Asymptotic L^1-decay of Solutions of the Porous Medium Equation to Self-similarity

J. A. CARRILLO & G. TOSCANI

ABSTRACT. We consider the flow of gas in an N-dimensional porous medium with initial density $v_0(x) \geq 0$. The density $v(x,t)$ then satisfies the nonlinear degenerate parabolic equation $v_t = \Delta v^m$ where $m > 1$ is a physical constant. Assuming that $\int (1 + |x|^2) v_0(x)\, dx < \infty$, we prove that $v(x,t)$ behaves asymptotically, as $t \to \infty$, like the Barenblatt-Pattle solution $V(|x|,t)$. We prove that the L^1-distance decays at a rate $t^{1/((N+2)m-N)}$. Moreover, if $N = 1$, we obtain an explicit time decay for the L^∞-distance at a suboptimal rate. The method we use is based on recent results we obtained for the Fokker-Planck equation [2], [3].

1. INTRODUCTION. This paper is intended to study the rate of convergence of solutions to the porous medium equation to the self-similarity solution. The flow of gas in an N-dimensional porous medium equation is classically described by the solution to the Cauchy problem

$$\frac{\partial v}{\partial t} = \Delta v^m, \qquad (x \in \mathbb{R}^N,\ t > 0), \tag{1.1}$$

$$v(x, t = 0) = v_0(x) \geq 0, \qquad (x \in \mathbb{R}^N). \tag{1.2}$$

The function v represents the density of the gas in the porous medium and $m > 1$ is a physical constant.

Instead of working on (1.1) directly, we will study the asymptotic decay towards its equilibrium state of solutions to the (nonlinear) Fokker-Planck type equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(xu + \nabla u^m), \qquad (x \in \mathbb{R}^N,\ t > 0), \tag{1.3}$$

$$u(x, t = 0) = u_0(x) \geq 0, \quad (x \in \mathbb{R}^N). \tag{1.4}$$

The reason relies on the following fundamental remark: there exists a time dependent scaling which transforms (1.3) into the porous medium (1.1); moreover, we can fix the time scaling in order that the initial data for (1.3) after rescaling are the same as for the original equation. The exact expression of this time transformation will be given in the next section, together with a short derivation.

As can be easily seen, the kinetic equation (1.3) has a unique compactly supported equilibrium state $u_\infty(x)$, which coincides with the similarity solution of (1.1) evaluated at time $t = 1$. Hence, any result on time asymptotics of the Fokker-Planck like equation, by application of the time-dependent scaling, translates into a result on the time asymptotics of the porous medium equation.

The main advantage in working with (1.3) is that there is a natural method to deal with, namely the entropy method, where the convergence towards equilibrium is concluded using the time monotonicity of the physical entropy. This method has been recently developped by the authors for the classical Fokker-Planck equation [3], [19], and for general linear Fokker-Planck type equations by Anton, Markowich, Toscani and Unterreiter [2]. All these results show exponential convergence towards equilibrium in relative entropy. Then, exponential convergence in L^1 follows by Csiszar-Kullback type inequalities [5], [12].

The entropy approach consists essentially in deriving an equation for the evolution of a convex (relative) entropy. This equation connects the entropy with the (nonnegative) entropy production. From this equation one usually concludes with the convergence to zero of the relative entropy, but not with the rate of convergence. The rate of convergence (when possible) comes out by studying the time evolution of the entropy production. This analysis, from now on called the entropy-entropy method has been fruitfully employed in [2], [3], [19].

A Lyapunov functional for the porous medium equation (1.1) has been introduced by Newman in [13]. This functional is the energy for the rescaled equation (1.3),

$$H(u) = \int_{\mathbb{R}^N} \left(|x|^2 u + \frac{2}{m-1} u^m \right) dx. \tag{1.5}$$

It can be seen that $L(u)$ can be used to build the relative entropy $H(u|u_\infty)$, where

$$H(u|u_\infty) = H(u) - H(u_\infty) \geq 0 \tag{1.6}$$

bounds the L^1-distance between u and u_∞. The Lyapunov functional (1.5) has been used by Ralston [15] to prove L^1-convergence of the solution to (1.1) towards the similarity solution. His proof does not use the Fokker-Planck type equation, and is essentially based on compactness arguments. Vazquez proved the L^1-convergence for $N = 1$ in [22], and subsequently simplified the proof by the four step method introduced in [11].

The entropy production for $H(u|u_\infty)$ is given by $2I(u)$, where

$$I(u) = \int_{\mathbb{R}^N} u \left| x + \frac{m}{m-1} \nabla u^{m-1} \right|^2 dx. \tag{1.7}$$

Together with the entropy equation

$$\frac{d}{dt} H(u(t)|u_\infty) = -2I(u(t)), \tag{1.8}$$

one obtains the equation for the entropy production

$$\frac{d}{dt} I(u(t)) = -2I(u(t)) - R(t), \tag{1.9}$$

where $R(t) \geq 0$. Combining (1.8) and (1.9) one obtains

$$0 \leq H(u(t)|u_\infty) \leq I(u(t)), \quad t \geq 0 \tag{1.10}$$

and, substituting (1.10) into (1.8), one concludes with the exponential decay of the relative entropy to zero at a rate $2t$.

Inequality (1.10) is sharp. There is equality in it if and only if $u(t)$ is a multiple and translate of u_∞. This implies that the rate of convergence in relative entropy is sharp.

As far as the porous medium equation is concerned, the asymptotic behaviour of the solution has been described in dimension 1 by Kamin in [9], [10] in a L^∞-setting. This result has been subsequently generalized to the case $N > 1$ by Friedman and Kamin [7]. The approach used in these papers is completely different, and uses similarity transformations for (1.1). Recently, most of the existing results have been collected by Vazquez in an excellent survey [23]. Later on in this paper we will review the existing results about rates of convergence of the solution.

Independently, Otto [14] has recently studied the porous medium equation from a Riemannian geometrical point of view obtaining some results related to ours concerning the exponential convergence of the relative entropy. Also, Dolbeault and del Pino [6] obtained some results on the asymptotic behavior by proving directly generalized Sobolev inequalities.

The organization of the paper is as follows. Section 2 is devoted to some preliminary material concerning the time-dependent scaling of parabolic equations. Here, the main example is furnished by the connection between the heat equation and the linear Fokker-Planck equation. In section 3 we study the time evolution of the entropy functional (1.6) and of the entropy production (1.8) for the nonlinear Fokker-Planck type equation (1.3). This analysis shows that the entropy decays exponentially fast. In section 4 we prove several results for the relative entropy, including a new Csiszar-Kullback type inequality [5], [12]. The main consequence of this inequality is the exponential convergence in L^1 of the solution to (1.3) towards the stationary solution. Section 5 deals with the L^∞ convergence, while section 6 contains the results on the asymptotic behaviour for the porous medium equation.

2. TIME-DEPENDENT SCALING. We shall discuss here the main idea which enables us to look for a "sharp" rate of convergence towards the similarity solution of the solution to the porous medium equation. A brief recall of the linear case will clarify the procedure.

Consider the linear heat equation

$$\frac{\partial v}{\partial t} = \Delta v, \tag{2.1}$$

and the linear Fokker-Planck equation for u

$$\frac{\partial u}{\partial t} = \operatorname{div}(xu + \nabla u). \tag{2.2}$$

It is known that both equations have fundamental solutions given by

$$\mathcal{N}(x,t) = (4\pi t)^{-N/2} \exp\left\{-\frac{|x|^2}{4t}\right\},$$

and

$$G(x;t) = e^{Nt}\mathcal{N}(e^t x, \beta(t)),$$

centered at zero, respectively, where $\beta(t) = \frac{1}{2}(e^{2t} - 1)$. It is easy to see that, if we consider the change of variables

$$u(x,t) = e^{Nt} v(e^t x, \beta(t)), \tag{2.3}$$

then u is a solution of (2.2) for any solution v of (2.1). Or, equivalently,

$$v(x,t) = (2t+1)^{-N/2} u\left(\frac{x}{\sqrt{2t+1}}, \frac{1}{2}\log(2t+1)\right) \tag{2.4}$$

is a solution of (2.1) for any solution u of (2.2). Therefore, the change of variables (2.3)-(2.4) produces an isomorphism between the sets of solutions of (2.1) and (2.2).

