Distributional Inequalities for Non-harmonic Functions

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ABSTRACT. The relationship between the non-tangential maximal function and convenient versions of the area function of a general (non harmonic) function in a upper-half space are studied.

1. INTRODUCTION

Let u be a harmonic function in the upper half space $\mathbb{R}^{n+1}_+ = \{(w,y) : w \in \mathbb{R}^n,\ y > 0\}$ of $\mathbb{R}^n$. Given $\alpha > 0$, we define the area function of u at a point $x \in \mathbb{R}^n$ as the integral of $|\nabla u|^2$ over the cone $\Gamma_\alpha(x)$ in the x_{n+1}-direction:

$$(A_\alpha u)(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(w,y)|^2 y^{1-n}\, dw\, dy \right)^{1/2},$$

where

$$\Gamma_\alpha(x) = \{(w,y) \in \mathbb{R}^{n+1}_+ : |x-w| < \alpha y\},$$

and the non-tangential maximal function of u at x as

$$(N_\alpha u)(x) = \sup_{\Gamma_\alpha(x)} |u(w,y)|.$$

We will also consider the (doubly) truncated cone

$$\Gamma_{\alpha,t,s}(x) = \{(w,y) \in \mathbb{R}^{n+1}_+ : |x-w| < \alpha y,\ t < y < s\}, \quad 0 \le t < s \le \infty$$

and the corresponding (doubly) truncated area function $(A_{\alpha,t,s}u)(x)$ and non-tangential maximal function $(N_{\alpha,t,s}u)(x)$ obtained by replacing $\Gamma_\alpha(x)$ by $\Gamma_{\alpha,t,s}(x)$ in the previous definitions. Note that $(A_\alpha u)$, $(N_\alpha u)$ correspond to $(A_{\alpha,0,\infty}u)$, $(N_{\alpha,0,\infty}u)$, respectively.

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We say that u is non-tangentially bounded at x if $(N_{\alpha,0,s}u)(x) < \infty$ for some $\alpha > 0$ and some (all) $\infty > s > 0$. Calderón ([7]) proved that, if u is a harmonic function in $\mathbb{R}^{n+1}_+$ and E is the set of points in $\mathbb{R}^n$ on which u is non-tangentially bounded, then for almost every point $x \in E$, $(A_{\alpha,0,s}u)(x) < \infty$ for all $\alpha > 0$, $s > 0$. Stein [24] proved the converse: if $F \subset \mathbb{R}^n$ is a set with the property that for every $x \in F$, there exists $\alpha = \alpha(x) > 0$, $s = s(x) > 0$ such that $(A_{\alpha,0,s}u)(x) < \infty$, then for almost every point $x \in F$, the function u is non-tangentially bounded at x. These results were first proved in dimension 1 by Marcinkiewicz and Zygmund [20] and Spencer [23]. Calderón ([8]) also showed that a harmonic function in $\mathbb{R}^{n+1}_+$ has a non-tangential limit at almost every point where it is non-tangentially bounded. Summarizing: if u is a harmonic function in the upper half space $\mathbb{R}^{n+1}_+$, the sets

$$\{x \in \mathbb{R}^n : u \text{ has non-tangential limit at } x\},$$
$$\{x \in \mathbb{R}^n : \text{there exist } s = s(x),\ \alpha = \alpha(x) \text{ such that } (A_{\alpha,0,s}u)(x) < \infty\},$$

can only differ in a set of measure 0.

The results of Calderón and Stein above on the connection between the area integral and the non-tangential maximal function extend to the L^p-setting. By a result of Fefferman and Stein, for all α, β, $p > 0$, the L^p-norm of A_α is dominated by the L^p-norm of N_β. The converse estimate holds if one in addition assumes that $\lim_{y\to\infty} u(x,y) = 0$ for each $x \in \mathbb{R}^n$. For an Orlicz-norm extension of this result see the paper [5] by Burkholder and Gundy. The proofs are based on inequalities relating the distribution functions of the non-tangential maximal function and the area function. These inequalities, which came to be known as good-λ inequalities, were sharpened by Murai and Uchiyama [22] and Bañuelos and Moore [3]. The result of Murai and Uchiyama can be stated as follows: If u is a harmonic function in the upper half space $\mathbb{R}^{n+1}_+$ and $0 < \beta < \alpha$, then there exist constants $C_1, C_2 > 0$ (depending on α, β, n) such that for any $M > 1$, $\lambda > 0$ one has

$$\begin{aligned} |\{x \in \mathbb{R}^n : (A_\beta u)(x) > M\lambda,\ (N_\alpha u)(x) < \lambda\}| \\ \le C_1 \exp(-C_2 M^2)|\{x \in \mathbb{R}^n : (A_\beta u)(x) > \lambda\}|, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} |\{x \in \mathbb{R}^n : (N_\beta u)(x) > M\lambda,\ (A_\alpha u)(x) < \lambda\}| \\ \le C_1 \exp(-C_2 M)|\{x \in \mathbb{R}^n : (N_\beta u)(x) > \lambda\}|. \end{aligned} \tag{1.2}$$

The proof involves the construction of certain "sawtooth" regions and estimates on the BMO-norm of truncated versions of the area function in such regions. Although both estimates have an exponential decay, observe that (1.2) does not have the whole subgaussian decay. These results have been extended to other contexts. In [21], [28], [17] area integrals of subharmonic functions are considered and an L^p-estimate is proved. Area integrals for solutions of second order elliptic equations were considered by Dahlberg, Jerison and Kenig [13]. See also [17]. The result of Bañuelos-Moore is deeper than (1.2), and using the preceding notation can be stated as:

$$\begin{aligned}(1.3) \qquad &|\{x \in \mathbb{R}^n : (N_\beta u)(x) > M\lambda,\ (A_\alpha u)(x) < \lambda\}| \\ &\leq C_1 \exp(-C_2 M^2)|\{x \in \mathbb{R}^n : (N_\beta u)(x) > \lambda\}|.\end{aligned}$$

The proof is based on reducing the desired estimate to its dyadic martingale analog, which was proved by Chang, Wilson and Wolff ([9]). It is worth mentioning that Bañuelos and Moore ([3]) extended the Murai-Uchiyama result (1.1) to harmonic functions in Lipschitz domains, while the estimate (1.3) is not known in this generality (see [2, p. 98, p.113]). For an exposition and extension of these ideas see [2, Chapter IV] by Bañuelos and Moore.

Yet another result in similar vein is the Law of the Iterated Logarithm for harmonic functions that we next describe. Suppose that u is harmonic in the upper half plane and that $|\nabla u(x,y)| \leq C/y$ for each (x,y). Then, according to a result of Makarov ([19]),

$$(1.4) \qquad \limsup_{y\to 0} \frac{|u(x,y)|}{\sqrt{\log(1/y)\log\log\log(1/y)}} \leq 2C$$

for almost all $x \in \mathbb{R}$. This result was extended to harmonic functions in the upper half space in [1] and to Lipschitz domains in [18]. It can be considered as a Fatou type theorem: for each $x \in \mathbb{R}^n$, the estimate $|\nabla u(x,y)| \leq C/y$ gives a logarithmic upper bound on the growth of u when y tends to zero, but a substantially improved estimate holds for almost all x. The following area integral version of this Law of the Iterated Logarithm was established by Bañuelos, Klemeš and Moore [1], [2, Chapter III] by reducing it to the dyadic martingale setting. Fix $0 < \beta < \alpha$ and $0 < \gamma < 1$. There is a constant $C = C(\alpha, \beta, \gamma, n)$ such that if u is harmonic in $\mathbb{R}^{n+1}_+$, then

$$(1.5) \qquad \limsup_{(w,y)\to(x,0),\ (w,y)\in\Gamma_{\beta,0,1}} \frac{|u(w,y)|}{\sqrt{A^2_{\alpha,\gamma y,1}u(x)\log\log A_{\alpha,\gamma y,1}u(x)}} \leq C,$$

for almost every point $x \in \{x \in \mathbb{R}^n : A_\alpha u(x) = \infty\}$.

