Weighted Hardy inequalities and Hardy transforms of weights

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Abstract

Many problems in analysis are described as weigthed norm inequalities thet have given rise to different classes of weights, such as A_p– weights of Mukenhoupt, B_p–weights of Ario and Mukenhoupt, ect. Our purpose is to show that different classes of weigths are relaterd by mean of composition with classical transforms. Typical exemples are A_p–weights as indefinite integrals of B_{p-1}–weights, and M_p–weights (for which Hardy transform is bounded) as Hardy transforms of B_p–weights. We pay special atention to monotonic weights.

1 Introduction

Throughout this paper we shall use the following notacion. To indicate that T is a bounded operator between X and Y, we write $T : X \longrightarrow Y$. For a given function space X on $R^+ = [0, \infty)$, X^d will denote the set of all non–increasing and nonnegative functions (briefly, decreasing functions) of X. A weight is a non–zero Lebesgue–measurable and nonnegative function on R^+.

In recent years, many problems in Analysis have benn studied in terms of weigthted norm inequalities, which describe the boundedness of some classical transforms, such as Hardy and Maximal operators acting on functions spaces.

These inequalities give reise to several classes of weights. The starting and better kwon are the A_p classes, that for every $p \in (1, \infty)$ contain all weights w such that

$$\sup_I \Big(\frac{1}{|I|}\int_I w(x)\,dx\Big)\Big(\frac{1}{|I|}\int_I w(x)^{1-p'}\,dx\Big)^{p-1} < \infty, \qquad (A_p)$$

where the suppremun is taken over all intervals I and $p' = \frac{p}{p-1}$.

It was shown by Mukenhoupt [Mu1] that $w \in A_p$ if and only if the Hardy–Litllewood maximal function M satisfies

$$M : L^p(w) \longrightarrow L^p(w).$$

We refer to [GR] for the description of these weights, and to [Bu] for their relation with boundedness of classical operators.

In [Mu2], Mukenhoupt also caracterized the weights w such that the Hardy operator

$$S_1 f(t) = \frac{1}{t} \int_0^t f(x)\, dx$$

is bounded on $L^p(w)$ $(1 \leq p < \infty)$ as the weigths of the class M_p, defined by the estimates

$$\sup_{t>0} \Big(\int_t^\infty \frac{w(x)}{x^p}\, dx \Big)^{1/p} \Big(\int_0^t w(x)^{-p'/p}\, dx \Big) 1/p' < \infty \qquad (M_p)$$

if $1 < p < \infty$, and

$$\int_t^\infty \frac{w(x)}{x} \leq C w(t) \qquad (M_1)$$

if $p = 1$.

In the case of the conjugate Hardy operator, $S_2 f(x) = \int_x^\infty f(t)\, \frac{dt}{t}$,

$$S_2 : L^p(w) \longrightarrow L^p(w)$$

if, and only if

$$\sup_{t>0} \Big(\int_0^t w(x)\, dx \Big)^{1/p} \Big(\int_t^\infty \frac{w(x)^{-p'/p}}{x^{p'}}\, dx \Big)^{1/p'} < \infty \qquad (M_p^*)$$

when $1 < p < \infty$, and

$$S_1 w(t) := \frac{1}{t} \int_0^t f(x)\, dx \leq C w(x) \qquad (M_1^*)$$

when $p = 1$. These conditions define the classes M^p of weigths.

New classes were introduced by Ario and Munkenhoupt [AM] when solving the boundedness of the maximal operator of Hardy–Littlewood on Lorentz spaces. They Observed that this leads to study when

$$S_1 : L_p^d \longrightarrow L_p(w)$$

and proved that, for $1 < p < \infty$, this happens if, and only if w satisfies the condition

$$\int_t^\infty \frac{w(x)}{x^p} dx \leq \frac{C}{t^p} \int_0^t w(x)\, dx, \tag{B_p}$$

which defines the calss B_p (for any $p \in (0, \infty)$). If

$$W(x) = \int_0^x w(t)\, dt,$$

Soria [So] has shown that (B_p) is equivalent to

$$\frac{1}{t^p} \int_0^t \frac{x^{p-1}}{W(x)}\, dx \leq C \frac{1}{W(t)} \tag{$So1$}$$

and to

$$\int_t^\infty \frac{W(x)}{x^{p+1}}\, dx \leq C \frac{W(t)}{t^p} \tag{$So2$}$$

We shall give a new proof of this fact.