Moreover, the fundamental solution of the heat equation (2.1) at time $t = \frac{1}{2}$ is nothing but the stationary solution of (2.2),

$$u_\infty(x) = (2\pi)^{-N/2} \exp\left\{-\frac{|x|^2}{2}\right\}.$$

Also, Toscani [17] proved that the solution of the linear heat equation behaves as the heat kernel as $t \to \infty$ in $L^1(\mathbb{R}^N)$ with a decay rate of the order $t^{1/2}$, provided that the nonnegative initial datum $v_0(x)$ satisfies

$$\int_{\mathbb{R}^N} (1 + |x|^2 + |\log v_0(x)|) v_0(x)\, dx < \infty. \tag{2.5}$$

More exactly,

$$\|v(x,t) - M\mathcal{N}(x,t)\|_{L^1(\mathbb{R}^N)} \le \frac{C}{\sqrt{2t+1}}, \quad t \ge 0$$

and that this bound is sharp. Here, M is the initial mass of v.

As far as the Fokker-Planck equation is concerned, it was proved in [3](under the same condition (2.5)) that the solution of (2.2) behaves as $u_\infty(x)$ in $L^1(\mathbb{R}^N)$ as $t \to \infty$ with an exponential decay rate, that is

$$\|u(x,t) - Mu_\infty(x)\|_{L^1(\mathbb{R}^N)} \le Ce^{-t}, \quad t \ge 0.$$

Now, using the time-dependent scaling (2.3)-(2.4), we see that both results are equivalent, since any result about the asymptotic behavior of $u(x,t)$ translates into a result about the asymptotic behavior of $v(x,t)$ and vice versa.

We remark that condition (2.5) is quite natural for the Fokker-Planck equation, where the usual requirements for the particle density, in addition to the positivity, are exactly the initial bounds on mass, energy, and entropy, while the same condition is in principle not so natural for the heat equation.

The previous example shows that the "kinetic" equation (1.3) is the natural one to look for the asymptotic decay. This is the point of view we will adopt to treat the porous medium equation (1.1).

It is well-known that equation (1.1) admits a family of self-similar solutions (in a weak sense) called Barenblatt-Pattle solutions given by

$$V(|x|,t) = t^{-kN}\left(C_1 - \frac{(m-1)k}{2m}|x|^2 t^{-2k}\right)_+^{1/(m-1)}, \tag{2.6}$$

where $k = (N(m-1)+2)^{-1}$ and C_1 is fixed by mass conservation

$$\int_{\mathbb{R}^N} V(x,t)\,dx = M \geq 0.$$

Following the line of the linear case $m = 1$, we look for a (mass-conserving) change of variables of the type

$$u(x,t) = \alpha(t)^N v(\alpha(t)x, \beta(t))$$

such that $u(x,t)$ satisfies the Fokker-Planck type equation. After some computations, one verifies that

$$\alpha(t) = e^t \quad \text{and} \quad \beta(t) = k(e^{t/k} - 1)$$

works, obtaining the same relationship between the sets of solutions of equations (1.1) and (1.3) than for $m = 1$. Thus, if v is a solution of (1.1), then

$$u(x,t) = e^{Nt} v(e^t x, k(e^{t/k} - 1)) \tag{2.7}$$

is a solution of (1.3) and vice versa, if u is a solution of (1.3), then

$$v(x,t) = \left(1 + \frac{t}{k}\right)^{-Nk} u\left(\left(1 + \frac{t}{k}\right)^{-k} x\, ,\, k\log\left(1 + \frac{t}{k}\right)\right) \tag{2.8}$$

is a solution of (1.1).

We outline here that the unique stationary solution of (1.3) is given by the Barenblatt-Pattle type formula

$$u_\infty(x) = \left(C_2 - \frac{m-1}{2m}|x|^2\right)_+^{1/(m-1)} \tag{2.9}$$

for a C_2 such that u_∞ has M mass, and this solution is found easily simply owing to positivity and mass conservation. In fact, the stationary solution of eq. (1.3) is obtained by taking the flux $xu + \nabla u^m$ to be zero. Since

$$
\begin{aligned}
(2.10)\quad xu + \nabla u^m &= u\left(\frac{m}{m-1}\nabla u^{m-1} + x\right)\\
&= \frac{m}{m-1} u \nabla\left(u^{m-1} - \left(C_2 - \frac{m-1}{2m}|x|^2\right)_+\right),
\end{aligned}
$$

the flux is zero if and only if $u(x,t) = u_\infty(x)$.

Also, $u_\infty(x)$ corresponds to $V(|x|, 1+k)$ with mass M through the change of variables (2.7)-(2.8). One can verify that the constants C_1 and C_2 scale in the right way.

As a conclusion, if we are able to derive any property about the asymptotic behavior of $u(x,t)$ towards $u_\infty(x)$ we can translate it into a result about the asymptotic behavior of $v(x,t)$ towards the Barenblatt-Pattle profile $MV(|x|,t)$.

For instance, we will prove in the next sections that in some cases

$$\|u(x,t) - u_\infty(x)\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty$$

or

$$\|u(x,t) - u_\infty(x)\|_{L^p(\mathbb{R}^N)} \le Ce^{-\gamma t}, \quad t \ge 0,$$

where $\gamma > 0$, $1 \le p \le \infty$. Therefore, we obtain using (2.7)-(2.8) that

$$\|v(x,t) - MV(|x|,t)\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty$$

or

$$\|v(x,t) - MV(|x|,t)\|_{L^p(\mathbb{R}^N)} \le \frac{C}{(1+t/k)^{(\gamma+N/p')k}}, \quad t \ge 0.$$

Remark 2.1. The previous scaling and the consequent results are also valid for the fast-diffusion equation ($N/(N+2) < m < 1$). In this case, the similarity solution is not compactly supported.

Remark 2.2. Time-dependent scalings can be obtained for fast diffusion equations

$$\frac{\partial v}{\partial t} = \Delta(\varphi(v)),$$

where $\varphi(v) = v^m$ for $0 < m < 1$ with $2 + N(m-1) > 0$, and $\varphi(u) = \log(u)$ if $N = 1$. Corresponding Barenblatt-Pattle profiles are translated into stationary states.

3. EXPONENTIAL DECAY OF RELATIVE ENTROPY. From now on, we will focus on solutions of the Fokker-Planck equation (1.3). The following properties are just easy consequences of analogous properties for the porous medium equation using the change of variables (2.7)-(2.8). It is known that the Cauchy problem for (1.3) is well-posed for any initial data $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$. This notion of solution, called strong solution, gives rise to the following properties:

(1) $u \in C([0,\infty), L^1(\mathbb{R}^N))$ with $u(0) = u_0$.

(2) The functions u^m, $u_t + div(xu)$ and Δu^m belong to $L^1((t_1,t_2), L^1(\mathbb{R}^N))$ for all $0 < t_1 < t_2$ and u solves (1.3) in distributional sense.

(3) If the initial data $u_0(x) \geq 0$, $u(x,t) \geq 0$, for any $t > 0$.

(4) Conservation of mass $\|u(t)\|_{L^1} = \|u_0\|_{L^1} = M$.

(5) The solution is uniformly bounded for any $t \geq \tau > 0$. Moreover, there exists a constant depending on N and m such that

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C(1 - e^{-t/k})^{-Nk} \|u\|_{L^1(\mathbb{R}^N)}^{2k}. \tag{3.1}$$

If the initial data is uniformly bounded, then $u \in L^\infty((0,\infty) \times \mathbb{R}^N)$.

(6) The solutions are uniformly α-Hölder continuous for $t \geq \tau > 0$. Furthermore, for any compact set Q in $(0,\infty) \times \mathbb{R}^N$, there exists a constant C such that

$$\|u(t)\|_{C^\alpha(Q)} \leq C(\|u\|_{L^\infty(Q)}, Q). \tag{3.2}$$

(7) If the initial data $u_0(x)$ is compactly supported so are $u(x,t)$, for any $t > 0$. Let $\mathcal{R}(u)$ be

$$\mathcal{R}(u) = \inf\{r > 0 \text{ such that the support of } u(x) \text{ lies in } B_r(0)\},$$

where $B_r(0)$ is the euclidean ball of centre 0 and radius r. Then, there exists a constant C such that

$$\mathcal{R}(u(t)) \leq C\mathcal{R}(u_\infty).$$

(8) The Aronson-Bénilan estimates for the porous medium equation can be written for the solution of the Fokker-Planck equation as

$$-\Delta(u^{m-1}) \leq N\frac{m-1}{m}(1 - e^{-t/k})^{-1}$$

and

$$-\Delta(u^m) \leq N(1 - e^{-t/k})^{-1} u,$$

in the sense of distributions.

(9) The nonlinear semigroup is an order-preserving contraction in $L^1(\mathbb{R}^N)$, that is, if u_1 and u_2 are solutions of equation (1.3), then

$$\|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^N)} \le \|u_1(0) - u_2(0)\|_{L^1(\mathbb{R}^N)}.$$

Much more information on the solutions can be found in [22], [23] and the references therein. Let us remark that some of these properties are not needed in this paper.

