On the Analytic Capacity γ_+

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ABSTRACT. The analytic capacity γ_+ is a version of the usual analytic capacity γ which is generated by Cauchy potentials of positive measures. Some recent results have shown the importance of γ_+ for the understanding of the metric-geometric properties of γ. This paper is devoted to the study of γ_+. Among other things, it is shown that although this capacity is not originated by a positive symmetric kernel, it satisfies some properties usually fulfilled this other type of capacities (such as Riesz capacities).

1. INTRODUCTION

The *analytic capacity* of a compact set $E \subset \mathbb{C}$ is defined as

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$ and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$. On the other hand, the *analytic capacity* γ_+ (or *capacity* γ_+) of E is

$$\gamma_+(E) = \sup_{\mu} \mu(E),$$

where the supremum is taken over all positive Radon measures μ supported on E such that the Cauchy transform $f = (1/z) * \mu$ is a function in $L^\infty(\mathbb{C})$ with $\|f\|_\infty \leq 1$. Since $[(1/z) * \mu]'(\infty) = \mu(E)$, we have $\gamma_+(E) \leq \gamma(E)$.

The notion of analytic capacity was first introduced by Ahlfors [Ah] in order to study removable singularities of bounded analytic functions. He showed that a compact set E is removable for all bounded analytic functions if and only if $\gamma(E) = 0$. Later, Vitushkin rediscovered analytic capacity and he used it for problems of rational approximation on compact sets [Vi]. The main drawback of his techniques arises from the fact that there is not a complete description of analytic capacity in metric or geometric terms.

As far as we know, the capacity γ_+ was introduced by Murai [Mu, pp. 71-72]. He introduced γ_+ only for sets supported on rectifiable curves, and he showed its relationship with the weak (1,1) boundedness of the Cauchy transform on these curves.

The main objective of this paper is to study the capacity γ_+. We will show that although γ_+ is a capacity which is not originated by a positive symmetric kernel, it satisfies some properties analogous to the properties fulfilled by this other kind of capacities (for example, Riesz capacities). In particular, γ_+ can be described in terms of an energy or a potential, and it admits a dual characterization in terms of this potential.

There are several reasons why the capacity γ_+ deserves some attention. As Murai noticed, γ_+ has a strong connection with the L^2-boundedness of the Cauchy transform. Moreover, Melnikov, Paramonov and Verdera have shown recently that γ_+ is useful to deal with problems of C^1-approximation of subharmonic functions.

On the other hand, it is easily seen that a conjecture of Melnikov about the characterization of sets with zero analytic capacity (see [Da2, Section 12]) is equivalent to the assertion $\gamma \approx \gamma_+$, that is to say, there exists some positive absolute constant C such that

$$C^{-1}\gamma_+(E) \le \gamma(E) \le C\gamma_+(E) \tag{1.1}$$

for all compact sets $E \subset \mathbb{C}$. A positive answer to this conjecture would yield a description of γ with a metric-geometric flavour, and would imply that γ is semiadditive, by the results on γ_+ in [To1]. Let us also mention that in [MTV] it has been shown that (1.1) holds for a big class of Cantor type sets.

We need to introduce some notation and definitions. We let $M(\mathbb{C})$ be the set of all complex Radon measures on $\mathbb{C}$. $M_+(\mathbb{C})$ stands for the set of all positive finite Radon measures on $\mathbb{C}$, and $\Sigma(E)$ is the set of all $\mu \in M_+(\mathbb{C})$ supported on E with $\mu(B(x,r)) \le r$ for all $x \in \mathbb{C}$.

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $R(x,y,z) = \infty$, $c(x,y,z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we let $c(x,y,z) = 0$. For $\mu \in M_+(\mathbb{C})$, we set

$$c_\mu^2(x) = \iint c(x,y,z)^2 \, d\mu(y)\, d\mu(z),$$

and we define the *curvature of* μ as

$$c^2(\mu) = \int c_\mu^2(x)\, d\mu(x) = \iiint c(x,y,z)^2 \, d\mu(x)\, d\mu(y)\, d\mu(z).$$

The notion of curvature of a measure was introduced by Melnikov [Me] when he was studying a discrete version of analytic capacity. He proved the following inequality:

$$\gamma(E) \geq C \sup_{\mu \in \Sigma(E)} \frac{\|\mu\|^{3/2}}{(\|\mu\| + c^2(\mu))^{1/2}}, \tag{1.2}$$

where $C > 0$ is some absolute constant. In [To1] it was shown that, indeed,

$$\gamma_+(E) \approx \sup_{\mu \in \Sigma(E)} \frac{\|\mu\|^{3/2}}{(\|\mu\| + c^2(\mu))^{1/2}}. \tag{1.3}$$

The arguments in [To1] also show that

$$\gamma_+(E) \approx \sup_{\mu \in \Sigma(E)} \frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)} \tag{1.4}$$

$$\approx \sup\{\|\mu\| : \mu \in \Sigma(E),\ c_\mu^2(x) \leq 1\ \forall x \in E\}. \tag{1.5}$$

Moreover, in (1.5) we can replace the condition "$\forall x \in E$" by "$\forall x \in \mathbb{C}$".

Let M be the maximal radial operator:

$$M\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{r},$$

and let $c_\mu(x) = \left(c_\mu^2(x)\right)^{1/2}$. From (1.5), it easily follows that we also have

$$\gamma_+(E) \approx \sup\{\|\mu\| : \mathrm{supp}(\mu) \subset E,\ M\mu(x) + c_\mu(x) \leq 1\ \forall x \in E\}. \tag{1.6}$$

As above, the condition "$\forall x \in E$" can be replaced by "$\forall x \in \mathbb{C}$" in (1.6). So, if we set

$$U_\mu(x) = M\mu(x) + c_\mu(x),$$

then (1.6) suggests that the function U_μ behaves as a potential for the capacity γ_+. In this paper we will obtain other results which support this assertion.

The idea of introducing the potential U_μ is due to Verdera. In [Ve2] it is shown that if one defines the energy associated to μ as $E(\mu) := \int U_\mu \, d\mu$, then

$$\gamma(E) \geq C \sup\{E(\mu)^{-1} : \mathrm{supp}(\mu) \subset E,\ \|\mu\| = 1\},$$

with $C > 0$. In fact, notice that, using our characterization (1.6) of γ_+, it follows that

$$\gamma_+(E) \approx \sup\{E(\mu)^{-1} : \mathrm{supp}(\mu) \subset E,\ \|\mu\| = 1\}.$$

The plan of the paper is the following. In Section 2 we obtain some results which show the close connection between γ_+ and the L^2 and weak $(1,1)$ boundedness of the Cauchy transform. In Section 3 we obtain a dual characterization of γ_+ in terms of an infimum involving U_μ. As a corollary, we prove that $\gamma_+(E) = 0$ if and only if there exists some finite Radon measure μ such that $U_\mu(x) = \infty$ for all $x \in E$. Also, it is shown that the capacity γ_+ of some Cantor sets can be estimated easily using these results.

In Section 4 we show that U_μ satisfies a minimum principle which implies that

$$\gamma_+(E) \approx \gamma_+(\partial_{\text{out}} E), \tag{1.7}$$

where $\partial_{\text{out}} E$ is the outer boundary of E (for the definition of $\partial_{\text{out}} E$ and additional details see Section 4). Let us remark that in the case of the analytic capacity, we have $\gamma(E) = \gamma(\partial_{\text{out}} E)$. So the estimate (1.7) is implied by the conjecture $\gamma \approx \gamma_+$.

In Section 5 we show that the assumption $\mu \in \Sigma(E)$ in the characterizations (1.3), (1.4) and (1.5) of γ_+ can be weakened. It is enough to ask $\limsup_{r\to 0} \mu(B(x,r))/r \leq 1$. This result will be used in Section 6, where some density estimates involving γ_+ related to instability will be obtained.

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2. Relationship between γ_+ and the Cauchy Transform

Given a complex Radon measure ν on $\mathbb{C}$, the *Cauchy transform* of ν is

$$C\nu(z) = \int \frac{1}{\xi - z}\, d\nu(\xi).$$

This definition does not make sense, in general, for $z \in \operatorname{supp}(\nu)$. In fact, one can see that the integral above is convergent at a.e. $(\mathcal{H}^2)$ $z \in \mathbb{C}$ (that is, at almost every $z \in \mathbb{C}$, with respect to $\mathcal{H}^2$, where $\mathcal{H}^s$ stands for the s-dimensional Hausdorff measure). This is the reason why one considers the *truncated Cauchy transform* of ν, which is defined as

$$C_\varepsilon \nu(z) = \int_{|\xi - z| > \varepsilon} \frac{1}{\xi - z}\, d\nu(\xi),$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$. Given a μ-measurable function f on $\mathbb{C}$ (where μ is a positive Radon measure on $\mathbb{C}$), we also denote $Cf \equiv C(f\,d\mu)$ and $C_\varepsilon f \equiv C_\varepsilon(f\,d\mu)$ for any $\varepsilon > 0$. It is said that the Cauchy transform is bounded on $L^2(\mu)$ if the operators C_ε are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

Using (1.5) and the main result of [To1], it is not difficult to check that

$$\gamma_+(E) \approx \sup\{\|\mu\| : \operatorname{supp}(\mu) \subset E,\ \mu \text{non-atomic},\ \|C\|_{L^2(\mu)} \leq 1\},$$

where $\|C\|_{L^2(\mu)}$ stands for the norm of the Cauchy transform on $L^2(\mu)$, that is,

$$\|C\|_{L^2(\mu)} = \sup_{\varepsilon>0} \|C_\varepsilon\|_{L^2(\mu)}.$$

Another result that shows the close relationship between γ_+ and the Cauchy transform is the following.

