MAXIMAL DOMAIN OF PREFERENCES

IN THE DIVISION PROBLEM*

by

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Abstract: The division problem consists of allocating an amount of a perfectly divisible good among a group of n agents. Sprumont (1991) showed that if agents have single-peaked preferences over their shares, the uniform allocation rule is the unique strategy-proof, efficient, and anonymous rule. We identify the maximal set of preferences, containing the set of single-peaked preferences, under which there exists at least one rule satisfying the properties of strategy-proofness, efficiency, and strong symmetry. In addition, we show that our characterization implies a slightly weaker version of Ching and Serizawa's (1998) result.

1. Introduction

The division problem consists of allocating an amount M of a perfectly divisible good among a group of n agents. A rule maps preference profiles into n shares of the amount M. Sprumont (1991) shows that, given M, if agents have single-peaked preferences over their shares, the "uniform" allocation rule is the unique strategy-proof, efficient, and anonymous rule. This is a nice example of a large literature that, by restricting the domain of preferences, investigates the possibility of designing strategy-proof rules.[1] Moreover, in this case, single-peakedness does not only allow strategy-proof rules but also efficient ones.

In this paper we ask how much the set of single-peaked preferences can be enlarged to still allow for rules satisfying interesting properties. In particular, we show that there is a unique maximal domain of preferences that includes the set of single-peaked preferences for which there exists at least one rule satisfying strategy-proofness, efficiency, and strong symmetry. Moreover, we characterize it as the set of feebly single-plateaued preferences.

It turns out that this maximal domain depends crucially on both M and n. Indeed the egalitarian share M/n plays, as a consequence of the strong symmetry requirement, a fundamental role in its description. In particular, our domain

includes only preferences whose set of best shares is an interval and which are weakly monotonic on an interval Θ defined by the relative position of M/n and the set of best shares. Our set departs from the single-peaked domain in two significant directions. First, shares outside Θ can be ordered freely. Second, special intervals of indifference are allowed on Θ. The set of these preferences, given M and n, is much larger than the single-plateaued domain studied by Moulin (1984) and Berga (1998) in a public good context, since single-plateaued preferences are strictly monotonic on both sides of the plateau. We do not claim that the domain identified here has economic relevance; rather, we understand our result as giving a precise and definite answer to an interesting and economically relevant question.

Furthermore, the intersection of all of our maximal domains, when M varies from 0 to ∞, coincides with the single-plateaued domain. This also implies that, when the rule depends not only on preferences but also on the amount M to be allocated, the maximal domain coincides with the set of single-plateaued preferences as already shown by Ching and Serizawa (1998). Notice that in their setting, M is treated as a variable of the problem rather than one of its data. We want to emphasize though, that in spite of their result, our analysis with a fixed amount M is meaningful since there are many allocation problems where to assume the contrary would be senseless.

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A number of papers have also identified maximal domains of preferences allowing for strategy-proof social choice functions in voting environments. Barberà, Sonneschein, and Zhou (1991) show that the set of separable preferences is the maximal domain that preserves strategy-proofness of voting by committees without dummies and vetoers. Serizawa (1995), Barberà, Massó, and Neme (1999), Berga and Serizawa (2000), and Berga (1997) improve upon this result in several directions; for instance, by either looking at a more general voting model and/or by admitting larger classes of social choice functions.

Finally, it is worth mentioning that, in contrast to all the papers mentioned above, the rule that we exhibit when showing our maximality result is not "tops-only" in the sense that it does not depend exclusively on the n sets of best shares. Efficiency forces the rule to be sensitive to intervals of indifference away from the "top".

The paper is organized as follows. Section 2 contains notation, definitions, and the statement of our result. This is proven in Section 3. Section 4 concludes by deriving a weaker version of Ching and Serizawa's (1998) result as a corollary of our theorem and by relating our maximal domains to the "option" sets associated with strategy-proof, efficient, and strongly symmetric rules.

2. Preliminaries, Definitions, and the Theorem

Agents are indexed by the elements of a finite set $N = \{1,..,n\}$ where $n \geq 2$. They have to share the amount $M \in \mathbb{R}_{++}$ of a perfectly divisible good. An *allocation* is a vector $(x_1,...,x_n) \in \mathbb{R}^n_+$ such that $\sum x_i = M$. We denote by Z the set of allocations. Each agent $i \in N$ has a complete preorder R_i over $[0,M]$, his *preference relation*. Let P_i be the strict preference relation associated with R_i and let I_i be the corresponding indifference relation. We assume that preferences are continuous in the sense that for each $x \in [0,M]$ the sets $\{y \in [0,M] \mid xR_iy\}$ and $\{y \in [0,M] \mid yR_ix\}$ are closed. We denote by $\mathcal{R}$ the set of continuous preferences on $[0,M]$ and by $\mathcal{V}$ a generic subset of $\mathcal{R}$. *Preference profiles* are n-tuples of continuous preferences on $[0,M]$ and they are denoted by $\mathbf{R} = (R_1,...,R_n) \in \mathcal{R}^n$. When we want to stress the role of agent i's preference we will represent a preference profile by (R_i, R_{-i}).

A *rule on* $\mathcal{V}^n \subseteq \mathcal{R}^n$ is a function $\Phi : \mathcal{V}^n \longrightarrow Z$; that is, $\sum \Phi_i(\mathbf{R}) = M$ for all $\mathbf{R} \in \mathcal{V}^n$.

Rules require each agent to report a preference. A rule is strategy-proof if it is always in the best interest of an agent to reveal his preferences truthfully. Formally,

Definition 1. *A rule on $\mathcal{V}^n$, Φ, is strategy-proof if for all $(R_1, ..., R_n) \in \mathcal{V}^n$, all $i \in N$, and all $R'_i \in \mathcal{V}$ we have $\Phi_i(R_i, R_{-i}) R_i \Phi_i(R'_i, R_{-i})$.*

Given a preference profile $\mathbf{R} \in \mathcal{V}^n$, an allocation $x \in Z$ is *efficient* if there is no $z \in Z$ such that for all $i \in N$, $z_i R_i x_i$, and for at least one $j \in N$ we have $z_j P_j x_j$. Denote by $E(\mathbf{R})$ the set of efficient allocations.

A rule is efficient if it selects an efficient allocation. Formally,

Definition 2. *A rule on $\mathcal{V}^n$, Φ, is efficient if for all $\mathbf{R} \in \mathcal{V}^n$, we have $\Phi(\mathbf{R}) \in E(\mathbf{R})$.*

We are also interested in rules satisfying the following property.

Definition 3. *A rule on $\mathcal{V}^n$, Φ, is strongly symmetric if for all $\mathbf{R} \in \mathcal{V}^n$ and all $i,j \in N$ such that $R_i = R_j$, we have $\Phi_i(\mathbf{R}) = \Phi_j(\mathbf{R})$.*[2]

We will consider different subsets of preferences, all of them related to single-peakedness. For the definitions we need the following notation. Given a preference $R_i \in \mathcal{R}$ we denote the set of preferred shares according to R_i as $p(R_i) = \{x \in [0, M] \mid x R_i y \text{ for all } y \in [0, M]\}$. Let $\underline{p}(R_i) = \min p(R_i)$ and $\overline{p}(R_i) = \max p(R_i)$. Abusing notation, we also denote by $p(R_i)$ the unique element of the set $p(R_i)$ whenever $\underline{p}(R_i) = \overline{p}(R_i)$.

The first definition is the classical notion of single-peakedness. It requires that the preference R_i has a *unique* maximal element $p(R_i)$ and on each side preference is monotonic and strict. Formally,

Definition 4. *A preference $R_i \in \mathcal{R}$ is single-peaked if $p(R_i)$ is a singleton and for all $x, y \in [0, M]$ we have xP_iy whenever $y < x < p(R_i)$ or $p(R_i) < x < y$.*

Let $\mathcal{R}_s$ be the set of single-peaked preferences on $[0, M]$. The following rule on $\mathcal{R}_s$, the uniform allocation rule, has been extensively studied.

