Asymptotic evaluation of the Poisson measures for tubes around a jump curves

by

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Abstract

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1 Introduction

This paper deals with the asymptotic evolution of the Poisson measures for tubes around jump curves.

This problem has been widely studied for Wiener measures. Set $\{W_t\}_{t\geq 0}$ a Wiener process in $\mathbb{R}^d$ and $\phi \in L^2([0,1],\mathbb{R}^d)$. It is known that

$$P(\|W-\phi\|_\infty \leq \varepsilon) \sim \exp\big(-\frac{1}{2\varepsilon^2}|\phi|^2\big).$$

A related problem is the obtention of Onsager-Machlup functionals: given a process X, we consider a norm $\|\cdot\|$ and two smooth curves ϕ and ψ. If the

$$\lim_{\varepsilon\downarrow 0} \frac{P(\|X-\phi\| \leq \varepsilon)}{P(\|X-\psi\| \leq \varepsilon)}$$

exists and can be expressed as:

$$\exp\big[\int_0^T L(\dot{\phi(s)},\phi(s))ds - \int_0^T L(\dot{\psi(s)},\psi(s))ds\big]$$

by some function $L(\dot{x},x)$, this function is called the Onsager-Machlup. For instance, if X is a diffusion process solution of the stochastic differential equation

$$dX(t) = b(X(t))dt + dW(t),\ X(0)=x_0,\ X(t)\in\mathbb{R}^d,$$

where $x_0 \in \mathbb{R}^d$ and with some regularity of the coefficient b, for large classes of norms in Wiener space, and for functions in the Cameron-Martin space, the Onsager-Machlup function exists and is given by

$$L(\dot{\phi},\phi) = -\frac{1}{2}\sum_{i=1}^d \big|\dot{\phi}_i - b_i(\phi)\big|^2 - \frac{1}{2}\sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(\phi).$$

(See, for instance [2]).

Our aim is to study this kind of problems when the process N is a standard Poisson process. That is, we will study, using L^1 and L^2 norms,

$$P\{\|N-h\| \leq \varepsilon\},$$

when h is a jump function, and

$$P\{\|X-\phi\| \leq \varepsilon\},$$

where X is a diffusion process of the type

$$X_t = \int_0^T X_s ds + N_t,$$

and, given a jump function h, ϕ is the solution of

$$\phi_t^h = \int_0^T \phi_s^h ds + h_t.$$

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ a complete probability space,

Definition 2.1 *A Poisson process is a càdlàg process $N = \{N_s,\, s \in [0,1]\}$, such that:*

- $N_0 = 0$.
- *Given $0 \le t_1 < t_2 < \cdots < t_n \le 1$ the increments*

  $$N_{t_n} - N_{t_{n-1}}, N_{t_{n-1}} - N_{t_{n-2}}, \ldots, N_{t_2} - N_{t_1}, N_{t_1}$$

  are independent.
- *Given $s < t$, the increment $N_t - N_s$ has a Poisson law of parameter $(t-s)$.*

Given a Poisson process, we will denote $T_1, T_2, \ldots, T_n, \ldots$ the successive jump points of the Poisson process.

In this situation it is well known the following result (see for instance [1, Proposition 4.5.6]):

Proposition 2.2 *The conditional distribution of $(T_1, T_2, \ldots, T_k)$ given $\{N_t = k\}$ is the same as that of k increasingly ordered independent random variables each having the uniform distribution on $(0, t]$. That is, the conditional density of the vector $(T_1, T_2, \ldots, T_k)$ given $\{N_t = k\}$ is*

$$f_{(T_1, T_2, \ldots, T_k)} = k! I_{\{0 < t_1 < \cdots < t_k < t\}}.$$

In this paper, given a standard Poisson process N_s, $s \in [0,1]$ we will consider a diffusion process of the form

$$X_t = \int_0^t X_s ds + N_t, \quad t \in [0,1].$$

It is easy to check that the solution of such equation can be written in terms of the jumps points of the Poisson process as

$$X(t) = \sum_{i=1}^{\infty} e^{t-T_i} I_{[T_i,1)}(t_i).$$

3 The case of the Poisson process

When we consider a standard Poisson process, we have the following result:

Theorem 3.1 *Let $\{N_s,\, s \in [0,1]\}$ be a standard Poisson process and $h : [0,1] \longrightarrow \mathbb{R}$ a jump function with expression*

$$h_t = \sum_{i=1}^{k} I_{[S_i,1)}(t),$$

where $0 < S_1 < S_2 < \cdots < S_{k-1} < S_k < 1$ are the jump points.

Then, for $\varepsilon > 0$ small enough,

$$P\{\|N-h\|_1 \le \varepsilon\} = 2^k e^{-1}(e^{\varepsilon} - \sum_{j=0}^{k-1} \frac{\varepsilon^j}{j!}) = 2^k e^{-1}\frac{\varepsilon^k}{k!} + O(\varepsilon^{k+1})$$

and

$$P\{\|N-h\|_2 \le \varepsilon\} = 2^k e^{-1}\frac{\varepsilon^{2k}}{k!} + O(\varepsilon^{2k+2}).$$

Remark 3.2 *Notice that the probability depends only on the number of jumps of the function h.*

Proof:

We will only develop the L^2 case. The proof for the L^1 norm can be done using the same arguments.

Since $0 < S_1 < S_2 < \cdots < S_{k-1} < S_k < 1$, there exists ε_0 such that $S_{i+1} - S_i > \varepsilon_0^2$ for all $i \in \{1, \ldots, k-1\}$, $S_1 > \varepsilon_0^2$ and $1 - S_k > \varepsilon_0^2$. Along the proof we will consider $\varepsilon < \varepsilon_0$.

We have that,

$$P\{\|N-h\|_2 \leq \varepsilon\} = \sum_{j=0}^{\infty} P\{\|N-h\|_2 \leq \varepsilon | N_1 = j\} \frac{e^{-1}}{j!} \; .$$

In order to compute the probabilities involved in the sum we will consider three cases:

- If $j < k$, then $P\{\|N-h\|_2 \leq \varepsilon | N_1 = j\} = 0$. Indeed, in this case,

$$\begin{aligned} \|N-h\|_2^2 &\geq \int_{S_k}^1 (N_t - h_t)^2 dt \geq \int_{S_k}^1 dt \\ &= 1 - S_k > \varepsilon^2. \end{aligned}$$

- If $j = k$, then $P\{\|N-h\|_2 \leq \varepsilon | N_1 = k\} = (2\varepsilon^2)^k$. We will prove this result by induction on k. Notice that, if for some $i \in \{1, \ldots, k\}$, $|T_i(\omega) - S_i| > \varepsilon^2$, it is easy to check that $\|N(\omega) - h\|_2^2 > \varepsilon^2$. Otherwise, if for all $i \in \{1, \ldots, k\}$, $|T_i(\omega) - S_i| < \varepsilon^2$ then

$$\|N(\omega) - h\|_2^2 = \sum_{i=1}^{k} |T_i - S_i|.$$

So

$$P\{\|N-h\|_2 \leq \varepsilon | N_1 = k\} = P\{\sum_{i=1}^{k} |T_i - S_i| \leq \varepsilon^2 | N_1 = k\}.$$

If $k = 1$, using Proposition 2.2 we have that

$$\begin{aligned} P\{|T_1 - S_1| \leq \varepsilon^2 | N_1 = 1\} &= P\{S_1 - \varepsilon^2 \leq T_1 \leq S_1 + \varepsilon^2 | N_1 = 1\} \\ &= \int_{S_1 - \varepsilon^2}^{S_1 + \varepsilon^2} dt = 2\varepsilon^2. \end{aligned}$$