Assume that $u_0 \ge 0$ belongs to $(1 + |x|^2)u_0(x) \in L^1(\mathbb{R}^N)$ and $u_0 \in L^m(\mathbb{R}^N)$. In the sequel, we will work with classical solutions of (1.3), the general result being justified by density arguments, so we assume that u_0 is strictly positive on $\mathbb{R}^N$, $(1 + |x|^{2+\delta})u_0(x) \in L^1(\mathbb{R}^N)$ for some $\delta > 0$, and $u_0(x) \in L^\infty(\mathbb{R}^N)$. In this case, the solution of the porous medium equation is C^∞ for any $t > 0$ and the following computations are rigorously derived (see [23]). These hypotheses on u_0 will be removed later on.

Given any constant $c > 0$, let us study the time evolution of the Lyapunov functional, first considered by Newman and Ralston (see [13], [15]), evaluated on a solution of (1.3)

$$H(u) = \int_{\mathbb{R}^N} (|x|^2 u + c u^m)\, dx.$$

If $H(u_0)$ is bounded, we have

$$\frac{d}{dt} H(u) = \int_{\mathbb{R}^N} \operatorname{div}\left[u \left(x + \frac{m}{m-1} \nabla u^{m-1} \right) \right] (|x|^2 + c m u^{m-1})\, dx .$$

This comes from (1.3).

$$\frac{\partial u}{\partial t} = \operatorname{div}(xu + \nabla u^m) = \operatorname{div}\left[u \left(x + \frac{m}{m-1} \nabla u^{m-1} \right) \right].$$

Setting $c = 2/(m-1)$, after integration by parts we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \left(|x|^2 u + \frac{2}{m-1} u^m \right) dx \\ = \ -2 \int_{\mathbb{R}^N} u \left| x + \frac{m}{m-1} \nabla u^{m-1} \right|^2 dx \le 0. \end{aligned} \tag{3.3}$$

Let $u_\infty(x)$ be the stationary solution defined by 2.9. The relative entropy $H(u|u_\infty)$ is easily found by the position

$$\begin{aligned} H(u|u_\infty) &= H(u) - H(u_\infty) \\ &= \int_{\mathbb{R}^N} \left[|x|^2 u + \frac{2}{m-1} u^m - |x|^2 u_\infty - \frac{2}{m-1} u_\infty^m \right] dx. \end{aligned}$$

Since $H(u_\infty)$ is a constant, the previous argument shows that $H(u|u_\infty)$ is non-increasing.

It is remarkable that $H(u|u_\infty) \geq 0$ whenever $u \geq 0$ and $u_\infty \geq 0$ have the same mass. To prove this, considering that $H(u|u_\infty)$ is convex in u, we can make use of the usual variational argument with Lagrange multipliers [20]. The functional

$$H(u) + \lambda \int_{\mathbb{R}^N} u\,dx$$

has variation

$$\int_{\mathbb{R}^N} \left[|x|^2 + \lambda + \frac{2m}{m-1} u^{m-1} \right] \delta u\,dx = 0.$$

Thus, the extremals are solutions of

$$|x|^2 + \lambda + \frac{2m}{m-1} u^{m-1} = 0, \qquad u \geq 0,$$

and λ must be such that the mass of the extremal is equal to the mass of $u(x)$. Hence, u_∞ is an extremal of $H(u)$. This is obviously a minimum, since one can construct $u_1(x)$ with mass equal to the mass of $u_\infty(x)$ and with support contained in a ball of unit measure located very far from the origin. In this way we can get $H(u_1)$ larger than any fixed positive constant.

A deeper study of the properties of $H(u|u_\infty)$ is postponed to the next section. For the present purposes, we only prove that the convergence to zero of the relative entropy is a consequence of (3.3).

Theorem 3.1. *Let the initial condition for the Fokker-Planck equation* (1.3) *satisfy* $0 < u_0(x) \in L^1 \cap L^\infty(\mathbb{R}^N)$ *with* $|x|^{2+\delta} u_0(x) \in L^1(\mathbb{R}^N)$ *for some* $\delta > 0$. *Then the relative entropy* $H(u(t)|u_\infty)$ *is monotonically decreasing, and converges to zero as* $t \to \infty$.

Proof. Using (3.3), $H(u(t)|u_\infty)$ is decreasing and bounded from below so $H(u(t)|u_\infty) \to H^*$ as $t \to \infty$. We have to prove that $H^* = 0$. Now, remark that the entropy production for (1.3) given in (3.3)

$$I(u) = \int_{\mathbb{R}^N} u \left| x + \frac{m}{m-1} \nabla u^{m-1} \right|^2 dx$$

is summable over the time interval $[0, +\infty)$. In fact, (3.3) can be written as

$$\frac{d}{dt} H(u(t)|u_\infty) = -2I(u(t)), \tag{3.4}$$

which implies

$$\int_0^{+\infty} I(u(s))\, ds \leq \frac{1}{2} H(u_0|u_\infty).$$

Hence, there exists a sequence of times $\{t_k\}$, with $t_k \to \infty$ when $k \to \infty$, such that $I(u(t_k)) \to 0$ as $k \to \infty$. Let us denote $u(t_k)$ by u_k. Since $H(u(t_k)|u_\infty)$ is bounded, thus

$$\int_{\mathbb{R}^N} |x|^2 u_k\, dx \quad \text{and} \quad \int_{\mathbb{R}^N} u_k^m\, dx$$

are bounded.

Now, developing the value of $I(u)$ and applying the divergence theorem, we deduce

$$I(u) = \int_{\mathbb{R}} |x|^2 u\, dx - 2N \int_{\mathbb{R}^N} u^m\, dx + \left(\frac{m}{m-1}\right)^2 \int_{\mathbb{R}^N} u |\nabla u^{m-1}|^2\, dx.$$

Hence

$$\begin{aligned} I(u_k) + N(m-1)H(u_k) &= [Nm - N + 1] \int_{\mathbb{R}^N} |x|^2 u\, dx \\ &\quad + \left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^N} |\nabla u_k^{m-(1/2)}|^2\, dx. \end{aligned}$$

Since $I(u_k) \to 0$ and $H(u_k)$ is monotone nonincreasing, the sequence

$$\int_{\mathbb{R}^N} |\nabla u_k^{m-(1/2)}|^2\, dx$$

is bounded. On the other hand, we have that

$$\nabla u^m = \frac{m}{m-(1/2)} u^{1/2} \nabla u^{m-(1/2)},$$

and thus we can apply Hölder inequality to deduce that

$$\int_{\mathbb{R}^N} |\nabla u_k^m|\, dx \leq \frac{m}{m-(1/2)} \left[\int_{\mathbb{R}^N} u_k\, dx\right]^{1/2} \left[\int_{\mathbb{R}^N} |\nabla u_k^{m-(1/2)}|^2\, dx\right]^{1/2},$$

and hence the sequence u_k^m is bounded in $W^{1,1}(\mathbb{R}^N)$.

Taking into account that the initial data is uniformly bounded, we deduce that u is uniformly bounded in $(0,\infty) \times \mathbb{R}^N$ and thus

$$\int_{\mathbb{R}^N} |x|^\delta u^m\, dx \leq C \int_{\mathbb{R}^N} |x|^\delta u\, dx, \tag{3.5}$$

for any $t \geq 0$ and any $\delta \leq 2$, and hence, the sequence $|x|u_k^m$ is bounded in $L^1(\mathbb{R}^N)$.

Using the previous facts we deduce that $u_k^m \to g^m$ in $L^1(\mathbb{R}^N)$ (after passing to a subsequence). Since $0 \leq I(u_k) \to 0$, we have $I(g) = 0$ by lower semicontinuity arguments and the Fatou Lemma. Let

$$B(g) = \{x \in \mathbb{R}^N : g(x) > 0\}.$$

Then, $I(g) = 0$ implies $g(x) = c - (m-1)|x|/2m^2$ for all $x \in B(g)$, where c is constant. It is now immediate to conclude, by positivity and mass conservation, that $g(x) = u_\infty(x)$ a.e.