Weak tye estimates have also been considered. In [AnM] it was proved that

$$S_1 : L_1(w) \longrightarrow L_{1,\infty}(w)$$

if, and only if w lelongs to $M_{1,\infty}$, the class of weights w such taht

$$\int_t^\infty (\frac{t}{x})^\alpha \frac{w(x)}{x}\, dx \leq C(\alpha) \inf_{0 \leq x \leq t} w(x) \tag{$M_{1,\infty}$}$$

Only this case $p = 1$ is interesting, since $S_1 : L_p(w) \longrightarrow L_{p,\infty}(w)$ if, and only if $S_1 : L_p(w) \longrightarrow L_p(w)$ when $p > 1$ ([AnM, Theorem 3]).

The corresponding problem for the restriction of S_1 to decreasing functions was studied in [Ne1]. Again, for $1 < p < \infty$, $S_1 : L_p^d(w) \longrightarrow L_{p,\infty}(w)$ if, and only if $S_1 : L_p^d(w) \longrightarrow L_p(w)$.

If $p = 1$, it is proved in [CGS] that $S_1 : L_1^d(w) \longrightarrow L_{1,\infty}(w)$ if, and only if w belongs tho the class $B_{1,\infty}$ defined by the condition

$$\frac{1}{t} \int_0^t w(x)\, dx \leq C \frac{1}{s} \int_0^s w(x)\, dx \ \text{ if } \ s \leq t \tag{$B_{1,\infty}$}$$

i.e. $S_1 w(t) \leq C S_1 w(s)$ if $s \leq t$.

Remark 1.1 *In the ca¡se of the conjugate Hardy operator, Neugebauer [Ne2] proved that the property*

$$S_2 : L_p^d(w) \longrightarrow L_p(w)$$

doesn't depend on $p \in [1, \infty)$ *and it holds if, and only if*

$$\inf_0^t (S_1 w)(x)\, dx \leq C \inf_0^t w(x)\, dx \qquad (B^*)$$

i.e. $S_1 S_1 \leq C S_1 w$.

In his paper [Ne1], Neugebauer presented somo properties of B_p–weights suggested by the analogous properties of A_p–weights, and gave short proofs of facts such as B_p imply $B_{p-\varepsilon}$.

The purpose of this paper is to show taht differents classes of weights are in fact related by means composition with Hardy transforms and indefinite integrals.

In the section 2 the main result states that A_p–weigths are the indefinitive integral of B_{p-1}–weights and this fact is used to give easy proofs of some results, such as the above mentined property B_p imply $B_{p-\varepsilon}$, froma the properties of A_p.

Section 3 is mainly devoted to describe M_p as the Hardy transform of B_p, and also to see that $M^1 = S_1(B^1)$ and $M^{1,\infty} = S_1(B^{1,\infty})$.

In the brief section 4 we apply the above results to see that weights such that Calderón's operatos $S = S_1 + S_2$ is L_p–bounded are the S_1–images of weigths w such that $S : L_p^d(w) \longrightarrow L_p(w)$.

Finally, section 5 is devoted to decribe similar porperties of the special case of monotonic weights, and to give increase and decrease criteria for these weights belong to differents classes.

Cruz–Uribe's work [CU] is an important reference for this secction.

2 B_p–weigths as derivatives of increasing A_{p+1}–weigths

Theorem 2.1 *Let* w *be a weight on* R^+ *ans* $0 < p\infty$*, Then* $w \in B^p$ *if, and only if* $W \in A_{p+1}$

