the increments are uncorrelated and moreover, independent. ■

The following proposition gives us the identification of the limit.

Proposition 3.5 *Let $\{P_n\}_n$ be the laws in $\mathcal{C}([0, S], \mathbb{R})$ of processes y_n defined in (1). Assume that $\{P_{n_i}\}_i$ is a subsequence weakly convergent to P . Let X*

be the canonical process and let $\{\mathcal{F}_s\}$ be its natural filtration. Then, for all $0 < s \leq t$, $E_P(\Delta_s X(t) | \mathcal{F}_s^T) = 0$ and $E_P((\Delta_s X(t))^2 | \mathcal{F}_s^1) = \prod_{i=1}^d (t_i - s_i)$.

Proof: We follow the method of the two parameter case (see Bardina and Jolis). Fixed $\delta > 0$ and $z^1, \dots, z^m \in [0, S]$ such that for each $j = 1, \dots, m$ there exists $i \in \{1, \dots, d\}$ such that $z_i^j \leq s_i - \delta$.

To prove that $E_P(\Delta_s X(t) | \mathcal{F}_s^T) = 0$, it suffices to check that, for any bounded continuous function $\varphi : \mathbb{R}^m \mapsto \mathbb{R}$, $E_P[\varphi(X(z^1), \dots, X(z^m)) \Delta_s X(t)] = 0$. Since $P_n \xrightarrow{w} P$ and taking into account the bound obtained in Lemma 3.2, it suffices to prove that $\lim_{n \rightarrow \infty} E_{P_n}[\varphi(X(z^1), \dots, X(z^m)) \Delta_s X(t)] = 0$. Notice that using that φ is bounded

$$\begin{aligned} |E_{P_n}[\varphi(X(z^1), \dots, X(z^m)) \Delta_s X(t)]| &= |E[\varphi(y_n(z^1), \dots, y_n(z^m)) \Delta_s y_n(t)]| \\ &= |E[\varphi(y_n(z^1), \dots, y_n(z^m)) E[\Delta_s y_n(t) | \mathcal{G}_{s,\delta}^n]]| \leq K(E(Y_n^2))^{\frac{1}{2}}, \end{aligned}$$

with $\mathcal{G}_{s,\delta}^n = \mathcal{F}_{s-\delta}^T$ and

$$\begin{aligned} Y_n &:= E\left[n^{\frac{d}{2}} \int_{\prod_{i=1}^d [s_i, t_i]} \left(\prod_{i=1}^d x_i\right)^{\frac{d-1}{2}} (-1)^{N_n(x)} dx | \mathcal{G}_{s,\delta}^n\right] = E\left[(-1)^{\Delta_{s-\delta} N_n(s)}\right] \\ &\quad \times E\left[n^{\frac{d}{2}} (-1)^{\Delta_{s-\delta} N_n(s)} \int_{\prod_{i=1}^d [s_i, t_i]} \left(\prod_{i=1}^d x_i\right)^{\frac{d-1}{2}} (-1)^{N_n(x)} dx | \mathcal{G}_{s,\delta}^n\right], \end{aligned}$$

since $\Delta_{s-\delta} N_n(s)$ is independent of $\mathcal{G}_{s,\delta}^n$. We have that Y_n converges to zero in L^2 as n tends to infinity because $E[(-1)^{\Delta_{s-\delta} N_n(s)}] = \exp(-2\delta^d n)$ and the conditional expectation is L^2 bounded (see Lemma 3.2).

Let us check now the second part of the Proposition. Following similar arguments it is enough to check that $E((\Delta_s y_n(t))^2 | \mathcal{F}_s^1)$ converges in L^2 to $\prod_{i=1}^d (t_i - s_i)$, as n tends to infinity. To prove it, observe that

$$\begin{aligned} 0 &\leq E\left[\left(E((\Delta_s y_n(t))^2 | \mathcal{F}_s^1) - \prod_{i=1}^d (t_i - s_i)\right)^2\right] \\ &= E\left[\left(E((\Delta_s y_n(t))^2 | \mathcal{F}_s^1)\right)^2\right] - 2 \prod_{i=1}^d (t_i - s_i) E((\Delta_s y_n(t))^2) + \prod_{i=1}^d (t_i - s_i)^2. \end{aligned}$$

The following Lemmas, 3.6 and 3.7, will show that the last term converges to 0 when n tends to infinity. The proof is now complete. \blacksquare

Lemma 3.6 For any $0 \leq s < t$, $\lim_{n \rightarrow \infty} E \left[(\Delta_s y_n(t))^2 \right] = \prod_{i=1}^d (t_i - s_i)$.