It remains to show that $H(u_k|u_\infty) \to 0$. It suffices to prove that

$$\int_{\mathbb{R}^N} |x|^2 u_k \, dx \to \int_{\mathbb{R}^N} |x|^2 u_\infty \, dx \quad \text{as } k \to \infty.$$

It is clear that we cannot prove this, unless we have some extra moment bounded. Applying the divergence theorem it is easy to see that

$$\frac{d}{dt}\int_{\mathbb{R}^N} |x|^{2+\delta} u \, dx = (2+\delta)(N+\delta)\int_{\mathbb{R}^N} |x|^\delta u^m \, dx - (2+\delta)\int_{\mathbb{R}^N} |x|^{2+\delta} u \, dx.$$

Using (3.5) and the variation of constants formula, we deduce that

$$\int_{\mathbb{R}^N} |x|^{2+\delta} u \, dx$$

is bounded uniformly on t. As a consequence,

$$\int_{\mathbb{R}^N} |x|^2 u_k \, dx \to \int_{\mathbb{R}^N} |x|^2 u_\infty \, dx \quad \text{as } k \to \infty.$$

and $H^* = 0$. □

Eq. (3.3) relates the relative entropy to the entropy production. From this relation we just concluded with the convergence to zero of the solution in relative entropy, without any rate. As discussed in the introduction, to find the rate of convergence requires a further step. Having this in mind, let us compute the time evolutions of $I(u(t))$,

$$\begin{aligned}\frac{d}{dt} I(u) = & \int_{\mathbb{R}^N} \frac{\partial u}{\partial t}\left| x + \frac{m}{m-1}\nabla u^{m-1}\right|^2 dx \\ + & \; 2\int_{\mathbb{R}^N} u\left(x + \frac{m}{m-1}\nabla u^{m-1}\right)\cdot \frac{\partial}{\partial t}\left[\frac{m}{m-1}\nabla u^{m-1}\right] dx = I_1 + I_2.\end{aligned}$$

Let us set $y = x + m\nabla u^{m-1}/(m-1)$. The second term can be written as

$$\begin{aligned} I_2(u) &= -2\frac{m}{m-1}\int_{\mathbb{R}^N} \operatorname{div}(uy)\frac{\partial}{\partial t}u^{m-1}\,dx \\ &= -2m\int_{\mathbb{R}^N} u^{m-2}\operatorname{div}(uy)\frac{\partial u}{\partial t}\,dx \\ &= -2m\int_{\mathbb{R}^N} u^{m-2}\operatorname{div}(uy)^2\,dx. \end{aligned}$$

The first term can be written as

$$I_1(u) = -2\int_{\mathbb{R}^N} u(y\cdot \text{Jacob}(y)\cdot y^T)\,dx.$$

Since $\text{Jacob}(y) = I_N + m/(m-1)\ \text{Hess}(u^{m-1})$, we have

$$I_1(u) = -2\int_{\mathbb{R}^N} u|y|^2\,dx - 2\frac{m}{m-1}\int_{\mathbb{R}^N} u(y\cdot\ \text{Hess}(u^{m-1})\cdot y^T)\,dx.$$

Now, consider that the last integral can be written in the following way

$$\begin{aligned} \int_{\mathbb{R}^N} u(y\cdot \text{Hess}(u^{m-1})\cdot y^T)\,dx &= \sum_{i,j=1}^{N}\int_{\mathbb{R}^N} uy_iy_j\frac{\partial^2 u^{m-1}}{\partial x_i\partial x_j}\,dx \\ &= \sum_{i,j=1}^{N}\int_{\mathbb{R}^N} \frac{1}{u}(uy_i)(uy_j)\frac{\partial^2 u^{m-1}}{\partial x_i\partial x_j}\,dx. \end{aligned}$$

Using the divergence theorem, it is straightforward to check that

$$\begin{aligned} &\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} \frac{1}{u}(uy_i)(uy_j)\frac{\partial^2 u^{m-1}}{\partial x_i\partial x_j}\,dx \\ &= -\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} \frac{\partial u^{m-1}}{\partial x_i}\left\{-\frac{1}{u^2}(uy_i)(uy_j)\frac{\partial u}{\partial x_j} + \frac{1}{u}\frac{\partial[(uy_i)(uy_j)]}{\partial x_j}\right\}dx \\ &= (m-1)\left\{\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} u^{m-2}y_iy_j\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx\right. \\ &\quad \left. - \sum_{i,j=1}^{N}\int_{\mathbb{R}^N} u^{m-2}\frac{\partial u}{\partial x_i}\left[y_i\frac{\partial(uy_j)}{\partial x_j} + y_j\frac{\partial(uy_i)}{\partial x_j}\right]dx\right\} \quad = \end{aligned}$$

$$
\begin{aligned}
= \quad & (m-1)\Big\{ \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} u^{m-2} y_i y_j \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx - \\
& \quad - 2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} u^{m-2} y_i \frac{\partial u}{\partial x_i}\frac{\partial (u y_j)}{\partial x_j}\, dx \Big\} \\
& \quad + (m-1) \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} u^{m-2} \frac{\partial u}{\partial x_i} \left[y_i \frac{\partial (u y_j)}{\partial x_j} - y_j \frac{\partial (u y_i)}{\partial x_j} \right] dx.
\end{aligned}
$$

Owing to the identity $\partial y_i / \partial x_j = \partial y_j / \partial x_i$, we obtain

$$
\begin{aligned}
\sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \left[y_i \frac{\partial (u y_j)}{\partial x_j} - y_j \frac{\partial (u y_i)}{\partial x_j} \right] &= \sum_{i,j=1}^{N} u \left[y_i \frac{\partial u}{\partial x_i}\frac{\partial y_j}{\partial x_j} - y_j \frac{\partial u}{\partial x_i}\frac{\partial y_i}{\partial x_j} \right] \\
&= u \sum_{i,j=1}^{N} \left[y_i \frac{\partial u}{\partial x_i}\frac{\partial y_j}{\partial x_j} - \frac{1}{2}\frac{\partial u}{\partial x_i}\frac{\partial y_j^2}{\partial x_i} \right] \\
&= u \left[(y \cdot \nabla u) \mathrm{div} y - \frac{1}{2} \nabla |y|^2 \cdot \nabla u \right].
\end{aligned}
$$

Thus, simplifying and collecting terms we obtain

$$
\begin{aligned}
I_1(u) + I_2(u) \quad = \quad & -2I(u) - 2m \int_{\mathbb{R}^N} u^m [\mathrm{div} y]^2\, dx \\
& - 2m \int_{\mathbb{R}^N} u^{m-1} \left[(y \cdot \nabla u) \mathrm{div} y - \frac{1}{2} \nabla |y|^2 \cdot \nabla u \right] dx,
\end{aligned}
$$

that we can write as

$$
\begin{aligned}
I_1(u) + I_2(u) \quad = \quad & -2I(u) - 2m \int_{\mathbb{R}^N} u^m [\mathrm{div} y]^2\, dx \\
& - 2 \int_{\mathbb{R}^N} \left[(y \cdot \nabla u^m) \mathrm{div} y - \frac{1}{2} \nabla |y|^2 \cdot \nabla u^m \right] dx.
\end{aligned}
$$

Applying the divergence theorem in the last two terms and taking into account that

$$
\mathrm{div}(y \mathrm{div} y) = [\mathrm{div} y]^2 + (y \cdot \nabla(\mathrm{div} y)),
$$

we deduce

$$
\begin{aligned}
\frac{d}{dt} I(u) \quad = \quad & -2I(u) - 2(m-1) \int_{\mathbb{R}^N} u^m [\mathrm{div} y]^2\, dx \\
& - 2 \int_{\mathbb{R}^N} u^m \left[\frac{1}{2} \Delta |y|^2 - (y \cdot \nabla(\mathrm{div} y)) \right] dx.
\end{aligned}
$$

Using that $\partial y_i/\partial x_j = \partial y_j/\partial x_i$, we obtain that

$$\frac{1}{2}\Delta|y|^2 - (y \cdot \nabla(\mathrm{div} y)) = \sum_{i,j=1}^{N} \left(\frac{\partial y_i}{\partial x_j}\right)^2,$$

and as a consequence

$$\begin{aligned} \frac{d}{dt}I(u) &= -2I(u) - 2(m-1)\int_{\mathbb{R}^N} u^m [\mathrm{div} y]^2\,dx \\ &\quad - 2\int_{\mathbb{R}^N} u^m \Big[\sum_{i,j=1}^{N}\left(\frac{\partial y_i}{\partial x_j}\right)^2\Big]\,dx. \end{aligned} \tag{3.6}$$

As a result, we have the following theorem.