Definition 5. *The uniform allocation rule on $\mathcal{R}_s^n$, U, is defined as follows: for all $\mathbf{R} \in \mathcal{R}_s^n$ and all $i \in N$,*

$$
U_i(\mathbf{R}) =
\begin{cases}
\min\{p(R_i), \lambda(\mathbf{R})\} & \text{if } M \leq \sum p(R_j), \\
\max\{p(R_i), \lambda(\mathbf{R})\} & \text{if } M \geq \sum p(R_j),
\end{cases}
$$

where $\lambda(\mathbf{R})$ solves $\sum U_j(\mathbf{R}) = M$.

Ching (1994) characterized the uniform allocation rule on $\mathcal{R}_s^n$ as the unique one satisfying strategy-proofness, efficiency, and symmetry.[3]

The second definition of preferences is a bit weaker since it allows for indifferences at the top.

Definition 6. *A preference $R_i \in \mathcal{R}$ is single-plateaued if $p(R_i) = \left[\underline{p}(R_i), \overline{p}(R_i)\right]$ and for all $x, y \in [0, M]$ we have $x P_i y$ whenever $y < x < \underline{p}(R_i)$ or $\overline{p}(R_i) < x < y$.*[4]

Let $\mathcal{R}_{sp}$ be the set of single-plateaued preferences. The following rule on $\mathcal{R}_{sp}^n$ constitutes a natural extension of the uniform allocation rule to the domain of single-plateaued preferences.

Definition 7. *The uniform allocation rule on $\mathcal{R}_{sp}^n$, ψ, is defined as follows: for all $\mathbf{R} \in \mathcal{R}_{sp}^n$ and all $i \in N$,*

$$
\psi_i(\mathbf{R}) =
\begin{cases}
\min\left\{\underline{p}(R_i), \lambda(\mathbf{R})\right\} & \text{if } M \leq \sum_j \underline{p}(R_j), \\[2mm]
\min\left\{\overline{p}(R_i), \underline{p}(R_i) + \lambda(\mathbf{R})\right\} & \text{if } \sum_j \underline{p}(R_j) \leq M \leq \sum_j \overline{p}(R_j), \\[2mm]
\max\left\{\overline{p}(R_i), \lambda(\mathbf{R})\right\} & \text{if } \sum_j \overline{p}(R_j) \leq M,
\end{cases}
$$

where $\lambda(\mathbf{R})$ solves $\sum \psi_j(\mathbf{R}) = M$.

Finally, our third definition of preferences, the weakest one, refers to the following interval $\Theta(R_i)$, which will play a fundamental role in the sequel:

$$
\Theta(R_i) = \left[\min\left\{\frac{M}{n}, \underline{p}(R_i)\right\}, \max\left\{\frac{M}{n}, \overline{p}(R_i)\right\}\right].
$$

Before stating the formal definition, it seems useful to give a verbal explanation

of the set of feebly single-plateaued preferences. A preference relation $R_i \in \mathcal{R}$ is feebly single-plateaued if its set of best shares is an interval and the following additional properties are satisfied.

(a) If $\Theta(R_i) = \left[\frac{M}{n}, \bar{p}(R_i)\right]$, then the preference has to be "increasing" between M/n and its smallest best share $\underline{p}(R_i)$, although it may have intervals of indifference provided these intervals are sufficiently large in relation to M. Moreover, the egalitarian share M/n has to be at least as good as all smaller shares, but all orderings are possible among them.

(b) If $\Theta(R_i) = \left[\underline{p}(R_i), \frac{M}{n}\right]$, then the preference has to be "decreasing" between its largest best share $\bar{p}(R_i)$ and M/n, although it may have intervals of indifference provided these intervals are sufficiently small in relation to M.[5] Moreover, the egalitarian share M/n has to be at least as good as all larger shares, but also all orderings are possible among them.

Finally, if $\Theta(R_i) = \left[\underline{p}(R_i), \bar{p}(R_i)\right]$, then, no additional requirement is imposed. Formally,

Definition 8. *A preference $R_i \in \mathcal{R}$ is feebly single-plateaued if for all $x, y \in [0, M]$:*

(a) If $[x < y$ and $M/n \leq y \leq \underline{p}(R_i)]$ then $[yR_ix$ and if yI_ix then there exists $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 > M$ and $x'I_iy_0$ for all $x' \in [x_0, y_0]]$.

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(b) If $[x < y$ and $\overline{p}(R_i) \le x \le M/n]$ then $[xR_iy$ and if xI_iy then there exists $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 < M$ and $x'I_iy_0$ for all $x' \in [x_0, y_0]]$.

(c) If $x \in \left[\underline{p}(R_i), \overline{p}(R_i)\right]$ then $xI_i\overline{p}(R_i)$.

We denote by $\mathcal{R}_{fsp}$ *the set of feebly single-plateaued preferences.* Notice that this preference restriction implies a "weak-monotonicity" condition on the corresponding intervals $\Theta(\cdot)$ and that the number of agents n also appears in conditions (a) and (b). Theorem 1 states that the domain of feebly single-plateaued preferences is the unique maximal domain admitting strategy-proof, efficient, and strongly symmetric rules. Figure 1 illustrates three possible types of feebly single-plateaued preferences depending on whether $M/n \le \underline{p}(R_i)$, $\overline{p}(R_i) \le M/n$, or $\underline{p}(R_i) \le M/n \le \overline{p}(R_i)$.

<div style="border:1px solid black; display:inline-block; padding:8px;">Insert Figure 1 here</div>

Following Ching and Serizawa (1998) we can define, given a list of properties that a rule may satisfy, the concept of "a maximal domain of preferences for this list".

Definition 9. *A set $\mathcal{R}_m$ of preferences is a maximal domain for a list of properties if: (1) $\mathcal{R}_m \subseteq \mathcal{R}$; (2) there exists a rule on $\mathcal{R}_m^n$ satisfying the properties; and (3) there is no rule on $\mathcal{Q}^n$ satisfying the same properties such that $\mathcal{R}_m \subsetneq \mathcal{Q} \subseteq \mathcal{R}$.*

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Theorem 1. *The set of feebly single-plateaued preferences, $\mathcal{R}_{fsp}$, is the unique maximal domain including $\mathcal{R}_s$ for the properties of strategy-proofness, efficiency, and strong symmetry.*

Before proving Theorem 1 we illustrate, in Example 1 below, the reason why efficiency and strong symmetry together force the domain to contain only preferences with intervals of indifference of a very special type away from the set of best shares.

Example 1. Let $M = 8$ and $N = \{1, 2\}$. Let Φ be any efficient and strongly symmetric rule. Consider the preference $\bar{R}$ on $[0, 8]$ defined by:

$$y\bar{P}x \text{ for all } 0 \leq x < y \leq 2 \text{ and all } 5 \leq x < y \leq 8,$$

$$y\bar{I}x \text{ for all } x, y \in [2, 5].$$

Note that condition (a) of Definition 8 is not satisfied because $2\bar{I}5$ and there is no an interval of indifference $[x_0, y_0] \supseteq [2, 5]$ such that $x_0 + y_0 > 8$. Therefore $\bar{R} \notin \mathcal{R}_{fsp}$. A maximal domain of preferences cannot contain $\bar{R}$ because by strong symmetry $\Phi\left(\bar{R}, \bar{R}\right) = (4, 4)$ but the existence of the allocation $(2, 6)$, which is such that $2\bar{I}4$ and $6\bar{P}4$, indicates that $\Phi\left(\bar{R}, \bar{R}\right) \notin E\left(\bar{R}, \bar{R}\right)$, contradicting the efficiency of Φ. Consider now the preference $\hat{R}$ on $[0, 8]$ defined by:

$$y\hat{P}x \text{ for all } 0 \leq x < y \leq 3 \text{ and all } 6 \leq x < y \leq 8,$$

$$y\hat{I}x \text{ for all } x, y \in [3, 6].$$

Note that condition (a) of Definition 8 is satisfied because the sum of the extremes of the indifference interval $[3, 6]$ is larger than 8. Therefore $\hat{R} \in \mathcal{R}_{fsp}$. In contrast, the allocation $\Phi\left(\hat{R}, \hat{R}\right) = (4, 4)$ belongs to $E\left(\hat{R}, \hat{R}\right)$.