Proposition 2.1. *If μ is any positive Radon measure on $\mathbb{C}$, then*

$$\gamma_+(\{x : C_*\mu(x) > \lambda\}) \le C\frac{\|\mu\|}{\lambda}. \tag{2.1}$$

Let us remark that the capacity γ_+ of a general set $A \subset \mathbb{C}$ is defined as $\gamma_+(A) = \sup\{\gamma_+(E) : E \subset A,\ E \text{ compact}\}$.

In [Ve1] Verdera asked if the estimate (2.1) is true with γ_+ replaced by γ. Obviously, one obtains a positive answer if the conjecture $\gamma \approx \gamma_+$ holds.

Proof. Let $G = \{x : C_*\mu(x) > \lambda\}$. Take $\nu \in \Sigma(G)$ such that $\gamma_+(G) \approx \|\nu\|$ and $c_\nu^2(x) \le 1$ for all $x \in \mathbb{C}$. Since the Cauchy transform is bounded on $L^2(\nu)$, the maximal Cauchy transform is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\nu)$ (see [NTV] or [To2]) and so

$$\gamma_+(G) \le C\nu(G) = C\nu(\{C_*\mu(x) > \lambda\}) \le C\frac{\|\mu\|}{\lambda}. \qquad \square$$

3. A Dual Version for the Analytic Capacity γ_+

In this section we will show a characterization of γ_+ in terms of the potential U which can be understood as the dual version of (1.6). The following estimate, which was suggested to the author by Joan Verdera, is a first step towards this characterization.

Theorem 3.1. *If μ is any positive Radon measure on $\mathbb{C}$, then*

$$\gamma_+(\{x : U_\mu(x) > \lambda\}) \le C\frac{\|\mu\|}{\lambda}. \tag{3.1}$$

To obtain Theorem 3.1, we will need the following result, whose proof is similar to [To1, Lemma 3]:

Lemma 3.2. *Let μ and ν be positive Radon measures such that*

$$c^2(x,\nu,\mu) \le \beta \quad \text{for all } x \in \operatorname{supp}(\nu).$$

Then,

$$c^2(x,\nu,\mu) \le C_1 M\nu(x) M\mu(x) + 2\beta \quad \text{for all } x \in \mathbb{C}. \tag{3.2}$$

where C_1 is some absolute constant. We also have

$$c^2(x,\nu,\mu) \le C_1 M\nu(x) \cdot \sup_{z\in\operatorname{supp}(\nu)} M\mu(z) + 2\beta \quad \text{for all } x \in \mathbb{C}. \tag{3.3}$$

Proof of Theorem 3.1. We set

$$\begin{aligned} E &= \{x : U_\mu(x) > \lambda\}, \\ E_1 &= \left\{x : M\mu(x) > \frac{\lambda}{2}\right\}, \quad \text{and} \\ E_2 &= \left\{x : M\mu(x) \le \frac{\lambda}{2} \text{ and } c_\mu(x) > \frac{\lambda}{2}\right\}. \end{aligned}$$

Hence, $E \subset E_1 \cup E_2$, and $\gamma_+(E) \le C(\gamma_+(E_1) + \gamma_+(E_2))$.

It is not difficult to check that

$$\gamma_+(E_1) \le C\frac{\|\mu\|}{\lambda}. \tag{3.4}$$

We only have to take $\nu \in \Sigma(E_1)$ such that $\gamma_+(E_1) \approx \|\nu\|$ and $c_\nu^2(x) \le 1$ for all $x \in \mathbb{C}$. Since the maximal operator M is bounded from $M(\mathbb{C})$ to $L^{1,\infty}(\nu)$, we get

$$\gamma_+(E_1) \le C\nu(E_1) = \nu\left(\left\{M\mu(x) > \frac{\lambda}{2}\right\}\right) \le C\frac{\|\mu\|}{\lambda}.$$

Now we will show that

$$\gamma_+(E_2) \le C\frac{\|\mu\|}{\lambda}. \tag{3.5}$$

Let us take $\nu \in \Sigma(E_2)$ such that $\gamma_+(E_2) \approx \|\nu\|$ and $c_\nu^2(x) \le 1$ for all $x \in \mathbb{C}$. Observe that

$$c^2(\mu,\nu,\nu) \le \|\mu\| = \beta\|\nu\|,$$

where $\beta = \|\mu\|/\|\nu\|$. Then there exists a closed subset $F \subset \operatorname{supp}(\nu)$ such that $\nu(F) \ge \|\nu\|/4$ and $c^2(x,\mu,\nu) \le 2\beta$ for all $x \in F$. Therefore, $c^2(x,\mu,\nu_{|F}) \le 2\beta$ for all $x \in F$, and by Lemma 3.2,

$$c^2(x,\mu,\nu_{|F}) \le C\lambda + 4\beta$$

for all $x \in \mathbb{C}$, since $M\mu(z) \le \lambda/2$ for all $z \in \operatorname{supp}(\nu)$. Thus,

$$\begin{aligned} \gamma_+(E_2) &\le C\int d\nu_{|F} \le \frac{C}{\lambda^2}\int c^2(x,\mu,\mu)\, d\nu_{|F}(x) \\ &= \frac{C}{\lambda^2}\int c^2(y,\mu,\nu_{|F})\, d\mu(y) \le \frac{C}{\lambda^2}\int (C\lambda + 4\beta)\, d\mu \\ &\le \frac{C}{\lambda^2}\left(\lambda\|\mu\| + \frac{\|\mu\|^2}{\|\nu\|}\right). \end{aligned}$$

So we obtain

$$\gamma_+(E_2) \leq C_2 \left(\frac{\|\mu\|}{\lambda} + \frac{\|\mu\|^2}{\lambda^2 \gamma_+(E_2)} \right). \tag{3.6}$$

From this inequality we derive (3.5). Indeed, if $\|\mu\|/\lambda \geq \|\mu\|^2/(\lambda^2\gamma_+(E_2))$, then (3.6) implies $\gamma_+(E_2) \leq 2C_2(1/\lambda)\|\mu\|$, and if $\|\mu\|/\lambda \leq \|\mu\|^2/(\lambda^2\gamma_+(E_2))$ we also get (3.5). □

Theorem 3.3. *For $E \subset \mathbb{C}$ compact, we have*

$$\gamma_+(E) \approx \inf\{\|\mu\| : \mu \in M_+(\mathbb{C}),\ U_\mu(x) \geq 1\ \forall x \in E\}. \tag{3.7}$$

Before proving Theorem 3.3 we will state and prove the following corollary.

Corollary 3.4. *Let $E \subset \mathbb{C}$ be compact. Then, $\gamma_+(E) = 0$ if and only if there exists some measure $\mu \in M_+(\mathbb{C})$ such that $U_\mu(x) = \infty$ for all $x \in E$.*

Proof. If $U_\mu(x) = \infty$ for all $x \in E$, then $\gamma_+(E) = 0$. This follows easily from Theorem 3.3 or directly from (3.1).

Suppose that $\gamma_+(E) = 0$. By Theorem 3.3, for each n there exists some measure $\mu_n \in M_+(\mathbb{C})$ with $\|\mu_n\| \leq 2^{-n}$ and such that $M\mu_n(x) + c_{\mu_n}(x) \geq 1$ for all $x \in E$. We set $\mu = \sum_{n=1}^\infty n\mu_n$. Then $\mu \in M_+(\mathbb{C})$ and, for each n, $M\mu(x) + c_\mu(x) \geq n(M\mu_n(x) + c_{\mu_n}(x)) \geq n$ for $x \in E$. □

Let us remark that the measure μ constructed in the proof of the corollary also satisfies $U_\mu(y) < \infty$ for all $y \in E^c$.

We will need the following estimate for the proof of Theorem 3.3:

Lemma 3.5. *Let $\nu \in M_+(\mathbb{C})$. Let $Q \subset \mathbb{C}$ be a closed square with side length δ. Let L be a closed segment of length $\delta/2$ parallel to one of sides of Q and centered in Q. Suppose that $\nu_{|Q} = a \cdot \mathcal{H}^1_{|L}$ and $\nu(\frac{3}{2}Q \setminus Q) = 0$. Then, there exists some absolute constant C_3 such that for all $x, y \in Q$,*

$$c_\nu^2(x) \leq \frac{10}{9} c_\nu^2(y) + C_3 M\nu(y)^2,$$

and

$$c_\nu^2(x) \leq \frac{10}{9} c_\nu^2(y) + C_3 M\nu(x)^2.$$

The proof of Lemma (3.5) follows by standard arguments.

To prove Theorem 3.3 we will need to apply a variational argument on some 'nice' approximation of E by another compact $\tilde{E}$. The following lemma will be very useful.

Lemma 3.6. *Consider a grid of squares of side length $\delta > 0$ in $\mathbb{C}$ with sides parallel to the axes. Take a finite collection of closed squares $\{Q_i\}_{i\in I}$ of the grid. For each $i \in I$, let L_i be the closed segment of length $\delta/2$ centered in Q_i and parallel to the x axis. Set $\tilde{E} = \bigcup_{i\in I} L_i$. Let $\Sigma_0(\tilde{E})$ be the subset of $\Sigma(\tilde{E})$ of measures μ of the form $\mu = \sum_{i\in I} a_i \mathcal{H}^1_{|L_i}$. There exists a measure $\nu \in \Sigma_0(\tilde{E})$ such that*

$$\frac{\|\nu\|^2}{\|\nu\| + c^2(\nu)} = \sup_{\mu\in\Sigma_0(\tilde{E})} \frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)}. \tag{3.8}$$

The maximal measure ν satisfies

$$c^2(\nu) \le 2\|\nu\| \tag{3.9}$$

and

$$U_\nu(x) \ge C_4, \tag{3.10}$$

for all $x \in \tilde{E}$, where $C_4 > 0$ is some absolute constant.