Theorem 3.2. *Let the initial condition* $0 < u_0(x)$ *belong to* $L^1 \cap L^\infty(\mathbb{R}^N)$, *with* $|x|^{2+\delta}u_0(x) \in L^1(\mathbb{R})$ *for some* $\delta > 0$. *Then, for all* $t \geq t_0 > 0$, *the entropy production* $I(u(t))$ *is bounded and exponentially decreasing, with*

$$I(u(t)) \leq I(u(t_0))e^{-2(t-t_0)}, \quad t_0 > 0. \tag{3.7}$$

Moreover,

$$0 \leq H(u(t)|u_\infty) \leq I(u(t)), \quad t \geq 0, \tag{3.8}$$

and $H(u(t)|u_\infty)$ *is exponentially decreasing, with*

$$H(u(t)|u_\infty) \leq H(u_0|u_\infty)e^{-2t}, \quad t > 0. \tag{3.9}$$

Proof. The time evolution for the entropy production takes the form

$$\frac{d}{dt}I(u(t)) = -2I(u(t)) - R(u), \tag{3.10}$$

where

$$R(u) = 2(m-1)\int_{\mathbb{R}^N} u^m[\mathrm{div} y]^2\,dx + 2\int_{\mathbb{R}^N} u^m\Big[\sum_{i,j=1}^{N}\left(\frac{\partial y_i}{\partial x_j}\right)^2\Big]\,dx \geq 0.$$

This proves (3.7). Now we recover $I(u)$ from (3.10), and we plug it into (3.4) to have

$$\frac{d}{dt}H(u(t)|u_\infty) = \frac{d}{dt}I(u(t)) + R(u).$$

Using the result of theorem 3.1 we integrate between $t_0 > 0$ and $+\infty$ and we obtain (3.8). Finally, we use inequality (3.8) into (3.4) to conclude with (3.9). □

Remark 3.3. The results in Theorems 3.1 and 3.2 are also true with some minor changes in the proof for the fast diffusion equation with $N/(N+2) < m < 1$, with $m > (N-1)/N$ [4], [14]. In this case we do not need to assume that the initial data is strictly positive, because all the solutions are strictly positive and C^∞ for any $t > 0$.

Inequality (3.8) can be rewritten using a density argument as a differential inequality (of Gross logarithmic Sobolev type). We have the following result:

Corollary 3.4. *Let $m > 1$. For all functions $f \in L^1(\mathbb{R}^N)$ such that the distributional gradient of $f^{m-1/2}$ is square integrable, $f \in L^m(\mathbb{R}^N)$, and*

$$\left(N + \frac{1}{m-1}\right) \int_{\mathbb{R}^N} |f|^m \, dx \le \frac{1}{2} \left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^N} |\nabla f^{m-1/2}|^2 \, dx + A_m(\|f\|_1), \tag{3.11}$$

where

$$A_m(M) = \int_{\mathbb{R}^N} \left[\frac{|x|^2}{2} u_\infty + \frac{1}{m-1} u_\infty^m \right] dx,$$

being u_∞ the Barenblatt-Pattle solution of order m and mass M. Moreover, there is equality in (3.11) if and only if f is a multiple and translate of $f = u_\infty$.

Proof. First, suppose f satisfies the additional hypotheses of Theorem 3.1 for the initial data. Since the entropy production can be written as

$$I(f) = \int_{\mathbb{R}^N} |x|^2 f(x) \, dx - 2N \int_{\mathbb{R}^N} f^m \, dx + \left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^N} |\nabla f^{m-1/2}|^2 \, dx,$$

from (3.8) we obtain (3.11). Next, the cases of equality follow from theorem 3.2, considering that the only possibility for this to happen is that

$$\int_0^\infty R(f(s)) ds = 0,$$

where $f(t)$ is the solution of the equation (1.3) with initial data f. The general case follows easily by a standard limit procedure. □

Remark 3.5. The classical logarithmic Sobolev inequality by Gross [8] can be obtained by (3.11) simply subtracting to both sides $\|f\|_1/(m-1)$, and letting $m \to 1$. This gives an alternative way to find the remainder first obtained in [18].

4. RELATIVE ENTROPY AND L^1-ESTIMATES. We shall now be concerned with proving a generalized Csiszar-Kullback inequality [5] [12], which is an estimate for the L^1-distance of two functions in terms of their relative entropy. Classical entropies are defined in terms of a convex function ψ, with $\psi(1) = 0$ by the formula

$$L(u_1|u_2) = \int_{\mathbb{R}^N} \psi\left(\frac{u_1}{u_2}\right) u_2\,dx,$$

where $0 \le u_1,\, u_2 \in L^1$ and $\int u_1 = \int u_2$. For these entropies, the theory is well understood [2]. In our case, the relative entropy $H(u|u_\infty)$ has a slightly different form, and for $m > 1$, u_∞ has compact support. Hence, the classical theory is not directly applicable. In what follows, for any $m > 1$ let

$$B(M) = \left\{|x|^2 \le \frac{2m}{m-1} C_2(M)\right\}$$

be the support of the stationary solution of eq. (1.3). The key result is provided by the following.

Lemma 4.1. *Let $u \ge 0$ and u_∞ have equal mass M and support $B(M)$, and let us consider the entropy functional*

$$H(u|u_\infty) = \int_{B(M)} \left[|x|^2(u-u_\infty) + \frac{2}{m-1}(u^m - u_\infty^m)\right] dx.$$

Then if $1 < m \le 2$ for all $p \ge 1$,

$$\int_{u<u_\infty} |u-u_\infty|^{2/p}\,dx \le \left\{\frac{1}{m} H(u|u_\infty)\right\}^{1/p} \left\{\int_{B(M)} u_\infty^{-(m-2)(q-1)}\,dx\right\}^{1/q}.$$

If $m > 2$, for all $p > (2m-3)/(m-1)$,

$$\int_{u>u_\infty} |u-u_\infty|^{2/p}\,dx \le \left\{\frac{1}{m} H(u|u_\infty)\right\}^{1/p} \left\{\int_{B(M)} u_\infty^{(2-m)}\,dx\right\}^{1/q}$$

(p and q are conjugate exponents).

Proof. We can write $H(u|u_\infty)$ as

$$H(u|u_\infty) = \int_{B(M)} \left\{|x|^2\left(\frac{u}{u_\infty}-1\right) + \frac{2}{m-1}u_\infty^{m-1}\left[\left(\frac{u}{u_\infty}\right)^m - 1\right]\right\} u_\infty \, dx.$$

Using that $u_\infty^{m-1} = C_2(M) - |x|^2(m-1)/2m$, we have

$$H(u|u_\infty) = \int_{B(M)} u_\infty \left\{|x|^2\left(\frac{u}{u_\infty}-1\right) + \left(\frac{2C_2(M)}{m-1} - \frac{|x|^2}{m}\right)\left[\left(\frac{u}{u_\infty}\right)^m - 1\right]\right\}.$$

Let

$$\psi(x,t) = |x|^2(t-1) + \left(\frac{2C_2(M)}{m-1} - \frac{|x|^2}{m}\right)(t^m - 1), \quad t \geq 0, \ x \in B(M).$$

Then, $\psi(x,1) = 0$. Expanding $\psi(x,t)$ in Taylor's series of t up to order two we obtain

$$\psi(x,t) = \frac{2m}{m-1}C_2(M)(t-1) + m u_\infty^{m-1}(1+\theta(t-1))^{m-2}(t-1)^2,$$

with $\theta \in (0,1)$. Using that u and u_∞ have equal mass, we deduce that

$$\begin{aligned} H(u|u_\infty) &= \int_{B(M)} \psi\left(x, \frac{u}{u_\infty}\right) u_\infty \, dx \\ &= m\int_{B(M)} u_\infty^{m-2}\left(1+\theta\left(\frac{u}{u_\infty}-1\right)\right)^{m-2}(u-u_\infty)^2 \, dx. \end{aligned}$$

Taking $m \geq 2$ we have that

$$H(u|u_\infty) \geq m\int_{u>u_\infty} u_\infty^{m-2}(u-u_\infty)^2 \, dx.$$

If $1 < m < 2$, analogously we have

$$H(u|u_\infty) \geq m\int_{u<u_\infty} u_\infty^{m-2}(u-u_\infty)^2 \, dx.$$

If $1 < m \leq 2$, using Hölder's inequality

$$\begin{aligned} &\int_{u<u_\infty} |u-u_\infty|^{2/p} \, dx \\ &\quad \leq \left(\int_{u<u_\infty} (u-u_\infty)^2 u_\infty^{m-2} \, dx\right)^{1/p}\left(\int_{u<u_\infty} u_\infty^{q(2-m)/p} \, dx\right)^{1/q}. \end{aligned}$$

The first integral is bounded by $1/mH(u|u_\infty)$ while the second is a bounded constant for all $p \geq 1$. If now $m > 2$, Hölder inequality gives a bounded right-hand side only if $u_\infty^{(2-m)(q-1)}$ is integrable. This is clearly true for any $p > (2m-3)/(m-1)$. ◻

A simple consequence of the previous lemma is the following Csiszar-Kullback inequality:

Lemma 4.2. *For $m > 1$, let u_∞ be the stationary solution of eq.* (1.3), *and let $u \geq 0$ and u_∞ have the same mass M and support $B(M)$. Then, there exists a positive constant $D = D(m, M)$ such that*

$$\left(\int_{B(M)} |u - u_\infty| \, dx \right)^2 \leq DH(u|u_\infty). \tag{4.1}$$

Proof. Using that u and u_∞ have equal mass we obtain

$$\frac{1}{2} \int_{B(M)} |u - u_\infty| \, dx = \int_{u > u_\infty} |u - u_\infty| = \int_{u < u_\infty} |u - u_\infty|.$$

Thus, we use the previous lemma with $p = 2$ and we finish. □

Remark 4.3. In view of Corollary 3.4, the rate of convergence to the steady state in relative entropy is sharp. On the other hand, when u and u_∞ have the same support, the Csiszar-Kullback type inequality (4.1) is in the standard optimal form. Hence, when u and u_∞ have the same support, the decay in L^1 we obtain with our method is optimal.