Proof: Given x^1 and x^2 , Lemma 2.3 implies that $E \left[(-1)^{\sum_{j=1}^2 N_n(x^j)} \right]$ is in the interval

$$\exp \left[-2n \left[\sum_{i=1}^d |x_i^2 - x_i^1| \prod_{k \neq i} x_k^{1 \vee 2} \right] \right], \exp \left[-2n \left[\sum_{i=1}^d |x_i^2 - x_i^1| \prod_{k \neq i} x_k^{1 \wedge 2} \right] \right].$$

Using symmetry arguments it is then easy to check that $2^d I_1 \leq E \left[(\Delta_s y_n(t))^2 \right] \leq 2^d I_2$, where

$$\begin{aligned} I_1 &:= n^d \int_{\prod_{i=1}^d [s_i, t_i]^2} \left(\prod_{i=1, j=1}^{d, 2} x_i^j \right)^{\frac{d-1}{2}} \exp \left[-2n \sum_{i=1}^d [(x_i^2 - x_i^1) \prod_{k \neq i} x_k^2] \right] \\ &\quad \times \left(\prod_{i=1}^d \mathbf{1}_{\{x_i^1 \leq x_i^2\}} \right) \prod_{j=1}^2 dx^j \\ I_2 &:= n^d \int_{\prod_{i=1}^d [s_i, t_i]^2} \left(\prod_{i=1, j=1}^{d, 2} x_i^j \right)^{\frac{d-1}{2}} \exp \left[-2n \sum_{i=1}^d [(x_i^2 - x_i^1) \prod_{k \neq i} x_k^1] \right] \\ &\quad \times \left(\prod_{i=1}^d \mathbf{1}_{\{x_i^1 \leq x_i^2\}} \right) \prod_{j=1}^2 dx^j. \end{aligned}$$

We will study I_1 . Notice that

$$\begin{aligned} I_1 &= \frac{1}{2^d} \int_{\prod_{i=1}^d [s_i, t_i]} \left(\prod_{i=1}^d x_i^2 \right)^{-\frac{d-1}{2}} \prod_{i=1}^d \left(\int_{s_i}^{x_i^2} 2n \left(\prod_{k \neq i} x_k^2 \right) \right. \\ &\quad \times \exp \left[-2n [(x_i^2 - x_i^1) \prod_{k \neq i} x_k^2] \right] (x_i^1)^{\frac{d-1}{2}} dx_i^1 \Big) dx^2. \end{aligned}$$

Since $2n \left(\prod_{k \neq i} x_k^2 \right) \exp \left[-2n [(x_i^2 - x_i^1) \prod_{k \neq i} x_k^2] \right] \mathbf{1}_{(-\infty, x_i^2)}(x_i^1)$ is a probability density that gives an approximation of the identity as n tends to ∞ , we obtain that

$$\int_{s_i}^{x_i^2} 2n \left(\prod_{k \neq i} x_k^2 \right) \exp \left[-2n [(x_i^2 - x_i^1) \prod_{k \neq i} x_k^2] \right] (x_i^1)^{\frac{d-1}{2}} dx_i^1$$

tends to $(x_i^2)^{\frac{d-1}{2}}$. Moreover, by the dominated convergence theorem we get $\lim_{n \rightarrow \infty} I_1 = \frac{1}{2^d} \prod_{i=1}^d (t_i - s_i)$. Interchanging the roles of x_i^1 and x_i^2 for each i , we obtain the same result for I_2 . This fact finishes the proof of the lemma. ■

Lemma 3.7 For any $0 < s < t$, $\limsup_{n \rightarrow \infty} E \left[\left(E \left[(\Delta_s y_n(t))^2 | \mathcal{F}_s^1 \right] \right)^2 \right] \leq \prod_{i=1}^d (t_i - s_i)^2$.