To illustrate the role of condition (b) in Definition 8 consider the preference $\tilde{R}$ on $[0, 8]$ defined by:

$$x\tilde{P}y \text{ for all } 0 \leq x < y \leq 3 \text{ and all } 6 \leq x < y \leq 8,$$

$$x\tilde{I}y \text{ for all } x, y \in [3, 6].$$

In this case $\tilde{R} \notin \mathcal{R}_{fsp}$ because now the sum of the extremes of the indifference interval is larger than 8. By strong symmetry $\Phi\left(\tilde{R}, \tilde{R}\right) = (4, 4)$ but $2\tilde{P}4$ and $6\tilde{I}4$ which indicates that $\Phi\left(\tilde{R}, \tilde{R}\right) \notin E\left(\tilde{R}, \tilde{R}\right)$, contradicting the efficiency of Φ. Finally, consider the preference R' on $[0, 8]$ defined by:

$$xP'y \text{ for all } 0 \leq x < y \leq 2 \text{ and all } 5 \leq x < y \leq 8,$$

$$xI'y \text{ for all } x, y \in [2, 5].$$

Now condition (b) of Definition 8 is satisfied because the sum of the extremes of the indifference interval $[2, 5]$ is smaller than 8. Therefore $R' \in \mathcal{R}_{fsp}$. In this case the allocation $\Phi\left(R', R'\right) = (4, 4)$ belongs to $E\left(R', R'\right)$.

3. The Proof of Theorem 1

Before proving Theorem 1 we state, in the following remark, a consequence of Ching's characterization (Ching, 1994) that we will repeatedly use in this section.

Remark 1. *Let Φ be any rule on $\mathcal{V}^n$ $(\supseteq \mathcal{R}_s^n)$ satisfying strategy-proofness, efficiency, and strong symmetry. If $\mathbf{R} \in \mathcal{R}_s^n$ then $\Phi(\mathbf{R}) = U(\mathbf{R})$; that is, Φ coincides with the uniform allocation rule on the subset of single-peaked preferences.*

Let $\mathcal{R}_m$ be a subset of preferences satisfying the following condition: $\mathcal{R}_s \subsetneq \mathcal{R}_m \subseteq \mathcal{R}$. Suppose that there exists a rule on $\mathcal{R}_m^n$, Φ, satisfying strategy-proofness, efficiency, and strong symmetry. Assume $\mathcal{R}_m$ is a maximal domain in $\mathcal{R}$ satisfying these properties. To show that $\mathcal{R}_m = \mathcal{R}_{fsp}$ we will use the following Lemmata, where $R^0, R^M \in \mathcal{R}_s$ will denote the two single-peaked preferences such that $p(R^0) = 0$ and $p(R^M) = M$.

Lemma 1. *Let $R_0 \in \mathcal{R}_m$ and $x, y \in [0, M]$ be arbitrary.*

Case 1: $M/n \leq x < y \leq \overline{p}(R_0)$. Then yR_0x.

Case 2: $\underline{p}(R_0) \leq y < x \leq M/n$. Then yR_0x.

Proof of Lemma 1. <u>Case 1</u>: Suppose otherwise; that is, there exist $R_0 \in \mathcal{R}_m$ and $\overline{x}, \overline{y} \in [0, M]$ such that $M/n \leq \overline{x} < \overline{y} \leq \overline{p}(R_0)$ and $\overline{x}P_0\overline{y}$. We can also find (see Figure 2) $x_0, y_0 \in [0, M]$ such that:

(a.1) $M/n \leq x_0 < y_0 \leq \bar{p}(R_0)$,

(a.2) $x_0 I_0 y_0$,

(a.3) $x_0 R_0 x$ for all $x \in [M/n, x_0]$, and

(a.4) $x_0 P_0 x$ for all $x \in (x_0, y_0)$.

Insert Figure 2 here

Notice that x_0 is the smallest value below $\bar{p}(R_0)$ and above M/n at which R_0 starts decreasing to its right.[6] Since R_0 is continuous and $\bar{p}(R_0) R_0 \bar{x}$, the existence of such y_0 follows. Obviously, x_0 could be equal to M/n, y_0 equal to $\bar{p}(R_0)$, or both.

Note that for all $z_0 \in (x_0, y_0)$ the following inequalities hold:

$$\frac{M - y_0}{n - 1} < \frac{M - z_0}{n - 1} < \frac{M - x_0}{n - 1} \leq \frac{M}{n}. \tag{3.1}$$

Now, fix $z_0 \in (x_0, y_0)$ and let $\bar{R} \in \mathcal{R}_s$ be such that $p(\bar{R}) = \frac{M - z_0}{n - 1}$ and $\left(\frac{M - y_0}{n - 1}\right) \bar{P} \left(\frac{M - x_0}{n - 1}\right)$. The existence of such a preference $\bar{R}$ follows from (3.1).

Let $\hat{R} \in \mathcal{R}_s$ be any preference such that $p(\hat{R}) = x_0$. By Remark 1, $\Phi\left(\hat{R}, \bar{R}, ..., \bar{R}\right) = U\left(\hat{R}, \bar{R}, ..., \bar{R}\right)$, and since

$$x_0 + (n - 1) \cdot \frac{M - z_0}{n - 1} < M,$$

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we have that $\Phi_1\left(\hat{R}, \bar{R}, ..., \bar{R}\right) = x_0$. By strategy-proofness of Φ,

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) R_0 x_0. \tag{3.2}$$

Again, by Remark 1, $\Phi_1\left(R^M, \bar{R}, .., \bar{R}\right) = z_0$ and by strategy-proofness of Φ, $z_0 R^M \Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right)$, implying that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \leq z_0. \tag{3.3}$$

Finally, by Remark 1, $\Phi_1\left(R^0, \bar{R}, ..., \bar{R}\right) = M/n$ and by strategy-proofness of Φ, $M/n R^0 \Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right)$, implying that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \geq \frac{M}{n}. \tag{3.4}$$

Then, by (3.2), (3.3), and (3.4),

$$\Phi\left(R_0, \bar{R}, ..., \bar{R}\right) = \left(x_1, \frac{M - x_1}{n - 1}, ..., \frac{M - x_1}{n - 1}\right) \tag{3.5}$$

with $M/n \leq x_1 \leq x_0$ and $x_1 I_0 x_0$. But the existence of the allocation $\left(y_0, \frac{M-y_0}{n-1}, ..., \frac{M-y_0}{n-1}\right)$ and (3.5) imply that $\Phi\left(R_0, \bar{R}, ..., \bar{R}\right) \notin E\left(R_0, \bar{R}, ..., \bar{R}\right)$, contradicting the effi-

ciency of Φ.

Case 2: Its proof is omitted since it follows an argument which is symmetric to the one used to prove Case 1. ■

Lemma 2. *Let $R_0 \in \mathcal{R}_m$ and $x \in [0, M]$ be arbitrary.*

Case 1: $x < M/n \leq \bar{p}(R_0)$. Then M/nR_0x.

Case 2: $\underline{p}(R_0) \leq M/n < x$. Then M/nR_0x.