The next step consists in proving that the relative entropy gives a bound on the L^1-distance between u and u_∞ even if u has not the same support of u_∞. This follows from the fact we can write the relative entropy as the sum of two positive terms, one of them being defined only out of the support of u_∞.

Lemma 4.4. *Let $u \geq 0$, such that $(1 + |x|^2)u \in L^1(\mathbb{R}^N)$, and suppose u_∞ has the same mass of u. Then*

$$H(u|u_\infty) = H_1(u|u_\infty) + H_2(u|u_\infty),$$

where

$$H_1(u|u_\infty) = \int_{|x|^2 > C} \left[(|x|^2 - C)u + \frac{2}{m-1} u^m \right] dx,$$

and

$$C = \frac{2m}{m-1} C_2(M).$$

Both $H_1(u|u_\infty)$ and $H_2(u|u_\infty)$ are nonnegative.

Proof. Let us consider the sets

$$\begin{aligned} A_1 &= \{x \in \mathbb{R}^N : u \geq u_\infty,\ |x|^2 < C\}, \\ A_2 &= \{x \in \mathbb{R}^N : u < u_\infty,\ |x|^2 < C\}, \\ A_3 &= \{x \in \mathbb{R}^N : |x|^2 > C\}. \end{aligned}$$

We write

$$\begin{aligned} H(u|u_\infty) &= \int_{A_3} \left(|x|^2 u + \frac{2}{m-1} u^m \right) dx \\ &\quad + \int_{A_2} \left[|x|^2 (u - u_\infty) + \frac{2}{m-1} (u^m - u_\infty^m) \right] dx \\ &\quad + \int_{A_1} \left[|x|^2 (u - u_\infty) + \frac{2}{m-1} (u^m - u_\infty^m) \right] dx. \end{aligned}$$

Using the same Taylor expansion as in Lemma 4.1 we deduce that

$$\begin{aligned} &\int_{A_2} \left[|x|^2 (u - u_\infty) + \frac{2}{m-1} (u^m - u_\infty^m) \right] dx \\ &= C \int_{A_2} (u - u_\infty)\, dx + m \int_{A_2} u_\infty^{m-2} \left(1 + \theta \left(\frac{u}{u_\infty} - 1 \right) \right)^{m-2} (u - u_\infty)^2\, dx \\ &= C \int_{A_2} (u - u_\infty) + L(u), \end{aligned}$$

where $L(u) \geq 0$. Since u and u_∞ have equal mass and the support of u_∞ is $|x|^2 \leq C$, then

$$\int_{A_2} (u - u_\infty)\, dx = - \int_{u > u_\infty} (u - u_\infty)\, dx,$$

and

$$\begin{aligned} H(u|u_\infty) &= H_1(u|u_\infty) - C \int_{A_1} (u - u_\infty)\, dx \\ &\quad + \int_{A_1} \left[|x|^2 (u - u_\infty) + \frac{2}{m-1} (u^m - u_\infty^m) \right] dx + L(u). \end{aligned}$$

It is clear that $H_1(u|u_\infty)$ and $L(u)$ are positive. Let us prove that the sum of the other two terms is also positive. This sum can be written as

$$-\frac{2m}{m-1} \int_{A_1} u_\infty^{m-1} (u - u_\infty)\, dx + \frac{2}{m-1} \int_{A_1} (u^m - u_\infty^m)\, dx.$$

Since u^m is convex for $m \geq 1$,

$$u^m - u_\infty^m \geq m u_\infty^{m-1}(u - u_\infty) \quad \text{in } A_1,$$

then $H_2(u|u_\infty)$ is non negative and the lemma is proved. □

We now prove our main result.

Theorem 4.5. *Let the initial condition for the Fokker-Planck equation* (1.3) *satisfy* $0 \leq u_0(x) \in L^1(\mathbb{R}^N)$ *and* $H(u_0) < \infty$. *Then there exists a positive constant* $C = C(m, H(u_0))$ *such that*

$$\|u(t,x) - u_\infty(x)\|_{L^1(\mathbb{R}^N)} \leq C \exp\left\{-\frac{N(m-1)+2}{(N+2)m-N} t\right\}, \quad t \geq 0.$$

Proof. Firstly, assume that the initial data satisfies the additional hypotheses needed in Theorems 3.1 and 3.2. Let $B = H(u_0|u_\infty)$. We know that $H(u(t)|u_\infty) \leq Be^{-2t}$, $t \geq 0$. Therefore,

$$H_1(u|u_\infty) = \int_{|x|^2>C} \left[(|x|^2 - C)u + \frac{2}{m-1}u^m\right] dx \leq Be^{-2t}, \quad t \geq 0.$$

First, we prove that

$$\int_{|x|^2>C} u(x,t)\, dx \leq De^{-\gamma t}, \quad t \geq 0, \tag{4.2}$$

where $D(m) > 1$ is a bounded constant, and

$$\gamma(m) = \frac{2N(m-1)+4}{(N+2)m-N}.$$

For any given $\rho > 0$ we write

$$\begin{aligned}\int_{|x|^2>C} u(x,t)\, dx \quad &\leq \int_{C<|x|^2<(\sqrt{C}+\rho/2)^2} u(x,t)\, dx \\ &\quad + \frac{4}{\rho^2}\int_{|x|^2>(\sqrt{C}+\rho/2)^2} (|x|^2 - C)u(x,t)\, dx,\end{aligned}$$

thus, by Hölder inequality

$$\begin{aligned}\int_{|x|^2>C} u(x,t)\, dx &\leq A_N\left(\int_{C<|x|^2<(\sqrt{C}+\rho/2)^2} u^m\right)^{1/m} \rho^{N(m-1)/m} \\ &\quad + \frac{4}{\rho^2}\int_{|x|^2>C} (|x|^2 - C)u(x,t)\, dx \\ &\leq A_N\left(\frac{m-1}{2}H_1(u|u_\infty)\right)^{1/m} \rho^{N(m-1)/m} + \frac{4}{\rho^2}H_1(u|u_\infty).\end{aligned}$$

Optimizing over ρ we obtain (4.2). Now, let us choose $\alpha(t) \in \mathbb{R}^+$ such that

$$\alpha(t) \int_{|x|^2<C} u(x,t)\,dx = M = \int_{\mathbb{R}^N} u_0(x)\,dx.$$

It is clear that $\alpha(t) \geq 1$. Since

$$\int_{|x|^2<C} u(x,t)\,dx \geq M - De^{-\gamma t}$$

if $t_1 = 1/\gamma \log(2D/M)$, for all $t \geq t_1$ we deduce that

$$M = \alpha(t) \int_{|x|^2<C} u(x,t)\,dx \geq \alpha(t)(M - De^{-\gamma t}),$$

and

$$0 \leq \alpha(t) - 1 \leq \frac{De^{-\gamma t}}{M - De^{-\gamma t}} \leq \frac{2D}{M} e^{-\gamma t}, \quad t \geq t_1. \tag{4.3}$$

Let us define

$$\tilde{u}(x,t) = \begin{cases} \alpha(t)u(x,t) & \text{if } |x|^2 < C, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \|u - u_\infty\|_{L^1(\mathbb{R}^N)} &\leq \|u - \tilde{u}\|_{L^1(\mathbb{R}^N)} + \|\tilde{u} - u_\infty\|_{L^1(\mathbb{R}^N)} \\ &= \int_{|x|^2>C} u\,dx + (\alpha(t) - 1) \int_{|x|^2<C} u\,dx \\ &\qquad + \int_{|x|^2<C} |\tilde{u} - u_\infty|\,dx. \end{aligned}$$

Thanks to (4.2) and (4.3), the first two terms go to zero exponentially. Using Lemma 4.2, the last term is bounded by $H(\tilde{u}|u_\infty)^{1/2}$ so it goes exponentially to zero as well if we prove that $H(\tilde{u}|u_\infty)$ goes to zero exponentially. We have

$$\begin{aligned} H(\tilde{u}|u_\infty) &= \int_{|x|^2<C} \left[|x|^2(\alpha(t)u - u_\infty) + \frac{2}{m-1}(\alpha(t)^m u^m - u_\infty^m) \right] dx \\ &= \int_{|x|^2<C} \left[|x|^2(\alpha(t) - 1)u + \frac{2}{m-1}(\alpha(t)^m - 1)u^m \right] dx \\ &\qquad + \int_{|x|^2<C} \left[|x|^2(u - u_\infty) + \frac{2}{m-1}(u^m - u_\infty^m) \right] dx. \end{aligned}$$

Thus, since the support of u_∞ is $|x|^2 < C$, we conclude that

$$\begin{aligned} H(\tilde{u}|u_\infty) \leq\ & H(u|u_\infty) + (\alpha(t) - 1) \int_{|x|^2<C} |x|^2 u\,dx \\ &+ \frac{2}{m-1}(\alpha(t)^m - 1) \int_{|x|^2<C} u^m\,dx, \end{aligned}$$

and we finish the proof.