The results of Calderón and Stein cited above state that the sets where u is non-tangentially bounded and where the area function is finite can only differ on a set of Lebesgue measure zero. In the complement of this set, that is, at almost every point where the function is not non-tangentially bounded, this Law of the Iterated Logarithm (LIL) measures the relative growth of these quantities.

It is clear that the results above do not hold for all smooth functions. Very recently two of us showed in [14] that the result of Makarov on the growth of harmonic functions has the following analog for solutions to the Poisson equation $\Delta u = f$ in the unit ball B^n of $\mathbb{R}^n$ under a growth condition on f: Let u be a C^2 function in the unit ball B^n that satisfies $|\nabla u(x)| \le C/(1-|x|)$ and assume in addition that

$$|\Delta u(x)| \le \frac{C}{(1-|x|)^2\left(\log \frac{2}{1-|x|}\right)},$$

for all $x \in B^n$. Then

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\left(\log \frac{1}{1-r}\right) \log\log \frac{1}{1-r}}} \le c$$

for almost all $\zeta \in S^{n-1}$. Here c depends only on C and n.

Notice that the growth order of $|u(x)\Delta u(x)|$ above is at most $C(1-|x|)^{-2}$, which is no more than the worst possible growth of $|\nabla u(x)|^2$. Because of this, the contribution of the Laplacian can be embedded in the "gradient" estimates and the indicated version of the LIL can be proven.

In this paper, we continue the analysis of the relation between the non-tangential maximal function and the area function of non-harmonic functions. To control the non-tangential maximal function by the area function, very light assumptions on the function are needed. More precisely, we say that a C^2-function u in $\mathbb{R}^{n+1}_+$ has φ-controlled oscillation if there exists a constant $0 \le \eta < 1$ and an increasing positive function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, $\sup\{\varphi(2t)/\varphi(t) : t > 0\} < \infty$, such that for any ball $B \subset \mathbb{R}^{n+1}_+$ of radius r_B satisfying $2B \subset \mathbb{R}^{n+1}_+$, the estimate:

$$(*) \quad \max\{|u(w_1) - u(w)| : w_1,\ w \in B\} \\ \le \varphi\left(\left(r_B^{1-n} \int_{(1+\eta)B} |\nabla u(x,y)|^2 + |u(x,y)|\,|\Delta u(x,y)|\,dx\,dy\right)^{1/2}\right)$$

holds, where $(1+\eta)B$ is the ball with the same center as B and radius $(1+\eta)r_B$. We will often refer to this condition as condition $(*)$.

A harmonic function u in $\mathbb{R}^{n+1}_+$ satisfies condition $(*)$ for any $0 < \eta < 1$ with $\varphi(t) = Ct$, where C is a constant depending on η. This is a simple consequence of the subharmonicity of $|\nabla u|$. Condition $(*)$ holds for many functions. For instance, we will prove that a function $u \in C^2(\mathbb{R}^{n+1}_+)$ satisfying

$$(1.6) \qquad |u\Delta u| \le C|\nabla u|^2$$

on $\mathbb{R}^{n+1}_+$, for a fixed constant C, satisfies condition $(*)$ with the function $\varphi(t) = At$, where $A = A(C)$ is a constant. In the general context of functions u satisfying

$(*)$, we consider the area function $S_\alpha u$ defined as

$$(S_\alpha u)(x) = \left(\int_{\Gamma_\alpha(x)} (|\nabla u(w,y)|^2 + |u(w,y)|\,|\Delta u(w,y)|) y^{1-n}\, dw\, dy \right)^{1/2}.$$

Clearly, for functions satisfying (1.6), the area function $(S_\alpha u)$ is comparable to the usual area function $(A_\alpha u)$. Also, if $x \in \mathbb{R}^n$ and $y > 0$, we denote by $(S_\alpha u)(x,y)$ (resp. $(N_\alpha u)(x,y)$) the area function (resp. the non tangential maximal function), obtained by replacing the cone $\Gamma_\alpha(x)$ by its translation $\Gamma_\alpha(x,y) = y + \Gamma_\alpha(x)$ so that its vertex is at (x,y).

We prove a good-λ inequality analogous to (1.2), relating the non-tangential maximal function and the area function S_α of a function satisfying condition $(*)$. We do not know if a subgaussian estimate of the type of (1.3) holds in this more general setting. The good-λ inequality we prove leads to a Fatou-type theorem, to L^p-estimates and to a certain LIL, which we collect in the following result.

Theorem 1.1. *Let u be a C^2-function in $\mathbb{R}^{n+1}_+$ having φ-controlled oscillation. Assume*

$$\liminf_{\lambda\to\infty} \frac{\varphi(\lambda)}{\lambda} > 0.$$

Fix $0 < \alpha < \beta$ and assume there exists $x_0 \in \mathbb{R}^n$ such that $(N_{\alpha,1,\infty} u)(x_0) < \infty$. Then,

(a) *For* a.e. $x \in \{x \in \mathbb{R}^n : (S_\beta u)(x) < \infty\}$, *the function $u(w,y)$ has a finite limit when $(w,y) \in \Gamma_\alpha(x)$ tends to x.*

(b) *Assume $\varphi(t) \geq mt$ for any $t \in (0,\infty)$ and* $\lim u(x,y) = 0$ *as $\|(x,y)\| \to \infty$. For $0 < p < \infty$, there exists a constant C depending on p, α, β, n, φ, m such that*

$$\|N_\alpha u\|_{L^p(\mathbb{R}^n)} \leq C \|\varphi(S_\beta u)\|_{L^p(\mathbb{R}^n)}.$$

(c) *There exists a constant C depending on α, β, φ, n such that*

$$\limsup_{t\to 0} \frac{(N_\alpha u)(x,t)}{\varphi((S_\beta u)(x,t)) \log\log \varphi((S_\beta u)(x,t))} \leq C,$$

for a.e. $x \in \{x \in \mathbb{R}^n : (S_\beta u)(x) = \infty\}$.

As mentioned before, in the harmonic case one can take $\varphi(t) = ct$. So, in this case, comparing (c) with the LIL (1.5) which holds for harmonic functions, there is a square root missing in the term $\log\log S_\alpha$. This is due to the fact that the good-λ inequality we prove does not have the whole subgaussian decay. On the other hand, our results hold on Lipschitz domains, while the LIL and the good-λ inequality with subgaussian decay for harmonic functions on Lipschitz domains are open problems. See [2, p. 98]. Also, for solutions of second order elliptic equations an estimate analogous to (1.3) is only known in very concrete cases. See [27] and [2, p. 49].