Proof of Lemma 2. <u>Case 1</u>: Suppose otherwise; that is, there exist $R_0 \in \mathcal{R}_m$ and $x_0 < M/n \leq \bar{p}(R_0)$ such that $x_0 P_0 M/n$.

First, assume that M/n is a minimal element on $[x_0, M/n]$ relative to R_0; that is

$$yR_0\frac{M}{n} \text{ for all } y \in \left[x_0, \frac{M}{n}\right]. \tag{3.6}$$

Since Φ is strongly symmetric,

$$\Phi(R_0, ..., R_0) = (M/n, ..., M/n). \tag{3.7}$$

By (3.6) and Lemma 1 we have that for all $\varepsilon \in \left(0, \min\left\{\frac{M}{n} - x_0, \bar{p}(R_0) - \frac{M}{n}\right\}\right]$,

$$\left(\frac{M}{n} - \varepsilon\right)R_0\frac{M}{n} \text{ and } \left(\frac{M}{n} + \varepsilon\right)R_0\frac{M}{n}.$$

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Let $\bar{\varepsilon} = \min\left\{\frac{M}{n} - x_0, \bar{p}(R_0) - \frac{M}{n}\right\}$. Then, either

$$\left(\frac{M}{n} - \bar{\varepsilon}\right) P_0 \frac{M}{n} \text{ or } \left(\frac{M}{n} + \bar{\varepsilon}\right) P_0 \frac{M}{n},$$

depending on whether $\bar{\varepsilon}$ is either equal to $\frac{M}{n} - x_0$ or to $\bar{p}(R_0) - \frac{M}{n}$, respectively. Then the allocation $\left(\left(\frac{M}{n} + \bar{\varepsilon}\right), \left(\frac{M}{n} - \bar{\varepsilon}\right), M/n, ..., M/n\right)$ and (3.7) imply that $\Phi(R_0, ..., R_0) \notin E(R_0, ..., R_0)$, contradicting the efficiency of Φ.

Second, assume that there exists $y_0 \in (x_0, M/n)$ such that $M/n P_0 y_0$. Then, there exist x_1, y_1 and z_1 such that:

(a.1') $0 \leq x_1 < z_1 < y_1 \leq M/n$,

(a.2') $x_1 I_0 y_1 I_0 M/n$,

(a.3') $x_1 P_0 x$ for all $x \in (x_1, y_1)$, and

(a.4') $y_1 I_0 x$ for all $x \in [y_1, M/n]$.

Note that

$$\frac{M}{n} \leq \frac{M - y_1}{n - 1} < \frac{M - z_1}{n - 1} < \frac{M - x_1}{n - 1}.$$

Now, let $\bar{R} \in \mathcal{R}_s$ be any single-peaked preference such that $p(\bar{R}) = \frac{M - z_1}{n - 1}$ and $\frac{M - x_1}{n - 1} \bar{P} \frac{M - y_1}{n - 1}$. By Remark 1, $\Phi_1(R^M, \bar{R}, ..., \bar{R}) = U_1(R^M, \bar{R}, ..., \bar{R}) = M/n$, the

uniform allocation. By strategy-proofness of Φ,

$$\Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right) R_0 \frac{M}{n}. \tag{3.8}$$

Again, by Remark 1, $\Phi_1 \left(R^0, \bar{R}, ..., \bar{R} \right) = z_1$ and by strategy-proofness of Φ, $z_1 R^0 \Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right)$, implying that

$$\Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right) \geq z_1. \tag{3.9}$$

Then, by (3.8) and (3.9),

$$\Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right) \geq y_1. \tag{3.10}$$

Finally, by Remark 1, $\Phi_1 \left(R^M, \bar{R}, ..., \bar{R} \right) = M/n$ and by strategy-proofness of Φ, $M/n R^M \Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right)$, implying that

$$\Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right) \leq \frac{M}{n}. \tag{3.11}$$

Then by (3.10) and (3.11),

$$\Phi\left(R_0, \bar{R}, ..., \bar{R}\right) = \left(x_2, \frac{M - x_2}{n - 1}, ..., \frac{M - x_2}{n - 1}\right) \qquad (3.12)$$

with $y_1 \leq x_2 \leq M/n$ and $x_2 I_0 M/n$ (by construction). But then, since $\frac{M - x_2}{n - 1} \leq$ $\frac{M - y_1}{n - 1}$, $\bar{R} \in \mathcal{R}_s$, and all preference orderings are transitive, the allocation $\left(x_1, \frac{M - x_1}{n - 1}, ..., \frac{M - x_1}{n - 1}\right)$ and (3.12) imply that $\Phi\left(R_0, \bar{R}, ..., \bar{R}\right) \notin E\left(R_0, \bar{R}, ..., \bar{R}\right)$ contradicting the efficiency of Φ.

Case 2: Its proof is omitted since it follows an argument which is symmetric to the one used to prove Case 1. ∎

Lemma 3. *Let $R_0 \in \mathcal{R}_m$ and $x \in [0, M]$ be arbitrary.*

Case 1: $x < M/n \leq \bar{p}(R_0)$ and $x I_0 M/n$. Then $M/n I_0 x'$ for all $x' \in [x, M/n]$.

Case 2: $\underline{p}(R_0) \leq M/n < x$ and $x I_0 M/n$. Then $M/n I_0 x'$ for all $x' \in [M/n, x]$.

Proof of Lemma 3. <u>Case 1</u>: Suppose otherwise; that is, there exist $R_0 \in \mathcal{R}_m$ and $x_1 < M/n \leq \bar{p}(R_0)$ such that $x_1 I_0 M/n$ and $M/n P_0 z_1$ for at least one $z_1 \in (x_1, M/n)$. Notice that by Lemma 2 we already know that $M/n R_0 z_1$. Without loss of generality we can assume that there exists $y_1 \in [x_1, M/n]$ such that $M/n I_0 y$

for all $y \in [y_1, M/n]$, M/nP_0y for all $y \in (x_1, y_1)$, and $z_1 \in (x_1, y_1)$. Note that

$$\frac{M}{n} \leq \frac{M - y_1}{n - 1} < \frac{M - z_1}{n - 1} < \frac{M - x_1}{n - 1}.$$

Now, let $\bar{R} \in \mathcal{R}_s$ be any single-peaked preference such that $p(\bar{R}) = \frac{M - z_1}{n - 1}$ and

$\frac{M - x_1}{n - 1} \bar{P} \frac{M - y_1}{n - 1}$.

By Remark 1, $\Phi\left(R^M, \bar{R}, ..., \bar{R}\right) = U\left(R^M, \bar{R}, ..., \bar{R}\right)$; therefore, $\Phi_1\left(R^M, \bar{R}, ..., \bar{R}\right) = M/n$. By strategy-proofness of Φ,

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) R_0 \frac{M}{n}. \tag{3.13}$$

Again, by Remark 1, $\Phi_1\left(R^0, \bar{R}, ..., \bar{R}\right) = z_1$ and by strategy-proofness of Φ, $z_1 R^0 \Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right)$, implying that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \geq z_1. \tag{3.14}$$

Then, by (3.13) and (3.14),

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \geq y_1. \tag{3.15}$$

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Finally, since $\Phi_1 \left(R^M, \bar{R}, ..., \bar{R} \right) = M/n$, by strategy-proofness of Φ,

$M/n R^M \Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right)$, implying that

$$\Phi_1 \left(R_0, \bar{R}, ..., \bar{R} \right) \leq \frac{M}{n}. \qquad (3.16)$$

Then by (3.15) and (3.16),

$$\Phi \left(R_0, \bar{R}, ..., \bar{R} \right) = \left(x_2, \frac{M - x_2}{n - 1}, ..., \frac{M - x_2}{n - 1} \right) \qquad (3.17)$$

with $y_1 \leq x_2 \leq M/n$ and $\frac{M-x_2}{n-1} \leq \frac{M-y_1}{n-1}$. Because $\frac{M-x_1}{n-1} \bar{P} \frac{M-y_1}{n-1} \bar{P} \frac{M-x_2}{n-1}$ and $x_1 I_0 x_2$

we have that the allocation $\left(x_1, \frac{M-x_1}{n-1}, ..., \frac{M-x_1}{n-1} \right)$ and (3.17) imply that $\Phi \left(R_0, \bar{R}, ..., \bar{R} \right) \notin$

$E \left(R_0, \bar{R}, ..., \bar{R} \right)$, contradicting the efficiency of Φ.