Assume now that if $k = n$, then

$$P\{\sum_{i=1}^{n} |T_i - S_i| \leq \varepsilon^2 | N_1 = n\} = (2\varepsilon^2)^n.$$

Consider now $k = n+1$. Using Proposition 2.2, if ε is small enough,

$$\begin{aligned}
& P\{\sum_{i=1}^{n+1} |T_i - S_i| \le \varepsilon^2 | N_1 = n+1\} \\
= & \int_{\{(t_1 < \cdots < t_{n+1}) : \sum_{i=1}^{n+1} |t_i - S_i| \le \varepsilon^2\}} (n+1)! dt_1 \cdots dt_{n+1} \\
= & (n+1) \int_{S_{n+1}-\varepsilon^2}^{S_{n+1}+\varepsilon^2} \Big(n! \int_{\{(t_1 < \cdots < t_n) : \sum_{i=1}^{n} |t_i - S_i| \le \varepsilon^2 - |t_{n+1} - S_{n+1}|\}} dt_1 \cdots dt_n \Big) dt_{n+1}
\end{aligned}$$

but, by hypothesis of induction the last expression is equal to

$$\begin{aligned}
& (n+1) 2^n \int_{S_{n+1}-\varepsilon^2}^{S_{n+1}+\varepsilon^2} \big(\varepsilon^2 - |t_{n+1} - S_{n+1}|\big)^n dt_{n+1} \\
= & (n+1) 2^n 2 \int_0^{\varepsilon^2} (\varepsilon^2 - u)^n du \\
= & 2^{n+1} \varepsilon^{2(n+1)},
\end{aligned}$$

which is the desired conclusion.

- Finally, if $j > k$ then $P\{\|N - h\|_2 \le \varepsilon | N_1 = j\} \le 2^k \varepsilon^{2j}$.

  Notice that if for some $i \in \{1, \ldots, k\}$, $|T_i(\omega) - S_i| > \varepsilon^2$ or if for some $i \in \{k+1, \ldots, j\}$, $1 - T_i(\omega) > \varepsilon^2$, then $\|N(\omega) - h\|_2^2 > \varepsilon^2$. Otherwise,

$$\|N(\omega) - h\|_2^2 = \sum_{i=1}^{k} |T_i(\omega) - S_i| + \sum_{i=k+1}^{j} (T_{i+1}(\omega) - T_i(\omega))(i-k)^2,$$

  where in order to simplify the notation we assume $T_{j+1} \equiv 1$.

  But,

$$\begin{aligned}
\sum_{i=k+1}^{j} (T_{i+1} - T_i)(i-k)^2 &= \sum_{i=k+1}^{j} \big[(1 - T_i) - (1 - T_{i+1})\big](i-k)^2 \\
&= \sum_{i=k+1}^{j} (1 - T_i)(2(i-k) - 1) \\
&\ge \sum_{i=k+1}^{j} (1 - T_i).
\end{aligned}$$

So, in this case, for ε small enough

$$\begin{aligned} & P\{\|N-h\|_2 \le \varepsilon | N_1 = j\} \\ = \;& P\{\sum_{i=1}^{k} |T_i - S_i| + \sum_{i=k+1}^{j} (1-T_i)(2(i-k)-1) < \varepsilon^2 | N_1 = j\} \\ \le \;& P\{\sum_{i=1}^{k} |T_i - S_i| + \sum_{i=k+1}^{j} (1-T_i) \le \varepsilon^2 | N_1 = j\}. \end{aligned}$$

Using the result for the case $j = k$ last expression is equal to

$$\begin{aligned} & \frac{j!}{k!} \int_{1-\varepsilon^2}^{1} \int_{1-(\varepsilon^2-(1-t_j))}^{t_j} \cdots \int_{1-(\varepsilon^2-\sum_{i=k+2}^{j}(1-t_i))}^{t_{k+2}} 2^k \big(\varepsilon^2 - \sum_{i=k+1}^{j} (1-t_i)\big)^k dt_{k+1} \cdots dt_j \\ = \;& 2^k \varepsilon^{2j} \end{aligned}$$

