Tests of Large N_c QCD from Hadronic τ Decay

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(Received 28 July 2000)

We use the ALEPH Collaboration data on vector and axial-vector spectral functions to test simple duality properties of QCD in the large N_c limit, which emerge in the approximation of a *minimal hadronic ansatz* of a spectrum of narrow states. These duality properties relate the short- and long-distance behaviors of specific correlation functions, which are order parameters of spontaneous chiral symmetry breaking, in a way that we find well supported by the data.

DOI: 10.1103/PhysRevLett.86.14 PACS numbers: 12.38.Aw, 11.15.Pg, 12.20.Fv

At first sight, the *hadronic world* predicted by QCD in the limit of a large number of colors N_c [1] may seem rather different from the real world. The hadronic spectrum of vector and axial-vector states, observed, e.g., in e^+e^- annihilations and in τ decays, has certainly much more structure than the infinite set of narrow states predicted by large N_c QCD [2] (QCD_∞). There are, however, many instances in particle physics where one is interested only in certain weighted integrals of hadronic spectral functions. In these cases, it may be enough to know a few *global* properties of the hadronic spectrum; one does not expect the integrals to depend crucially on the details of the spectrum at all energies. Typical examples of that are the coupling constants of the effective chiral Lagrangian of QCD at low energies, as well as the coupling constants of the effective chiral Lagrangian of the electroweak interactions of pseudoscalar particles in the standard model, which are needed to understand K physics in particular (see, e.g., the review article in Ref. [3] and references therein). It is in these examples that the *hadronic world* predicted by QCD_∞ may provide a good approximation to the real hadronic spectrum. If so, QCD_∞ could then become a useful phenomenological approach for understanding nonperturbative QCD physics at low energies.

There are indeed a number of successful calculations which have already been made within the framework of QCD_∞ (see Ref. [4] and references therein). The picture that emerges from these applications is one of remarkable simplicity. It is found that, when dealing with Green's functions that are *order parameters* of spontaneous chiral symmetry breaking, the restriction of the infinite set of large N_c narrow states to a *minimal hadronic ansatz,* which is needed to satisfy the leading short- and long-distance behaviors of the relevant Green's functions, provides already a very good approximation to the observables one computes. The purpose of this Letter is to investigate this minimal hadronic ansatz approximation in a case where one can compare, in detail, the theoretical predictions to the phenomenological results evaluated with experimental data.

Of particular interest for our purposes is the correlation function ($Q^2 \equiv -q^2 \geq 0$ for q^2 spacelike)

$$\Pi_{LR}^{\mu\nu}(q) = 2i \int d^4x\, e^{iqx} \langle 0|T[L^\mu(x) R^\nu(0)^\dagger]|0\rangle\,, \qquad (1)$$

with color singlet currents

$$R^\mu(L^\mu) = \overline{d}(x)\gamma^\mu \tfrac{1}{2}(1 \pm \gamma_5) u(x)\,. \qquad (2)$$

In the chiral limit, $m_{u,d,s} \to 0$, this correlation function has only a transverse component,

$$\Pi_{LR}^{\mu\nu}(Q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi_{LR}(Q^2)\,. \qquad (3)$$

The self-energy-like function $\Pi_{LR}(Q^2)$ vanishes order by order in perturbative QCD (pQCD) and is an order parameter of spontaneous chiral symmetry breaking for all values of Q^2; therefore, it obeys an unsubtracted dispersion relation,

$$\Pi_{LR}(Q^2) = \int_0^\infty dt\, \frac{1}{t + Q^2} \frac{1}{\pi} \mathrm{Im}\Pi_{LR}(t)\,. \qquad (4)$$

In QCD_∞ the spectral function $\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t)$ consists of the difference between an infinite number of narrow vector and axial-vector states, together with the Goldstone pole of the pion:

$$\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = \sum_V f_V^2 M_V^2 \delta(t - M_V^2) - F_0^2 \delta(t) - \sum_A f_A^2 M_A^2 \delta(t - M_A^2)\,. \qquad (5)$$