Case 2: Its proof is omitted since it follows an argument which is symmetric

to the one used to prove Case 1. ∎

Lemma 4. Let $R_0 \in \mathcal{R}_m$ and $x, y \in [0, M]$ be arbitrary.

Case 1: $M/n \leq x < y \leq \underline{p} \left(R_0 \right)$ and $x I_0 y$. Then there exists an interval $[x_0, y_0] \supseteq$

$[x, y]$ such that $x_0 + y_0 > M$ and $x' I_0 y_0$ for all $x' \in [x_0, y_0]$.

Case 2: $\bar{p} \left(R_0 \right) \leq x < y \leq M/n$ and $x I_0 y$. Then there exists an interval $[x_0, y_0] \supseteq$

$[x, y]$ such that $x_0 + y_0 < M$ and $x' I_0 y_0$ for all $x' \in [x_0, y_0]$.

To prove Lemma 4 we need the following definition.

Definition 10. *Given a preference $R_0 \in \mathcal{R}$ we say that the interval $[x_0, y_0]$ is a maximal interval of indifference for R_0 if $x' I_0 x_0$ for all $x' \in [x_0, y_0]$ and if $[x_1, y_1] \supseteq [x_0, y_0]$ is such that $x I_0 x_0$ for all $x \in [x_1, y_1]$, then $[x_0, y_0] = [x_1, y_1]$.*

Proof of Lemma 4. <u>Case 1</u>: Let $R_0 \in \mathcal{R}_m$ and suppose that x and y are such that $M/n \leq x < y \leq \underline{p}(R_0)$ and $x I_0 y$. By Lemmata 1, 2, and 3 there exists a maximal interval of indifference for R_0, $[x_0, y_0]$, containing $[x, y]$. Notice that $x' I_0 y_0$ for all $x' \in [x_0, y_0]$ and $M/n < y_0$.

In order to obtain a contradiction, assume that $x_0 + y_0 \leq M$. Let $z_0 \in (x_0, y_0)$ be any share such that $M/n \leq z_0$ and

$$(z_0 - x_0) > (y_0 - z_0).$$

<u>Subcase 1.1</u>: $n \geq 3$ and there exists an integer n' such that $n \geq n' \geq 3$ and

$$(n' - 1) z_0 \leq M \leq n' z_0.$$

Notice that the latter condition is only possible if $x_0 + y_0 \leq M$.

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Let $\bar{R} \in \mathcal{R}_s$ be such that

$$p(\bar{R}) = M - (n' - 1)z_0 = z_1 \text{ and } \frac{M}{n'}\bar{P}y_1 = M - (n' - 1)\,y_0.$$

Notice that $M/n' \le z_0$ implies $z_1 = M - (n' - 1)z_0 \le M/n'$. Therefore,

$$y_1 = M - (n' - 1)\,y_0 < M - (n' - 1)z_0 = z_1 \le \frac{M}{n'}.$$

Define $\mathbf{R_0} = (\underbrace{R_0, ..., R_0}_{(n'-1)-times}, R^0, ..., R^0, \bar{R}) \in \mathcal{R}_m^n$. In order to show that
$\Phi\,(\mathbf{R_0}) = (\underbrace{z_0, ..., z_0}_{(n'-1)-times}, 0, ..., 0, z_1)$ suppose first that

$$\Phi\,(\mathbf{R_0}) = (t_1, ..., t_1, t_2, ..., t_2, t_3) \tag{3.18}$$

with $t_2 > 0$. Since

$$\Phi\left(R_0, ..., R_0, R^0, ..., R^0, R_0\right) = (M/n', ..., M/n', 0, ..., 0, M/n') \tag{3.19}$$

we have that $\Phi_n\,(\mathbf{R_0}) = t_3\bar{R}M/n'$, which implies

$$M - (n' - 1)\,y_0 < t_3 \le \frac{M}{n'}. \tag{3.20}$$

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But the existence of the allocation $\left(t'_1, ..., t'_1, 0, ..., 0, t_3\right)$ and (3.18) imply that $\Phi\left(\mathbf{R}_0\right) \notin E\left(\mathbf{R}_0\right)$. To see this, first notice that $0P^0 t_2$. Moreover, condition (3.20) implies $M/n' \leq t'_1 \leq y_0$. Therefore, since $t_1 < t'_1 \leq y_0$, Lemmata 1 and 2 imply that $t'_1 R_0 t_1$, contradicting the efficiency of Φ.

Now assume that

$$\Phi\left(\mathbf{R}_0\right) = \left(\hat{t}_1, ..., \hat{t}_1, 0, ..., 0, \hat{t}_3\right) \tag{3.21}$$

and $\hat{t}_3 \neq z_1 = M - \left(n' - 1\right) z_0$. Since (3.19), $\Phi_n\left(\mathbf{R}_0\right) = \hat{t}_3 \bar{R} M/n'$, which implies

$$M - \left(n' - 1\right) y_0 < \hat{t}_3 \leq \frac{M}{n'}. \tag{3.22}$$

But the existence of the allocation $\left(z_0, ..., z_0, 0, ..., 0, z_1\right)$ and (3.21) imply that $\Phi\left(\mathbf{R}_0\right) \notin E\left(\mathbf{R}_0\right)$. To see this, first notice that $z_1 \bar{P} \hat{t}_3$ since $p\left(\bar{R}\right) = z_1$ and $z_1 \neq \hat{t}_3$. Moreover, (3.22) implies $M/n' \leq \hat{t}_1 \leq y_0$. Therefore, since $\hat{t}_1 \leq y_0$, we have that $z_0 I_0 y_0 R_0 \hat{t}_1$. Hence, $z_0 R_0 \hat{t}_1$, contradicting the efficiency of Φ. Therefore,

$$\Phi\left(\mathbf{R}_0\right) = (\ \underbrace{z_0, ..., z_0}_{\left(n'-1\right)-times}\ , 0, ..., 0, z_1). \tag{3.23}$$

To finish with Subcase 1.1, suppose first that $y_0 < M$. Let $\varepsilon > 0$ be such that $\left(z_0 - x_0\right) > \varepsilon > \left(y_0 - z_0\right)$. Because $\left(z_0 - \varepsilon\right) \in \left[x_0, z_0\right]$, $\left(z_0 + \varepsilon\right) > y_0$, and by

Lemma 1, we have that $(z_0 - \varepsilon) \, I_0 z_0$ and $(z_0 + \varepsilon) P_0 z_0$, since $[x_0, y_0]$ is a maximal interval of indifference for R_0. Therefore, the existence of the allocation

$$((z_0 - \varepsilon), (z_0 + \varepsilon), \underbrace{z_0, ..., z_0}_{(n'-3)-times}, 0, ..., 0, z_1)$$

and (3.23) imply that $\Phi(\mathbf{R}_0) \notin E(\mathbf{R}_0)$, contradicting the efficiency of Φ. Now, assume that the extreme case $y_0 = M$ holds. Then, $x_0 = 0$ because our contradiction hypothesis says that $x_0 + y_0 \leq M$. In this case R_0 is such that $x I_0 y$ for all $x, y \in [0, M]$. But then, Lemma 4 follows, since for any $x_0' \in (0, M/n)$ we have that $[x_0', y_0] \supseteq [x, y]$, $x_0' + y_0 > M$, and $x' I_0 y_0$ for all $x' \in [x_0', y_0]$.