Proof: For $x \geq s$, $\Delta_0 N_n(x) = \Delta_0 N_n(s_1, x_2, \dots, x_d) + \Delta_{(s_1, 0, \dots, 0)} N_n(x)$.

Then by (i) of Lemma 2.3 we have

$$\begin{aligned} & E \left[\left(E \left[(\Delta_s y_n(t))^2 | \mathcal{F}_s^1 \right] \right)^2 \right] \\ &= E \left[\left(n^d \int_{\prod_{i=1}^d [s_i, t_i]^2} \left(\prod_{i=1, j=1}^{d, 2} x_i^j \right)^{\frac{d-1}{2}} (-1)^{\sum_{j=1}^2 \Delta_0 N_n(s_1, x_2^j, \dots, x_d^j)} \right. \right. \\ &\quad \left. \left. \times E \left[(-1)^{\sum_{j=1}^2 \Delta_{(s_1, 0, \dots, 0)} N_n(x^j)} \right] \prod_{j=1}^2 dx^j \right)^2 \right] \\ &= n^{2d} \int_{\prod_{i=1}^d [s_i, t_i]^4} \left(\prod_{i=1, j=1}^{d, 2} x_i^j \right)^{\frac{d-1}{2}} T_1(x^{1,2,3,4}) T_2(x^{1,2,3,4}) \prod_{j=1}^4 dx^j \quad (4) \end{aligned}$$

with $T_1(x^{1,2,3,4}) := E \left[(-1)^{\sum_{j=1}^4 \Delta_0 N_n(s_1, x_2^j, \dots, x_d^j)} \right]$ and

$$\begin{aligned} T_2(x^{1,2,3,4}) &:= \exp \left[-2n \left((x_1^1 - s_1) \prod_{i \neq 1} x_i^1 + (x_1^2 - s_1) \prod_{i \neq 1} x_i^2 - 2(x_1^{1\wedge 2} - s_1) \right. \right. \\ &\quad \left. \left. \times \prod_{i \neq 1} x_i^{1\wedge 2} \right) + [(x_1^3 - s_1) \prod_{i \neq 1} x_i^3 + (x_1^4 - s_1) \prod_{i \neq 1} x_i^4 - 2(x_1^{3\wedge 4} - s_1) \prod_{i \neq 1} x_i^{3\wedge 4}] \right]. \end{aligned}$$

We can then divide the integral of (4) into two parts: the integral over the set $D := \{(x^1, x^2, x^3, x^4) \in \prod_{i=1}^d [s_i, t_i]^4, x_i^{1\vee 2} \leq x_i^{3\wedge 4} \text{ or } x_i^{3\vee 4} \leq x_i^{1\wedge 2} \text{ for all } i \in \{2, \dots, d\}\}$ and the integral over D^c .

Integral over D^c . Given $(x^1, x^2, x^3, x^4) \in D^c$, there exists $l \in \{2, \dots, d\}$ such that $x_l^{1\vee 2} > x_l^{3\wedge 4}$ and $x_l^{3\vee 4} > x_l^{1\wedge 2}$. Using (iii) of Lemma 2.3 we can bound $T_2(x^{1,2,3,4})$ by

$$\exp \left[-2n \left([(x_1^{(4)} - x_1^{(3)}) + (x_1^{(2)} - x_1^{(1)})] \prod_{i \neq 1} s_i + (x_1^{(1)} - s_1)(x_l^{(3)} - x_l^{(2)}) \prod_{i \neq 1, l} s_i \right) \right],$$

where $x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, x_i^{(4)}$ denotes the usual reordenation of $x_i^1, x_i^2, x_i^3, x_i^4$. Notice that we have bounded by 1 the other terms obtained from (iii) of Lemma 2.3. On the other hand, by similar arguments and Lemma 2.3 ((ii), (iii)),

$$T_1(x^{1,2,3,4}) \leq \prod_{i=2}^d \exp \left[-2n [(x_i^{(4)} - x_i^{(3)}) + (x_i^{(2)} - x_i^{(1)})] \prod_{k \neq i} s_k \right].$$