Proof. The existence of ν follows easily by a compactness argument.

Let us see that (3.9) holds. Since ν is maximal and $\nu/2$ is also in $\Sigma_0(\tilde{E})$, we have

$$\frac{\|\nu\|^2}{\|\nu\| + c^2(\nu)} \ge \frac{\|\nu\|^2/4}{\|\nu\|/2 + c^2(\nu)/8}.$$

Therefore,

$$\frac{\|\nu\|}{2} + \frac{c^2(\nu)}{8} \ge \frac{\|\nu\|}{4} + \frac{c^2(\nu)}{4}.$$

That is, $c^2(\nu) \le 2\|\nu\|$.

Now we turn our attention to (3.10). Suppose that $M\nu(x) \le \frac{1}{10}$ for some $x \in L_i \subset \tilde{E}$. Since ν is a constant multiple of $\mathcal{H}^1_{|L_i}$ on L_i, $M\nu(x) \le \frac{1}{5}$ for all $x \in L_i$. For each $\lambda > 0$, we define the measure $\nu_\lambda = \nu + \lambda\mathcal{H}^1_{|L_i}$. It is not difficult to check that if λ is small enough, then $M\nu_\lambda(y) \le 1$ for all $y \in \tilde{E}$. This is clear if $y \in L_i$. Otherwise, if $d = |x - y| \ge \delta$, then

$$\frac{\nu_\lambda(B(y,r))}{r} = \frac{\nu(B(y,r))}{r} \le 1$$

for $r \le d - \delta/2$. If $r > d - \delta/2 \ge d/2$, then

$$\frac{\nu_\lambda(B(y,r))}{r} \le \frac{\nu(B(x,4r)) + \lambda\delta/2}{r} \le 4(M\nu(x) + \lambda) \le \frac{4}{10} + 4\lambda.$$

Therefore, $\nu_\lambda \in \Sigma_0(\tilde{E})$ for $\lambda > 0$ small enough. Since ν satisfies (3.8), if we denote

$$f(\lambda) = \frac{\|\nu_\lambda\|^2}{\|\nu_\lambda\| + c^2(\nu_\lambda)},$$

then $f'(0) \le 0$. Observe that $f(\lambda)$ equals

$$\frac{(\|\nu\| + \lambda\delta/2)^2}{\|\nu\| + \lambda\delta/2 + c^2(\nu) + 3\lambda c^2(\mathcal{H}^1_{|L_i}, \nu, \nu) + 3\lambda^2 c^2(\nu, \mathcal{H}^1_{|L_i}, \mathcal{H}^1_{|L_i}) + \lambda^3 c^2(\mathcal{H}^1_{|L_i})}.$$

So,

$$f'(0) = \frac{\|\nu\|\delta(\|\nu\| + c^2(\nu)) - \|\nu\|^2(\delta/2 + 3c^2(\mathcal{H}^1_{|L_i}, \nu, \nu))}{(\|\nu\| + c^2(\nu))^2}.$$

Therefore, $f'(0) \le 0$ if and only if

$$\delta(\|\nu\| + c^2(\nu)) \le \|\nu\| \left(\frac{\delta}{2} + 3c^2(\mathcal{H}^1_{|L_i}, \nu, \nu)\right).$$

That is,

$$\frac{\|\nu\| + 2c^2(\nu)}{\|\nu\|} \le \frac{6c^2(\mathcal{H}^1_{|L_i}, \nu, \nu)}{\delta}.$$

Therefore, $c^2(\mathcal{H}^1_{|L_i}, \nu, \nu)/\delta \ge \frac{1}{6}$. So there exists some $x' \in L_i$ such that $c^2(x', \nu, \nu) \ge \frac{1}{3}$. By Lemma 3.5

$$c^2_\nu(x) \ge \frac{9}{10}\left(\frac{1}{3} - C_3 M\nu(x)^2\right).$$

If $C_3 \cdot M\nu(x)^2 \le \frac{1}{6}$, then

$$c^2_\nu(x) \ge \frac{3}{20}. \tag{3.11}$$

So we have proved that if $M\nu(x) \le \min(\frac{1}{10}, 1/(6C_3)^{1/2})$, then (3.11) holds, and the lemma follows. ◻

Proof of Theorem 3.3. Let $\mu \in M_+(\mathbb{C})$ be such that $M\mu(x) + c_\mu(x) \ge 1$ for all $x \in E$. By Theorem 3.1, $\gamma_+(E) \le C\|\mu\|$.

So we must show that for each $\varepsilon > 0$ there exists some measure $\mu \in M_+(\mathbb{C})$ such that $\gamma_+(E) \ge C\|\mu\| - \varepsilon$, where $C > 0$ is some absolute constant, with μ satisfying $M\mu(x) + c_\mu(x) \ge 1$ on E. Let $\delta > 0$ be such that $\gamma_+(E) \ge \gamma_+(V_\delta(E)) - \varepsilon$, where $V_\delta(E)$ stands for the δ-neighborhood of E. Consider a grid of squares of side length $\delta > 0$ with sides parallel to the axes. Let $\{Q_i\}_{i\in I}$ the closed squares of the grid that intersect E. For each $i \in I$, let L_i be the closed segment of length $\delta/2$ centered in Q_i and parallel to the x axis. Let $\tilde{E} = \bigcup_{i\in I} L_i$. Notice that $\tilde{E} \subset V_\delta(E)$. Thus $\gamma_+(E) \ge \gamma_+(\tilde{E}) - \varepsilon$.

Let $\nu \in \Sigma_0(\tilde{E})$ be the maximal measure of Lemma 3.6 for the compact set $\tilde{E}$. Since $c^2(\nu) \le 2\|\nu\|$, we have $\gamma_+(\tilde{E}) \ge C\|\nu\|$. We know that $M\nu(x) + c_\nu(x) \ge$

C_4 for all $x \in \tilde{E}$, with $C_4 > 0$. We are going to show that this also holds for $x \in E$, changing the constant C_4 by another constant C_4' smaller enough. Clearly, this will will finish the proof of the theorem: Only has to take $\mu = \nu / C_4'$.

Let $x \in E$. Suppose that $M\nu(x) \leq C_5$, where $C_5 > 0$ is some constant much smaller than C_4, which will be defined later. Let Q_i be a square of the grid such that $x \in Q_i$. It is not difficult to check that for $y \in L_i$, $M\nu(y) \leq 4C_5$. Thus for $y \in L_i$, $c_\nu(y) \geq C_4 - 4C_5 \geq C_4/2$, choosing C_5 small enough. Now, by Lemma 3.5

$$c_\nu^2(x) \geq \frac{9}{10}\left(\frac{C_4^2}{4} - C_3 M\nu(x)^2\right) \geq \frac{9}{10}\left(\frac{C_4^2}{4} - C_3 \cdot C_5^2\right) \geq \frac{C_4^2}{16},$$

choosing C_5 small enough again. □

3.1. An application to the study of the analytic capacity γ_+ of some Cantor sets. We construct a Cantor set $E(\lambda) \subset \mathbb{C}$ as follows. Let $\lambda := \{\lambda_n\}_n \subset \mathbb{R}$ be a non-increasing sequence with limit $\frac{1}{4}$, satisfying $\frac{1}{4} \leq \lambda_n \leq \frac{1}{3}$ for all n. In $\mathbb{R}$, we consider the sets $K_0 = [0,1]$, $K_1 = [0,\lambda_1] \cup [1-\lambda_1, 1]$, and for each n, K_n made up of a finite union of closed intervals which have been obtained from K_{n-1} in the following way. We replace each connected component $K_{n-1,j}$ of K_{n-1} by the two endmost closed intervals contained in this same component, each one with length equal to λ_n times the length of connected component $K_{n-1,j}$. We set

$$K(\lambda) = \bigcap_{n=1}^{\infty} K_n, \quad \text{and then} \quad E(\lambda) = K(\lambda) \times K(\lambda).$$

Let us observe that K_n the union of 2^n closed intervals of length $\sigma_n = \lambda_1 \cdots \lambda_n$. So, if we denote $E_n = K_n \times K_n$, we have

$$E(\lambda) = \bigcap_{n=1}^{\infty} E_n, \quad \text{where } E_n = \bigcup_{j=1}^{4^n} E_{n,j},$$

with $E_{n,j}$ being a square with side length σ_n for every $j = 1, \ldots, 4^n$.

The Cantor sets $E(\lambda)$ were introduced by Garnett in [Gar], where he asked for what sequences λ the set $E(\lambda)$ has non zero analytic capacity. He proved that if $\sum_n \frac{4^{-n}}{\sigma_n} < \infty$, then $\gamma(E(\lambda)) > 0$.

Now it is known that

$$C^{-1}\left(\sum_n \left(\frac{4^{-n}}{\sigma_n}\right)^2\right)^{-1/2} \leq \gamma_+(E(\lambda)) \leq C\left(\sum_n \left(\frac{4^{-n}}{\sigma_n}\right)^2\right)^{-1/2}. \tag{3.12}$$

The left inequality follows from an estimate obtained by Mattila [Ma] and from the results about γ_+ in [To1]. Indeed, if μ is a measure supported on $E(\lambda)$ such that $\mu(E_{n,j}) = 4^{-n}$ for all n, j, it is shown in [Ma] that $\mu(B(x,r)) \leq Cr$ for all x, r and

$$c^2(\mu) \leq C \sum_n \left(\frac{4^{-n}}{\sigma_n}\right)^2. \tag{3.13}$$

This inequality and the characterization of $\gamma_+(E)$ in (1.3) yield the left inequality in (3.12).