Subcase 1.2: $n \geq 3$ and z_0 satisfies the following inequalities: $z_0 < M < 2 z_0$. Using arguments similar to the ones used in Subcase 1.1 it is possible to show that

$$\Phi(R_0, R_0, R^0, ..., R^0) = (M/2, M/2, 0, ..., 0). \tag{3.24}$$

Since $x_0 + y_0 \leq M$ and $y_0 > z_0 > \frac{M}{2}$ we have that $0 < y_0 - \frac{M}{2} \leq \frac{M}{2} - x_0$, which implies that we can find an $\varepsilon > 0$ such that $\frac{M}{2} + \varepsilon > y_0$ and $\frac{M}{2} - \varepsilon > x_0$. As before, we can assume that $\frac{M}{2} + \varepsilon \leq M$ because if $y_0 = M$ the statement follows trivially as in Subcase 1.1. By Lemma 1, $\left(\frac{M}{2} - \varepsilon\right) I_0 M/2$ and $\left(\frac{M}{2} + \varepsilon\right) P_0 M/2$

hold since $[x_0, y_0]$ is a maximal interval of indifference for R_0. Therefore, the existence of $\varepsilon > 0$ such that $\left(\frac{M}{2} - \varepsilon, \frac{M}{2} + \varepsilon, 0, ..., 0\right) \in Z$ and (3.24) imply that $\Phi\left(R_0, R_0, R^0, ..., R^0\right) \notin E\left(R_0, R_0, R^0, ..., R^0\right)$, contradicting the efficiency of Φ.

Subcase 1.3: $n = 2$. Remember that we can suppose that $M/n < y_0 < M$. By strong symmetry,

$$\Phi\left(R_0, R_0\right) = \left(M/2, M/2\right). \tag{3.25}$$

We can also find $\varepsilon > 0$ such that $y_0 < z_0 + \varepsilon$, $x_0 < z_0 - \varepsilon$, $(z_0 + \varepsilon) P_0 M/2$, and $(z_0 - \varepsilon) I_0 M/2$. Therefore, the existence of $\varepsilon > 0$ such that $(z_0 - \varepsilon, z_0 + \varepsilon) \in Z$ and (3.25) imply that $\Phi\left(R_0, R_0\right) \notin E\left(R_0, R_0\right)$, contradicting the efficiency of Φ.

Case 2: Its proof is omitted since it follows an argument which is symmetric to the one used to prove Case 1. ∎

Proof of Theorem 1: Let $R_0 \in \mathcal{R}_m$ be arbitrary. We have to show that R_0 is feebly single-plateaued. Consider the following cases:

Case A: Assume that $M/n \leq \underline{p}(R_0)$. Then, $\Theta(R_0) = [M/n, \overline{p}(R_0)]$. To show that property (a) of Definition 8 holds, suppose first that $M/n \leq x < y \leq \underline{p}(R_0)$. Then, by Lemma 1 (Case 1), yR_0x. If yI_0x then, by Lemma 4 (Case 1), there exists an interval $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 > M$ and $x'I_0y_0$ for all $x' \in [x_0, y_0]$. Assume now that $x < M/n \leq y \leq \underline{p}(R_0)$. Then, by Lemma 2 (Case

1), M/nR_0x. Moreover, by Lemma 1 (Case 1), yR_0M/n. Therefore, since R_0 is transitive, yR_0x. If yI_0x then, by Lemma 4 (Case 1), there exists an interval $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 > M$ and $x'I_0y_0$ for all $x' \in [x_0, y_0]$. To show that property (c) of Definition 8 holds, suppose that $x \in \left(\underline{p}(R_0), \overline{p}(R_0)\right)$. Then, $M/n \leq \underline{p}(R_0) < x < \overline{p}(R_0)$ which implies, by Lemma 1 (Case 1), that $xR_0\underline{p}(R_0)$, and hence $xI_0\overline{P}(R_0)$.

Case B: Assume that $\underline{p}(R_0) \leq M/n \leq \overline{p}(R_0)$. Then, $\Theta(R_0) = \left[\underline{p}(R_0), \overline{p}(R_0)\right]$. To show that property (c) of Definition 8 holds, assume first that $\underline{p}(R_0) = M/n$ and let x be any share such that $\underline{p}(R_0) < x \leq \overline{p}(R_0)$. By Lemma 1 (Case 1) $xR_0\underline{p}(R_0)$ which implies that $xI_0\overline{p}(R_0)$. Assume now that $\underline{p}(R_0) < M/n \leq \overline{p}(R_0)$. By Lemma 2 (Case 1),

$$\frac{M}{n}R_0\underline{p}(R_0). \tag{3.26}$$

First, let x be any share such that $\underline{p}(R_0) < x < M/n \leq \overline{p}(R_0)$. By Lemma 1 (Case 2) xR_0M/n and by (3.26), $xI_0\overline{p}(R_0)$. Second, let x be any share such that $\underline{p}(R_0) < M/n < x \leq \overline{p}(R_0)$. By Lemma 1 (Case 1) xR_0M/n and by (3.26), $xI_0\overline{p}(R_0)$.

Case C: Assume that $\overline{p}(R_0) \leq M/n$. Then, $\Theta(R_0) = \left[\underline{p}(R_0), M/n\right]$. The proof that properties (b) and (c) of Definition 8 hold is symmetrical to that of

Case A, using Cases 2's of Lemmata 1, 2, and 4, respectively.

The proof of Theorem 1 is completed by exhibiting a rule on the set of feebly single-plateaued preferences , $(\mathcal{R}_{fsp})^n$, that satisfies the properties of strategy-proofness, efficiency, and strong symmetry. We obtain such a rule by extending the uniform allocation rule ψ on the domain of single-plateaued preferences, $\mathcal{R}_{sp}$, to this larger domain.

The *extended uniform rule on* $(\mathcal{R}_{fsp})^n$, Ψ, is defined by the following algorithm: let $\mathbf{R} = (R_1, ..., R_n) \in (\mathcal{R}_{fsp})^n$ be any profile of feebly single-plateaued preferences.

<u>Stage 0</u>: Let $\bar{\mathbf{R}} = (\bar{R}_1, ..., \bar{R}_n) \in \mathcal{R}_{sp}^n$ be any profile of single-plateaued preferences such that $[\underline{p}(R_i), \overline{p}(R_i)] = [\underline{p}(\bar{R}_i), \overline{p}(\bar{R}_i)]$ for all $i \in N$. Compute $\psi(\bar{\mathbf{R}})$ and let S^0 be the set of agents receiving an amount in the interior of a maximal interval of indifference for R_i (the original preference), denoted by $[x_i^0, y_i^0]$, such that $[x_i^0, y_i^0] \neq [\underline{p}(R_i), \overline{p}(R_i)]$; that is,

$$
S^0 = \left\{ i \in N \ \middle| \ \begin{array}{l} \psi_i(\bar{\mathbf{R}}) \in (x_i^0, y_i^0) \text{ where } [x_i^0, y_i^0] \text{ is a maximal interval} \\ \text{of indifference for } R_i \text{ and } \underline{p}(R_i) \, P_i x \text{ for all } x \in [x_i^0, y_i^0] \end{array} \right\}.
$$

If $S^0 = \emptyset$ then define $\Psi(\mathbf{R}) = \psi(\bar{\mathbf{R}})$ and stop. If $S^0 \neq \emptyset$ then select any profile

$\mathbf{R}^1 = (R_1^1, ..., R_n^1) \in (\mathcal{R}_{fsp})^n$ such that $R_i^1 = R_i$ for all $i \notin S^0$ and for all $i \in S^0$

$$R_i^1 = \begin{cases} R_i \text{ on } [0, y_i^0] \text{ and } y_i^0 P_i^1 x \text{ for all } x > y_i^0 & \text{if } M \leq \sum \underline{p}(R_j) \\ R_i \text{ on } [x_i^0, M] \text{ and } x_i^0 P_i^1 x \text{ for all } x < x_i^0 & \text{if } \sum \overline{p}(R_j) \leq M \end{cases} \quad .7$$

Go to stage 1.