Thus,

$$\begin{aligned} P\{\|N-h\|_2 \le \varepsilon\} &= \sum_{j=0}^{\infty} P\{\|N-h\|_2 \le \varepsilon | N_1 = j\} \frac{e^{-1}}{j!} \\ &= 2^k e^{-1} \frac{\varepsilon^{2k}}{k!} + O(\varepsilon^{2k+2}), \end{aligned}$$

which completes the proof of the theorem.

$\square$

4 The case of diffusion Poisson process

Theorem 4.1 *Given $N = \{N_s,\, s \in [0,1]\}$ a standard Poisson process, and given a jump function $h_t = \sum_{i=1}^{k} I_{[S_i,1)}(t)$, $t \in [0,1]$, $0 < S_1 < \cdots < S_k < 1$, consider the diffusion process*

$$X_t = \int_0^t X_s ds + N_t, \quad t \in [0,1]$$

and the jump curve

$$\phi_t^h = \int_0^t \phi_s^h ds + h_t, \quad t \in [0,1].$$

Then, for $\varepsilon > 0$ small enough,

$$P\{\|X-\phi^h\|_1 \le \varepsilon\} = 2^k e^{-1}\frac{\varepsilon^k}{k!} + O(\varepsilon^{k+1})$$

and

$$P\{\|X-\phi^h\|_2 \le \varepsilon\} = 2^k e^{-1}\frac{\varepsilon^{2k}}{k!} + O(\varepsilon^{2k+2}).$$

The solution of such equations can be expressed, using the jump points of the Poisson process and the jump function, as

$$X(t) = \sum_{i=1}^{\infty} e^{t-T_i} I_{[T_i,1)}(t)$$

and

$$\phi_t^h = \sum_{i=1}^{k} e^{t-S_i} I_{[S_i,1)}(t).$$

As in the case of Poisson process, we can assume that there exists $\varepsilon_0 > 0$ such that $S_{i+1} - S_i > \varepsilon_0^2$, for all $i \in \{1,\ldots,k-1\}$, $S_1 > \varepsilon_0^2$ and $1 - S_k > \varepsilon_0^2$.

Before the proof of the theorem we will see a previous lemmas.

Lemma 4.2 *(a) Fixed $\alpha < \varepsilon_0$, if there exists $i \in \{1,\ldots,k\}$ such that $|S_i - T_i| > \alpha^2$, then*

$$\|X-\phi^h\|_2^2 > C_1\alpha^2$$

where C_1 denotes an universal constant.

(b) There exists $\varepsilon_1 > 0$, depending only on k, such that fixed $\alpha < \varepsilon_1$, if for some $i > k$, $T_i < 1$ and $1 - T_i > \alpha^2$, then

$$\|X-\phi^h\|_2^2 > C_2\alpha^2$$

where C_2 is another universal constant.

Proof:

We prove first a). Consider $n := \inf\{i \in \{1,\ldots,k\} : |S_i - T_i| > \alpha^2\}$, then

$$\|X-\phi^h\|_2^2 \ge \int_{S_n-\alpha^2}^{S_n+\alpha^2} |X(t)-\phi^h(t)|^2 dt.$$