The right inequality in (3.12) was obtained recently by Eiderman [Ei]. Let us show that this inequality can be obtained also by a straightforward application of Theorem 3.3. Indeed, it is not difficult to check that, for any $x \in E(\lambda)$,

$$c_\mu^2(x, E(\lambda), E(\lambda)) \approx \sum_{n=0}^{\infty} \left(\frac{4^{-n}}{\sigma_n}\right)^2,$$

where μ is the measure defined above. Then, applying Theorem 3.3 to the measure

$$\tilde{\mu} = \mu \cdot \left(\sum_{n=0}^{\infty} \left(\frac{4^{-n}}{\sigma_n}\right)^2\right)^{-1/2},$$

we deduce the right inequality of (3.12).

Let us notice that recently, in [MTV], it has been shown that

$$\gamma(E(\lambda)) \approx \gamma_+(E(\lambda))$$

for all the sequences λ as above. The estimates (3.12) are necessary for the proof, which is based on the use of the local $T(b)$ theorem of M. Christ.

4. A Minimum Principle and the Capacity γ_+ of the Outer Boundary

From Lemma 3.2, it easily checked that the potential U_μ satisfies the following maximum principle: *If* $U_\mu(x) \leq \lambda$ *for all* $x \in \operatorname{supp}(\mu)$, *then* $U_\mu(y) \leq C\lambda$ *for all* $y \in \mathbb{C}$. In this Section we will see that U_μ also satisfies a minimum principle. However, this result (and its subsequent corollary) depend on the following result of P. Jones, whose proof, to the best of our knowledge, has not been published up to now.

Claim 4.1 (P. Jones, 1998). *If* $E \subset \mathbb{C}$ *is compact and connected, then*

$$\gamma_+(E) \approx \operatorname{diam}(E).$$

Given a compact set $E \subset \mathbb{C}$, we denote by $\hat{E}$ the union of E and the bounded connected components of E^c. The intersection between E and the closure of the unbounded component of E^c is the outer boundary of E, and we write it as $\partial_{\text{out}} E$. Recall that $\partial \hat{E} = \partial_{\text{out}} E$.

Let us state our *minimum principle*.

Theorem 4.2 (Minimum principle). *Assume that Claim 4.1 holds. Let* $\lambda > 0$ *be some fixed constant. If* $E \subset \mathbb{C}$ *is compact and* μ *is a positive Radon measure on* $\mathbb{C}$ *such that* $U_\mu(x) \geq \lambda$ *for all* $x \in \partial_{\text{out}} E$*, then* $U_\mu(y) \geq C^{-1}\lambda$ *for all* $y \in E$.

Before the proof, let us state and prove the following corollary:

Corollary 4.3. *Assume that Claim 4.1 holds. If* $E \subset \mathbb{C}$ *is compact, then*

$$\gamma_+(E) \approx \gamma_+(\partial_{\text{out}} E).$$

Proof. Obviously, $\gamma_+(\partial_{\text{out}} E) \leq \gamma_+(E)$. On the other hand, by Theorem 3.3 there exists some measure μ such that $\gamma_+(\partial_{\text{out}} E) \approx \|\mu\|$ and $U_\mu(x) \geq 1$ for all $x \in \partial_{\text{out}} E$. By Theorem 4.2, $U_\mu(y) \geq C^{-1}$ for all $y \in E$, and so $\gamma_+(E) \leq C\|\mu\|$, by Theorem 3.1. Therefore, $\gamma_+(E) \leq C\gamma_+(\partial_{\text{out}} E)$. ☐

Let us remark that in the case of the analytic capacity, the proof of the identity $\gamma(E) = \gamma(\partial_{\text{out}} E)$ is straightforward. In the case of γ_+, the proof of $\gamma_+(E) \approx \gamma_+(\partial_{\text{out}} E)$ relies on Claim 4.1, which seems to be a difficult result.

Proof of Theorem 4.2. We will show that there exists some $\varepsilon > 0$ (small enough) such that if $y \in \mathring{\hat{E}}$ is such that $M\mu(y) \leq \varepsilon\lambda$, then $U_\mu(y) \geq C^{-1}\lambda$.

Let $y \in \mathring{\hat{E}}$, $d = \operatorname{dist}(y, \partial_{\text{out}} E)$ and $\Delta = B(y, d)$. Then, it is not difficult to check that, for $x \in 2\Delta$,

$$|c_{\mu_{|(4\Delta)^c}}(x) - c_{\mu_{|(4\Delta)^c}}(y)| \leq CM\mu(y) \leq C\varepsilon\lambda. \tag{4.1}$$

This follows from the inequality

$$|c(x,z,t) - c(y,z,t)| \leq C\frac{d}{|y-z|\,|y-t|}$$

for $x \in 2\Delta$ and z, $t \in (4\Delta)^c$ (see Lemma 2.4 of [To1], for example) and some standard estimates.

Assume that

$$U_{\mu_{|(4\Delta)^c}}(x_0) \geq \frac{\lambda}{10} \tag{4.2}$$

for some $x_0 \in (\partial_{\text{out}} E) \cap 2\Delta$. If $M\mu_{|(4\Delta)^c}(x_0) \geq \lambda/20$, then one easily gets that $M_{|(4\Delta)^c}(y) \geq C^{-1}\lambda$, which implies $U_\mu(y) \geq C^{-1}\lambda$. If $c_{\mu_{|(4\Delta)^c}}(x_0) \geq \lambda/20$, then $c_{\mu_{|(4\Delta)^c}}(y) \geq \lambda/40$ by (4.1), if ε has been chosen small enough. Thus if (4.2) holds, $U_\mu(y) \geq C^{-1}\lambda$ in any case.

Suppose now that (4.2) does not hold for any $x \in (\partial_{\text{out}} E) \cap 2\Delta$, that is,

$$U_{\mu_{|(4\Delta)^c}}(x) \leq \frac{\lambda}{10} \tag{4.3}$$

for all $x \in (\partial_{\text{out}} E) \cap 2\Delta$. We will show that this implies that

$$U_{\mu_{|4\Delta}}(x) \geq C^{-1}\lambda \tag{4.4}$$

for all $x \in (\partial_{\text{out}} E) \cap 2\Delta$. Assuming this for the moment, by Theorem 3.1,

$$\gamma_+((\partial_{\text{out}} E) \cap 2\Delta) \leq C\frac{\mu(4\Delta)}{\lambda}. \tag{4.5}$$

If Ω is the connected component of $\mathring{\hat{E}}$ which contains y, then Ω is simply connected and thus $\partial\Omega$ is connected. Thus $\partial\Omega \cap 2\Delta$ has at least one connected component with diameter $\geq d$. Since $\partial\Omega \subset \partial_{\text{out}} E$, by Claim 4.1,

$$\gamma_+((\partial_{\text{out}} E) \cap 2\Delta) \geq \gamma_+(\partial\Omega \cap 2\Delta) \geq C^{-1}d.$$

Therefore, by (4.5), $d \leq C\mu(4\Delta)/\lambda$, and so $U_\mu(y) \geq M\mu(y) \geq C^{-1}\lambda$.

Now we only have to show that (4.3) implies (4.4). Since $U_\mu(x) \geq \lambda$, we have either $M\mu(x) \geq \lambda/2$ or $c_\mu(x) \geq \lambda/2$. Notice that

$$M\mu_{|(4\Delta)^c}(x) \leq U_{\mu_{|(4\Delta)^c}}(x) \leq \frac{\lambda}{10},$$

and so if $M\mu(x) \geq \lambda/2$, then

$$U_{\mu_{|4\Delta}}(x) \geq M\mu_{|4\Delta}(x) \geq \left(\frac{1}{2} - \frac{1}{10}\right)\lambda,$$

and (4.4) holds. If $c_\mu(x) \geq \lambda/2$, then we write

$$c_\mu^2(x) = c^2_{\mu_{|4\Delta}}(x) + c^2_{\mu_{|(4\Delta)^c}}(x) + 2c^2_{\mu_{|4\Delta},\mu_{|(4\Delta)^c}}(x).$$

Observe that $c_\mu^2(x) \geq \lambda^2/4$, $c^2_{\mu_{|(4\Delta)^c}}(x) \leq \lambda^2/100$ and, also,

$$\begin{aligned} c^2_{\mu_{|4\Delta},\mu_{|(4\Delta)^c}}(x) &\leq C\int_{z\in 4\Delta}\int_{t\in(4\Delta)^c} \frac{1}{|y-t|^2}\,d\mu(t)\,d\mu(z) \\ &\leq C\int_{z\in 4\Delta} \frac{M\mu(y)}{d}\,d\mu(z) \\ &\leq CM\mu(y)^2 \leq C\varepsilon^2\lambda^2 \leq \frac{\lambda^2}{100}, \end{aligned}$$

choosing ε small enough in the last inequality. Therefore we get

$$U_{\mu_{|4\Delta}}(x) \geq c_{\mu_{|4\Delta}}(x) \geq C^{-1}\lambda.$$

So in any case (4.4) holds. □

Remark 4.4. The main tool to prove the minimum principle of Theorem 4.2 and its corollary is Claim 4.1. Let us see that the assertion in this claim also can be obtained from Corollary 4.3. Let $\Gamma_0 \subset \mathbb{C}$ be some simple curve. We may assume that the endpoints a, b of Γ_0 are on the x axis, with $a < b$ and $\operatorname{diam}(\Gamma_0) = |a-b|$. We denote by Γ_n the curve $\Gamma_0 + n|a-b|$, (a translation of Γ_0 in the x direction). Let Γ^n be the curve obtained joining $\Gamma_0, \Gamma_1, \dots, \Gamma_n$. Let $E_n \subset \mathbb{C}$ be the region enclosed by $\Gamma^n, \Gamma^n + i4|a-b|$ and the two vertical segments joining the endpoints of Γ^n and $\Gamma^n + i4|a-b|$. Since E_n contains a horizontal segment of length $(n+1)|a-b|$, we get

$$\gamma_+(E_n) \geq C_6^{-1}(n+1)|a-b|. \tag{4.6}$$

By Corollary 4.3 and the countable semiadditivity of γ_+, we have

$$\gamma_+(E_n) \leq C\gamma_+(\partial E_n) \leq C_7(2(n+1)\gamma_+(\Gamma_0) + 8|a-b|). \tag{4.7}$$