The low Q^2 behavior of $\Pi_{LR}(Q^2)$, i.e., the long-distance behavior of the correlation function in Eq. (1), is governed by the chiral perturbation theory:

$$-Q^2 \Pi_{LR}(Q^2)|_{Q^2\to 0} = F_0^2 + 4L_{10}Q^2 + \mathcal{O}(Q^4)\,, \qquad (6)$$

where F_0 is the pion coupling constant in the chiral limit, and L_{10} is one of the coupling constants of the $\mathcal{O}(p^4)$ effective chiral Lagrangian. The high Q^2 behavior of $\Pi_{LR}(Q^2)$, i.e., the short-distance behavior of the correlation function in Eq. (1), is governed by the operator product expansion (OPE) of the two local currents in Eq. (1) [5],

$$\lim_{Q^2\to\infty} Q^6\Pi_{LR}(Q^2) = \left[-4\pi^2\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right]\langle\bar\psi\psi\rangle^2, \quad (7)$$

which implies the two Weinberg sum rules

$$\int_0^\infty dt\,\mathrm{Im}\Pi_{LR}(t) = \sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 - F_0^2 = 0, \quad (8)$$

$$\int_0^\infty dt\,t\,\mathrm{Im}\Pi_{LR}(t) = \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0, \quad (9)$$

as well as the sum rule [6]

$$\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 = [-4\pi\alpha_s + \mathcal{O}(\alpha_s^2)]\langle\bar\psi\psi\rangle^2. \quad (10)$$

In fact, as pointed out in Ref. [6], in QCD_∞ there exist an infinite number of Weinberg-like sum rules. In full generality, the moments of the Π_{LR} spectral function with $n = 3, 4, \ldots,$

$$\int_0^\infty dt\,t^{n-1}\left[\frac{1}{\pi}\mathrm{Im}\Pi_V(t) - \frac{1}{\pi}\mathrm{Im}\Pi_A(t)\right] = \sum_V f_V^2 M_V^{2n} - \sum_A f_A^2 M_A^{2n}, \quad (11)$$

govern the short-distance expansion of the $\Pi_{LR}(Q^2)$ function,

$$\Pi_{LR}(Q^2)|_{Q^2\to\infty} = \left(\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6\right)\frac{1}{Q^6} + \left(\sum_V f_V^2 M_V^8 - \sum_A f_A^2 M_A^8\right)\frac{1}{Q^8} + \cdots. \quad (12)$$

On the other hand, inverse moments of the Π_{LR} spectral function with the pion pole removed [which we denote by $\mathrm{Im}\tilde\Pi_A(t)$] determine a class of coupling constants of the low-energy effective chiral Lagrangian. For example,

$$\int_0^\infty dt\,\frac{1}{t}\left[\frac{1}{\pi}\mathrm{Im}\Pi_V(t) - \frac{1}{\pi}\mathrm{Im}\tilde\Pi_A(t)\right] = \sum_V f_V^2 - \sum_A f_A^2 = -4L_{10}. \quad (13)$$

Moments with higher inverse powers of t are associated with couplings of composite operators of higher dimension in the chiral Lagrangian. Tests of the two Weinberg sum rules in Eqs. (8) and (9) and of the L_{10} sum rule in Eq. (13), in a different context from the one we are interested in here, have often appeared in the literature (see, e.g., Refs. [7] and [8] for recent discussions where earlier references can also be found).

The minimal hadronic ansatz which satisfies the two Weinberg sum rules in Eqs. (8) and (9) is a spectrum of one vector state V, one axial-vector state A, and the Goldstone pion, with the ordering [6] $M_V < M_A$. In this approximation, $\Pi_{LR}(Q^2)$ has a very simple form

$$-Q^2\Pi_{LR}(Q^2) = \frac{F_0^2}{(1+\frac{Q^2}{M_V^2})(1+\frac{Q^2}{M_A^2})} \quad (14)$$

$$= \frac{M_A^2 M_V^2}{Q^4}\frac{F_0^2}{(1+\frac{M_V^2}{Q^2})(1+\frac{M_A^2}{Q^2})}. \quad (15)$$

