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The Tamagama number conjecture for CM elliptic curves defined over $\mathbb{Q}$

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Abstract

In the present paper we will prove, under the assumption that the Soulé regulator is not zero, that the predicted p-valuations for the L-function $L(E^+, k+2)$ for $k \geq 0$ coming from the Bloch-Kato conjecture are true, where E^+ is an elliptic curve defined over $\mathbb{Q}$ with complex multiplication $\mathcal{O}_K$ the ring of integers of $End_{\overline{\mathbb{Q}}}(E^+ \times_{\mathbb{Q}} \overline{\mathbb{Q}}) \otimes \mathbb{Q}$.

1 Introduction

In the Bloch-Kato paper [2] is presented a conjecture relating special values of the L-function of a pure motive or more generally a motivic pair of a varietyin terms of Tamagama mesures coming from exponential maps of Galois representations of the motivic pair. After a reciprocity law (sketching in Perrin-Rieu-Fontain paper [7]) the Bloch-Kato conjecture is rewrite in terms of the values of the Deligne regulator (Beilinson conjecture) and Soulè regulator [9]. This last conjecture relates the value of L-function to the construction of a space inside K-theory and the computacion of the Deligne and Soule regulator (for precise estatement see next section).
There are only basically two cases that is can be proved. The first correspon to the trivial motive that it corresponds the Riemann zeta function (see [2]6). The second knowing case is basically for elliptic curves with complex multiplication in some particular case. Bloch-Kato in [2] proved the local Bloch-Kato conjecture for the values of the L-function of an elliptic curve with CM $\mathcal{O}_K$ that is defined over $\mathbb{Q}$ for regular primes evaluated for $s = 2$. In an actual work Kings [11] proved the same result for an elliptic curve defined over the quadratic field of the field of endomorphism of the CM elliptic curve, but without hypothesis of regularity. Then the paper proves how with these result one can take the hypotesis of regularity on primes and comprove the conjecture for all the values $k+2$ with $k \geq 0$, with the hypothesis that the Soulé regulator do not kill our element.

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*Actual address: Depart.Matem'àtiques, Facultat Ciencies, 08193 Bellaterra, Barcelona, Catalonia, francesc@mat.uab.cat prelimary proof for the determinant part that after the discussion was already more clear. I also want to gratefull Christopher Deninger who give me the posibility to introduce me inside the fascinating world of the L-functions and her problems.

2 The Tamagama number conjecture (d'après Kato) and the main theorem

The section will give the formulation of the local Tamagama number conjecture in the formulation of Kato [9],[10]. We review only for our proposes.
Let X/K be a smooth proper variety over a number fiel K with ring of integers $\mathcal{O}_K$. Fix integers $m \geq 0$ and r such that $m-2r \leq -3$ and $r > inf(m, dim(X))$. Let p be a prime number not equal to 2. Denote by S the set of finite primes of K lying over p or where X has bad reduction. Write $\mathcal{O}_S = \mathcal{O}_K[1/S]$. Define the $Gal(\overline{K}/K) - modules$:

$$V_p := H^m_{et}(X_K \times_K \overline{K}, \mathbb{Q}_p(r))$$

$$T_p := H^m_{et}(X_K \times_K \overline{K}, \mathbb{Z}_p(r))$$

Let $j : K \to Spec\mathcal{O}_S$ and define the p-adic realitations to be

$$H^i_p := H^i_{et}(\mathcal{O}_S, j_* T_p)$$

Write

$$H_{h,\mathbb{Z}} := H^m_{sing}(X \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{r-1}\mathbb{Z})^+$$

where + denotes the fixed part under $Gal(\mathbb{C}/\mathbb{R})$ of the singular cohomology of X, where the galois group acts on $\mathbb{C}$ and on $(2\pi i)^{r-1}$. Let

$$H_M := (K_{2r-m-1}(X) \otimes \mathbb{Q})^{(r)}$$

be the r-th Adams eigenspace of the $2r-m-1$-th Quillen K-theory of X. There are regulator maps due to Beilinson and Soulé:

$$r_D : H_M \otimes_{\mathbb{Q}} \mathbb{R} \to H_{h,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R} \quad [1]$$

$$r_p : H_M \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^1_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad [14]$$