So, by (4.6) and (4.7) we get

$$\gamma_+(\Gamma_0) \geq \frac{C_6^{-1}(n+1)|a-b| - C_7^{-1}8|a-b|}{2(n+1)}.$$

Letting $n \to \infty$, we obtain

$$\gamma_+(\Gamma_0) \geq C_8^{-1}|a-b|.$$

Hence the minimum principle of Theorem 4.2 and the fact that the capacity γ_+ of a continuum is comparable to its diameter are equivalent.

5. Characterization of γ_+ by Means of Measures with Bounded Upper Density

Given some Radon measure μ on $\mathbb{C}$, we denote by $\Theta_\mu^*(x)$ its 1-dimensional upper density at x:

$$\Theta_\mu^*(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{2r}.$$

Let $\Lambda(E)$ be the set of Radon measures supported on E such that $\Theta_\mu^*(x) \leq 1$ for μ-a.e. $x \in E$. The main objective of this section is to prove the following result.

Theorem 5.1. *Let $E \subset \mathbb{C}$ be compact. Then,*

$$\gamma_+(E) \approx \sup_{\mu \in \Lambda(E)} \frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)}.$$

Let us remark that analogous results can be obtained in connection with the characterizations (1.3), (1.5) and (1.6) of γ_+. For example, one can describe γ_+ in terms of the potential $\tilde{U}_\mu(x) := \Theta^*_\mu(x) + c_\mu(x)$:

$$\gamma_+(E) \approx \sup\{\|\mu\| : \text{supp}(\mu) \subset E,\ \tilde{U}_\mu(x) \le 1\ \forall x \in E\}.$$

To prove Theorem 5.1, notice that $\Sigma(E) \subset \Lambda(E)$, and so the inequality

$$\gamma_+(E) \le C \sup_{\mu \in \Lambda(E)} \frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)}$$

is trivial. For the converse inequality we will need some lemmas. The first one is an estimate involving $c^2(\mu)$.

Lemma 5.2. *Let μ be some Radon measure supported on $B(x_0, R)$, with $\Theta^*_\mu(x) \le 1$ for μ-a.e. $x \in \mathbb{C}$. If $c^2(\mu) \le C_9\mu(B(x_0, R))$, then $\mu(B(x_0, R)) \le MR$, where M is some constant depending only on C_9.*

To prove this lemma we will use the following remarkable result due to G. David and J.C. Léger [Lé, Proposition 1.2].

Theorem 5.3 (David, Léger). *For any constant $C_{10} \ge 10$, there exists a number $\eta > 0$ such that if μ is any positive Radon measure supported on $\bar{B}(x_0, R)$ satisfying $\mu(B(x_0, R)) \ge R$, $\mu(B(x, r)) \le C_{10}r$ for all $x \in \mathbb{C}$, $r > 0$, and $c^2(\mu) \le \eta R$, then there exists a Lipschitz graph Γ such that $\mu(\Gamma) \ge \frac{99}{100}\mu(\mathbb{C})$. Moreover, the Lipschitz constant of Γ depends only on C_{10} and η.*

Proof of Lemma 5.2. Suppose that $\mu(B(x_0, R)) = MR$ for some big $M > 10^6$. Let

$$E = \{x \in B(x_0, R) : c^2_\mu(x) \le 2C_9\}.$$

By Chebychev, $\mu(E) \ge \mu(B(x_0, R))/2$. We set

$$E_n = \left\{x \in E : \mu(B(x, r)) \le 3r \text{ if } 0 < r \le \frac{1}{n}\right\}.$$

Since $\mu(\mathbb{C} \setminus \bigcup_{n=1}^\infty E_n) = 0$, we can choose n big enough so that $\mu(E_n) \ge \|\mu\|/3$. We denote $\nu = \mu_{|E_n}$.

We claim that there exists some ball $B(x_1, R_1)$, with $x_1 \in B(x_0, R)$, such that

$$\nu(B(x_1, R_1)) \ge \frac{M}{100} R_1 \tag{5.1}$$

and

$$\nu(B(x, r)) \le 100Mr \quad \text{for all } x \in \mathbb{C} \text{ and } 0 < r \le R_1. \tag{5.2}$$

Indeed, let Q_0 be the square concentric with $B(x_0, R)$ and side length $4R$. Let $\mathcal{D}(Q_0)$ be the collection of dyadic squares generated by Q_0 which are contained in Q_0. That is, $\mathcal{D}(Q_0) = \bigcup_{k=0}^{\infty} \mathcal{D}_k(Q_0)$, where $\mathcal{D}_k(Q_0)$ stands for the family of 2^{2k} squares with disjoint interiors and side length $2^{-k}\ell(Q_0)$ contained in Q_0.

For each square $Q \in \mathcal{D}(Q_0)$ with side length $\le \frac{1}{10}n$ we have $\nu(Q) \le 10\ell(Q) < \frac{M}{20}\ell(Q)$. On the other hand, $\nu(Q_0) \ge \frac{M}{10}\ell(Q_0)$. So there exists some square $Q_1 \in \mathcal{D}(Q_0)$ such that $\nu(Q_1) \ge \frac{M}{20}\ell(Q_1)$, whose side length is minimal. If we choose x_1 as the center of Q_1 and we set $R_1 = \ell(Q_1)$, then $B(x_1, R_1)$ fulfils the required properties.

Now we consider the measure $\sigma = \frac{100}{M}\nu_{|B(x_1,R_1)}$. Notice that $\operatorname{supp}(\sigma) \subset \bar{B}(x_1, R_1)$, $\|\sigma\| \ge R_1$, $\sigma(B(x,r)) \le 10^4 r$ for all $x \in \mathbb{C}$, and

$$c_\sigma^2(x) \le \frac{10^4}{M^2}\, c_\nu^2(x) \le \frac{10^4}{M^2}\, c_\mu^2(x) \le \frac{10^4\, 2C_9}{M^2}$$

for all $x \in E \cap B(x_1, R_1)$. Hence,

$$c^2(\sigma) \le \frac{C}{M^2}\, \|\sigma\| \le \frac{C_{11}}{M^2}\, R_1.$$

If $M^2 \ge C_{11}/\eta$, by Theorem 5.3 we derive that there exists a Lipschitz graph Γ, with Lipschitz constant L bounded above, such that $\sigma(\Gamma) \ge \frac{99}{100}\|\sigma\|$. Therefore,

$$\mu(\Gamma \cap B(x_1, R_1)) \ge \frac{99}{100}\, \mu(B(x_1, R_1) \cap E_n) \ge \frac{99}{10000}\, M R_1.$$

Since $\Theta_\mu^*(x) \le 1$ μ-a.e., we have

$$\mu(\Gamma \cap B(x_1, R_1)) \le 2L R_1.$$

Thus, $M \le CL$. □

We will prove Theorem 5.1 by an application of a variational argument on a set $\tilde{E}$ which approximates E, as in Lemma 3.6.

Lemma 5.4. *Consider a grid of squares of side length $\delta > 0$ in $\mathbb{C}$ with sides parallel to the axes. Take a finite collection of closed squares $\{Q_i\}_{i\in I}$ of the grid. For each $i \in I$, let L_i be the closed segment of length $\delta/2$ centered in Q_i and parallel to the x axis. Set $\tilde{E} = \bigcup_{i\in I} L_i$. Let $\Lambda_0(\tilde{E})$ be the subset of measures μ of the form $\mu = \sum_{i\in I} a_i\, \mathcal{H}^1_{|L_i}$, with $0 \le a_i \le 1$. There exists a measure $\nu \in \Lambda_0(\tilde{E})$ such that*

$$\frac{\|\nu\|^2}{\|\nu\| + c^2(\nu)} = \sup_{\mu \in \Lambda_0(\tilde{E})} \frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)} =: \gamma_{+,0}(\tilde{E}). \tag{5.3}$$

The maximal measure ν *satisfies*

$$c^2(\nu) \le 2\|\nu\| \tag{5.4}$$

and

$$\nu(B(x,r)) \le C_{12} r \quad \text{for all } x \in \tilde{E}, r > 0, \tag{5.5}$$

where $C_{12} > 0$ *is some absolute constant. Also,* $\gamma_+(\tilde{E}) \ge C^{-1}\gamma_{+,0}(\tilde{E})$.

Proof. The existence of ν follows easily by a compactness argument. Moreover, the same argument used in Lemma 3.6 to prove (3.9) applies to present situation and yields (5.4). On the other hand, the inequality $\gamma_+(\tilde{E}) \ge C^{-1}\gamma_{+,0}(\tilde{E})$ is a direct consequence of (5.5), which implies that the maximal measure ν belongs to $\Sigma(\tilde{E})$, after dividing by some absolute constant.