This equation shows, explicitly, a remarkable short-distance $\rightleftharpoons$ long-distance duality [9]. Indeed, with g_A defined so that $M_V^2 = g_A M_A^2$ and $z \equiv \frac{Q^2}{M_V^2}$, the nonlocal order parameters corresponding to the long-distance expansion for $z \to 0$, which are couplings of the effective chiral Lagrangian, i.e.,

$$-Q^2\Pi_{LR}(Q^2)|_{z\to 0} = F_0^2\{1 - (1+g_A)z + (1 + g_A + g_A^2)z^2 + \cdots\}, \quad (16)$$

are correlated to the local order parameters of the short-distance OPE for $z \to \infty$ in a very simple way:

$$-Q^2\Pi_{LR}(Q^2)|_{z\to\infty} = F_0^2\frac{1}{g_A}\frac{1}{z^2}\left\{1 - \left(1+\frac{1}{g_A}\right)\frac{1}{z} + \left(1 + \frac{1}{g_A} + \frac{1}{g_A^2}\right)\frac{1}{z^2} + \cdots\right\}; \quad (17)$$

in other words, there is a one-to-one correspondence between the two expansions by changing

$$g_A \rightleftharpoons \frac{1}{g_A} \quad \text{and} \quad z^n \rightleftharpoons \frac{1}{g_A}\frac{1}{z^{n+2}}. \quad (18)$$

The moments of the Π_{LR} spectral function, when evaluated in the minimal hadronic ansatz approximation, can be converted into a very simple set of finite energy sum rules (FESR's), corresponding to the OPE in Eq. (17):

$$\mathcal{M}_2 = \int_0^{s_0} dt\,t^2\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = -F_0^2 M_V^4\frac{1}{g_A}, \quad (19)$$

$$\mathcal{M}_3 = \int_0^{s_0} dt\,t^3\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = -F_0^2 M_V^6\frac{1+\frac{1}{g_A}}{g_A}, \quad (20)$$

$$\mathcal{M}_4 = \int_0^{s_0} dt\,t^4\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = -F_0^2 M_V^8\frac{1+\frac{1}{g_A}+\frac{1}{g_A^2}}{g_A}, \quad \cdots \quad \cdots, \quad (21)$$

where the upper limit of integration s_0 denotes the onset of the pQCD continuum which, in the chiral limit, is common to the vector and axial-vector spectral functions. It is important to realize that s_0 is not a free parameter. Its value is fixed by the requirement that the OPE of the correlation function of two vector currents (or two axial-vector currents) in the chiral limit have no $1/Q^2$ term, which results in an implicit equation for s_0 [10,11]. In the minimal hadronic ansatz approximation the onset of the pQCD continuum, which we call s_0^*, is then fixed by the equation

$$\frac{N_c}{16\pi^2}\,\frac{2}{3}\,s_0^*[1+\mathcal{O}(\alpha_s)] = F_0^2\frac{1}{1-g_A}\,. \qquad (22)$$

Also, the moments which correspond to the chiral expansion in Eq. (16) are given by another simple set of FESR's:

$$\mathcal{M}_0 = \int_0^{s_0} dt\,\frac{1}{\pi}\,\mathrm{Im}\tilde{\Pi}_{LR}(t) = F_0^2, \qquad (23)$$

$$\mathcal{M}_{-1} = \int_0^{s_0}\frac{dt}{t}\,\frac{1}{\pi}\,\mathrm{Im}\tilde{\Pi}_{LR}(t) = \frac{F_0^2}{M_V^2}(1+g_A), \qquad (24)$$

$$\mathcal{M}_{-2} = \int_0^{s_0}\frac{dt}{t^2}\,\frac{1}{\pi}\,\mathrm{Im}\tilde{\Pi}_{LR}(t) = \frac{F_0^2}{M_V^4}(1+g_A+g_A^2),$$
$$\cdots \qquad\qquad \cdots. \qquad (25)$$

We propose to test these duality relations by comparing moments of the *physical* spectral function $\frac{1}{\pi}\,\mathrm{Im}\Pi_{LR}(t)$ to the predictions of the minimal hadronic ansatz.

The ALEPH Collaboration at LEP has measured the inclusive invariant mass-squared distribution of hadronic τ decays [12] into nonstrange particles. They have been able to extract from their data both the vector current spectral function $\frac{1}{\pi}\,\mathrm{Im}\Pi_V^{\mathrm{exp}}(t)$ and the axial-vector current spectral function $\frac{1}{\pi}\,\mathrm{Im}\Pi_A^{\mathrm{exp}}(t)$ up to $t \simeq 3\ \mathrm{GeV}^2$. In fact, in the real world, the correlation function in Eq. (3) has a nontransverse term as well, which is dominated by the pion pole contribution to the axial-vector component. The vector contribution to the nontransverse term vanishes in the limit of isospin invariance.