Define the local Euler factors for a prime $\mathfrak{p} \nmid p$ in $\mathcal{O}_K$

$$P_{\mathfrak{p}}(V_p, s) := det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{p}} N\mathfrak{p}^{-s} | V_p^{I_{\mathfrak{p}}})$$

be the characteristic polynomial of the geometric Frobenius $Fr_{\mathfrak{p}}$ at $\mathfrak{p}$ on the invariants by the inertia group at $\mathfrak{p}$ in V_p. For $\mathfrak{p} \mid p$

$$P_{\mathfrak{p}}(V_p, s) := det_{\mathbb{Q}_p}(1 - \psi_{\mathfrak{p}}^{-1} N\mathfrak{p}^{-s} | D_{cris}(V_p))$$

where $D_{cris}(V_p) := (V_p \otimes_{\mathbb{Q}_p} B_{cris})^{Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ and $\psi_{\mathfrak{p}}$ is the arithmetic Frobenius. Define the L-function of X as

$$L_S(V_p, s) := \prod_{\mathfrak{p} \notin S} P_{\mathfrak{p}}(V_p, s)^{-1}.$$

independent of the choise of p. Let V_p^* the dual Galois module.

Conjecture 2.1. *([10]) Let $p \neq 2$, r, m be as above and let S be the set of places where X has bad reduction or which lie over p. Assume that*

$$P_{\mathfrak{p}}(V_p^*(1), 0) \neq 0$$

for all $\mathfrak{p} \in S$ and that $L_S(V_p^*(1), s)$ has an analytic continuation to all $\mathbb{C}$, then:

1. *The maps r_D and r_p are isomorphisms and H_p^2 is finite.*
2. *$dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=0} L_S(V_p^*(1), s)$ write this number l.*
3. *Let $\eta \in det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$ be a $\mathbb{Z}$-basis. There is an element $\xi \in det_{\mathbb{Q}}(H_M)$ such that*
$$r_D(\xi) = (lim_{s \to 0} s^{-l} L_S(V_p^*(1), s))\eta$$
(Beilinson conjecture)
4. *Consider $r_p(\xi) \in det_{\mathbb{Q}_p}(H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Then $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1}$$
i.e.
$$[det_{\mathbb{Z}_p}(H_p^1) : r_p(\xi)\mathbb{Z}_p] = \#(H_p^2) = det_{\mathbb{Z}_p}(H_p^2)$$

Remark 2.2. *The assumption in the conjecture is true for abelian varieties with CM.*

As our limited knowledge of K-theory, we take a weak version of the conjecture,

Conjecture 2.3. *([11]) There is a subspace H_M^{constr} in H_M such that:*

1. *r_D and r_p restricted to H_M^{constr} are isomorphisms and H_p^2 is finite.*
2. *same as 2) in 2.1.*
3. *There is an element $\xi \in det_{\mathbb{Q}}(H_M^{constr})$ such that*
$$r_D(\xi) = (lim_{s \to 0} s^{-l} L_S(V_p^*(1), s))\eta.$$
4. *The element $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_p)[-1])$$

We are going to state the main result of the paper. For this we will fix our representation of motive that we will prove some part of the conjecture 2.3. We take $X = E^+$ an elliptic curve with CM $\mathcal{O}_K$ where K is a quadratic field, but we supose E^+ is defined over $\mathbb{Q}$. We can consider then $E := E^+ \times_{\mathbb{Q}} K$ elliptic curve with CM $\mathcal{O}_K$. Let us then denote by

$$\psi : \mathbb{A}_K^* \to K^* \subset \mathbb{C}^*$$

the CM-character or Serre-Tate character of E and let $\mathfrak{f}$ be its conductor.
Fix a prime number p. In our situation S is the set of primes in K dividing $Norm_{K/\mathbb{Q}}\mathfrak{f}p$, for has E precisely bad reduction on the primes dividint $\mathfrak{f}$ and for E^+ only difer with the ramified primes of $K/\mathbb{Q}$ with local L-serie is 1.
Remember the following result of Deuring:

Theorem 2.4. *(see [12]II 10.5)*

1. *Let* $L_S(E^+/\mathbb{Q}, s) := L_S(V_p, s)$ *be the L-series of the Galois representation* $V_p := H^1(E^+ \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$ *then*

$$L_S(E^+/\mathbb{Q}, s) = L_S(\psi, s)$$

where $L_S(\psi, s) = \prod_{\mathfrak{p} \nmid p\mathfrak{f}} \frac{1}{1 - \frac{\psi \mathfrak{p}}{N\mathfrak{p}^s}}$