It remains to see that (5.5) holds. Consider a fixed ball $B(x_0, R)$. We only need to deal with the case $R \ge 10\delta$, say. Let $I_0 = \{i \in I : L_i \cap B(x_0,R) \neq \varnothing\}$ and $\tilde{E}_0 = \bigcup_{i\in I_0} L_i$. Given $\lambda \in [0,1]$, let $\nu_\lambda = \nu - \lambda \nu_{|\tilde{E}_0}$. It is clear that $\nu_\lambda \in \Lambda_0(\tilde{E})$. Let $f(\lambda)$ be the function

$$f(\lambda) = \frac{\|\nu_\lambda\|^2}{\|\nu_\lambda\| + c^2(\nu_\lambda)}.$$

By the maximality of ν, $f'(0) \le 0$.

Operating as in the proof of Lemma 3.6 (we only need to replace $\mathcal{H}^1_{|L_i}$ by $-\nu_{|\tilde{E}_0}$, and $\delta/2$ by $-\nu(\tilde{E}_0)$), we deduce that $f'(0) \le 0$ is equivalent to

$$2\nu(\tilde{E}_0)(\|\nu\| + c^2(\nu)) \ge \|\nu\|(\nu(\tilde{E}_0) + 3c^2(\nu_{|\tilde{E}_0}, \nu, \nu)),$$

that is,

$$\nu(\tilde{E}_0)\frac{\|\nu\| + 2c^2(\nu)}{3\|\nu\|} \ge c^2(\nu_{|\tilde{E}_0}, \nu, \nu).$$

Since $c^2(\nu) \le 2\|\nu\|$, this implies

$$c^2(\nu_{|\tilde{E}_0}) \le \frac{5}{3}\nu(\tilde{E}_0).$$

From Lemma 5.2, we deduce $\nu(\tilde{E}_0) \le CR$. □

Proof of Theorem 5.1. Given any measure $\mu \in \Lambda(E)$, we have to show that

$$\gamma_+(E) \ge C^{-1}\frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)}. \tag{5.6}$$

Let $E_n = \{x \in \operatorname{supp}(\mu) : \mu(B(x,r)) \leq 3r \text{ if } 0 < r \leq 1/n\}$. We choose n big enough so that $\mu(E_n) \geq \|\mu\|/2$. Given $\varepsilon_0 > 0$ arbitrarily small, let $\delta_0 > 0$ be such that $\gamma_+(E) \geq \gamma_+(V_{\delta_0}(E)) - \varepsilon_0$.

Notice that there exists some (big) constant C_{13} such that $\mu(B(x,r) \cap E_n) \leq C_{13} r$ for all $r > 0$, with C_{13} depending on n and $\operatorname{diam}(E)$. We let $N \geq 10$ be the least integer such that $C_{13} \leq 2^N$.

Consider a grid of squares with side length $\delta := \min(\delta_0/2, 2^{-2N}/n)$, with sides parallel to the axes. Let $\{Q_i\}_{i\in I}$ be the closed squares of the grid that intersect E_n. For each $i \in I$, let L_i be the closed segment of length $\delta/2$ centered in Q_i and parallel to the x axis. Let $\tilde{E} = \bigcup_{i\in I} L_i$, and let ν be the measure supported on $\tilde{E}$ which coincides with some multiple of $\mathcal{H}^1_{|L_i}$ on each segment L_i, and which satisfies $\nu(L_i) = \mu(Q_i \cap E_n)$. Notice that by the definitions of E_n and ν, we have $\|\nu\| \geq \|\mu\|/2$ and $\nu(L_i) \leq 6\ell(Q_i)$ for each $i \in I$. We will show that

$$c^2(\nu) \leq C(c^2(\mu) + \|\mu\|), \tag{5.7}$$

where C is some absolute constant. This estimate implies 5.6, because $\frac{1}{12}\nu \in \Lambda_0(\tilde{E})$, and by Lemma 5.4 we have

$$\begin{aligned}\gamma_+(E) &\geq \gamma_+(V_{\delta_0}(E)) - \varepsilon_0 \geq \gamma_+(\tilde{E}) - \varepsilon_0 \\ &\geq C^{-1}\frac{\|\nu\|^2}{\|\nu\| + c^2(\nu)} - \varepsilon_0 \geq C^{-1}\frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)} - \varepsilon_0,\end{aligned}$$

for any $\varepsilon_0 > 0$.

Let us prove 5.7. We will use the notation

$$c^2_\nu(A,B,C) := \int_A \int_B \int_C c(x,y,z)^2\, d\nu(x)\, d\nu(y)\, d\nu(z).$$

We have

$$c^2(\nu) = \sum_{i\in I} c^2_\nu(Q_i, \mathbb{C}, \mathbb{C}) = \sum_{i\in I} c^2_\nu(Q_i, 3Q_i, \mathbb{C}) + \sum_{i\in I} c^2_\nu(Q_i, \mathbb{C}\setminus 3Q_i, \mathbb{C}). \tag{5.8}$$

First we estimate $c^2_\nu(Q_i, 3Q_i, \mathbb{C})$ for a fixed i. Given $x \in L_i$, we have

$$\begin{aligned}c^2(x, \nu_{|3Q_i}, \nu_{|\mathbb{C}\setminus Q_i}) &\leq C\int_{y\in 3Q_i}\int_{z\notin Q_i} \frac{1}{|x-z|^2}\, d\nu(y)\, d\nu(z) \\ &\leq C\delta \int_{|x-z|\geq \delta/2} \frac{1}{|x-z|^2}\, d\nu(z).\end{aligned}$$

Notice that $\nu(2^k Q_i) \leq 6\,\ell(2^k Q_i)$ if $0 \leq k \leq N$, and $\nu(2^k Q_i) \leq C_{13}\,\ell(2^k Q_i)$ for $k > N$. Then we obtain

$$\begin{aligned} (5.9)\quad & \int_{|x-z|\geq\delta/2} \frac{1}{|x-z|^2}\,d\nu(z) \\ &= \sum_{k=0}^{N-1} \int_{\delta 2^{k-1}\leq|x-z|\leq\delta 2^k} \frac{1}{|x-z|^2}\,d\nu(z) + \sum_{k=N}^{\infty} \int_{\delta 2^{k-1}\leq|x-z|\leq\delta 2^k} \frac{1}{|x-z|^2}\,d\nu(z) \\ &\leq C\left(\frac{1}{\delta} + \frac{C_{13}}{2^N\delta}\right) \leq \frac{C}{\delta}. \end{aligned}$$

Observe that the last constant C above does not depend on C_{13} because of the choice of N. Thus, we get $c_\nu^2(Q_i, 3Q_i, \mathbb{C}\setminus Q_i) \leq C\nu(Q_i)$. It is easy to check that we also have $c_\nu^2(Q_i, 3Q_i, Q_i) \leq C\nu(Q_i)$. Thus,

$$(5.10)\qquad c_\nu^2(Q_i, 3Q_i, \mathbb{C}) \leq C\nu(Q_i).$$

Now we have to deal with the last term on the right of 5.8. We write

$$\begin{aligned} c_\nu^2(Q_i, \mathbb{C}\setminus 3Q_i, \mathbb{C}) = & \sum_{j:Q_j\cap 3Q_i=\varnothing} c_\nu^2(Q_i, Q_j, \mathbb{C}\setminus(3Q_i\cup 3Q_j)) \\ & + \sum_{j:Q_j\cap 3Q_i=\varnothing} c_\nu^2(Q_i, Q_j, 3Q_j) =: A_i + B_i. \end{aligned}$$

Using 5.10, we deduce

$$\begin{aligned} (5.11)\quad \sum_{i\in I} B_i &\leq \sum_{i\in I}\sum_{j\in I} c_\nu^2(Q_i, Q_j, 3Q_j) \\ &\leq \sum_{j\in I} c_\nu^2(\mathbb{C}, Q_j, 3Q_j) \leq C\sum_{j\in I}\nu(Q_j) \leq C\|\nu\|. \end{aligned}$$

It remains to estimate A_i. Let Q_j, $j \in I$, be such that $Q_j \cap 3Q_i = \varnothing$, and $Q_k \subset \mathbb{C}\setminus(3Q_i \cup 3Q_j)$. Let $y \in Q_j$, $z \in Q_k$, and denote by q_h the center of each square Q_h. Notice that $|x-y| \approx |q_i - q_j|$, $|x-z| \approx |q_i - q_k|$ and $|y-z| \approx |q_j - q_k|$. By [To1, Lemma 2.4], we have

$$\begin{aligned} & |c(x,y,z) - c(q_i,q_j,q_k)| \\ & \qquad \leq \frac{C\delta}{|x-y|\,|y-z|} + \frac{C\delta}{|y-x|\,|y-z|} + \frac{C\delta}{|z-x|\,|z-y|}. \end{aligned}$$

Thus,

$$(5.12)\quad c(x,y,z)^2 \le 2c(q_i,q_j,q_k)^2 + \frac{C\delta^2}{|x-y|^2\,|y-z|^2} + \frac{C\delta^2}{|y-x|^2\,|y-z|^2} + \frac{C\delta^2}{|z-x|^2\,|z-y|^2} = 2c(q_i,q_j,q_k)^2 + S_1 + S_2 + S_3.$$