Notice that we can bound the integrand in (4), over the set D^c , by a symmetric function on x_i^1, x_i^2, x_i^3 and x_i^4 for all i . So, by some changes of variables we can bound the integral given in (4), over the set D^c , by

$$\begin{aligned} K & \int_{\prod_{i=1}^d [s_i, t_i]^2} \left(\prod_{i=1}^d \int_{x_i^3}^{t_i} n \exp[-2n(x_i^4 - x_i^3) \prod_{k \neq i} s_k] dx_i^4 \right) \\ & \times \left(\prod_{i \neq l} \int_{x_i^1}^{t_i} n \exp[-2n(x_i^2 - x_i^1) \prod_{k \neq i} s_k] dx_i^2 \right) \int_{s_1}^{x_l^2} n \exp[-2n(x_l^2 - x_l^1) \prod_{k \neq l} s_k] dx_l^1 \\ & \times \exp[-2n(x_l^1 - s_1)(x_l^3 - x_l^2) \prod_{k \neq 1, l} s_k] dx_l^2 \left(\prod_{k \neq l} dx_k^1 \right) dx^3 \end{aligned}$$

that clearly goes to zero by dominated convergence.

Integral over D . Fixed $l \in \{2, \dots, d+1\}$, assume first that $x_i^{1 \vee 2} \leq x_i^{3 \wedge 4}$ for all $i \in \{2, \dots, l-1\}$ and $x_i^{3 \vee 4} \leq x_i^{1 \wedge 2}$ for all $i \in \{l, \dots, d\}$. Using (ii) of Lemma 2.3 we get $T_2(x^{1,2,3,4})T_1(x^{1,2,3,4}) = \exp[-2nR_1(x^{1,2,3,4})]$ where

$$\begin{aligned} R_1(x^{1,2,3,4}) &= \prod_{i=1}^d x_i^1 + \prod_{i=1}^d x_i^2 + \prod_{i=1}^d x_i^3 + \prod_{i=1}^d x_i^4 - 2 \prod_{i=1}^d x_i^{1 \wedge 2} - 2 \prod_{i=1}^d x_i^{3 \wedge 4} \\ &\quad - 2s_1 \left(\prod_{i=2}^{l-1} x_i^1 + \prod_{i=2}^{l-1} x_i^2 - 2 \prod_{i=2}^{l-1} x_i^{1 \wedge 2} \right) \left(\prod_{i=l}^d x_i^3 + \prod_{i=l}^d x_i^4 - 2 \prod_{i=l}^d x_i^{3 \wedge 4} \right). \end{aligned}$$

Using again Lemma 2.3, we have $R_1(x^{1,2,3,4}) = \sum_{j=1}^5 R_{1,j}(x^{1,2,3,4})$ with

$$\begin{aligned} R_{1,1}(x^{1,2,3,4}) &= |x_1^2 - x_1^1| \prod_{i=2}^d x_i^{1 \vee 2(1)} + \sum_{i=l}^d \left(\prod_{k=1}^{i-1} x_k^{1 \wedge 2} \right) |x_i^2 - x_i^1| \left(\prod_{k=i+1}^d x_k^{1 \vee 2(i)} \right) \\ &\quad + \sum_{i=1}^{l-1} \left(\prod_{k=1}^{i-1} x_k^{3 \wedge 4} \right) |x_i^4 - x_i^3| \left(\prod_{k=i+1}^d x_k^{3 \vee 4(i)} \right), \\ R_{1,2}(x^{1,2,3,4}) &= \sum_{i=2}^{l-1} (x_1^{1 \wedge 2} - s_1) \left(\prod_{k=2}^{i-1} x_k^{1 \wedge 2} \right) |x_i^2 - x_i^1| \left(\prod_{k=i+1}^d x_k^{1 \vee 2(i)} \right), \\ R_{1,3}(x^{1,2,3,4}) &= \sum_{i=l}^d (x_1^{3 \wedge 4} - s_1) \left(\prod_{k=2}^{i-1} x_k^{3 \wedge 4} \right) |x_i^4 - x_i^3| \left(\prod_{k=i+1}^d x_k^{3 \vee 4(i)} \right), \\ R_{1,4}(x^{1,2,3,4}) &= s_1 \sum_{i=2}^{l-1} \left(\prod_{k=2}^{i-1} x_k^{1 \wedge 2} \right) |x_i^2 - x_i^1| \left(\prod_{k=i+1}^{l-1} x_k^{1 \vee 2(i)} \right) \\ &\quad \times \left[\prod_{k=l}^d x_k^{1 \vee 2(i)} - \left(\prod_{k=l}^d x_k^3 + \prod_{k=l}^d x_k^4 - 2 \prod_{k=l}^d x_k^{3 \wedge 4} \right) \right], \end{aligned}$$