If we denote

$$d_n = \sum_{i=1}^{n-1} (e^{-T_i} - e^{-S_i}) + I_{\{T_n < S_n\}} e^{-T_n},$$

the last integral is equal to

$$\begin{aligned}
& \int_{S_n-\alpha^2}^{S_n} e^{2t} d_n^2 dt + \int_{S_n}^{S_n+\alpha^2} e^{2t} (d_n - e^{-S_n})^2 dt \\
= \;& \frac{d_n^2}{2} (e^{2S_n} - e^{2(S_n-\alpha^2)}) + \frac{(d_n - e^{-S_n})^2}{2} (e^{2S_n+\alpha^2} - e^{2S_n}) \\
\geq \;& [\frac{d_n^2}{2} + \frac{(d_n - e^{-S_n})^2}{2}] 2\alpha^2 \geq \frac{e^{-2S_n}}{2} \alpha^2 > \frac{e^{-2}}{2} \alpha^2
\end{aligned}$$

using that for $0 < y < x$, $e^x - e^y \geq (x - y)$ and that for all $a, b \in \mathbb{R}$, $a^2 + b^2 \geq \frac{(a+b)^2}{2}$.

Let us prove now b). We can assume that $|S_i - T_i| \leq \alpha^2$ for all $i \in \{1, \ldots, k\}$. Notice

$$\begin{aligned}
d_\infty(t) &= |\sum_{i=1}^{k} (e^{-T_i} - e^{-S_i}) + \sum_{i=k+1}^{\infty} e^{-T_i} I_{[T_i,1)}(t)| \\
&\geq e^{-T_{k+1}} - k\alpha^2 \geq e^{-1} - k\alpha^2.
\end{aligned}$$

Then

$$\begin{aligned}
\|X - \phi^h\|_2^2 &> \int_{1-\alpha^2}^{1} |X(t) - \phi(t)|^2 dt = \int_{1-\alpha^2}^{1} e^{2t} |d_\infty(t)|^2 dt \\
&\geq \frac{(e^{-1} - k\alpha^2)^2}{2} \left(e^2 - e^{2(1-\alpha^2)}\right) \geq C_2 \alpha^2,
\end{aligned}$$

using similar arguments as in the proof of a) and for α small enough.

□

Remark 4.3 *Notice that from Lemma 4.2 we can assume, for ε small enough, that for all $i, l \in \{1, \ldots, k\}$ such that $i < l$, $S_i < T_l$ and $T_i < S_l$.*

Proof of Theorem 4.1:

As in the case of Poisson process we only develop the L^2 case.

We have that

$$P\{\|X-\phi^h\|_2 \le \varepsilon\} = \sum_{j=0}^{\infty} P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = j\} \frac{e^{-1}}{j!} \; .$$

As in the case of Poisson process, to compute the probabilities involved in the sum we will consider three cases:

- If $j < k$, then

$$P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = j\} = 0 \tag{1}$$

  for ε small enough by Lemma 4.2.

- If $j = k$, we will proof that $P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = k\} = (2\varepsilon^2)^k + O(\varepsilon^{2k+2})$.

  Notice first that by Lemma 4.2 we only need to consider the ω such that $|S_i - T_i(\omega)| < \frac{\varepsilon^2}{C_1}$ for all $i \in \{1, \dots, k\}$.

  Using the expressions of X and ϕ^h involving the jump points, we have that if $N_1 = k$,

$$\begin{aligned}
\|X-\phi^h\|_2^2 &= \int_0^1 e^{2t}\Big(\sum_{i,l=1}^{k} (e^{-T_i} I_{[T_i,1)}(t) - e^{-S_i} I_{[S_i,1)}(t)) \\
&\quad \times (e^{-T_l} I_{[T_l,1)}(t) - e^{-S_l} I_{[S_l,1)}(t))\Big) dt \\
&= \frac{1}{2}\Big[\sum_{i,l=1}^{k} \big\{e^{-T_i-T_l}\big(e^2 - e^{2(T_i \vee T_l)}\big) - e^{-T_i-S_l}\big(e^2 - e^{2(T_i \vee S_l)}\big) \\
&\quad - e^{-S_i-T_l}\big(e^2 - e^{2(S_i \vee T_l)}\big) + e^{-S_i-S_l}\big(e^2 - e^{2(S_i \vee S_l)}\big)\big\}\Big] \\
&= \frac{1}{2}\Big[\sum_{i,l=1}^{k} \big\{e^{2-T_i-T_l} - e^{|T_i-T_l|} - 2e^{2-T_i-S_l} + 2e^{|T_i-S_l|} \\
&\quad + e^{2-S_i-S_l} - e^{|S_i-S_l|}\big\}\Big].
\end{aligned} \tag{2}$$