Finally, we have to get rid of the additional hypotheses on the initial data u_0. We use a standard density argument (see [22]). Consider an initial data $0 \le u_0(x) \in L^1(\mathbb{R}^N)$ with $H(u_0) < \infty$. Let

$$u_0^\varepsilon = (\omega_\varepsilon * u_0) + \varepsilon e^{-|x|^2} \quad \text{for any } \varepsilon > 0,$$

where ω_ε is a regularizing sequence. Now, u_0^ε satisfies all the hypotheses in Theorems 3.1 and 3.2. Moreover, we can assume that

$$\|u_0^\varepsilon - u_0\|_{L^1(\mathbb{R}^N)} \le \varepsilon',$$

with $\varepsilon' \to 0$ as $\varepsilon \to 0$. Using now the L^1 contraction property of the solutions of (1.3), we deduce that

$$\|u^\varepsilon(t) - u(t)\|_{L^1(\mathbb{R}^N)} \le \|u_0^\varepsilon - u_0\|_{L^1(\mathbb{R}^N)}.$$

As a consequence, we have that

$$\|u(t) - u_\infty\|_{L^1(\mathbb{R}^N)} \le \|u^\varepsilon(t) - u_\infty^\varepsilon\|_{L^1(\mathbb{R}^N)} + \|u_0^\varepsilon - u_0\|_{L^1(\mathbb{R}^N)} + \|u_\infty^\varepsilon - u_\infty\|_{L^1(\mathbb{R}^N)}.$$

Taking into account the first part of the theorem, then the proof is easy to finish. □

Remark 4.6. For $m > 1$, the decay towards the steady state of the nonlinear Fokker-Planck type equation (1.3) is exponential, at a rate that for $m \to 1$ coincides with the rate of convergence of the classical Fokker-Planck equation [3], and tends to $N/(N+2)$ as $m \to \infty$.

5. Entropy production and $L^\infty(\mathbb{R})$-estimates. This last section is devoted to find the rate of exponential convergence towards equilibrium in $L^\infty(\mathbb{R})$. The main result is given by the following

Theorem 5.1. *Let the initial condition for the Fokker-Planck equation* (1.3) *satisfy* $0 \le u_0(x) \in L^1(\mathbb{R})$ *and* $H(u_0) < \infty$. *Then, for all* $t \ge t_0 > 0$, $u(t) \in L^\infty(\mathbb{R})$ *and there exists a positive constant* $C = C(m, H(u_0), I(u(t_0)))$ *such that*

$$\|u(t) - u_\infty\|_{L^\infty(\mathbb{R})} \le C \exp\left\{-\frac{m+1}{m(3m-1)}t\right\}, \quad \text{for any } t \ge t_0.$$

Proof. Firstly, assume that the initial data satisfies the additional hypotheses needed in Theorems 3.1 and 3.2. We can write the entropy production as

$$I(u) = \int_{\mathbb{R}} x^2 u \, dx - 2 \int_{\mathbb{R}} u^m \, dx + \int_{\mathbb{R}} \frac{1}{u} \left(\frac{\partial}{\partial x} u^m \right)^2 dx.$$

Let as usual $B(M) = \left\{ |x|^2 \le C \right\}$ be the support of u_∞, and let $A^2 = C$. If $x \le -A$,

$$\begin{aligned} u^m(x) &= \int_{-\infty}^{x} \frac{\partial u^m}{\partial y} \, dy \le \int_{-\infty}^{-A} \left| \frac{\partial u^m}{\partial y} \right| dy \\ &\le \left(\int_{-\infty}^{-A} \frac{1}{u} \left(\frac{\partial u^m}{\partial y} \right)^2 dy \right)^{1/2} \left(\int_{-\infty}^{-A} u(y) \, dy \right)^{1/2}. \end{aligned}$$

Using the definition of $I(u)$, we obtain

$$\begin{aligned} \int_{-\infty}^{-A} \frac{1}{u} \left(\frac{\partial u^m}{\partial x} \right)^2 dx &\le I(u) + 2 \int_{\mathbb{R}} u^m \, dx \\ &\le I(u) + (m-1) \left[\frac{2}{m-1} \int_{\mathbb{R}} u^m \, dx + \int_{\mathbb{R}} x^2 u \, dx \right] \\ &\le I(u) + (m-1) H(u|u_\infty) \le D e^{-2t}, \qquad t \ge 0. \end{aligned}$$

In addition, if $x \le -A$

$$\int_{-\infty}^{-A} u \, dx \le \frac{1}{A^2} \int_{-\infty}^{-A} x^2 u \le \frac{1}{A^2} H(u|u_\infty).$$

Finally, if $t \ge t_0 > 0$ by Theorem 3.2, eq. (3.7) we know that $I(u(t))$ is bounded and exponentially decreasing. Collecting the previous inequalities we conclude that there exists a positive constant such that

$$u^m(x) \le D e^{-2t}.$$

Analogously for $x > A$. Thus, outside the support of u_∞ we have

$$|u(x,t) - u_\infty(x)| \le D e^{-2t/m}. \tag{5.1}$$

Suppose now that $|x|^2 \le C$. We use the identity

$$\begin{aligned} & u^m(x) - u_\infty^m(x) \\ &\quad = u^m(-A) + \int_{-A}^{x} \frac{\partial}{\partial y} [u^m - u_\infty^m] \, dy \quad = \end{aligned}$$

$$= \quad u^m(-A) + \int_{-A}^{x} m\left\{u^{m-1}\frac{\partial u}{\partial y} - u_\infty^{m-1}\frac{\partial u_\infty}{\partial y}\right\} dy$$

$$= \quad u^m(-A) + \frac{m}{m-1}\int_{-A}^{x}\left\{u\left[\frac{\partial u^{m-1}}{\partial y} - \frac{\partial u_\infty^{m-1}}{\partial y}\right] + (u-u_\infty)\frac{\partial u_\infty^{m-1}}{\partial y}\right\} dy.$$

Therefore

$$\begin{aligned} |u^m(x) - u_\infty(x)^m| \quad \le \quad & |u^m(-A)| + \frac{m}{m-1}\int_{-A}^{A} u\left|\frac{\partial u^{m-1}}{\partial x} - \frac{\partial u_\infty^{m-1}}{\partial x}\right| dx \\ & + \frac{m}{m-1}\int_{-A}^{A} |u-u_\infty|\left|\frac{\partial u_\infty^{m-1}}{\partial x}\right| dx. \end{aligned}$$

Now, consider that

$$\frac{\partial u_\infty^{m-1}}{\partial x} = -\frac{m-1}{m}x.$$

Thus, applying Cauchy-Schwarz inequality we obtain

$$\begin{aligned} & |u^m(x) - u_\infty^m(x)| \\ & \le |u^m(-A)| + \left[\int_{-A}^{A} u\left(x + \frac{m}{m-1}\frac{\partial u^{m-1}}{\partial x}\right)^2 dx\right]^{1/2}\left(\int_{-A}^{A} u\,dx\right)^{1/2} \\ & \qquad + \left(\int_{-A}^{A}|u-u_\infty|\,dx\right)^{1/2}\left(\int_{-A}^{A} x^2|u-u_\infty|\,dx\right)^{1/2}. \end{aligned}$$

Using (5.1), we have

$$|u^m(x,t) - u_\infty^m(x)| \le De^{-2t} + I(u)^{1/2}M^{1/2} + A^2\|u-u_\infty\|_1. \tag{5.2}$$

Since $(x-y)^m + y^m$ is convex for $m > 1$, $x, y > 0$, $(x-y)^m + y^m \le x^m$ whenever $x \ge y$. Hence, if $u \ge u_\infty$, $(u-u_\infty)^m \le u^m - u_\infty^m$. The same argument works for $u \le u_\infty$. Therefore

$$|u - u_\infty| \le |u^m - u_\infty^m|^{1/m}, \quad t \ge t_0 > 0.$$

By Theorem 3.2 we recover the sharp decay of $I(u(t))$, while Theorem 4.5 furnishes the decay of the L^1-distance. Hence, substituting into (5.2) we conclude the proof for the initial data as in Theorems 3.1 and 3.2.