Proof As remarked in [So], it easily seen that $w \in B_p$ iff

$$\int_t^\infty \frac{W(x)}{x^{p+1}}\,dx \le C\frac{W(t)}{t^p} \qquad (So1)$$

since $w \in B_p$ iff

$$\int_t^s \frac{w(x)}{x^{p+1}}\,dx \le C\frac{W(t)}{t^p} \quad (s > t) \qquad (So11)$$

with

$$\int_t^s \frac{w(x)}{x^p}\,dx = \frac{W(s)}{s} - \frac{W(t)}{t} + p\int_t^s \frac{W(x)}{x^{p+1}}\,dx,$$

and $(So11)$ is equivalent to $(So1)$. We observe that it follows from $(So1)$ that, for the increasing weight W,

$$\frac{W(s)}{s^p} \le C\frac{W(t)}{t^p} \quad (s > t) \qquad (So12)$$

It is Known (cf [CU, Corollary 6.3]) that for an increasing weight W, $(So1)$ holds iff $W \in A_{p+1}$. □

As apllications we otain very easy proofs of two know important properties of B_p weights from properties of A_p weights.

Corollary 2.1 *If $w \in B_p$ $(0 < p < \infty)$, there exists $\varepsilon \in (0, p)$ such that $w \in B_{p-\varepsilon}$*

Proof Let $\varepsilon \in (0, p)$ such that $W \in A_{p+1-\varepsilon}$ (cf [GR]). From Theorem 2.1 we obtain that $w \in B_{p-\varepsilon}$ □

Remark 2.1 *From this propertiy it is easily proved, as in [Ne1; Theorem 6.5]) that, if $w \in B_p$ and $\alpha > 0$, $W^\alpha \in B_{p\alpha+1}$, $W^\alpha(t) = \alpha\int_0^t W^{\alpha-1}w$ and $W^{\alpha-1}w \in B_{p\alpha}$ $(Ne*)$*

Corollary 2.2 *Let $p \in (0,\infty)$ and let w be a weight on R^+. Then $w \in B_p$ if, and only if*

$$\int_0^t \frac{dx}{W(x)^{1/p}} \simeq \frac{t}{W(x)^{1/p}} \qquad (So3)$$

Proof (Compare with the proof of the equivalence of (i) and (ii) in [So, Theorem 2.5], where Sagher equivalence and a type condition p imply $p-\varepsilon$ is used).

It is kwonn (See [GR]) that $\omega in A_q$ iff $w^{1-q'} \in A_{q'}$ (for $1 \leq q < \infty$). Thus, in our case, $W \in A_{p+1}$ iff $W^{1-(p+1)'} \in A_{(p+1)'}$, which means that $W^{-1/p} \in A_{1+1/p}$.

But for a decreasing weight, $w \in A_q$ $(1 < q < \infty)$ iff

$$\sup_{t>0} \Big(\int_0^t w(x)\, dx \Big) \Big(\int_t^\infty \frac{w(x)^{1-q'}}{x^{q'}}\, dx \Big) q - 1 < \infty$$

(Cf [CU, Theorem 6.1]), which is the same (M_p^*) so $S_2 : L_q(w) \longrightarrow L_q(w)$, this means that

$$S_2 : L_{1+1/p}(W^{-1/p}) \longrightarrow L_{1+1/p}(W^{-1/p})$$

and we know (Cf [CM]) that this property is equivalent to $(So3)$ □

Remark 2.2 bis and Corollary 2.3 allow to improve Theorem 2.1:

Preposition 2.1 *Let* $0 < p < \infty$ *and* $0 < \alpha < \infty$*, Then* $w \in B_p$ *if, and only if* $W^\alpha \in A_{p\alpha+1}$

Proof We may assume $\alpha \neq 1$.

If $w \in B_p$, it follows from $(Ne*($ and Theorem 2.1 that $W^\alpha \in A_{p\alpha+1}$.

Conversly, if $W^\alpha \in A_{p\alpha+1}$, we use [CU; Theorem 6.1] that gives for this increasing weight the estimate

$$\Big(\int_t^\infty \frac{W(x)^\alpha}{t^{\alpha p+1}}\, dx \Big) 1/(\alpha p + 1) \Big(\int_0^t (W(x)^\alpha)^{-1/\alpha p}\, dx \Big) \alpha p/(\alpha p + 1) \leq C$$

with

$$\frac{W(t)^\alpha}{\alpha p t^{\alpha p}} \leq \int_t^\infty \frac{W(x)^\alpha}{t^{\alpha p+1}}\, dx,$$

and we obtain the B_p condition

$$\int_0^t \frac{dx}{W(x)^{1/p}} \simeq \frac{t}{W(t)^{1/p}}$$

□

Another application of Theorem 2.1 is the following caracterization of the class $B_\infty = \cup_{p>0} B_p$ through class Δ_2.