  Notice that if we put $\delta_i := T_i - S_i$,

$$\begin{aligned}
& e^{2-S_i-S_l} + e^{2-T_i-T_l} - 2e^{2-T_i-S_l} \\
&= e^{2-S_i-S_l}\big(1 + e^{-\delta_i-\delta_l} - 2e^{-\delta_i}\big) \\
&= e^{2-(S_i+S_l)}\big(\delta_i - \delta_l + \frac{(\delta_i+\delta_l)^2}{2} e^{\eta^1_{i,l}} - \delta_i^2 e^{\eta^1_i}\big),
\end{aligned}$$

using Taylor's decomposition, where $\eta_i^1, \eta_{i,l}^1 \in [-2\frac{\varepsilon^2}{C_1}, 2\frac{\varepsilon^2}{C_1}]$.

On the other hand, by Remmark 4.3, for the other term involved in the sum (2) we have three situations

– If $i < l$ then,

$$\begin{aligned} & 2e^{|T_i - S_l|} - e^{|T_i - T_l|} - e^{|S_i - S_l|} \\ = \; & e^{S_l - S_i}\big(- e^{\delta_l - \delta_i} + 2e^{-\delta_i} - 1\big) \\ = \; & e^{S_l - S_i}\big(- \delta_l - \delta_i - \frac{(\delta_l - \delta_i)^2}{2} e^{\eta_{i,l}^2} + \delta_i^2 e^{\eta_i^2}\big), \end{aligned}$$

where $\eta_i^2, \eta_{i,l}^2 \in [-2\frac{\varepsilon^2}{C_1}, 2\frac{\varepsilon^2}{C_1}]$.

– If $i = l$

$$\begin{aligned} & 2e^{|T_i - S_l|} - e^{|T_i - T_l|} - e^{|S_i - S_l|} \\ = \; & 2e^{|T_i - S_i|} - 2 \\ = \; & 2|\delta_i| + \delta_i^2 e^{\eta_i^3}, \end{aligned}$$

where $\eta_i^3 \in [-2\frac{\varepsilon^2}{C_1}, 2\frac{\varepsilon^2}{C_1}]$.

– Finally, if $i > l$, then

$$\begin{aligned} & 2e^{|T_i - S_l|} - e^{|T_i - T_l|} - e^{|S_i - S_l|} \\ = \; & e^{S_i - S_l}\big(- e^{\delta_i - \delta_l} + 2e^{\delta_i} - 1\big) \\ = \; & e^{S_i - S_l}\big(\delta_i + \delta_l - \frac{(\delta_i - \delta_l)^2}{2} e^{\eta_{i,l}^4} + \delta_i^2 e^{\eta_i^4}\big). \end{aligned}$$

where $\eta_i^4, \eta_{i,l}^4 \in [-2\frac{\varepsilon^2}{C_1}, 2\frac{\varepsilon^2}{C_1}]$.

So, when $N_1 = k$,

$$\|X - \phi^h\|_2^2 = \sum_{i=1}^{k} |\delta_i| + Z, \tag{3}$$

where

$$
\begin{aligned}
Z \;=\; & \sum_{i,l=1}^{k} e^{-2(S_i+S_l)}\Big(\frac{(\delta_i+\delta_l)^2}{2}e^{\eta^1_{i,l}} - \delta_i^2 e^{\eta^1_i}\Big) \\
& + \sum_{i=1}^{k}\delta_i^2 e^{\eta^3_i} + \sum_{i<l} e^{S_l-S_i}\Big(-\frac{(\delta_l-\delta_i)^2}{2}e^{\eta^2_{i,l}} + \delta_i^2 e^{\eta^2_i}\Big) \\
& + \sum_{i>l} e^{S_i-S_l}\Big(-\frac{(\delta_i-\delta_l)^2}{2}e^{\eta^4_{i,l}} + \delta_i^2 e^{\eta^4_i}\Big)\Big].
\end{aligned}
$$