2. *Let* $L_S(E/K, s) := L_S(V_p, s)$ *be the L-series of the Galois representation* $V_p := H^1(E \times_K \overline{\mathbb{Q}}, \mathbb{Q}_p)$ *then*

$$L_S(E/K, s) = L_S(\psi, s)^2 = L_S(\psi, s) L_S(\overline{\psi}, s)$$

Let $T_pE^+ = lim_{\leftarrow} E^+[p^n]$ the Tate-module of E^+ a $Gal(\overline{\mathbb{Q}}/Q)$-module. Then $H^1(E^+ \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_p) \cong T_pE^+(-1)$ Then for our situation, take $m = 1$, $r = k + 2$ with $k \geq 0$ and

$$H^i_p = H^i(Spec(\mathbb{Z}[1/S]), T_pE(k+1)) = H^i(\mathbb{Q}, T_pE(k+1))$$

$$H_{h,\mathbb{Z}} = H^1_{sing}(E^+ \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{r-1}\mathbb{Z})^+$$

$$H_M = H^2_M(E^+, k+2)$$

where $H^i_M(X, j) := (K(X)_{2j-i} \otimes \mathbb{Q})^{(j)}$. State then the main theorem:

Theorem 2.5. *Let* $p \neq 2, 3$ *and* $p \notin N_{K/\mathbb{Q}}\mathfrak{f}$ *and* $k \geq 0$. *Then, there is a submodul* $\mathcal{R}_\psi \subset H_M$ *of rank 1 such that:*

1. $det_{\mathbb{Z}_p}(r_D(\mathcal{R}_\psi)) \cong L^*_S(\psi, -k) det_{\mathbb{Z}_p}(H_{h,\mathbb{Z}}) = L^*_S(E^+, -k) det_{\mathbb{Z}_p}(H_{h,\mathbb{Z}})$ *in* $det_{\mathbb{Z}_p}(H_{h,\mathbb{Z}} \otimes \mathbb{R})$ *and*

2. *If the map* r_p *is injective on* $\mathcal{R}_\psi$ *then:*

$$det_{\mathbb{Z}_p}(r_p(\mathcal{R}_\psi)) \cong det_{\mathbb{Z}_p}(R\Gamma(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1)))^{-1}.$$

Here $L^*(\psi, -k) = lim_{s \to -k} \frac{L(\psi, s)}{s+k}$.

Remark 2.6. *The part 1) of the theorem is proven by Deninger in [5], Beilinson conjecture for Hecke characters.*
The part 2) for $k = 0$ *and regular primes* p *is proven by Bloch-Kato in [2]. See the last section for more details and study of the injectivity condition on the Soulé regulator.*

The proof of the theorem will be completed in the following sections. The idea is descend over E^+ the statement of the theorem of E proved over K by Kings [11], see next section.

3 The Tamagama number conjecture for E(d'après Kings)

The point of work of our result is the following result of Kings:

Theorem 3.1 (Kings[11]). *Write $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$. Let $p \neq 2,3$ and $p \nmid N_{K/\mathbb{Q}}\mathfrak{f}$ and $k \geq 0$. Then there is an $\mathcal{O}_K$ submodul $\tilde{\mathcal{R}_\psi} \subset H^2_M(E, k+2)$ of rank 1 such that*

1. *$det_{\mathcal{O}_K}(r_D(\tilde{\mathcal{R}_\psi}) \cong L^*_S(\overline{\psi}, -k) det_{\mathcal{O}_K}(H^1(E \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^r \mathbb{Z})^+)$ in $det_{\mathcal{O}_K \otimes \mathbb{R}}(H^1(E \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^r \mathbb{Z})^+ \otimes \mathbb{R})$.*
2. *If r_p is injective in $\tilde{\mathcal{R}_\psi}$ then*

$$det_{\mathcal{O}_p}(r_p(\tilde{\mathcal{R}_\psi})) \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, T_pE(k+1)))^{-1}.$$

For our proposes we go to review the element that generates the Kings $\mathcal{O}_K$-subspace $\tilde{\mathcal{R}_\psi}$, constructed by Deninger for proving the Beilinson conjecture [4] and we will define our $\mathcal{R}_\psi$ that will satisfies the conditions of theorem 2.5.
Fix an algebraic differential $\omega \in H^0(E, \Omega_{E/K})$ that we will suppose that lies in $H^0(E^+, \Omega_{E^+/\mathbb{Q}})$. Let Γ its period lattice. We have