To finish the proof, we use again a density argument (see [22]). Consider an initial data $0 \le u_0(x) \in L^1(\mathbb{R}^N)$ with $H(u_0) < \infty$ and $I(u_0) < \infty$. Let

$$u_0^\varepsilon = (\omega_\varepsilon * u_0) + \varepsilon e^{-|x|^2} \quad \text{for any } \varepsilon > 0,$$

where ω_ε is a regularizing sequence. Now, u_0^ε satisfies all the hypotheses in Theorems 3.1 and 3.2. Therefore, there exists a positive constant $C(m, H(u_0), I(u_0))$ such that

$$\|u^\varepsilon(t) - u_\infty^\varepsilon\|_{L^\infty(\mathbb{R})} \leq C \exp\left\{-\frac{m+1}{m(3m-1)}t\right\}. \tag{5.3}$$

Using the Hölder estimate (3.2) in combination with the uniform bound (3.1) we deduce that the sequence $u^\varepsilon(t)$ is relatively compact in $C([t_1, t_2] \times \Omega)$ for any $0 < t_1 < t_2$ and any compact interval Ω. It is easy to see, using the uniqueness of strong solution, that the limit must be the solution of the equation (1.3) with initial data u_0. Therefore, taking a subsequence we deduce that $u^\varepsilon(t) - u_\infty^\varepsilon$ converges towards $u(t) - u_\infty$ in $C([t_1, t_2] \times \Omega)$ Using that the bound (5.3) is uniform in ε we have that $u^\varepsilon(t) - u_\infty^\varepsilon$ converges towards $u(t) - u_\infty$ in $C([t_0, \infty) \times \Omega)$ for any $0 < t_0$ and any compact interval Ω. As a consequence,

$$\|u(t) - u_\infty\|_{L^\infty(\Omega)} \leq C \exp\left\{-\frac{m+1}{m(3m-1)}t\right\},$$

for any compact interval Ω with C independent of Ω. Therefore, the proof is finished provided $I(u_0) < \infty$. To eliminate this hypothesis $I(u_0) < \infty$ it suffices to check that, for an initial data such that $0 \leq u_0(x) \in L^1(\mathbb{R})$ and $H(u_0) < \infty$, the solution satisfies that $I(u(t)) < \infty$ for any $t \geq t_0 > 0$. Therefore, you apply the previous result to $u(t_0)$. □

6. CONCLUSION. The relations (2.7) and (2.8), and the results of the previous section on the convergence of the solution to the Fokker-Planck equation (1.3) towards the unique stationary solution, give a description of the asymptotic behaviour of the solution to the porous medium equation (1.1). We have

Theorem 6.1. *Assume that v is a nonnegative solution of (1.1) in $L^1(\mathbb{R}^N)$, corresponding to a nonnegative initial datum v_0 of bounded mass M and entropy $H(v_0)$, for $m > 1$. Then, if $MV(|x|, t)$ is the Barenblatt-Pattle profile of mass M, there exists a constant $C = C(m, H(v_0))$ such that*

$$\|v(x,t) - MV(|x|,t)\|_{L^1(\mathbb{R}^N)} \leq C \left(\frac{1}{1 + (t/k)}\right)^{1/((N+2)m-N)}.$$

The case $N = 1$ gives stronger convergence results, in that we obtain the rate of convergence both in L^1 and L^∞. We have

Theorem 6.2. *Assume that v is a nonnegative solution of (1.1) in $L^1(\mathbb{R})$, corresponding to a nonnegative initial datum v_0 of bounded mass M and entropy $H(v_0)$, for $m > 1$. Then, if $MV(|x|,t)$ is the Barenblatt-Pattle profile of mass M, there exists a constant $C = C(m, H(v_0))$ such that*

$$\|v(x,t) - MV(|x|,t)\|_{L^1(\mathbb{R})} \le C\left(\frac{1}{1+(m+1)t}\right)^{1/(3m-1)}.$$

Moreover, for all $t \ge t_0 > 0$, $v(x,t) \in L^\infty(\mathbb{R})$, and there exists a constant $C_1 = C_1(m, H(v_0), I(u_{t_0}))$ such that

$$[1+(m+1)t]^{1/(m+1)}\|v(x,t) - MV(|x|,t)\|_{L^\infty(\mathbb{R})} \le C_1\left(\frac{1}{1+(m+1)t}\right)^{1/[m(3m-1)]}$$

Next, Theorem 3.2, which gives the exponential decay of the entropy production, can be translated into a decay result for the difference between the gradients of the solution and of the Barenblatt-Pattle profile. One has only to remark that, whenever x belongs to the support of u_∞,

$$x = -\frac{m}{m-1}\nabla u_\infty^{m-1}. \tag{6.4}$$

Hence

$$\int_{\mathrm{Supp}\{u_\infty(|x|)\}} u(x,t)|\nabla(u^{m-1}(x,t) - u_\infty^{m-1}(x))|^2\,dx \le I(u(t)). \tag{6.5}$$

Applying (2.8), we obtain the following result:

Theorem 6.3. *Assume that $v > 0$ is a solution of (1.1) in $L^1(\mathbb{R}^N)$, corresponding to a initial datum $v_0 > 0$ of bounded mass M and entropy $H(v_0)$, for $m > 1$. Then, if $MV(|x|,t)$ is the Barenblatt-Pattle profile of mass M, there exists a constant $C = C(m, H(v_0))$ such that for all $t \ge t_0 > 0$*

$$\int_{\mathrm{Supp}\{MV(|x|,t)\}} v(x,t)|\nabla(v^{m-1}(x,t) - MV^{m-1}(|x|,t))|^2\,dx \le C\left(1+\frac{t}{k}\right)^{-2}. \tag{6.6}$$

Table 1: Comparison of rates of convergence in1-D. RS is radially symmetric. CS is compact support.

	RSCS	CS $1<m<2$	CS $m>2$	Theorem 6.2
$L^1(\mathbb{R})$	1	$\frac{1}{m+1}$	$\frac{1}{m^2-1}$	$\frac{1}{3m-1}$
$L^\infty(\mathbb{R})$	$\frac{1}{m+1}+\frac{1}{m-1}$	$\frac{2}{m+1}$	$\frac{1}{m+1}+\frac{1}{m^2-1}$	$\frac{1}{m+1}+\frac{1}{m(3m-1)}$

Theorem 6.1 is the first result, as far as we are concerned, about general rates of decay of solutions for the porous medium equation in $N > 1$. Theorem 6.2 improves previous results of Kamin [9], [10], as far as the asymptotic behaviour in $L^\infty(\mathbb{R})$ is concerned. In [1], [21], [22] the rate of convergence in $L^\infty(\mathbb{R})$ has been studied for initial data with compact support. In this case one obtain better rates of convergence but the constants depend on the support of the solution. Also, the special case of radially symmetric compact support solutions is studied and it is proved that the rates obtained in this case are optimal. Again, the constants on the estimates depend on the symmetry hypothesis. From the rate of convergence in $L^\infty(\mathbb{R})$ and taking into account that the support remains compact for all times one obtains directly rates of convergence in $L^1(\mathbb{R})$. The situation is summarized in table 6, in which we write the orders of convergence a in the cases studied in [1], [21], [22] and Theorem 6.2 in the form t^{-a}. Our rates of convergence are worse than the rates for compact support initial data but are valid for non compact support initial data and we recover the same behavior for the heat equation when $m \to 1$. It is interesting to remark that in this case only, the rate of asymptotic decay in $L^1(\mathbb{R}^N)$ towards the similarity solution is independent of the dimension N.

Our analysis extends to cover the case $N/(N+2) < m < 1$ with $m > (N-1)/N$ [4], [14], since in this range of the parameter, one can easily show that the entropy of u_∞ is bounded. Let us remember that if $m < (N-2)/N$ we have the finite extinction phenomena. If $N = 1$ and $-\frac{1}{2} < m \leq \frac{1}{3}$, the Fokker-Planck type equation admits a unique stationary state u_∞, but we can no longer apply our entropy-entropy method in this general form, since now (for example) the energy of the stationary state is unbounded. On the other hand, for perturbations of the steady state such that the relative entropy is bounded, the analysis still works. This shows that when $m < \frac{1}{3}$, the domain of attraction of the stationary solution is given only by the class of functions that have a suitable decay at infinity. Finally, let us remark that the Theorem 6.3 is valid for any nonnegative initial data by an approximation argument [14].

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J. A. CARRILLO
Department of Mathematics
University of Texas at Austin
Austin, Texas, U.S.A.
EMAIL: carri@fireant.ma.utexas.edu

G. TOSCANI
Department of Mathematics,
University of Pavia
Pavia, ITALY.

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