Fixed ω, by part a) of Lemma 4.2, if $\|X(\omega)-\phi^h\|_2 \le \varepsilon$ we get $|T_i(\omega)-S_i| \le \frac{\varepsilon^2}{C_1}$ for all $i \in \{1,\dots,k\}$. Then $|Z(\omega)| < 6k^2\frac{\varepsilon^4}{C_1^2}$ and we get from (3) and using the results proved in Theorem 3.1

$$
\begin{aligned}
P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = k\} &\le P\{\sum_{i=1}^{k}|\delta_i| \le \varepsilon^2 + 6k^2\frac{\varepsilon^4}{C_1^2} | N_1 = k\} \\
&= \big(2\varepsilon^2(1+6k^2\frac{\varepsilon^2}{C_1^2})\big)^k \le 2^k\varepsilon^{2k} + c_{k,1}\varepsilon^{2k+2} \qquad (4)
\end{aligned}
$$

where $c_{k,1}$ depends only on k and C_1.

On the other hand, fixed ω such that $\sum_{i=1}^{k}|\delta_i| \le \varepsilon$ we clearly have $|Z(\omega)| < 6k^2\varepsilon^4$. So, again from (3) we get

$$
\begin{aligned}
P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = k\} &\ge P\{\sum_{i=1}^{k}|\delta_i| \le \varepsilon^2 - 6k^2\varepsilon^4 | N_1 = k\} \\
&= \big(2\varepsilon^2(1-6k^2\frac{\varepsilon^2}{C_1^2})\big)^k \ge 2^k\varepsilon^{2k} + c_{k,2}\varepsilon^{2k+2} \qquad (5)
\end{aligned}
$$

where $c_{k,2}$ depends only on k and C_2.

Putting together (4) and (5) we obtain

$$
P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = k\} = (2\varepsilon^2)^k + O(\varepsilon^{2k+2}). \qquad (6)
$$

- To deal with the case $j > k$ notice that, again by Lemma 4.2, if $\|X(\omega)-\phi^h\|_2 \le \varepsilon$ then $|T_i(\omega)-S_i| \le \frac{\varepsilon^2}{C_1}$ for all $i \in \{1,\dots,k\}$ and $|1-T_i(\omega)| \le \frac{\varepsilon^2}{C_2}$

for all $i \in \{k+1, \dots, j\}$. So, using Proposition 2.2 we have

$$
\begin{aligned}
&P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = j\} \\
&\quad \le P\{\{\max_{1\le i\le k} |T_i - S_i| \le \frac{\varepsilon^2}{C_1}\} \cap \{\max_{k+1\le i\le j} |1-T_i| \le \frac{\varepsilon^2}{C_2}\} | N_1 = j\} \\
&\quad \le \frac{j! 2^k \varepsilon^{2j}}{C_1^k C_2^{j-k}},
\end{aligned}
$$

and then

$$
\begin{aligned}
&\sum_{j=k+1}^{\infty} P\{\|X-\phi^h\|_2 \le \varepsilon | N_1 = j\} \frac{e^{-1}}{j!} \\
\le &\sum_{j=k+1}^{\infty} \frac{e^{-1} 2^k \varepsilon^{2j}}{C_1^k C_2^{j-k}} = e^{-1} 2^k \frac{C_2}{C_1^k} \frac{\varepsilon^{2k+2}}{(1-\varepsilon^2 C_2)}.
\end{aligned} \tag{7}
$$

Putting together (1), (6) and (7) we finish the proof of the theorem.

$\square$

References

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