$$E^+(\mathbb{C}) = E(\mathbb{C}) \to \mathbb{C}/\Gamma$$

$$z \mapsto \int_0^z \omega$$

with all the time a fixed embedding $K \subset \mathbb{C}$. We have $\Gamma = \alpha\mathcal{O}_K$ for some $\alpha \in \mathbb{C}^*$. Let $\mathbb{Z}[E[\mathfrak{f}] \setminus O]$ the group of divisors with support in the $\mathfrak{f}$-torsion points defined over K. Then Beilinson defines an Eisenstein symbol map

$$\mathcal{E}^{2k+1}_M : \mathbb{Z}[E[\mathfrak{f} \setminus 0]] \to H^{2k+2}_M(E^{2k+1}, 2k+2)$$

and Deninger constructs a projector

$$\mathcal{K}_M : H^{2k+2}_M(E^{2k+1}, 2k+2) \to H^2_M(E, k+2)$$

Let $K(\mathfrak{f}) = K(E[\mathfrak{f}])$ the ray class field, and let f a generator of $\mathfrak{f}$. Then

$$\Omega f^{-1} \in \mathfrak{f}^{-1}\Gamma$$

defines a divisor over $K(\mathfrak{f})$ take then

$$\beta := N_{K(\mathfrak{f})/K}((\Omega f^{-1})).$$

Fix also a $\mathcal{O}_K$ generator $\gamma \in H^1(E(\mathbb{C}), \mathbb{Z})$ where α is obtained by $\alpha = \int_\gamma \omega$.
Denote by η the $\mathcal{O}_K$ generator of $H^1(E \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{k+1}\mathbb{Z})^+$ corresponding to $(2\pi i)^k \gamma$ under the isomorphism:

$$H^1(E(\mathbb{C}), (2\pi i)^{k+1}\mathbb{Z}) \cong H^1(E(\mathbb{C}), (2\pi i)^k \mathbb{Z}).$$

Theorem 3.2 (Deninger [4][5]). *Let β and η as above and define*

$$\xi := (-1)^{k-1}\frac{(2k+1)!}{2^{k-1}}\frac{L_p(\overline{\psi},-k)^{-1}}{\psi(f)N_{K/\mathbb{Q}}\mathfrak{f}^k}\mathcal{K}_M\circ\mathcal{E}_M^{2k+1}(\beta)\in H^2_M(E,k+2)$$

where $L_p(\overline{\psi},-k)$ is the Euler factor of $\overline{\psi}$ at p evaluated at $-k$. Then

$$r_D(\xi) = L_S^*(\overline{\psi},-k)\eta \in H^1(E\times_Q\mathbb{C},(2\pi i)^{k+1}\mathbb{Z})^+.$$

Moreover we can take η^+ a generator of $H^1(E^+\times_{\mathbb{Q}}\mathbb{C},(2\pi i)^{k+1}\mathbb{Z})^+$ satysfiying

$$r_D(\xi) = L_S^*(\overline{\psi},-k)\eta^+$$

Then is defined $\tilde{\mathcal{R}_\psi} := \xi\mathcal{O}_K \subset H^2_M(E,k+2)$.

Definition 3.3. *We have the norm map $H^2_M(E,k+2)\to H^2_M(E^+,k+2)$ given by the action of F_∞ such that $\delta\mapsto\frac{1}{2}(\delta+F_\infty\delta)$. Then define*

$$\mathcal{R}_\psi := Norm(\tilde{\mathcal{R}_\psi})$$

Corollary 3.4. *With the above notation*

$$r_D(det_{\mathbb{Z}}(\mathcal{R}_\psi)) = L_S^*(E^+/\mathbb{Q},-k)det_{\mathbb{Z}}(H^1(E^+(\mathbb{C}),(2\pi i)^k\mathbb{Z})))$$

where S were the set of primes of $\mathbb{Q}$ dividing $pN_{K/\mathbb{Q}}\mathfrak{f}$.

Proof. Only note that $Norm(\xi)$ satisies that $r_D(Norm(\xi)) = L_S^*(E^+/\mathbb{Q},-k)\eta^+$ for good Galois descens in motivic cohomology, and taking determinants we conclude. □

Remark 3.5. *As the good Galois descent for the motivic cohomology ([4]) we have constructed a one dimensional submodul in H_M.*