The control of the area function by the non-tangential maximal function requires more specific assumptions on the function u. Let $u \in C^2(\mathbb{R}^{n+1}_+)$. We say that the function u satisfies condition (1.7) if there exists $0 < \theta < 1$ such that

$$|u(x,y)|\,|\Delta u(x,y)| \leq \theta |\nabla u(x,y)|^2, \tag{1.7}$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$. As before, for functions u satisfying condition (1.7), the area functions $A_\alpha u$ and $S_\alpha u$ are comparable. We prove a good-λ inequality which is analogous to the estimate (1.1) of Murai and Uchiyama. Again, the good-λ inequality leads to a Fatou type result, an L^p-estimate and to a Law of the Iterated Logarithm.

Theorem 1.2. *Let* $u \in C^2(\mathbb{R}^{n+1}_+)$ *be a function satisfying condition* (1.7) *for a constant* $0 < \theta < 1$. *Let* $0 < \alpha < \beta$ *and assume there exists* $x_0 \in \mathbb{R}^n$ *such that* $(A_{\alpha,1,\infty} u)(x_0) < \infty$. *Then:*

(a) *For* a.e. $x \in \{x \in \mathbb{R}^n : (N_\alpha u)(x) < \infty\}$, *one has* $(A_\alpha u)(x) < \infty$.

(b) *For* $0 < p < \infty$, *there exists a constant* C_1 *depending on* p, α, β, n, θ *such that*

$$\|A_\alpha u\|_{L^p(\mathbb{R}^n)} \leq C_1 \|N_\beta u\|_{L^p(\mathbb{R}^n)}.$$

(c) *There exists a constant* C_2 *depending on* α, β, n, θ *such that*

$$\limsup_{t\to 0} \frac{(A_\alpha u)(x,t)}{\sqrt{(N_\beta u)(x,t)\log\log(N_\beta u)(x,t)}} \leq C_2$$

for a.e. $x \in \{x \in \mathbb{R}^n : (N_\beta u)(x) = \infty\}$.

Local versions of Theorems 1.1 and 1.2 also hold. For instance, one can replace A_α, N_α in (a) in both results by $A_{\alpha,0,1}$, $N_{\alpha,0,1}$.

We do not have an example to show that the condition $0 \leq \theta < 1$ is essential, but we see no real hope in relaxing this condition because of the following: There is a bounded C^2-function u so that the area integral of u is infinite at every point, but still (1.6) holds in the following averaged sense for some constant $C \geq 1$:

$$\int_B |u\Delta u| \leq C \int_B |\nabla u|^2$$

for each ball $B = B((w,t),(1-\varepsilon)t)$, where $\varepsilon > 0$ is a small positive constant. This indicates that perhaps one should replace (1.7) by an averaged integral, and similarly replace u by the average of u. We have not been able to do this, but the reason might well be only technical.

The proofs of Theorems 1.1 and 1.2 are based on stopping time arguments and on Green's formula on sawtooth regions. In this sense, the work of Murai and Uchiyama [22] and of Bañuelos and Moore [3] provide not only an outline for our proof, but also important techniques. However, the fact that our function is not harmonic causes several difficulties, which are solved by a careful choice of the domains where we apply Green's formula, by stopping time arguments and Caccioppoli inequalities. Even if one only wishes to prove the results in the upper half space, the use of stopping time arguments and sawtooth regions leads to Lipschitz domains, and one needs to study the corresponding results in this context. It is actually on the setting of Lipschitz domains that we prove our results. The work of Dahlberg ([11]) provides the necessary estimates for the harmonic measure and Green's functions on such domains.

The paper is organized as follows. Section 2 contains notation and background material on Lipschitz domains. Section 3 is devoted to the results on the control of the non-tangential maximal function by the area function S. Section 4 is devoted to the converse results, that is, to the control of the area function by the non-tangential maximal function. Finally, Section 5 contains the proof that condition (1.6) implies $(*)$ with $\varphi(t) = At$.

We are indebted to the referee who pointed out several errors and proposed alternative arguments.

2. Notation, Definitions and Background on Lipschitz Domains

From now on, we will consider domains of the form

$$\Omega = \{(x, y) \in \mathbb{R}^{n+1}_+ : y > \phi(x)\},$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant M, that is,

$$|\phi(x) - \phi(z)| \leq M|x - z| \quad \text{if } x, z \in \mathbb{R}^n.$$

Given $x \in \mathbb{R}^n$, $\alpha > 0$, define $\Gamma_\alpha(x) = \{(z, y) \in \mathbb{R}^{n+1}_+ : |z - x| < \alpha(y - \phi(x))\}$. Since $y - \phi(x) \geq \operatorname{dist}((x, y), \partial\Omega) \geq (y - \phi(x))/(\sqrt{1 + M^2})$ if $(x, y) \in \Omega$, then $\Gamma_\alpha(x) \subset \Omega$ provided $0 < \alpha < 1/M$. Hereafter, we will only consider such values of α.

Now, if $x \in \mathbb{R}^n$, $0 < \alpha < 1/M$, $0 \leq t \leq s \leq +\infty$, we also introduce the truncated cones:

$$\Gamma_{\alpha,t,s}(x) = \Gamma_\alpha(x) \cap \{(z, y) : \phi(z) + t < y < \phi(z) + s\},$$

with special attention to the cases $t = 0$ or $s = \infty$.

Let Ω, ϕ, α be as above. Let $u : \Omega \to \mathbb{R}$, $f : \Omega \to [0, +\infty]$ be measurable functions, $x \in \mathbb{R}^n$ and $0 \leq t \leq s \leq \infty$. We define the non-tangential maximal function of u by: $(N_\alpha u)(x) = \sup_{\Gamma_\alpha(x)} |u|$, and also its truncated version: $(N_{\alpha,t,s} u)(x) = \sup_{\Gamma_{\alpha,t,s}(x)} |u|$. Furthermore, we define the area function associated to the density f by:

$$(\mathcal{A}_\alpha f)(x) = \left(\int_{\Gamma_\alpha(x)} f(z, y)(y - \phi(x))^{1-n} \, dz \, dy \right)^{1/2},$$

and also the truncated version:

$$(\mathcal{A}_{\alpha,t,s}f)(x) = \left(\int_{\Gamma_{\alpha,t,s}(x)} f(z,y)(y-\phi(x))^{1-n}\,dz\,dy\right)^{1/2}.$$

We are specially interested in the case $f = |\nabla u|^2$. Then, we will simply denote,

$$(A_\alpha u)(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(z,y)|^2 (y-\phi(x))^{1-n}\,dz\,dy\right)^{1/2},$$

$$(A_{\alpha,t,s}u)(x) = \left(\int_{\Gamma_{\alpha,t,s}(x)} |\nabla u(z,y)|^2 (y-\phi(x))^{1-n}\,dz\,dy\right)^{1/2}$$

and we will refer to them as the (truncated) area functions of u. When u is harmonic, these forms of the area functions were already used in [3]. If $(z,y) \in \Gamma_\alpha(x)$, the quantity $|y-\phi(x)|$ is comparable to $\operatorname{dist}((z,y),\partial\Omega)$. Hence $A_\alpha u(x)$, $A_{\alpha,t,s}u(x)$ are comparable to

$$\left(\int_{\Gamma_\alpha(x)} |\nabla u(w)|^2 \delta(w)^{1-n}\,dm(w)\right)^{1/2},$$

$$\left(\int_{\Gamma_{\alpha,t,s}(x)} |\nabla u(w)|^2 \delta(w)^{1-n}\,dm(w)\right)^{1/2},$$

where $\delta(w) = \operatorname{dist}(w,\partial\Omega)$, $w \in \Omega \subset \mathbb{R}^{n+1}_+$ and dm is the Lebesgue measure in $\mathbb{R}^{n+1}_+$.