Corollary 2.3 *The weight w belongs to B_∞ if, and only if $W \in \Delta_2$ i.e., $W(2t) \leq CW(t)$ for some constant $C > 1$.*

Proof If $w \in B_p$, then $w \in A_{p+1}$, and it belongs to Δ_2 and, conversely, if the increasing weight W is in Δ_2, then $W \in A_q$ for some $q > 1$ (see [CU, corollary 4.4 and Theorem 3.3]) □

3 M_p–weigths as Hardy trnaforms of B_p–weigths

Theorem 3.1 *If $1 \leq p < \infty$, $M_p = S_1(B_p)$. I.e.*

$$S_1 : L_p^d(w) \longrightarrow L_p(w) \quad \textit{iff} \quad S_1 : L_p(S_1 w) \longrightarrow L_p(S_1 w).$$

Proof First assume $S_1 w \in M_p$, i.e. $S_1 : L_p(S_1 w) \longrightarrow L_p(S_1 w)$. In the case $1 < p < \infty$, the weight

$$w_1(t)(S_1 w)(t) = \frac{W(t)}{t}$$

satisfies

$$\Big(\int_t^\infty \frac{w_1(x)}{x^p}\,dx\Big)^{1/p}\Big(\int_0^t w(x)_1^{-p'/p}\,dx\Big)1/p' \leq C$$

and, W beeing increasing,

$$\int_0^t w_1^{-p/p'}(x)\,dx = \int_0^t \Big(\frac{x}{(W(x))^{p/p'}}\,dx \geq \frac{1}{p'}\frac{t^{p'}}{W(t)^{p/p'}}.$$

Thus

$$\Big(\int_t^\infty \frac{W(x)}{x^{p+1}}\,dx\Big)^{1/p}\Big(\frac{t^{p'}}{W(t)^{p/p'}}\Big)^{1/p'} \leq p'^{1/p'}C$$

and from $(So1)$ we obtain $w \in B_p$.

In the case $p = 1$, the condition $w_1 = S_1 w \in M_1$ means that

$$S_2 w_1 \leq C w_1$$

and then $S_1 S_2 w_1 \leq C S_1 w$. Since $S_2 w$ is decreasing, $S_2 w \leq S_1 S_2 w_1 \leq C S_1 w$ and $w \in B_1$.

Suppose now that $w \in B_p$. If $p = 1$, this means that $S_2 w \leq C S_1 w$, hence $S_2 S_1 w = S_2 w + S_1 w \leq (C+1) S_1 w$ and $S_1 w \in M_1$.

In the case $1 < p < \infty$, to prove that $w_1 = S_1 w$ satisfies

$$I_1.I_2 := \Big(\int_t^\infty \frac{w_1(x)}{x^p}\,dx)^{1/p}.\Big(\int_0^t w_1(x)^{-p'/p}\,dx\Big)^{1/p'} \leq C \tag{1}$$

we observe that it follows from $(So1)$, for the first factor we have

$$I_1 := \Big(\int_t^\infty \frac{w_1(x)}{x^p}\,dx)^{1/p} \leq C\frac{W(t)^{1/p}}{t}.$$

On the other hand, we apply $(Ne*)$ with $\alpha = p'/p = p' - 1$ to the weight $\widetilde{w} := W^{\alpha-1} w \in B_{p'}$ to obtain

$$I_2^{p'} := \int_0^t \frac{x^{p'-1}}{W(x)^\alpha(x)}\,dx \simeq \int_0^t \frac{x^{p'-1}}{\int_0^x \widetilde{w(s)}\,ds}\,dx = \int_0^t \frac{x^{p'-1}}{\widetilde{W(x)}} \leq C'\frac{t^{p'}}{\widetilde{W(t)}} = C'\frac{t^{p'}}{W^\alpha(t)}$$