4 The Galois descent for the Soulé regulator

We concentrate in our situation. Denote by $G = Gal(K/\mathbb{Q})$, consider the following Soulé $\mathbb{Q}_p$-regulator maps:

$$r_{p,K}: K_{2n-2}(E)^{(n)}\otimes\mathbb{Q}_p \to H^1(G_K, H^1(E\times_K\overline{\mathbb{Q}},\mathbb{Q}_p(n)))$$

$$r_{p,\mathbb{Q}}: K_{2n-2}(E^+)^{(n)}\otimes\mathbb{Q}_p \to H^1(G_{\mathbb{Q}}, H^1(E^+\times_{\mathbb{Q}}\overline{\mathbb{Q}},\mathbb{Q}_p(n)))$$

with $2<n$, where G_L means the Galois group $Gal(\overline{L}/L)$.
We have an action of G in both members of $r_{p,K}$ we are goint to study if the action is compatible with the descent of the regulator map. For this we rewiew the construction of the higher regulator map. First of all there is a natural map ϕ_L between K-theory and continous étale theory see [14]

Lemma 4.1. *The following diagram commutes with the norm maps:*

$$\begin{array}{ccc} & \phi_K & \\ K_{2n-2}(E)^{(n)}\otimes\mathbb{Q}_p & \to & H^2_{cont}(E,\mathbb{Q}_p(n)) \\ \downarrow & & \downarrow \\ K_{2n-2}(E^+)^{(n)}\otimes\mathbb{Q}_p & \to & H^2_{cont}(E^+,\mathbb{Q}_p(n)) \\ & \phi_{\mathbb{Q}} & \end{array}$$

Proof. Is consequence of [6] that says that the regulator is compatible with norm maps, between K-theory and etale K-theory. $\square$

Then to the definition of the p-adic regulator we note that given a Galois covering $X' \to X$ with group G', we have a Hochschild-Serre spetral sequence in the continous etale cohomology with

$$E_2^{st} = H^s_{cont}(G', H^t(X'; \mathbb{Q}_p(i))$$

converges to $H^{r+s}_{cont}(G', \mathbb{Q}_p(i))$, using these for $G_K = G'$ and $X' = \overline{E}$ and $X = E$ is defined the p-adic regulator $r_{p,K}$ by (an the same for the natural elections for $r_{p,\mathbb{Q}}$):

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & H^2_{cont}(E, \mathbb{Q}_p(n))_0 & \xrightarrow{\pi} & H^1(G_K, H^1(\overline{E}, \mathbb{Q}_p(n))) \\
 & & \downarrow & \nearrow_{r_{p,K}} & \\
H^2_M(E, n) \otimes \mathbb{Q}_p & \longrightarrow & H^2_{cont}(E, \mathbb{Q}_p(n)) & & \\
 & & \downarrow & & \\
 & & H^2(\overline{E}, \mathbb{Q}_p(n))^{G_K} & &
\end{array}$$

where $H^2_{cont}(X, \mathbb{Q}_p(n))_0 = ker(H^2_{cont}(X, \mathbb{Q}_p(n)) \overset{res}{\to} H^2(\overline{X}, \mathbb{Q}_p(n)))$ and π comes from the HS-spectral sequence, and observe that $n = k + 2 \geq 2$ in our situation in particular we have $H^2(\overline{E}, \mathbb{Q}_p(n))^{G_K} = 0$ (is zero for $n \neq 1$).

Lemma 4.2. *The following diagram commutes:*

$$\begin{array}{ccc}
H^2_M(E, n) \otimes \mathbb{Q}_p & \overset{r_{p,K}}{\rightarrow} & H^1(K, H^1(\overline{E}, \mathbb{Q}_p(n))) \\
\downarrow & & \downarrow \\
H^2_M(E^+, n) \otimes \mathbb{Q}_p & \underset{r_{p,\mathbb{Q}}}{\rightarrow} & H^1(K, H^1(\overline{E}, \mathbb{Q}_p(n)))^G
\end{array}$$

where the left vertical map correspons to the norm map and the right vertical map to the corestriction map

Proof. First of all we note that $H^1(K, H^1(\overline{E}, \mathbb{Q}_p(n))) = H^1(K, V_pE(n+1))$ and as $H^1(K, V_pE^+(n+1)) = H^1(K, V_pE(n+1))$ we now that the restriction map in the $G_\mathbb{Q}$-mod $V_pE^+(n+1)$ induces and isomorphism

$$H^1(\mathbb{Q}, V_pE^+(n+1)) \cong H^1(K, H^1(\overline{E}, \mathbb{Q}_p(n)))^G = H^1(K, V_pE(n+1))^G$$

for be $(\#G, p) = 1$.
Using the previous lemma we can concentrate only in the continuos etale cohomology and as the naturaly of the HS-spectral sequence and the fact that galois recobrement of E^+ is factorized by E, $\overline{E} \to E \to E^+$ is proved the result. $\square$