Now, for k$\geq$ 1, and given that the algorithm has not stopped yet at stage k-1,

stage k is as follows.

<u>Stage k</u>: Given the preference profile $\mathbf{R}^k = (R_1^k, ..., R_n^k) \in (\mathcal{R}_{fsp})^n$, the out-

come of stage k-1, let $\bar{\mathbf{R}}^k = (\bar{R}_1^k, ..., \bar{R}_n^k) \in \mathcal{R}_{sp}^n$ be any profile of single-plateaued

preferences such that $[\underline{p}(R_i^k), \overline{p}(R_i^k)] = [\underline{p}(\bar{R}_i^k), \overline{p}(\bar{R}_i^k)]$ for all $i \in N$. Compute

$\psi(\bar{\mathbf{R}}^k)$. If $\psi(\bar{\mathbf{R}}^k) = \psi(\bar{\mathbf{R}}^{k-1})$, define $\Psi(\mathbf{R}) = \psi(\bar{\mathbf{R}}^k)$ and stop. Otherwise, let

S^k be the set of agents receiving an amount in the interior of a maximal interval

of indifference for R_i^k, denoted by $[x_i^k, y_i^k]$, such that $[x_i^k, y_i^k] \neq [\underline{p}(R_i^k), \overline{p}(R_i^k)]$;

that is,

$$S^k = \left\{ i \in N \mid \begin{array}{l} \psi_i(\bar{\mathbf{R}}^k) \in (x_i^k, y_i^k) \text{ where } [x_i^k, y_i^k] \text{ is a maximal interval} \\ \text{of indifference for } R_i^k \text{ and } \underline{p}(R_i^k) P_i^k x \text{ for all } x \in [x_i^k, y_i^k] \end{array} \right\}.$$

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If $S^k = \emptyset$ then define $\Psi\left(\mathbf{R}\right) = \psi\left(\bar{\mathbf{R}}^k\right)$ and stop. If $S^k \neq \emptyset$ then select any profile $\mathbf{R}^{k+1} = \left(R_1^{k+1}, ..., R_n^{k+1}\right) \in \left(\mathcal{R}_{fsp}\right)^n$ such that $R_i^{k+1} = R_i^k$ for all $i \notin S^k$ and for all $i \in S^k$

$$R_i^{k+1} = \begin{cases} R_i^k \text{ on } \left[0, y_i^k\right] \text{ and } y_i^k P_i^{k+1} x \text{ for all } x > y_i^k & \text{if } M \leq \sum \underline{p}\left(R_j^k\right) \\ R_i^k \text{ on } \left[x_i^k, M\right] \text{ and } x_i^k P_i^{k+1} x \text{ for all } x < x_i^k & \text{if } \sum \overline{p}\left(R_j^k\right) \leq M \end{cases}.$$

Go to stage k+1.

The algorithm stops after at most n stages. This is because the sets S^k only contain players whose stage k proposed shares are not maximal. Hence, for all $K \geq 2$

$$S^K \cap \left(\bigcup_{k=0}^{K-1} S^k\right) = \emptyset.$$

Note that the rule Ψ satisfies strategy-proofness and strong symmetry. To show that it satisfies efficiency, let $\mathbf{R} = \left(R_1, ..., R_n\right) \in \left(\mathcal{R}_{fsp}\right)^n$ be arbitrary and consider the following cases:

<u>Case 1</u>: $\sum \underline{p}(R_j) \leq M \leq \sum \overline{p}(R_j)$. Then, efficiency is clearly satisfied because $\psi_i\left(\bar{\mathbf{R}}^0\right) \in \left[\underline{p}(R_i), \overline{p}\left(R_i\right)\right]$ for all $i \in N$ implies that $S^0 = \emptyset$ and the process stops at stage 0 after setting $\Psi\left(\mathbf{R}\right) = \psi\left(\bar{\mathbf{R}}^0\right)$. Therefore, $\overline{p}(R_i) I_i \Psi_i\left(\mathbf{R}\right)$ for all $i \in N$, which means that $\Psi\left(\mathbf{R}\right) \in E\left(\mathbf{R}\right)$.

<u>Case 2</u>: $M \leq \sum \underline{p}(R_j)$. Then, it is easy to show that $\Psi_i(\mathbf{R}) \leq \underline{p}(R_i)$ for all $i \in N$. Let S be the subset of agents who are rationed; that is,

$$S = \left\{ i \in N \mid \Psi_i(\mathbf{R}) < \underline{p}(R_i) \right\}.$$

If $S = \emptyset$, then $\sum \underline{p}(R_j) = M$ and $\Psi_i(\mathbf{R}) = \underline{p}(R_i)$ for all $i \in N$, in which case, $\Psi(\mathbf{R}) \in E(\mathbf{R})$. Therefore, suppose $S \neq \emptyset$ and assume that $\Psi(\mathbf{R}) \notin E(\mathbf{R})$; that is, there exist a feasible allocation $r = (r_1, ..., r_n) \in Z$ and $j \in N$ such that:

$$r_i R_i \Psi_i(\mathbf{R}) \text{ for all } i \in N \text{ and}$$

$$r_j P_j \Psi_j(\mathbf{R}). \tag{3.27}$$

However, (3.27) and the definition of Ψ imply that $j \in S$ and $\Psi_j(\mathbf{R}) < r_j$. Denote by $[x_j, y_j]$ the maximal interval of indifference containing $\Psi_j(\mathbf{R})$. By definition of Ψ, for all $i \in S$,

$$\Psi_i(\mathbf{R}) = \Psi_j(\mathbf{R}).$$

Consider the preference profile $\bar{\mathbf{R}} \in (\mathcal{R}_{fsp})^n$ where $\bar{\mathbf{R}}_i = \mathbf{R}_i$ if $i \notin S$ and $\bar{\mathbf{R}}_i = \mathbf{R}_j$ if $i \in S$. By definition of Ψ,

$$\Psi(\bar{\mathbf{R}}) = \Psi(\mathbf{R}).$$

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For all $i \notin S$, $\Psi_i(\mathbf{R}) = \underline{p}(R_i)$ hold, hence $r_i \geq \Psi_i(\bar{\mathbf{R}})$, implying that $\sum_{i \notin S} r_i \geq$ $\sum_{i \notin S} \Psi_i(\bar{\mathbf{R}})$. Since $\Psi_j(\bar{\mathbf{R}}) < r_j$, there exists $k \in N$ such that $\Psi_k(\bar{\mathbf{R}}) > r_k$, because $\Psi(\bar{\mathbf{R}}) \in Z$. Then

$$\Psi_k(\bar{\mathbf{R}}) \, \bar{I}_k r_k.$$

Because $\Psi_k(\bar{\mathbf{R}}) \in [x_j, y_j]$, $r_k \in [x_j, y_j]$. Therefore,

$$M \geq \sum_{\substack{i \in S \\ i \neq j}} r_i + r_j + \sum_{i \notin S} \Psi_i(\bar{\mathbf{R}}) > \sum_{\substack{i \in S \\ i \neq j}} r_i + y_j \geq x_j + y_j,$$

a contradiction with the fact that $\bar{\mathbf{R}}$ satisfies Definition 8.