We also need to consider a new square function $(S_\alpha u)(x)$, in spirit very similar to the usual area function, $(A_\alpha u)(x)$. In fact, they both coincide when u is harmonic. So, following the notation above, we define

$$(S_\alpha u)(x) = \left(\int_{\Gamma_\alpha(x)} \left(|\nabla u(w)|^2 + |u(w)\Delta u(w)|\right) \delta(w)^{1-n}\,dm(w)\right)^{1/2}$$

and its truncated version

$$(S_{\alpha,t,s}u)(x) = \left(\int_{\Gamma_{\alpha,t,s}(x)} \left(|\nabla u(w)|^2 + |u(w)\Delta u(w)|\right) \delta(w)^{1-n}\,dm\,(w)\right)^{1/2}.$$

If Ω, ϕ are as above, then, whenever $E \subset \mathbb{R}^n$, we denote by $G_E = \{(x,\phi(x)) : x \in E\}$ the piece of the graph above E. If $Q \subset \mathbb{R}^n$ is a cube with side length $\ell(Q)$, we define

$$\hat{Q} = \{(x,y) \in \Omega : x \in Q,\ \phi(x) < y < \phi(x) + \ell(Q)\},$$

$$T(\hat{Q}) = \left\{(x,y) \in \Omega : x \in Q,\ \phi(x) + \tfrac{1}{2}\ell(Q) < y < \phi(x) + \ell(Q)\right\}.$$

As in the case of $\mathbb{R}^n$, it is useful to deal with dyadic decompositions in Ω. Denote by $\mathcal{F}_k$ the family of all dyadic cubes of the generation k, that is, all cubes Q of the form $Q = \prod_{i=1}^n [m_i 2^{-k}, (m_i+1)2^{-k}]$, where $m_i \in \mathbb{Z}$. Then $\{G_Q\}_{Q \in \mathcal{F}_k}$ is called a dyadic partition of $\partial\Omega$ and, for each $k \in \mathbb{N}$ and each $Q \in \mathcal{F}_k$, the collection $\{T(\widehat{Q'}) : Q' \in \mathcal{F}_j,\ Q' \subset Q,\ j \geq k\}$ is called a dyadic partition of $\widehat{Q}$.

Finally, we close this preliminary section with some properties of Green's function and the harmonic measure in Lipschitz domains that will be needed later.

Suppose that Q is a cube in $\mathbb{R}^n$, centered at $x_0 \in \mathbb{R}^n$, with side length $\ell(Q)$. Set $p_Q = (x_0, \phi(x_0) + \frac{1}{2}\ell(Q))$ and let g, ω be Green's function and the harmonic measure in $\widehat{Q}$, with respect to p_Q. We also denote by ω^* the projection of ω, restricted to the graph G_Q, that is, $\omega^*(E) = \omega(G_E, p_Q, \widehat{Q})$, for $E \subset Q$. The following theorem proved by Dahlberg [11] collects the central properties of ω.

Theorem 2.1 ([11], [17]). *With the notation above,*

(a) *ω is mutually absolutely continuous with respect to the surface measure of $\partial\widehat{Q}$. In particular, ω^* is mutually absolutely continuous with respect to the Lebesgue measure in Q.*

(b) *The density $d\omega^*/dx$ is an A_∞-weight. Actually, there exist positive constants, C, α, β depending only on M (the Lipschitz constant of Ω), such that*

$$C^{-1}\left(\frac{|E|}{|Q^*|}\right)^{\beta} \leq \frac{\omega^*(E)}{\omega^*(Q^*)} \leq C\left(\frac{|E|}{|Q^*|}\right)^{\alpha}$$

whenever $Q^* \subset Q$ is a cube and $E \subset Q^* \subset Q$.

(c) *Let Q^* in $\mathbb{R}^n$ be a cube with $Q^* \subset \frac{1}{2}Q$, where $\frac{1}{2}Q$ is the cube with the same center as Q and half its side length. Then*

$$C^{-1}\left(\frac{\ell(Q^*)}{\ell(Q)}\right)^{n-1} g(p_{Q^*}) \leq \omega^*(Q^*) \leq C\left(\frac{\ell(Q^*)}{\ell(Q)}\right)^{n-1} g(p_{Q^*}).$$

3. Control of the Non-tangential Maximal Function by the Area Function

Throughout this section we will consider functions $u \in C^2(\Omega)$ whose oscillation on hyperbolic balls is controlled by a quantity similar to the one defining $(S_\alpha u)(x)$. By a hyperbolic ball centered at a point $w_0 \in \Omega$, we will understand an euclidean ball centered at w_0 whose radius is $c\delta(w_0)$, where $0 < c < 1$, and $\delta(w_0) = \mathrm{dist}(w_0, \partial\Omega)$. Since Ω is a Lipschitz domain, these balls are actually comparable to the ones induced by the hyperbolic metric in Ω.

We recall the notation given at the introduction. We say that a function u satisfies condition $(*)$ if there exists a constant η, $0 \leq \eta < 1$, and an increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(2t) < C\varphi(t)$, such that for any ball $B \subset \Omega$ of radius r_B satisfying $2B \subset \Omega$, the following holds:

$$\begin{aligned}\operatorname{osc}(u, B) &= \max\{|u(w_1) - u(w_2)| : w_1, w_2 \in B\} \\ &\leq \varphi\left(\left(\int_{(1+\eta)B} (|\nabla u|^2 + |u\Delta u|) r_B^{1-n}\, dm\right)^{1/2}\right),\end{aligned}$$

where $(1+\eta)B$ is the ball with the same center as B and radius $(1+\eta)r_B$.

Note that, if u is harmonic in Ω, then it satisfies condition $(*)$ for any $1 > \eta > 0$ and $\varphi(t) = Ct$, where C is a constant depending on η. This is simply a consequence of the subharmonicity of the gradient. Actually, given w_1, $w_2 \in B$, let L be the line that joins them. Then, using that $|\nabla u|$ is subharmonic, Fubini's Theorem and Hölder's inequality, we get for $\rho = \rho(\eta) < \delta(w_0)$ that

$$\begin{aligned}|u(w_1) - u(w_2)| &\leq \int_L |\nabla u(s)|\, ds \leq \int_L \fint_{B(s,\rho)} |\nabla u(w)|\, dw\, ds \\ &\lesssim \left(\int_{(1+\eta)B} |\nabla u|^2 r_B^{1-n}\, dm\right)^{1/2},\end{aligned}$$

with comparison constants only depending on η. From now on, $\fint_B f$ will denote the average of the function f over the ball B.

We proceed now to state and prove a lemma that will become our main auxiliary result, but before that let us fix the notation.