Then we obtain

Corollary 4.3. $r_{p,\mathbb{Q}}(\mathcal{R}_\psi) = r_{p,K}(\tilde{\mathcal{R}_\psi})^{G=Gal(K/\mathbb{Q})}$

5 Relation between determinants

We will go to prove the second part of the main theorem 2.5. The first aim of these section, suposing that r_p is not zero in $\mathcal{R}_\psi$ prove that we have the determinant equality

$$det_{\mathbb{Z}_p}((H_p^1 := H^1(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1)))/r_{p,\mathbb{Q}}(\mathcal{R}_\psi)) =$$

$$det_{\mathbb{Z}_p}(H_p^2 := H^2(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1)).$$

The second aim will be to obtain the same equality without the hypotesis of the injectivity of the Soulé regulator.
First of all we observe that $H_p^i = H^i(\mathbb{Q}, T_pE^+(k+1))$ using the result of Serre-Tate and the action of inertia groups(see [8]). The same observation can be take for $H^i(Spec(\mathcal{O}_K[1/S]), T_pE(k+1)) = H^i(K, T_pE(k+1))$. Then we can consider $T_pE^+(k+1)$ as an G_K-module and then we have that $H^i(K, T_pE^+(k+1)) = H^i(K, T_pE(k+1))$. From the theorem 3.1 we have a comparation of the $\mathcal{O}_p$ determinants of $H^i(K, T_pE^+(k+1))$ and $r_{p,K}(\tilde{\mathcal{R}_\psi})$, suposing that the Soule regulator is not zero in ξ. Then coming from this situation we will deduce our determinant comparision for ours H_p^i. First of all observe the following

Lemma 5.1. $H^i(\mathbb{Q}, T_pE^+(k+1)) = H^i(K, T_pE(k+1))^{Gal(K/\mathbb{Q})=G}$

Proof. Is a clasical fact of cohomology of groups that when $\#G$ invertible in $T_pE^+(k+1)$ then the restriction map gives an isomorphism for the invariants. (see for example Prop 10 [3]), we suppose all the time $p \neq 2$. □

Observe that $H^i(K, T_pE^+(k+1))$ are $\mathcal{O}_p$-modules and also has a G-action, where the invariants for the last action is calculated in the last lemma. Moreover $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$ has a G-action coming for acting by $\sigma \otimes 1$ with $\sigma \in G$

Lemma 5.2. *Let $\alpha_i \in H^i(K, T_pE^+(k+1))$ and $\delta \in \mathcal{O}_p$, let $\sigma \in G$ then we have:*

$$\sigma(\delta\alpha_i) = \sigma(\delta)\sigma(\alpha_i)$$

Proof. Take F a projective resolution of $\mathbb{Z}_p$ over $\mathbb{Z}_pG_\mathbb{Q}$ the action of $\sigma \in G$ is induced in the terms of complex

$$\mathcal{H}om_{G_K}(F, T_pE^+(k+1)) \to \mathcal{H}om_{\sigma G_K \sigma^{-1} = G_K}(F, T_pE^+(k+1))$$

by

$$f \mapsto [x \mapsto \sigma f(\sigma^{-1}x)]$$

The action on $T_pE(k+1)$ of $\mathcal{O}_p$ correspon to multiplication by δ for the canonical isomorphism of CM elliptic curves $E^+[p^n](\overline{\mathbb{Q}}) \cong \mathcal{O}_K/p^n$. Then taking a representant of α_i in $\mathcal{H}om_{G_K}(F, T_pE^+(k+1))_i$ that we will note with the same name, then

$$\sigma(\delta\alpha_i) : x \mapsto \sigma(\delta f(\sigma^{-1}x)$$

as $\sigma\delta\sigma^{-1} = \overline{\delta} = \sigma(\delta)$ we obtain the result. □

Take then M a $\mathcal{O}_p$-module and a $\mathbb{Z}_pG$-module that the G-action satisfies

$$\sigma(rm) = \sigma(r)\sigma(m)$$

for all σ, r and m where $\sigma \in G$, $r \in R$ and $m \in M$. Denote by $M^+ = M^G$ the fixed module for the G-action. Write $\mathcal{O}_p = \mathbb{Z}_p[\sqrt{-D}]$ where D is the discriminant of K, for be $p \neq 2$. Writing for the following $\sigma \in G \setminus 1$ we obtain the following decomposition as $\mathbb{Z}_p$-modules of M:

$$M = (\frac{\sigma+1}{2})M \oplus (\frac{1-\sigma}{2})M$$

is clear $M^+ = (\frac{\sigma+1}{2})M$ we denote by $M^- = (\frac{1-\sigma}{2})M$. Observe then $\sqrt{-D}$ sends bijectivily $M^+ \to M^-$ and also $M^- \to M^+$ for be $(D, p) = 1$.