Therefore,

$$(5.13)\quad A_i \le 2 \sum_{\substack{j,k:Q_j\cap 3Q_i=\varnothing,\\ Q_k\cap(3Q_i\cup 3Q_j)=\varnothing}} \int_{x\in Q_i}\int_{y\in Q_j}\int_{z\in Q_k} c(q_i,q_j,q_k)^2\,d\nu(x)\,d\nu(y)\,d\nu(z) + \int_{x\in Q_i}\int_{|y-x|>\delta}\int_{\substack{|z-x|>\delta\\ |z-y|>\delta}} (S_1+S_2+S_3)\,d\nu(x)\,d\nu(y)\,d\nu(z).$$

Using the estimate 5.9, we get

$$\int_{|y-x|>\delta}\int_{\substack{|z-x|>\delta\\ |z-y|>\delta}} S_1\,d\nu(y)\,d\nu(z) \le C\delta^2 \int_{|y-x|>\delta} \frac{1}{|x-y|^2}\,d\nu(y) \int_{|z-x|>\delta} \frac{1}{|x-z|^2}\,d\nu(z) \le C.$$

Operating in analogous way, we obtain the same estimates for the integrals of S_2 and S_3 over the same domain. Therefore,

$$(5.14)\quad \int_{x\in Q_i}\int_{|y-x|>\delta}\int_{\substack{|z-x|>\delta\\ |z-y|>\delta}} (S_1+S_2+S_3)\,d\nu(x)\,d\nu(y)\,d\nu(z) \le C\,\nu(Q_i).$$

Let us consider the first term on the right hand side of (5.13). Since $\nu(Q_h) = \mu(Q_h)$ for each h, this term equals

$$D_i := 2 \sum_{\substack{j,k:Q_j\cap 3Q_i=\varnothing,\\ Q_k\cap(3Q_i\cup 3Q_j)=\varnothing}} \int_{x\in Q_i}\int_{y\in Q_j}\int_{z\in Q_k} c(q_i,q_j,q_k)^2\,d\mu(x)\,d\mu(y)\,d\mu(z).$$

By an estimate analogous to (5.12) and operating as above, we will derive

$$(5.15)\quad D_i \le 4 \sum_{\substack{j,k:Q_j\cap 3Q_i=\varnothing,\\ Q_k\cap(3Q_i\cup 3Q_j)=\varnothing}} \int_{x\in Q_i}\int_{y\in Q_j}\int_{z\in Q_k} c(x,y,z)^2\,d\mu(x)\,d\mu(y)\,d\mu(z) + C\nu(Q_i) \le 4c_\mu^2(Q_i,\mathbb{C},\mathbb{C}) + C\nu(Q_i).$$

Now, (5.7) follows from the estimates (5.10), (5.11), (5.13), (5.14), and (5.15). □

The following corollary follows from Theorem 5.1 and will be used in next section.

Corollary 5.5. *Let $E \subset \mathbb{C}$ be compact. If μ is a positive Radon measure supported on E such that $\Theta^*_\mu(x) = 0$ for μ-a.e. $x \in E$, then*

$$\gamma_+(E) \geq C^{-1} \frac{\|\mu\|^{3/2}}{c^2(\mu)^{1/2}}.$$

Proof. Notice that μ is not supported on a line (unless $\mu = 0$). As a consequence, $c^2(\mu) \neq 0$. We can apply Theorem 5.1 to the measure $\tilde{\mu} = (\|\mu\|/c^2(\mu))^{1/2}\mu$, which belongs to $\Lambda(E)$, and the corollary follows. □

6. Density Estimates for γ_+ and Instability

The main result of this section is the following.

Theorem 6.1. *Let $E \subset \mathbb{C}$ be compact with $\gamma_+(E) > 0$. Let μ be a positive Radon measure supported on E. Then, for γ_+-almost all $x \in$ supp(μ) one (and only one) of the following statements holds:*

$$\liminf_{r\to 0} \frac{r}{\mu(B(x,r))} < \infty; \tag{6.1a}$$

$$\lim_{r\to 0} \frac{\gamma_+(B(x,r)\cap E)}{\mu(B(x,r))} = \infty. \tag{6.1b}$$

This result should be understood as a kind of instability result (see [O'F], for instance). Notice that, if (6.1a) does not hold for some x, then clearly we have $\lim_{r\to 0} r/\mu(B(x,r)) = \infty$. The statement (6.1b) is sharper because $\gamma_+(B(x,r)\cap E) \leq r$. Indeed, it may happen $\gamma_+(B(x,r)\cap E)/r \to 0$ as $r \to 0$. This is the case, for example, when E coincides with the set $E(\lambda)$ of Section 3.1and $\sum_{n=0}^\infty 4^{-2n}\sigma_n^{-2} < \infty$, for any $x \in E(\lambda)$.

It is worth to compare Theorem 6.1 with the following result due to Mattila and Paramonov [MP] which also deals with densities for γ_+.

Theorem 6.2 (Mattila, Paramonov)**.** *Let $h : [0,+\infty) \to [0,+\infty)$ be a non decreasing continuous function with $h(0) = 0$ and $h(t) > 0$ for $t > 0$.*

(a) *Let $E \subset \mathbb{C}$ be compact with $\gamma_+(E) > 0$. If h satisfies $\int_0^1 h(t)^2/t^3\,dt < \infty$, then for γ_+-almost all $x \in E$,*

$$\limsup_{r\to 0} \frac{\gamma_+(B(x,r)\cap E)}{h(r)} = \infty.$$

(b) *Suppose that* $h(t) \le t$ *for* $t \ge 0$. *If* $\int_0^1 h^2(t)/t^3\,dt = \infty$ *and the function*

$$g(r) = h(r)\exp\left(-\int_r^1 \frac{h(t)^2}{2t^3}\,dt\right), \quad r \ge 0$$

satisfies $g^{-1}(4^{-j-1})/g^{-1}(4^{-j}) \in [\varepsilon, \frac{1}{2})$ *for some* $\varepsilon \in (0, \frac{1}{2})$ *and all* j *big enough, then there exists a Cantor set* $E_0 \subset \mathbb{R}^2$ *with*

$$C^{-1}h(r) \le \gamma_+(B(x,r) \cap E_0) \le Ch(r)$$

for all $x \in E_0$, $r \in (0,1)$, *and some* $C > 0$ *depending only on* h.

From Theorem 6.2 it follows that there exist compact sets $E_1 \subset \mathbb{C}$ and continuous non decreasing functions $h : [0,+\infty) \to [0,+\infty)$, with $h(0) = 0$ and $h(t) > 0$ for $t > 0$, which satisfy $\lim_{t\to 0} h(t)/t = 0$ and

$$\frac{\gamma_+(B(x,r) \cap E_1)}{h(r)} \le C \tag{6.2}$$

for all $x \in E_1$, $r > 0$. In view of (6.2), one can ask if

$$\limsup_{r\to 0} \frac{\gamma_+(B(x,r) \cap E_1)}{\mathcal{H}^h(B(x,r) \cap E_1)} < \infty, \tag{6.3}$$

where $\mathcal{H}^h$ is the generalized Hausdorff measure associated with the function h. However, notice that (6.1b) in Theorem 6.1 seems to suggest that (6.3) should be false. We will deal with this question in Proposition 6.4 below.

To prove Theorem 6.1 we need the following lemma.

Lemma 6.3. *Let* μ *be some positive finite Radon measure without atoms on* $\mathbb{C}$. *If* $c^2(\mu) < \infty$, *then, for* μ*-almost all* $x \in \mathbb{C}$,

$$\lim_{r\to 0} \frac{c^2(\mu_{|B(x,r)})}{\mu(B(x,r))} = 0.$$

Although we think that this result is known, we will show the detailed proof for the reader's convenience.

Proof. For each $m \ge 1$, let

$$A_m = \left\{x \in \mathbb{C} : \limsup_{r\to 0} c^2(\mu_{|B(x,r)})[\mu(B(x,r))]^{-1} > \frac{1}{m}\right\}.$$

For $r > 0$, we denote

$$c_r^2(\mu) = \iiint_{|x-y|\le r} c(x,y,z)^2\,d\mu(x)\,d\mu(y)\,d\mu(z).$$

Notice that $\lim_{r\to 0} c_r^2(\mu) = 0$, because $c^2(\mu) < \infty$ and μ has no atoms.

Given any $r > 0$ and $m \geq 1$, for each $x \in A_m$ there exists some ball $B(x,s)$ with $s < r/2$ such that $\mu(B(x,s)) \leq mc^2(\mu_{|B(x,s)})$. With this type of balls, we consider a Besicovitch covering of A_m. That is, $A_m \subset \bigcup_i B(x_i, s_i)$, with $\sum_i \chi_{B(x_i,s_i)} \leq C$. Then,

$$\mu(A_m) \leq \sum_i \mu(B(x_i, s_i)) \leq m \sum_i c^2(\mu_{|B(x_i,s_i)}) \leq Cmc_r^2(\mu),$$

which tends to 0 as $r \to 0$. Thus $\mu(A_m) = 0$ for each m. □

Proof of Theorem 6.1. It is easy to check that (6.1a) and (6.1b) cannot hold simultaneosly. Indeed, if (6.1a) is satisfied, using the estimate $\gamma_+(B(x,r) \cap E) \leq r$, we obtain

$$\liminf_{r\to 0} \frac{\gamma_+(B(x,r) \cap E)}{\mu(B(x,r) \cap E)} \leq \liminf_{r\to 0} \frac{r}{\mu(B(x,r))} < \infty.$$

Now we must prove that either (6.1a) or (6.1b) holds γ_+-a.e. That is, if we denote by A the set of $x \in \operatorname{supp}(\mu)$ such that both (6.1a) and (6.1b) fail, then we have to show that $\gamma_+(A) = 0$.