Let $\Omega = \{(x, y) \in \mathbb{R}^{n+1}_+,\ y > \phi(x)\}$ be a Lipschitz domain with Lipschitz constant M. Denote by Q a dyadic cube in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ of side length $\ell(Q)$, and by w_Q the point $w_Q = (x_0, \phi(x_0) + \ell(Q))$, which is contained in the boundary of $T(\widehat{Q})$. Also recall that the usual dyadic decomposition of Q gives a dyadic decomposition of $\widehat{Q} \subset \Omega$.

Lemma 3.1. *Let u be a function satisfying condition $(*)$ in Ω. Fix $\alpha > 0$. Assume that for each $x \in Q$,*

$$(S_\alpha u)(x) \leq a \leq C\varphi(a).$$

Then there exist constants $c = c(\alpha, M, n, C)$, $A = A(\alpha, M, n, C)$ such that: If $N \geq A$ and $\{Q_j\}$ are the maximal dyadic cubes of $\mathbb{R}^n$ contained in Q satisfying

$$\sup_{w \in T(\widehat{Q}_j)} |u(w) - u(w_Q)| \geq N\varphi(ca),$$

then

$$\frac{1}{(\ell(Q))^n} \sum_j (\ell(Q_j))^n \leq \varepsilon(N),$$

where $\varepsilon(N) \to 0$ as $N \to \infty$.

Proof. The idea is similar to the one given in [25] to prove the local version of Fatou's Theorem for harmonic functions. We will build a new Lipschitz domain $R \subset \Omega$ and we will apply Green's formula in R to the functions $(u - u(w_Q))^2$ and $g(z)$, where $g(z)$ is some Green's function to be specified later on. In the harmonic case, the boundary terms in Green's formula are controlled using properties of the Poisson Kernel. Since this tool is not available in our setting, we will need to choose very carefully the hyperplanes in the construction of ∂R, in order to overcome the technical difficulties that arise in estimating such boundary terms.

We will often use dm and $d\sigma$ to denote the Lebesgue measure and the surface measure, respectively. Also by $A \lesssim B$, $(A \gtrsim B)$ we will mean that $A \leq CB$ $(A \geq CB)$, where C is a constant. Moreover, $A \simeq B$ means that $A \lesssim B$ and $A \gtrsim B$. Without loss of generality we can assume that $\ell(Q) = 1$. Thus $\delta(w) \simeq 1$ if $w \in T(\hat{Q})$. Since $T(\hat{Q})$ can be covered by a fixed number of cones and $(S_\alpha u)(x) \leq a$ for all $x \in Q$, we get

$$\int_{T(\hat{Q})} (|\nabla u|^2 + |u\Delta u|)\, dm \lesssim a^2,$$

with comparison constant depending on α, M and n. By Chebychev's inequality, we can choose a "horizontal hyperplane" L such that

$$\int_{L\cap T(\hat{Q})} (|\nabla u|^2 + |u\Delta u|)\, d\sigma \lesssim a^2, \tag{3.1}$$

where $L \cap T(\hat{Q}) = \{(x, y) : x \in Q,\ y = \phi(x) + y_0\}$ with $\frac{1}{2} < y_0 < 1$.

Observe that condition $(*)$ and the bound on the function S_α give that $\operatorname{osc}(u, T(\hat{Q})) < C(\alpha, M, n)\varphi(a)$. Hence, if the constants c, A are chosen sufficiently large, one may assume that $\widehat{Q_j} \subset \hat{Q} \setminus T(\hat{Q})$, for $j = 1, 2, \ldots$.

Next, consider the domain

$$\mathcal{D} = \{(x, y) : x \in Q,\ \phi(x) < y < \phi(x) + 2y_0\} \cap \Big(\Omega \setminus \bigcup \widehat{Q_j}\Big).$$

Let $p_0 = (x_0, \phi(x_0) + \frac{3}{2}y_0)$ and let $g(\cdot) = g(\cdot, p_0)$ be Green's function of $\mathcal{D}$ with pole at p_0. Denote by $\omega(\cdot) = \omega(\cdot, p_0, \mathcal{D})$ the harmonic measure in $\mathcal{D}$ with respect to p_0. Even though $\mathcal{D}$ is not a Lipschitz domain, it is clear that the results concerning the harmonic measure and behavior of Green's function stated in Theorem 2.1 hold on $\mathcal{D}$ as well.

Applying now Green's formula in the region

$$\mathcal{R} = \{(x, y) : x \in Q,\ \phi(x) < y < \phi(x) + y_0\} \cap \mathcal{D}$$

to the functions $(u - u(w_Q))^2$ and g, we obtain

$$(3.2)\quad \int_{\partial\mathcal{R}} (u - u(w_Q))^2 \partial_{\vec{n}} g \, d\sigma - \int_{\partial\mathcal{R}} \partial_{\vec{n}} (u - u(w_Q))^2 g \, d\sigma = \int_{\mathcal{R}} (|\nabla u|^2 + (u - u(w_Q))\Delta u) g \, dm,$$

$\Delta(u^2) = 2(|\nabla u|^2 + u\Delta u)$ and g is harmonic in $\mathcal{R}$.

Note that $\partial\mathcal{R} = (\partial\mathcal{D} \cap \partial\mathcal{R}) \cup (L \cap T(\hat{Q}))$. Because g vanishes in $\partial\mathcal{D} \cap \partial\mathcal{R}$, the integrals on $\partial\mathcal{R}$ can be written as

$$\int_{\partial\mathcal{D}\cap\partial\mathcal{R}} (u - u(w_Q))^2 \partial_{\vec{n}} g \, d\sigma + \int_{L\cap T(\hat{Q})} (u - u(w_Q))^2 \partial_{\vec{n}} g \, d\sigma - \int_{L\cap T(\hat{Q})} \partial_{\vec{n}} (u - u(w_Q))^2 g \, d\sigma.$$

We proceed now to estimate the integrals on $L \cap T(\hat{Q})$. Observe that in the hyperbolic metric the diameter of $T(\widehat{Q_k})$, where Q_k is any dyadic cube, is bounded from above and below by constants depending only on M and n. In particular, $T(\hat{Q})$ can be covered by k balls B_i of center w_i and radius $\rho\delta(w_i)$, where k depends on ρ and on n. Fix $\rho_0 = \rho_0(\alpha)$ so that each such hyperbolic ball is covered by a bounded number of cones. Now condition $(*)$ implies that for any $w \in L \cap T(\hat{Q})$,

$$|u(w) - u(w_Q)| \le \sum_{i=1}^{k} \operatorname{osc}(u, B_i) \le \sum_{i=1}^{k} \varphi\Big(\int_{(1+\eta)B_i} (|\nabla u|^2 + |u\Delta u|) r_{B_i}^{1-n} \, dm \Big)^{1/2}.$$

Observe that one can assume that $\delta(w_i)$ is comparable to $\delta(w)$ for all $w \in (1+\eta)B_i$. Thus for some constant $c = c(\rho_0) > 1$ and for any $w \in L \cap T(\hat{Q})$,

$$(3.3)\quad |u(w) - u(w_Q)| \lesssim \varphi(ca).$$

Also, $g(z) \lesssim 1$ and $|\partial_{\vec{n}} g| \simeq y_0^{-n}$ on $L \cap T(\hat{Q})$ by the estimates related to Green's function in Theorem 2.1, therefore

$$(3.4)\quad \int_{L\cap T(\hat{Q})} (u - u(w_Q))^2 |\partial_{\vec{n}} g| \, d\sigma \lesssim \varphi^2(ca)$$

and

$$
\begin{aligned}
&\left| \int_{L\cap T(\hat{Q})} \partial_{\vec{n}}(u-u(w_Q))^2 g\, d\sigma \right| \\
&\qquad \lesssim \varphi(ca) \int_{L\cap T(\hat{Q})} |\nabla u|\, d\sigma \\
(3.5)\qquad &\qquad \le \varphi(ca)|\sigma(L\cap T(\hat{Q}))|^{1/2}\Big(\int_{L\cap T(\hat{Q})} |\nabla u|^2\, d\sigma\Big)^{1/2} \\
&\qquad \lesssim a\varphi(ca).
\end{aligned}
$$

Note that the last inequality is a consequence of (3.1).