$$\begin{aligned}
R_{1,5}(x^{1,2,3,4}) &= s_1 \sum_{i=l}^d \left[\prod_{k=2}^{l-1} x_k^{3\wedge 4} - \left(\prod_{k=2}^{l-1} x_k^1 + \prod_{k=2}^{l-1} x_k^2 - 2 \prod_{k=2}^{l-1} x_k^{1\wedge 2} \right) \right] \\
&\quad \times \prod_{k=l}^{i-1} x_k^{3\wedge 4} |x_i^4 - x_i^3| \left(\prod_{k=i+1}^d x_k^{3\vee 4(i)} \right).
\end{aligned}$$

Notice that all the terms appearing in the above decomposition are nonnegative (for instance, (iv) of Lemma 2.3 yields that $R_{1,4}(x^{1,2,3,4})$ is nonnegative).

Fix $\rho > 0$, consider the set $F_{\rho,l} := \{|x_i^2 - x_i^1| < \rho \text{ for all } i \in \{2, \dots, l-1\}, |x_i^4 - x_i^3| < \rho \text{ for all } i \in \{l, \dots, d\}\}$. It is easy to check, using (iv) of Lemma 2.3, that on the set $F_{\rho,l}^c$ we get $R_1(x^{1,2,3,4}) \geq \rho \min_i \{s_1 \prod_{j \neq i,1} s_j\}$. Then, by dominated convergence we get that the integral (4) over the set $D \cap F_{\rho,l}^c$ tends to zero when n goes to infinity. So, it is enough to study what happens over $D \cap F_{\rho,l}$.

On the other hand, $R_{1,1}(x^{1,2,3,4})$ is bigger than or equal to

$$|x_1^2 - x_1^1| \prod_{i=2}^d x_i^{1\wedge 2} + \sum_{i=l}^d \left(\prod_{k \neq i} x_k^{1\wedge 2} \right) |x_i^2 - x_i^1| + \sum_{i=1}^{l-1} \left(\prod_{k \neq i} x_k^{3\wedge 4} \right) |x_i^4 - x_i^3|.$$

Observe that using again Lemma 2.3, $R_{1,4}(x^{1,2,3,4})$ is greater than or equal to

$$s_1 \sum_{i=2}^{l-1} \left(\prod_{1 < k \leq l-1, k \neq i} x_k^{1\wedge 2} \right) |x_i^2 - x_i^1| \left[\prod_{k=l}^d x_k^{1\wedge 2} - \sum_{j=l}^d \left(\prod_{k=l}^{j-1} x_k^{3\wedge 4} \right) |x_j^4 - x_j^3| \left(\prod_{k=j+1}^d x_k^{1\wedge 2} \right) \right].$$

Finally,

$$R_{1,2}(x^{1,2,3,4}) \geq \sum_{i=2}^{l-1} (x_1^{1\wedge 2} - s_1) \left(\prod_{2 \leq k, k \neq i} x_k^{1\wedge 2} \right) |x_i^2 - x_i^1|.$$

Following similar computations for $R_{1,3}(x^{1,2,3,4})$ and $R_{1,5}(x^{1,2,3,4})$, we get

$$\begin{aligned}
R_1(x^{1,2,3,4}) &\geq \sum_{i=1}^d \left(\prod_{k \neq i} x_k^{1\wedge 2} \right) |x_i^2 - x_i^1| + \sum_{i=1}^d \left(\prod_{k \neq i} x_k^{3\wedge 4} \right) |x_i^4 - x_i^3| \\
&\quad - s_1 \sum_{i=2}^{l-1} \sum_{j=l}^d \left(\prod_{2 \leq k \leq l-1, k \neq i} x_k^{1\wedge 2} \right) |x_i^2 - x_i^1| |x_j^4 - x_j^3| \left(\prod_{k=l}^{j-1} x_k^{3\wedge 4} \right) \left(\prod_{k=j+1}^d x_k^{1\wedge 2} \right) \\
&\quad - s_1 \sum_{i=l}^d \sum_{j=2}^{l-1} \sum_{k=2}^{j-1} \left(\prod_{k \neq i} x_k^{1\wedge 2} \right) |x_j^2 - x_j^1| |x_i^4 - x_i^3| \left(\prod_{k=j+1}^{i-1} x_k^{3\wedge 4} \right) \left(\prod_{k=i+1}^d x_k^{1\wedge 2} \right).
\end{aligned}$$