In order to compare the moments of the experimental spectral function $\frac{1}{\pi}\,\mathrm{Im}\Pi_{LR}^{\mathrm{exp}}(t)$ to those in Eqs. (19)–(21) and (23)–(25) we still have to correct for the fact that the FESR's in these equations apply in the chiral limit where $m_{u,d} \to 0$. This we do by exploiting the analyticity properties of the two-point function Π_{LR} in the complex q^2 plane. Integration over a standard contour relates weighted integrals of the spectral function $\frac{1}{\pi}\,\mathrm{Im}\Pi_{LR}^{\mathrm{exp}}(t)$ in a finite interval on the real axis to integrals of $\Pi_{LR}(q^2)$ over a *small* circle $|q^2| = s_{\mathrm{th}}$ and a *large* circle $|q^2| = s_0$:

$$\int_{s_{\mathrm{th}}}^{s_0} dt\, f(t)\,\mathrm{Im}\Pi_{LR}(t) = \frac{1}{2i}\oint_{|q^2|=s_{\mathrm{th}}} dq^2\, f(q^2)\Pi_{LR}(q^2) - \frac{1}{2i}\oint_{|q^2|=s_0} dq^2\, f(q^2)\Pi_{LR}(q^2)\,, \qquad (26)$$

where the weight function $f(q^2)$ is a conveniently chosen analytic function inside the contour, in our case, simple powers and inverse powers of q^2. The chiral corrections in the *small* circle are particularly important in the evaluation of the inverse moments. We have evaluated them by taking into account the one loop expression of $\Pi_{LR}(z)$ in chiral perturbation theory [13]. The chiral corrections in the *large* circle are rather small. They appear as leading $1/Q^2$ and next-to-leading $1/Q^4$ power corrections in the OPE of $\Pi_{LR}(Q^2)$ at large Q^2, but their coefficients, proportional to quark masses, are small [14]. With these corrections incorporated, we proceed now to the comparison we are looking for. This is shown in Figs. 1 and 2 below where we show the various moments as a function of s_0. The six plots in Figs. 1 and 2 show the experimental moments on the left hand side of Eqs. (19)–(21) and Eqs. (23)–(25), respectively, as a function of s_0, extrapolated to the chiral limit as discussed above and normalized to the corresponding minimal hadronic ansatz predictions on the right hand side (rhs).

The horizontal bands on these plots correspond to the induced error of the minimal hadronic ansatz predictions from the input values: $F_0 = 87 \pm 3.5$ MeV, $M_V = 748 \pm 29$ MeV, and $g_A = 0.50 \pm 0.06$. These are the values favored by a global fit of the minimum

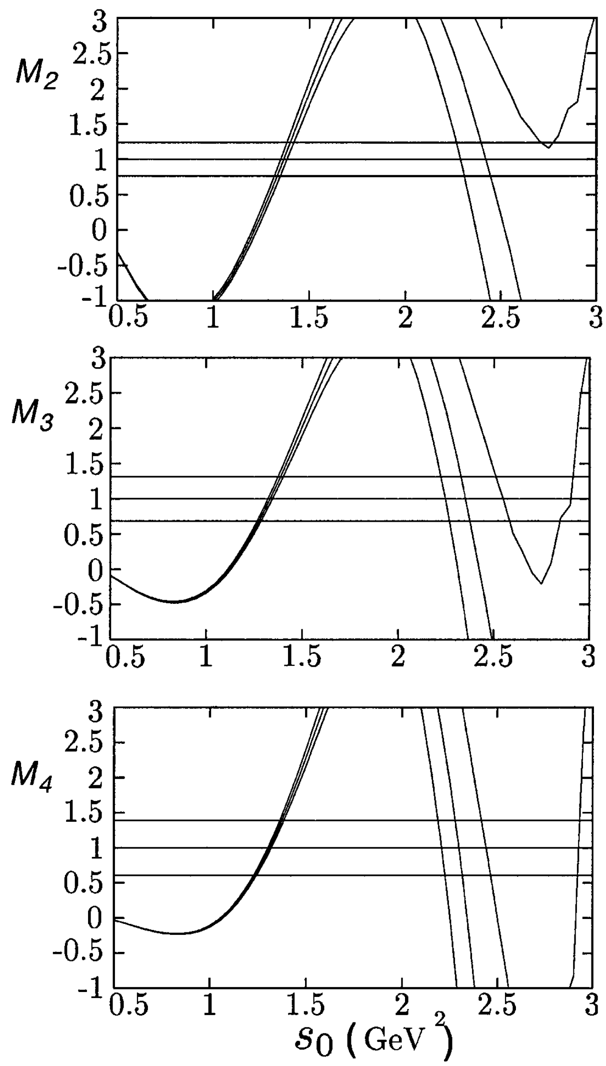

FIG. 1. Plot of the experimental moments in Eqs. (19), (20), and (21) normalized to the minimal hadronic ansatz predictions on the rhs.