<u>Case 3</u>: Assume that $\sum \bar{p}(R_j) \leq M$. Then, an argument symmetric to the one used in Case (2) proves that $\Psi(\mathbf{R}) \in E(\mathbf{R})$. ■

4. Concluding Remarks

We close with two remarks. First, we show how to derive a slightly weaker version of Ching and Serizawa (1998) result as an implication of our Theorem. While we have considered M as exogenous data, they formulate the division problem for all possible values of M by letting rules depend not only on preferences profiles but also on all possible amounts of the good to be allocated. This distinction has

important consequences for the maximality problem since their approach implies that preferences have to be defined over all positive shares, and consequently the same domain of preferences has to be maximal for *all* values of M, while we search for a maximal domain of preferences (on $[0, M]$) for *each* value M. To formulate the division problem in their setting, assume now that every agent $i \in N$ has a continuous preference ordering over the interval $\mathbb{R}_+$ and denote by $\mathcal{R}(\infty)$ the set of all these preference orderings.

A *rule on* $\mathcal{V}^n \subseteq \mathcal{R}(\infty)^n$ and $\mathbb{R}_{++}$ is a function $\Phi^\infty : \mathcal{V}^n \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_+^n$ such that $\sum \Phi_i^\infty(\mathbf{R}, M) = M$ for all $(\mathbf{R}, M) \in \mathcal{V}^n \times \mathbb{R}_{++}$.

Consider the natural extensions of strategy-proofness, efficiency, symmetry, and strong symmetry to this new setting, where rules are defined on $\mathcal{V}^n$ and $\mathbb{R}_{++}$.[8] Denote them by $sp(\infty)$, $eff(\infty)$, $sy(\infty)$, and $ssy(\infty)$.

The definition below adapts our concept of maximal domain of preferences to their setting.

Definition 11. *A set* $\mathcal{R}_m(\infty)$ *of preferences is a* maximal (infinite) domain *for a list of properties if: (1)* $\mathcal{R}_m(\infty) \subseteq \mathcal{R}(\infty)$; *(2) there exists a rule on* $\mathcal{R}_m(\infty)^n$ *and* $\mathbb{R}_{++}$ *satisfying the properties; and (3) there is no rule on* $\mathcal{Q}(\infty)^n$ *and* $\mathbb{R}_{++}$ *satisfying the same properties and such that* $\mathcal{R}_m(\infty) \subsetneq \mathcal{Q}(\infty) \subseteq \mathcal{R}(\infty)$.

Ching and Serizawa (1998) prove that the set of single-plateaued preferences is the unique maximal (infinite) domain including single-peaked preferences for $sp(\infty)$, $eff(\infty)$, and $sy(\infty)$. Theorem 2 below identifies the single-plateaued domain using the strong version of symmetry.[9]

Theorem 2. *The set of single-plateaued preferences, $\mathcal{R}_{sp}(\infty)$, is the unique maximal (infinite) domain including single-peaked preferences for $sp(\infty)$, $eff(\infty)$, and $ssy(\infty)$.*

Proof. Let $\mathcal{R}_a(\infty)$ be a domain on which there is a rule Φ^∞ on $\mathcal{R}_a(\infty)^n$ and $I\!R_{++}$ satisfying $sp(\infty)$, $eff(\infty)$, and $ssy(\infty)$. Assume also that $\mathcal{R}_s(\infty) \subsetneq \mathcal{R}_a(\infty)$. Given M, denote by $\mathcal{R}_{fsp}(M)$ the set of feebly single-plateaued preferences and by $Z(M)$ the set of allocations. Then, for each $M \in I\!R_{++}$, the rule $\Phi^M : \mathcal{R}_a(M)^n \longrightarrow Z(M)$ satisfies strategy-proofness, efficiency, and strong symmetry (where $\mathcal{R}_a(M)$ is the set of preferences on $[0, M]$ obtained by restricting to $[0, M]$ all preferences in $\mathcal{R}_a(\infty)$) after setting $\Phi^M(\mathbf{R}) = \Phi^\infty(\mathbf{R}, M)$. Then, by Theorem 1, $\mathcal{R}_a(M) = \mathcal{R}_{fsp}(M)$ for every $M \in I\!R_{++}$. Since this is true for every M it follows that $\mathcal{R}_a(\infty) = \bigcap_{M>0} \mathcal{R}_{fsp}(M)$. Finally, one sees immediately that $\mathcal{R}_{sp}(\infty) = \bigcap_{M>0} \mathcal{R}_{fsp}(M)$. Hence $\mathcal{R}_a(\infty) = \mathcal{R}_{sp}(\infty)$. ∎

Second, the interval $\Theta(R_i)$ is intimately related with "option" sets, where

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given a rule Φ on $\mathcal{V}^n$ and a preference $R_i \in \mathcal{V}$ we define the *set of options* left open to the other agents by i declaring R_i at Φ as

$$\sigma^\Phi(R_i) = \left\{ x \in [0, M] \mid \exists R_{-i} \in \mathcal{V}^{n-1} \text{ such that } \Phi_i(R_i, R_{-i}) = x \right\}.$$

This is not surprising, since option sets also play a fundamental role to describe maximal domains in voting environments. The main two ideas are the following. Given a preference R_i, alternatives at the left (right) of the top plateau and outside the option set have to be worse than the smallest (largest) alternative in the option set. Moreover, the preference R_i has to be single-plateaued on the option set.

It is easy to show here that, given a preference $R_i \in \mathcal{R}_{fsp}$ and a strategy-proof, efficient and strongly symmetric rule on $(\mathcal{R}_{fsp})^n$, the relationship between $\Theta(R_i)$ and $\sigma^\Phi(R_i)$ is as follows. Suppose that R_i is such that M/n does not belong to an indifference interval, then $\Theta(R_i) = \sigma^\Phi(R_i)$. However, if M/n belongs to an indifference interval, then $\sigma^\Phi(R_i) = \Theta(R_i) \cup [x_0, y_0]$, where $[x_0, y_0]$ is the maximal interval of indifference for R_i that contains M/n.

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6. Footnotes

[1] See Sprumont (1995) and Barberà (1996) for two comprehensive surveys of this literature as well as for two exhaustive bibliographies

[2] Ching (1994) names this property *equal treatment of equals.* Ching and Serizawa (1998) use the name of *symmetry* when the condition $\Phi_i(\mathbf{R}) = \Phi_j(\mathbf{R})$ is replaced by $\Phi_i(\mathbf{R}) I_i \Phi_j(\mathbf{R})$.

[3] See Ching (1992), Schummer and Thomson (1997), Sprumont (1991), Thomson (1994), (1995), and (1997) for alternative characterizations of the uniform allocation rule. In a recent paper, Weymark (1999) shows that Sprumont's characterization using efficiency, strategy-proofness, and anonymity still holds if continuity of the preferences is not required.

[4] See Moulin (1984) and Berga (1998) for characterizations of strategy-proof rules under this domain restriction in a public good context.

[5] See Example 1 at the end of Section 2 for an illustration of why efficiency imposes these conditions on the intervals of indifference.

[6] We often abuse language by using utility representation terminology to refer to properties of preference relations.

[7] Notice that the efficiency of ψ implies that if $M \leq \sum \underline{p}(R_j)$ then $\psi_i(\mathbf{R}) \leq \underline{p}(R_i)$

and therefore $y_i^0 < \underline{p}(R_i)$. Symmetrically, if $\sum \overline{p}(R_j) \leq M$ then $\overline{p}(R_i) \leq \psi_i(\mathbf{R})$

and therefore $x_i^0 > \overline{p}(R_i)$. The same argument will apply also in all stages.

[8] This means that we have to replace, in Definitions 1, 2, and 3, the expression

"for all $\mathbf{R} \in \mathcal{V}^n$" by the expression "for all $(\mathbf{R}, M) \in \mathcal{V}^n \times I\!R_{++}$".

[9] It is an open question whether the maximal domain identified in Theorem 1

becomes larger if we substitute strong symmetry by symmetry.