Next we estimate the right hand side term in (3.2). Consider g^* the Green's function for the domain $\mathcal{D}\cup(\bigcup\widehat{Q_j})$ with pole at the point p_0. Then by the maximum principle, $g^* \ge g$ in $\mathcal{D}$. Thus, changing the order of integration and using the left inequality in Theorem 2.1(c), one has

$$
\begin{aligned}
(3.6)\quad &\int_{\mathcal{R}} \Big(|\nabla u|^2 + |u-u(w_Q)|\,|\Delta u|\Big) g\, dm \\
&\le \int_{\mathcal{R}} (|\nabla u|^2 + |u-u(\omega_Q)|\,|\Delta u|) g^*\, dm \\
&\lesssim \int_Q \Big(\int_{\Gamma_\alpha(x)\cap\mathcal{R}} (|\nabla u|^2 + |u-u(w_Q)|\,|\Delta u|)\delta^{1-n}\, dm\Big)\, d\omega^*(x),
\end{aligned}
$$

where $\omega^*(E) = \omega(G_E, p_0, \mathcal{D})$ and G_E is the graph of $\partial\mathcal{D}$ above E. Let $N \ge A$. The argument below will indicate how large A must be chosen.

To estimate the last integral in (3.6) we will first assume that $|u(w_Q)| \ge 2N\varphi(ca)$ and consider the general case afterwards. So, let us assume that $|u(w_Q)| \ge 2N\varphi(ca)$. Since, by construction, $|u(w)-u(w_Q)| \le N\varphi(ca)$ in $\mathcal{R}$, we get for all $w\in\mathcal{R}$, $|u(w)-u(w_Q)| \le |u(w)|$. The last integral in (3.6) is then bounded by

$$
\int_Q (S_\alpha^2 u)(x)\, d\omega^*(x) \le a^2.
$$

Therefore, going back to (3.2) and using (3.4) and (3.5), we obtain

$$
\int_{\partial\mathcal{D}\cap\partial R} (u-u(w_Q))^2 \partial_{\vec{n}} g\, d\sigma \lesssim a^2 + a\varphi(ca) + \varphi^2(ca).
$$

The cubes Q_j are chosen so that $|u(w)-u(w_Q)| > N\varphi(ca)$ for some $w \in T(\widehat{Q_j})$. Since the oscillation on hyperbolic balls is controlled by $\varphi(ca)$, arguing as above we get that for all $w\in T(\widehat{Q_j})$

$$
(3.7)\qquad |u(w)-u(w_Q)| > (N-c_0)\varphi(ca),
$$

where $c_0 = c_0(\rho_0, \alpha, M, n)$. We deduce then that

$$
(3.8)\qquad ((N-c_0)\varphi(ca))^2 \sum_j \omega^*(Q_j) \lesssim a^2 + a\varphi(ca) + \varphi^2(ca).
$$

So, if $N > 2c_0$,

$$\sum_j \omega^*(Q_j) \lesssim \frac{1}{N^2}.$$

Now we use Theorem 2.1(b) with $Q^* = Q$ and $E = \bigcup_j Q_j$. Thus, since $\ell(Q) = 1$, we get

$$\Big(\sum_j \ell(Q_j)^n\Big)^\beta = \Big|\bigcup_j Q_j\Big|^\beta \lesssim \omega^*\Big(\bigcup_j Q_j\Big) \lesssim \frac{1}{N^2}.$$

Taking roots and using the assumption $\ell(Q) = 1$, this ends the proof in the case $|u(w_Q)| \geq 2N\varphi(ca)$.

Assume now that $|u(w_Q)| \leq 2N\varphi(ca)$. If $N\varphi(ca)/4 \leq |u(w_Q)| \leq 2N\varphi(ca)$, replace N by $N/8$ and apply the previous argument. If $N\varphi(ca) > 4|u(w_Q)|$ we consider an intermediate family of cubes. Denote by $\{Q_\ell^{(1)}\}$ the maximal dyadic cubes contained in Q satisfying

$$\sup_{w \in T(\widehat{Q_\ell^{(1)}})} |u(w) - u(w_Q)| \geq \frac{N\varphi(ca)}{2}.$$

Because of the maximality of $Q_\ell^{(1)}$, if we denote by $\widetilde{w}_\ell$ the point w_Q, where Q is the smallest dyadic cube which properly contains $Q_\ell^{(1)}$, we have

$$|u(\widetilde{w}_\ell) - u(w_Q)| < \frac{N\varphi(ca)}{2}.$$

Since $\widetilde{w}_\ell$ and $w_{Q_\ell^{(1)}}$ are at some fixed hyperbolic distance, the same argument as before gives:

$$\left(\frac{N}{2} + c_1\right)\varphi(ca) \geq |u(w_{Q_\ell^{(1)}}) - u(w_Q)| \geq \left(\frac{N}{2} - c_0\right)\varphi(ca). \tag{3.9}$$

Hence,

$$|u(w_{Q_\ell^{(1)}})| \geq \left(\frac{N}{4} - c_0\right)\varphi(ca)$$

and we can apply the previous case, that is, for each $Q_\ell^{(1)}$, choose the maximal dyadic cubes $\{Q_k^{(2)}\}$ contained in $Q_\ell^{(1)}$ such that

$$\sup_{T(\widehat{Q_k^{(2)}})} |u(w) - u(w_{Q_j^{(1)}})| > \left(\frac{N}{4} - c_0\right)\frac{\varphi(ca)}{2}.$$

So, proceeding as before we obtain

$$\frac{1}{(\ell(Q_\ell^{(1)}))^n}\sum_{Q_k^{(2)}\subset Q_\ell^{(1)}}(\ell(Q_k^{(2)}))^n \lesssim \varepsilon(N).$$

Finally, note that if $w\in \hat{Q}\setminus\bigcup_k \widehat{Q_k^{(2)}}$, because of the left inequality in (3.9) and the maximality of $\{Q_k^{(2)}\}$,

$$|u(w)-u(w_Q)|\le\left(\frac{N}{4}-c_0\right)\frac{\varphi(ca)}{2}+\left(\frac{N}{2}+c_1\right)\varphi(ca)+\frac{N}{4}\varphi(ca)\le N\varphi(ca)$$

if N is big enough. Hence $\bigcup_j Q_j\subset \cup Q_k^{(2)}$ and the proof is completed. □

Next, we shall prove a good-λ inequality between the distribution functions of the non-tangential maximal function $N_\beta u$ and the area function $S_\alpha u$, where u is a function having φ-controlled oscillation in an unbounded Lipschitz domain $\Omega\subset\mathbb{R}^{n+1}_+$ (with Lipschitz constant M). This type of a distribution inequality will lead to an L^p-inequality comparing both quantities, to a certain Law of the Iterated Logarithm and to a version of Fatou's Theorem in this setting.