Let

$$A_0 = \{x \in A : c_\mu^2(x) = \infty\},$$

and, for each $n \geq 1$,

$$A_n = \{x \in A : c_\mu^2(x) < n\}.$$

We have $A = \bigcup_{n=0}^{\infty} A_n$. Notice that, by Corollary 3.4, $\gamma_+(A_0) = 0$. We will show that, for $n \geq 1$, $\gamma_+(A_n) = 0$ too.

We denote

$$A_{n,m} = \left\{x \in A_n : \liminf_{r\to 0} \frac{\gamma_+(E \cap B(x,r))}{\mu(B(x,r))} < m\right\}.$$

Notice that $A_n = \bigcup_{m=1}^{\infty} A_{n,m}$. Now we set

$$A_{n,m,0} = \left\{x \in A_{n,m} : \liminf_{r\to 0} \frac{\mu(B(x,r) \cap A_{n,m})}{\mu(B(x,r))} = 0\right\},$$

and, for $k \geq 1$,

$$A_{n,m,k} = \left\{x \in A_{n,m} : \liminf_{r\to 0} \frac{\mu(B(x,r) \cap A_{n,m})}{\mu(B(x,r))} > \frac{1}{k}\right\}.$$

We will show that $\gamma_+(A_{n,m,k}) = 0$ for all $n, m \geq 1$ and $k \geq 0$.

First we deal with $A_{n,m,0}$. We have $\mu(A_{n,m,0}) = 0$. This is quite easy to check: Given any $\varepsilon > 0$, we take a Besicovitch covering of $A_{n,m,0}$ with balls $B(x_i, r_i)$ such that $x_i \in A_{n,m,0}$ and $\mu(B(x_i, r_i) \cap A_{n,m}) \le \varepsilon\mu(B(x_i, r_i))$. Then,

$$\mu(A_{n,m,0}) \le \sum_i \mu(B(x_i, r_i) \cap A_{n,m}) \le \varepsilon \sum_i \mu(B(x_i, r_i)) \le C\varepsilon\mu(E).$$

Letting $\varepsilon \to 0$, we derive $\mu(A_{n,m,0}) = 0$. This implies $\gamma_+(A_{n,m,0}) = 0$ because of the following claim.

Claim. *If $K \subset A_{n,m}$ is such that $\mu(K) = 0$, then $\gamma_+(K) = 0$.*

Proof of the claim. We may assume that K is compact. Given any fixed $\varepsilon > 0$, let K^ε be an open neighborhood of K with $\mu(K^\varepsilon) \le \varepsilon$. We consider a Besicovitch covering of K with balls $B(x_i, r_i)$ such htat $x_i \in K$, $B(x_i, r_i) \subset K^\varepsilon$ and $\gamma_+(B(x_i, r_i) \cap E) < m\mu(B(x_i, r_i))$ (which is possible by the definition of $A_{n,m}$). Then we have

$$\begin{aligned}\gamma_+(K) &\le C \textstyle\sum_i \gamma_+(B(x_i, r_i) \cap K) \le C \textstyle\sum_i \gamma_+(B(x_i, r_i) \cap E) \\ &\le Cm \textstyle\sum_i \mu(B(x_i, r_i)) \le Cm\mu(K^\varepsilon) \le Cm\varepsilon.\end{aligned}$$

Since ε is arbitrarily small, the claim is proved. □

Let us see that $\gamma_+(A_{n,m,k}) = 0$ if $k \ge 1$. For every $x \in A_{n,m,k}$ there exists a sequence of radii $\{r_j\}_j$, tending to 0, such that

$$\begin{aligned}\gamma_+(B(x, r_j) \cap A_{n,m}) &\le \gamma_+(B(x, r_j) \cap E) \\ &\le m\mu(B(x, r_j)) \le km\mu(B(x, r_j) \cap A_{n,m}).\end{aligned} \tag{6.4}$$

If we apply Corollary 5.5 to the set $B(x, r_j) \cap A_{n,m}$ and to the measure $\mu_{|B(x,r_j)\cap A_{n,m}}$, we get

$$\gamma_+(B(x, r_j) \cap A_{n,m}) \ge C^{-1} \frac{\mu(B(x, r_j) \cap A_{n,m})^{3/2}}{c^2(\mu_{|B(x,r_j)\cap A_{n,m}})^{1/2}}. \tag{6.5}$$

From 6.4 and 6.5 we deduce

$$\frac{c^2(\mu_{|B(x,r_j)\cap A_{n,m}})}{\mu(B(x, r_j) \cap A_{n,m})} \ge \frac{C^{-1}}{(km)^2}.$$

Therefore,

$$\limsup_{j\to\infty} \frac{c^2(\mu_{|B(x,r_j)\cap A_{n,m}})}{\mu(B(x, r_j) \cap A_{n,m})} > 0$$

for all $x \in A_{n,m,k}$. Since $c^2(\mu_{|A_{m,n}}) \le c^2(\mu_{|A_n}) < \infty$, from Lemma 6.3 we derive $\mu(A_{n,m,k}) = 0$. Finally, because of the claim above, we deduce $\gamma_+(A_{n,m,k}) = 0$, and the proof is complete. □

Proposition 6.4. *Let* $h : [0, +\infty) \to [0, +\infty)$ *be a continuous non decreasing function with* $h(0) = 0$ *and* $h(t) > 0$ *for* $t > 0$, *which satisfies* $\lim_{t\to 0} h(t)/t = 0$. *Let* $E \subset \mathbb{C}$ *be compact, and let*

$$A = \left\{ x \in E : \limsup_{r\to 0} \frac{\gamma_+(B(x,r)\cap E)}{h(r)} < \infty \right\}.$$

If $\gamma_+(A) > 0$, *then* $\mathcal{H}^h(A)$ *is non* σ*-finite.*

Notice that, in particular, the proposition says that the set E_1 in 6.2 has non σ-finite measure $\mathcal{H}^h$. Thus $\mathcal{H}^h_{|E_1}$ is not a Radon measure, and then Theorem 6.1 cannot be applied.

Proof. Assume that $\mathcal{H}^h(A)$ is finite or σ-finite. Then we can write $A = \bigcup_n A_n$, with $\mathcal{H}^h(A_n) < \infty$ for each n. Also, there exists some A_k such that $\gamma_+(A_k) > 0$. This implies that $\mathcal{H}^h(A_k) > 0$, because of the following claim.

Claim. *If* $K \subset A_k$ *is such that* $\gamma_+(K) > 0$, *then* $\mathcal{H}^h(K) > 0$.

Proof of the claim. We set $A_k = \bigcup_{m=1}^\infty A_{k,m}$, with

$$A_{k,m} = \left\{ x \in A_k : \limsup_{r\to 0} \frac{\gamma_+(B(x,r)\cap E)}{h(r)} < m \right\}.$$

Let m be such that $\gamma_+(A_{k,m}\cap K) > 0$. For each integer $p \geq 1$, we also denote

$$A_{k,m,p} = \left\{ x \in A_{k,m} : \frac{\gamma_+(B(x,r)\cap E)}{h(r)} < m \text{ if } r < \frac{1}{p} \right\}.$$

Since $A_{k,m} = \bigcup_{p=1}^\infty A_{k,m,p}$, we can choose some $p \geq 1$ such that $\gamma_+(A_{k,m,p} \cap K) > 0$.

Remember that $\mathcal{H}^h(A_{k,m,p}\cap K) = \lim_{\delta\to 0} \mathcal{H}^h_\delta(A_{k,m,p}\cap K)$, with

$$\mathcal{H}^h_\delta(A_{k,m,p}\cap K) \approx \inf \sum_i h(t_i),$$

where the infimum is taken over all the coverings $A_{k,m,p}\cap K \subset \bigcup_i B(x_i,t_i)$ with $0 < t_i \leq \delta$ and $x_i \in A_{k,m,p}\cap K$. Consider such a covering with $\delta \leq 1/p$. Then we have

$$\sum_i h(t_i) \geq \frac{1}{m}\sum_i \gamma_+(B(x_i,t_i)\cap A_{k,m,p}\cap K) \geq \frac{C^{-1}}{m}\gamma_+(A_{k,m,p}\cap K).$$

Therefore $\mathcal{H}^h_\delta(A_{k,m,p}\cap K) \geq C^{-1}m^{-1}\gamma_+(A_{k,m,p}\cap K)$ for all $\delta \leq 1/p$, and so $\mathcal{H}^h(A_{k,m,p}\cap K) > 0$. Thus $\mathcal{H}^h(K) > 0$. □

Let us continue with the proof of the proposition. Notice that $\mu := \mathcal{H}^h_{|A_k}$ is a (non zero) Radon measure. Moreover, for μ-a.e. $x \in A_k$, we have

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{h(r)} > 0. \tag{6.6}$$

This is also true for γ_+-a.e. $x \in A_k$, because for any $Z \subset A_k$, such that $\mathcal{H}^h(Z) = 0$ we have $\gamma_+(Z) = 0$, by the last claim. Then, for γ_+-a.e. $x \in A_k$, we deduce

$$\begin{aligned}\liminf_{r\to 0} &\frac{\gamma_+(B(x,r)\cap A_k)}{\mu(B(x,r))} \\ &\leq \left(\limsup_{r\to 0} \frac{\gamma_+(B(x,r)\cap E)}{h(r)}\right) \cdot \left(\liminf_{r\to 0} \frac{h(r)}{\mu(B(x,r))}\right) < \infty,\end{aligned}$$

since the first limit on the right hand side is finite by the definition of A, and the second is also finite because of (6.6). This is a contradiction with Theorem 6.1. □

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