Lemma 5.3. *We have that the following morphism:*

$$\tau : M^+ \otimes_{\mathbb{Z}_p} \mathcal{O}_p \to M$$

$$m^+ \otimes (a + b\sqrt{-D}) \mapsto am^+ + b\sqrt{-D}m^+$$

is an isomorphism of $\mathcal{O}_p$-modules

Proof. Define an map from M to $M^+ \otimes \mathcal{O}_p$ by

$$m = m^+ + m^- \mapsto m^+ \otimes 1 + \frac{1}{-D}\sqrt{-D}m^- \otimes \sqrt{-D}$$

where $m = m^+ + m^-$ correspon to the $\mathbb{Z}_p$-decomposition of $M = M^+ + M^-$. This last map is $\mathcal{O}_p$-lineal, and it defines the inverse of τ. ☐

Observe then that in this situation we have

$$det_{\mathcal{O}_p} M = (det_{\mathbb{Z}_p}(M^+)) \otimes_{\mathbb{Z}_p} \mathcal{O}_p$$

then always we have $det_{\mathcal{O}_p} M = p^j \mathcal{O}_p$ for some integer j, then it implies $det_{\mathbb{Z}_p}(M^+) = p^j \mathbb{Z}_p$ saying

$$det_{\mathcal{O}_p} M \cap \mathbb{Q}_p = det_{\mathbb{Z}_p} M^+$$

Moreover consider a exact sequence of $\mathcal{O}_p$-moduls M_i with the compatibility of G-action of the ring and the modul

$$0 \to M_1 \to M_2 \to M_2/M_1 \to 0$$

then we have $det_{\mathcal{O}_p}(M_2/M_1) = det_{\mathcal{O}_p}(M_2) det_{\mathcal{O}_p}(M_1)$ then

$$det_{\mathcal{O}_p}(M_2/M_1) = det_{\mathbb{Z}_p}(M_2^+/M_1^+) \otimes \mathcal{O}_p$$

for be $\mathcal{O}_p$ flat over $\mathbb{Z}_p$.
Consider then the following equality from theorem 3.1:

$$det_{\mathcal{O}_p}(H^1(K, T_pE(k+1))/r_p(\tilde{\mathcal{R}_\psi})) = det_{\mathcal{O}_p}(H^1(K, T_pE(k+1)))/det_{\mathcal{O}_p}(r_{p,K}(\tilde{\mathcal{R}_\psi}))$$

$$= det_{\mathcal{O}_p}(H^2(K, T_pE(k+1)))$$

We now by the lemma 5.2 that all the $\mathcal{O}_p$-modules involucrated in the previous equation are also G-modules with the compatibility with the action of $\mathcal{O}_p$, only note for $r_{p,K}(\tilde{\mathcal{R}_\psi})$ comes from be a $\mathcal{O}_p$-submodul of H^1_p that is Galois stable.

Corollary 5.4. *With the hypotesis that* $r_{p,K}(\xi) \neq 0$ *we have*

$$det_{\mathbb{Z}_p}(H_p^1/r_{p,\mathbb{Q}}(\mathcal{R}_\psi)) = det_{\mathbb{Z}_p}(H_p^2)$$

Proof. Only note that we have

$$H_p^i = H^i(K, T_pE(k+1))^+ \quad i = 1, 2$$

and

$$r_{p,\mathbb{Q}}(\mathcal{R}_\psi) = r_{p,K}(\tilde{\mathcal{R}_\psi})^+$$

by corollary 4.3. □

We are now interested in the pure motive $H^2(E, k+2)$. We observe that we will obtain our theorem 2.5 if we can see

$$det_{\mathcal{O}_p}\tilde{\mathcal{R}_\psi} = det_{\mathbb{Z}_p}\mathcal{R}_\psi \otimes_{\mathbb{Z}_p} \mathcal{O}_p$$