Theorem 3.2. *Let Ω be an unbounded Lipschitz domain with Lipschitz constant M. Let $u\in C^2(\Omega)$ be a function having φ-controlled oscillation in Ω for an increasing function φ satisfying*

$$\liminf_{\lambda\to\infty}\frac{\varphi(\lambda)}{\lambda}>0.$$

Fix α and β such that $0<\alpha<\beta<1/M$. Then there exist constants c_0, c_1 and c_2 depending only on $(\alpha,\beta,M,n,\varphi)$ such that, for any $\lambda\ge 1$ and any $\gamma>c_0$, one has

$$\begin{aligned}&\left|\{x\in\mathbb{R}^n:(N_\alpha u)(x)>\gamma\lambda,\ (S_\beta u)(x)<\varphi^{-1}(\lambda)\}\right|\\ &\qquad\le c_1e^{-c_2\gamma}\left|\{x\in\mathbb{R}^n:(N_\alpha u)(x)>\lambda\}\right|.\end{aligned}$$

Proof. Fix $\lambda>1$. We may assume that the set $\{x\in\mathbb{R}^n:(N_\alpha u)(x)>\lambda\}$ has a finite measure. Let $\varepsilon_0>0$ be a small positive number, to be fixed later. Let Q be a maximal dyadic cube such that

$$|\{x\in Q:(N_\alpha u)(x)>\lambda\}|\ge\varepsilon_0|Q|.$$

It is then enough to show that

$$|\{x\in Q:(N_\alpha u)(x)>\lambda,\ (S_\beta u)(x)<\varphi^{-1}(\lambda)\}|\le c_1e^{-c_2\gamma}|Q|. \tag{3.10}$$

Also, observe that, by maximality, on the parent of Q, call it $\tilde{Q}$, we have the inequality

$$|\{x \in \tilde{Q} : (N_\alpha u)(x) > \lambda\}| \le \varepsilon_0 |\tilde{Q}|.$$

Hence, if $\varepsilon_0 = \varepsilon_0(n, M)$ is chosen sufficiently small, the inequality $|u(w)| < \lambda$ holds for every point $w \in T(\hat{\tilde{Q}})$. Now choose $w \in T(\hat{\tilde{Q}})$ such that $Q \subset Q_w$. Since the volumes of Q and Q_ω are comparable, replacing Q by Q_w in (3.10) if necessary, it can be assumed that $|u(w_Q)| < \lambda$.

So the proof consists on showing (3.10) under the additional assumption that $|u(w_Q)| \le \lambda$.

We will mainly follow the proof in Lemma 3.1, instead of directly applying its conclusions.

Define the set $E = \{x \in Q : (S_\beta u)(x) \le \varphi^{-1}(\lambda)\}$ and the Lipschitz domain $\Omega' = \bigcup_{x\in E} \Gamma_{\beta'}(x)$, where β' is chosen so that $\alpha < \beta' < \beta$. Then Ω' is a Lipschitz domain contained in Ω with Lipschitz constant $1/\beta'$, and we have $(S_\beta u)(x) \le \varphi^{-1}(\lambda)$ for all $x \in E$. Hence the area function of u (in Ω') is bounded by a fixed multiple of $\varphi^{-1}(\lambda)$ at the points of $(\partial\Omega') \cap E$. A technical difficulty arises because we do not know such estimate in the whole $\partial\Omega'$. In the harmonic setting, it holds (see Lemma 4.2.9 in [2]), but this lemma does not seem to hold in our situation.

We consider now the dyadic decomposition of Q with respect to the domain Ω'. Denote the dyadic cubes in Ω' by $\{\widehat{Q'}\}$. Since Q was chosen to be big enough, we can assume as well that $|u| \le \lambda$ on $T(\widehat{Q'})$. The idea is to run a stopping time process in Ω'.

We are essentially in the setting of Lemma 3.1 with Ω' replacing Ω, except for the fact that the condition $(S_\beta u)(x) < \varphi^{-1}(\lambda)$ involves the distance to $\partial\Omega$ which could be quite different from the one to $\partial\Omega'$. This will create some technical difficulties that can be solved by adapting the proof of Lemma 3.1 to this situation.

Proceeding as in Lemma 3.1, consider the maximal dyadic cubes (with respect to Ω') $\{Q_j'\} \subset Q$ satisfying

$$\sup_{w\in T(\widehat{Q}_j')} |u(w)| \ge C_0\lambda$$

for some constant C_0 to be chosen later. Define the corresponding regions $\mathcal{D}'$ and $\mathcal{R}'$ and apply Green's formula to u^2 and g' (Green's function in $\mathcal{D}'$). Hence

$$\int_{\partial\mathcal{R}'} u^2\, d\omega' - \int_{\partial\mathcal{R}'} (\partial_{\vec{n}} u^2) g'\, d\sigma = \int_{\mathcal{R}'} (|\nabla u|^2 + u\Delta u) g'\, dm. \tag{3.11}$$

Since $\delta(w) = \operatorname{dist}(w, \partial\Omega)$ and $\delta'(w) = \operatorname{dist}(w, \partial\Omega')$ for $w \in \Omega'$, do not need to be comparable, we need to consider a new Green's function to estimate this last integral. Let $g_\beta(\cdot)$ be Green's function in $\Omega_\beta = \bigcup_{x\in E} \Gamma_\beta(x)$ with the same pole as g'. Note that $\Omega_\beta \supset \Omega'$, so by the maximum principle, $g_\beta(w) \ge g'(w)$ for all

$w \in \mathcal{R}'$. Therefore

$$\int_{\mathcal{R}'} (|\nabla u|^2 + |u\Delta u|) g' \, dm \le \int_{\mathcal{R}'} (|\nabla u|^2 + |u\Delta u|) g_\beta \, dm. \tag{3.12}$$

The point now is that, for points $w \in \mathcal{R}'$, $\delta_\beta(w) = \operatorname{dist}(w, \partial\Omega_\beta)$ is comparable to $\delta(w) = \operatorname{dist}(w, \partial\Omega)$, with constants depending on (β, β', n). So changing the order of integration as in Lemma 3.1 we can bound (3.12) by

$$\int_Q \Big(\int_{\Gamma_\beta(x) \cap \mathcal{R}'} (|\nabla u|^2 + |u\Delta u|) \delta_\beta^{1-n} \, dm \Big) \, d\omega_\beta^*.$$

Recall that $(S_\beta u)(x) \le \varphi^{-1}(\lambda)$ if $x \in E$. Since $\delta(w)$ is comparable to $\delta_\beta(w)$ for $w \in \mathcal{R}'$, we deduce that

$$\int_{\Gamma_\beta(x) \cap \mathcal{R}'} (|\nabla u|^2 + |u\Delta u|) \delta_\beta^{1-n} \, dm \lesssim \varphi^{-1}(\lambda)^2, \quad x \in E.$$