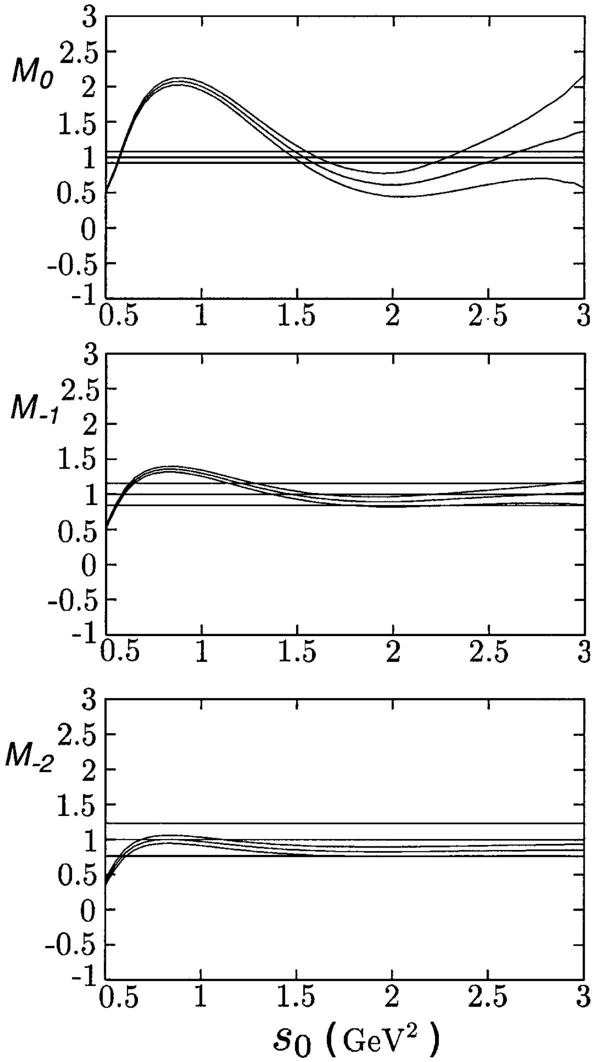

FIG. 2. Plot of the experimental moments in Eqs. (23), (24), and (25) normalized to the minimal hadronic ansatz predictions on the rhs.

hadronic ansatz to low-energy observables [11]. The moments $\mathcal{M}_n$, with the experimental error propagation included, are the curved bands in the figures.

The remarkable feature which the curves in Figs. 1 and 2 show is that, within errors, there is a crossing of *all* the experimental moments with the minimal hadronic ansatz band that takes place in the *same* s_0 region, around $s_0 \sim 1.4$ GeV^2 rather close indeed to the s_0^* value which follows from Eq. (22): $s_0^* = (1.2 \pm 0.2)$ GeV^2 [15]. We have also checked that for the second Weinberg sum rule in Eq. (9), not shown in the figures. In fact, the agreement for the inverse moments is excellent. This is due to the fact that inverse moments put more and more weight on the low-energy tail of the spectral function, which is known to be dominated by the ρ resonance [16]. By contrast, the positive moments are very sensitive to the cancellations between opposite parity hadronic states; this is why the experimental curves show larger and larger oscillations as one increases the power of the moment. In spite of that, it is quite impressive that, when restricted to the s_0 region of duality, the experimental moments agree well with the minimal hadronic ansatz prediction, even for rather large powers which correspond to vacuum expectation values of operators of higher and higher dimension in the OPE. It would be interesting to see how this would affect current determinations of these condensates.

We conclude that the experimental data from ALEPH is consistent with the simple pattern of duality properties between short and long distances which follow from the minimal hadronic ansatz of a narrow vector and axial-vector states plus the Goldstone pion in large-N_c QCD.

This work is supported in part by TMR, EC Contract No. ERBFMRX-CT980169 (EuroDaφne). The work of S. P. is also supported by CICYT-AEN99-0766.

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