Then, for any $\rho < \frac{1}{2(d-2)} \min_{i,j \neq 1} \{s_i \prod_{k \neq i,j,1} \frac{s_k}{t_k}\}$ (for any $\rho > 0$ if $d = 2$), doing some changes of variables the integral (4) over the set D is bounded by

$$\begin{aligned}
& 4^d \sum_{l=2}^{d+1} \binom{d-1}{l-2} \int_{\prod_{i=1}^d [s_i, t_i]^4 \cap F_{\rho, l}} n^{2d} \prod_{i=2}^{l-1} \mathbf{1}_{\{x_i^1 \leq x_i^2 \leq x_i^3 \leq x_i^4\}} \prod_{i=l}^d \mathbf{1}_{\{x_i^3 \leq x_i^4 \leq x_i^1 \leq x_i^2\}} \\
& \times \mathbf{1}_{\{x_1^1 \leq x_1^2\}} \mathbf{1}_{\{x_1^3 \leq x_1^4\}} \left(\left(\prod_{i=1}^d x_i^1 x_i^3 \right) \left(\prod_{i=2}^{l-1} (x_i^1 + \rho) x_i^4 \right) \left(\prod_{i=l}^d x_i^2 (x_i^3 + \rho) \right) x_1^2 x_1^4 \right)^{\frac{d-1}{2}} \\
& \times \exp \left[-2n \left(\sum_{i=1}^d \left(\prod_{k \neq i} x_k^1 \right) (x_i^2 - x_i^1) + \sum_{i=1}^d \left(\prod_{k \neq i} x_k^3 \right) (x_i^4 - x_i^3) \right. \right. \\
& - s_1 \sum_{i=2}^{l-1} \sum_{j=l}^d \left(\prod_{2 \leq k \leq l-1, k \neq i} x_k^1 \right) \rho (x_j^4 - x_j^3) \left(\prod_{k=l}^{j-1} x_k^3 \right) \left(\prod_{k=j+1}^d x_k^1 \right) \\
& \left. \left. - s_1 \sum_{i=l}^d \sum_{j=2}^{l-1} \left(\prod_{k=2}^j x_k^1 \right) \rho (x_i^4 - x_i^3) \left(\prod_{k=j+1}^{i-1} x_k^3 \right) \left(\prod_{k=i+1}^d x_k^1 \right) \right) \right] dx^2 dx^4 dx^1 dx^3, \quad (5)
\end{aligned}$$

plus some terms that we have seen that go to zero.

Integrating first with respect to $dx_i^2, i \in \{2, \dots, l-1\}$ and $dx_i^4, i \in \{l, \dots, d\}$, and using the approximation of the identity as in Lemma 3.6 to integrate with respect the rest of components of x^2 and x^4 , we can bound the lim sup as $n \rightarrow \infty$ of (5) by $\sum_{l=2}^{d+1} \binom{d-1}{l-2} \int_{\prod_{i=1}^d [s_i, t_i]^2} G_l(\rho, x^1, x^3) \prod_{i=2}^{l-1} \mathbf{1}_{\{x_i^1 \leq x_i^3\}} \prod_{i=l}^d \mathbf{1}_{\{x_i^3 \leq x_i^1\}} dx^1 dx^3$, where G_l does not depend on n and $\lim_{\rho \rightarrow 0} G_l(\rho, x^1, x^3) = 1$, for each l . So, the limit when $\rho \rightarrow 0$ of the last integral is equal to $\int_{\prod_{i=1}^d [s_i, t_i]^2} dx^1 dx^3$ and the proof is now completed. \blacksquare

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