Thus, $I_1 I_2 \leq CC'$ gives 1 □

A similar result holds for the weak type Hardy inequalities:

Theorem 3.2 *$M_{1,\infty} = S_1(B_{1,\infty})$, i.e.,*

$$S_1 : L_1^d(w) \longrightarrow L_{1,\infty}(w) \quad \textit{iff} \quad S_1 : L_1(S_1 w) \longrightarrow L_{1,\infty}(S_1 w).$$

Proof If $w \in B_{1,\infty}$, from

$$(S_1 w)(x) \leq C(S_1 w)(t) \quad (t < x)$$

we obtain

$$\int_t^\infty \frac{t}{x}(S_1 w)(x)\,\frac{dx}{x} \leq C(S_1 w)(t)\int_t^\infty \frac{t}{x^2}\,dx = C(S_1 w)(t)$$

and S_1 satisfies $(M_{1,\infty})$ with $\alpha = 1$.

Assume now $S_1 w \in M_{1,\infty}$. Then, as in [AnM; proof of Theorem 2],

$$\frac{1}{y}\int_t^y (S_1 w)(x)\,dx \leq \|S_1\| \inf_{0\leq s\leq t} (S_1 w)(s) \quad (0 < t < y),$$

and for $y = 2t$ we obtain

$$\frac{1}{2t}\int_t^{2t} (S_1 w)(x)\,dx \geq \frac{1}{2t}\int_t^{2t} (\frac{1}{x}\int^0 tw(s)\,ds)\,dx = \frac{\ln 2}{2}(S_1 w)(t).$$

Hence

$$(S_1w)(t) \leq \inf_{0\leq s\leq t}(S_1w)(s)$$

and $w \in B_{1,\infty}$. □

As we have recalled in $(B*)$, for operator S_2 we only need to consider the case $p = 1$.

Theorem 3.3 $M^1 = S_1(B^1)$, *i.e.*

$$S_2 : L_1^d(w) \longrightarrow L_1(w) \quad \textit{iff} \quad S_2 : L_1(S_1w) \longrightarrow L_1(S_1w).$$

Proof If $w \in B^1$ and in

$$\int_0^\infty f(x)(S_1w))(x)\,dx = \int_0^\infty (S_2f)(x)w(x)\,dx \leq C\int_0^\infty f(x)w(x)\,dx$$

(f decreasing) we take $f = \chi_[0,t]$, se obtain

$$\int_0^t (S_1w))(x)\,dx \leq C\int_0^t w(x)\,dx,$$

which means that $S_1w \in M^1$.

Conversly, if $S_1w \in M^1$,

$$\int_0^\infty f(x)(S_1S_1w))(x)\,dx = \int_0^\infty (S_2f)(x)(S_1w))(x)\,dx \leq C\int_0^\infty f(x)(S_1w))(x)\,dx$$

when $f \geq 0$, and then $S_1S_1w \leq CS_1w$, i.e. $w \in B^1$. □

Corollary 3.1 *If $S_2 : L_{p_0}^d(w) \longrightarrow L_{p_0}(w)$ for some $p_0 \in [1,\infty)$, then $S_2 : L_p(S_1w) \longrightarrow Lp(S_1w)$, for any $p \in [1,\infty)$.*

Proof Since $S_2 : L_1^d(w) \longrightarrow L_1(w)$ (Remark 1.1), $S_1w \in M^1$ and also $S_2 : L_p(S_1w) \longrightarrow L_p(S_1w)$ (cf. [BMR, Proposition 2.9 ii]). □

If $w \in \Delta_2$, we obtain a converse of corollary 3.4.