then will follow for the previous arguments our relation of determinants over $\mathbb{Z}_p$, because the $\mathcal{O}_p$-determinant of $R\Gamma(Spec(\mathbb{Z}[1/S], T_pE^+(k+1))$ comes from the $\mathbb{Z}_p$-determinant.
For prove the equality of determinants in this subspace of motivic cohomology, we only note that $\tilde{\mathcal{R}_\psi}$ is a $\mathcal{O}_p$-module of rank 1, and there is on $H^2(E, k+2)$ an $\mathbb{Z}/2$-action say F_∞^* coming from complex conjugation in the second factor of $E^+ \times_\mathbb{Q} K$ that our generator is fix by F_∞^*, and F_∞^* acts by complex conjugation on $\mathcal{O}_p$, then we are in the situation of lemma 5.3 with $\sigma = F_\infty^*$. For more detail of these actions on the K-theory, see pag 153-155 in [5].

6 About the non-vanishing of the Soulé regulator

This section wil be only a resum of the knowing conditions that are until now known about the condition for the no-vanishing for the generator element of $\mathcal{R}_\psi$ trought the Soulé regulator. In [11] is proved $r_{p,K}(\xi)$ is not zero proving then that r_p is injective r_p if $H^2(\mathcal{O}_K[1/S], T_pE(k+1))$ is finite.Observe for our particular case this vanishing will be enought for the injectivity of $r_{p,\mathbb{Q}}$ on $\mathcal{R}_\psi$ using the corollary 4.3.
About the finiteness of this galois group is a particular case of a conjecture of Jansen [8] that afirms the finitenes of the latter group. Moreover in our situation we have the following two results

Theorem 6.1. *([13] 1.5 proposition 3) For fixed p the group*

$$H^2(\mathcal{O}_K[1/S], T_pE(k+1))$$

is finite for almost all k

And for a regular prime p (see [14] 3.3.1 for the definition) then

Theorem 6.2. *([14] 3.3.2, [15]cor.2,[8]lem.1) Let p a regular prime for E, then*

$$H^2(\mathcal{O}_K[1/S], E[p^\infty](k+1)) = 0.$$

From this is obtain then that $H^2(\mathcal{O}_K[1/S], T_pE(k+1))$ *is finite.*

Then as a conclusion, we obtain for regular primes that $r_{p,\mathbb{Q}}$ is injective over $\mathcal{R}_\psi$ and then is no condition on the second part of theorem 2.5. When p is not regular, for allmost all twists we obtain that $r_{p,\mathbb{Q}}$ is injective and the condition in theorem 2.5 is satisfied.

References

[1] *A.Beilinson*, Higher regulators and values of L-functions; J.Sov.Math. 30, 2036-2070(1985).

[2] *Bloch and K. Kato*, L-functions and Tamagawa numbers; in Grothendieck Fest. I., Prog.Math. 86, Birkhuser (1990).

[3] *K.S.Brown*, Cohomology of groups, GTM 87, Springer.

[4] *C.Deninger*, Higher regulators and Hecke L-series of imaginary quadratic fields I; Invent.math.95, 1-69 (1989).

[5] *C.Deninger*, Higher regulators and Hecke L-series of imaginary quadratic field II; Annals of math. 132, 131-158 (1990).

[6] *D.Dwyer and E.Friedlander*, Algebraic and Etale K-theory; Trans.Amer.Math.Soc. 292 n1, 247-280 (1985).

[7] *Jean-Marc Fontaine and Bernadette Perrin-Riou*, Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L, Proceedings of Symposia in Pure Mathematics, Vol 55(1994), 599-706.

[8] *U.Jannsen*, On the l-adic cohomology of varieties over number fields and its Galois cohomology, in: Ihara et al.(eds.):Galois groups over $\mathbb{Q}$, MSRI Publication (1989).

[9] *K. Kato*, Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}; in Arithmetic Algebraic Geometry, LNM 1553.

[10] *K. Kato*, Iwasawa theory and p-adic Hodge theory; Kodai Math.J. 16, 1-31 (1993).

[11] *G. Kings*, The Tamagama number conjecture for elliptic curves with complex multiplication; (Preprint) Münster Universität, März 2000.

[12] *J.H.Silverman* Advanced topics in the arithmetic of elliptic curves. GTM 151, Springer 1994.

[13] *C.Soulé*, The rank of étale cohomology of varieties over p-adic or number fields; Comp.Math.53, 113-131 (1984).

[14] *C.Soulé*, p-adic K-theory of elliptic curves; Duke math. Journal 54, 249-269 (1987).

[15] *K.Wingberg*, On the étale K-theory of an elliptic curve with complex multiplication for regular primes; Canad.Math.Bull.33, 145-150(1990).