Preposition 3.1 *If $w \in \Delta_2$, and $S_2 : L_p(S_1w) \longrightarrow L_p(S_1w)$, for some $p \in [1,\infty)$, then $S_2 : L_q^d(w) \longrightarrow L_q(w)$ for any $q \in [1,\infty)$.*

Proof Let $1 < p\infty$ and $S_2 : L_p(S_1 w) \longrightarrow L_p(S_1 w)$ (the case $p = 1$ is contained in Theorem 3.3). Then

$$\Big(\int_0^t (S_1 w)(x)\,dx\Big)^{1/p}\Big(\int_t^\infty \frac{(S_1 w)(x))^{-p'/p}}{x^{p'}}\,dx\Big)^{1/p'} \le C$$

and in our case

$$\int_t^{2t} \frac{(S_1 w)(x))^{-p'/p}}{x^{p'}}\,dx = \Big(\int_t^{2t} \frac{W(x)^{-p'/p}}{x}\,dx \ge \ln 2 W(2t)^{-p'/p}.$$

Thus

$$\Big(\int_0^t (S_1 w)(x)\,dx\Big)^{1/p} \le \frac{C}{\Big(\int_t^\infty \frac{W(x)^{-p'/p}}{x}\,dx\Big)^{1/p}} \le \frac{C}{\Big(\int_t^{2t} \frac{W(x)^{-p'/p}}{x}\,dx\Big)^{1/p}} \le \frac{C}{(\ln 2)^{1/p'} W(2t)^{-1/p}}$$

and if follows from condition $W \in \Delta_2$ that

$$\int_0^t (S_1 w)(x)\,dx \le C' W(t),$$

which is property $(B*)$ □

Remark 3.1 *It is easy to obtain examples of weights w such that $S_2 : L_p(S_1 w) \longrightarrow L_p(S_1 w)$ but $S_2 : L_p^d(w) \not\longrightarrow L_p(w)$, i.e., $S_2 : L_q^d(w) \not\longrightarrow L_q(w)$ for some $q \in [1, \infty)$. It cannot be decreasing (See [CM]).*

4 Calderón weights

Another classical operator, which plays an important role in interpolation theory (cf [BRM]) is the Calderón operator

$$S = S_1 + S_2.$$

For $1 \le p\infty$, we define $C_p : 0M_p \cup M^p$, i.e. $w \in C_p$ means that

$$S = S_1 + S_2 = S_1 \circ S_2 :\ L_p(w) \longrightarrow L_p(w).$$

Simylarly, $C_p^d := B_p \cup B^p$, and $w \in C_p^d$ iff

$$S : L_p^d(w) \longrightarrow L_p(w).$$

If $w \in B_p$ it is known that $W \in \Delta_2$.

Theorem 4.1 *$C_p = S_1(C_p^d)$ for any $p \in [1, \infty)$.*

Proof If $S_1,\, S_2\,:\, L_p^d(w) \longrightarrow L_p(w)$, it follows from Theorem 3.1 and Corollary 3.4 that $S_1 w \in C_p$.

Conversely, if $S_1,\, S_2 : L_p(S_1 w) \longrightarrow L_p(S_1 w)$, it follows from Theorem 3.1 that $w \in B_p$ and by Corollary 2.5 $W \in \Delta_2$, we may apply Proposition 3.5 and also $w \in B^p$ □

5 Monotonic weights

Increasing and decreasing weights are important in apllications and easy to work with.

For example, monotonic weight w is an A_∞–weight ($A_\infty = \cup_{p>1} A_p$) iff it is doubling, i.e. $\omega(2I) \leq C\omega(I)$ if we denote $\omega(E) = \int_E w(x)\,dx$ and $2I = (c-2r, c+2r)$ for $I = (c-r, c+r)$. We refer to [CU] for a description of these weights.

If w is increasing, it is doubling iff $S_1 w \simeq w$ ([CU, Theorem 3.8]) and then, for $1 < p < \infty$, the following properties are equivalent:

(i) $w \in A_p$

(ii) $w \in M_p$

(iii) $\int_t^\infty x^{-p} w(x)\,dx \leq C t^{1-p} w(t)$ (See [CU, Theorem 6.1 and Corollary 6.3])

This properties are also equivalent to

(iv) $w \in B_p$

since in the case of an increasing weight, if $w \in B_p$

$$\int_t^\infty x^{-p} w(x)\,dx \leq \frac{C}{t^p} \int_0^t w(x)\,dx \leq \frac{Cw(t)}{t^{p-1}}$$

and we obtain (iii).

For this weights, it follows from $S_1 : L_p(w) \longrightarrow L_p(w)$ that

$$S_1 w \simeq w$$

since $w \in A_\infty$ is doubling and increasing (Cf [CU, Theorem 3.8])