To estimate the corresponding integral for points in $Q \setminus E$, we use an argument in [22], which uses the different apertures $\alpha < \beta' < \beta$, $(x_0, t_0) = P \in \partial\Omega_\beta$ and $x \in E$ such that P is in the closure of $\Gamma_\beta(x)$. Since $P \notin \Omega_\beta$, the vertical cone (in the negative direction) of aperture β with vertex at P,

$$\{(x,t) : |x - x_0| < \beta|t - t_0|, \ t < t_0\},$$

does not meet Ω_β. Therefore, since $\beta' < \beta$, the distances $\operatorname{dist}(P, \Omega')$, $\operatorname{dist}(x, \Omega')$ are comparable. Hence if $w \in \Gamma_\beta(P) \cap \Omega' \supset \Gamma_\beta(P) \cap \mathcal{R}'$, $\delta_\beta(w)$ is comparable to $|w - x|$. Hence

$$\int_{\Gamma_\beta(P) \cap \mathcal{R}'} (|\nabla u|^2 + |u\Delta u|) \delta_\beta^{1-n} \, dm \lesssim (S_\beta^2 u)(x) \le \varphi^{-1}(\lambda)^2.$$

Therefore

$$\int_Q \Big(\int_{\Gamma_\beta(x) \cap \mathcal{R}'} (|\nabla u|^2 + |u\Delta u|) \delta_\beta^{1-n} \, dm \Big) \, d\omega_\beta^* \lesssim (\varphi^{-1}(\lambda))^2. \tag{3.13}$$

The rest of the argument is exactly the same as in Lemma 3.1. Just note that the oscillations of u on tops of the cubes Q' will be controlled by the oscillation on hyperbolic balls in Ω'. The quantity that controls such oscillations depends on the euclidean radius of the balls. That might be very small compared to the distance of the ball to $\partial\Omega$. So, we need to consider some bigger balls that will help us control the oscillation in the smaller ones. Let B be a hyperbolic ball in Ω', that is, $B = B(w_0, \rho_0 \delta'(w_0)) \subset \Omega'$, where $\rho_0 < \frac{1}{2}$. Define $\widetilde{B}$ to be the ball

$\widetilde{B} = B(w_0, \rho_0 \delta_\beta(w_0))$. Then obviously $\widetilde{B} \supset B$ and the euclidean radius of $\widetilde{B}$ is comparable to $\delta(w)$ for all $w \in \widetilde{B}$. Fix $\rho_0 = \rho_0(\beta)$ so that $(1+\eta)\widetilde{B} \subset \Gamma_\beta(x)$ for some $x \in E$. Then condition $(*)$ gives

$$\operatorname{osc}(u, B) \le \operatorname{osc}(u, \widetilde{B}) \le \varphi\left(\int_{(1+\eta)\widetilde{B}} (|\nabla u|^2 + |u \Delta u|) r_{\widetilde{B}}^{1-n} \, dm \right)^{1/2},$$

and the same argument which leads to (3.3) implies that

$$\operatorname{osc}(u, B) \le \varphi(c\varphi^{-1}(\lambda)) \lesssim \lambda$$

for some $c = c(\beta, \beta', n)$. This is where the assumption $\varphi(2t) < C\varphi(t)$ is used. Next, since $\liminf_{\lambda \to \infty} \varphi(\lambda)/\lambda = c_\varphi > 0$, we can choose c_1 such that

$$c_1 \varphi(c\varphi^{-1}(\lambda)) > c_\varphi \varphi^{-1}(\lambda).$$

So the statement (3.8) (with $a = \varphi^{-1}(\lambda)$, $N = C_0$) becomes

$$((C_0 - c_0)\lambda)^2 \sum_j \omega^{*'}(Q'_j) \lesssim (c_1^2 c^{-2} + c_1 c^{-1} + 1)\lambda^2 \omega^{*'}(Q).$$

Choosing C_0 big enough, the proof of Lemma 3.1 gives:

$$\sum_j (\ell(Q'_j))^n < \frac{\ell(Q)^n}{2}.$$

Changing the notation now, we set $Q'_j = Q_j^{(1)}$. For each j, we repeat the construction, that is, we consider $\{Q_i^{(2)}\}$, the collection of the maximal dyadic subcubes (with respect to Ω') of $Q_j^{(1)}$ satisfying

$$\sup_{T(\widehat{Q_i^{(2)}})} |u(w) - u(w_{Q_j^{(1)}})| \ge C_0 \lambda.$$

Repeating the same process n_0 times, where $n_0 = n_0(\gamma)$ will be chosen later, we obtain nested families $\{Q_j^{(k)}\}_j$ of pairwise disjoint dyadic cubes in Q satisfying

$$\bigcup_j Q_j^{(k+1)} \subset \bigcup_j Q_j^{(k)}, \qquad \sum_j \left(\ell(Q_j^{(k+1)})\right)^n \le \tfrac{1}{2} \sum_j \left(\ell(Q_j^k)\right)^n,$$

for any $k = 1, \ldots, n_0$. Moreover, the same argument as in (3.9) yields

$$(C_0 + c_0)\lambda \ge |u(w_{Q_j^{(k+1)}}) - u(w_{Q_j^{(k)}})| \ge (C_0 - c_0)\lambda$$

and by the maximality of the families $\{Q_j^{(n)} : j\}$, we have that

$$|u(w) - u(w_{Q_j^{(k)}})| \le C_0\lambda,$$

whenever $w \in \widehat{Q_i^{(k)}} \setminus \bigcup_j \widehat{Q_j^{(k+1)}}$. Thus, if $n_0 \simeq \gamma/C_0$, then one has

$$\Big\{x \in E : \sup_{w\in\Gamma_\alpha(x)} |u(w)| > \gamma\lambda\Big\} \subseteq \bigcup CQ_j^{(n_0)},$$

where C is a constant depending on α, n, M. For $n_0 \simeq [\gamma/C_0]$, we can then deduce

$$\begin{aligned}\Big|\{x \in Q : \sup_{w\in\Gamma_\alpha(x)} \big|u(w)\big| > \gamma\lambda,\ (S_\beta u)(x) < \varphi^{-1}(\lambda)\}\Big| \\ \le c_1 e^{-c_2\gamma}\Big|\{x \in Q : \sup_{w\in\Gamma_\alpha(x)} \big|u(w)\big| > \lambda\}\Big|,\end{aligned}$$

which concludes the proof. □

Remark. Assume that the function u satisfies condition $(*)$ for a function φ such that

$$\varphi(t) \ge ct, \quad 0 < t < \infty,$$

for a fixed constant $c > 0$. Then, the conclusion of Theorem 3.2 holds for any $\lambda > \limsup_{|w|\to\infty} u(w)$.

Once this type of distributional inequalities is established, standard arguments lead to the following L^p-inequalities for functions u that vanish at infinity.

Theorem 3.3. *Under the notation of Theorem 3.2, assume that*

$$\lim_{\|(x,y)\|\to\infty} u(x,y) = 0.$$

Then for $0 < p < \infty$ *and* $0 < \alpha < \beta$ *there exists a constant* $C = C(p,\alpha,\beta,n,M,\varphi)$, *such that*

$$\|N_\alpha u\|_{L^p(\partial\Omega)} \le C\|\varphi(S_\beta u)\|_{L^p(\partial\Omega)}.$$

We can also obtain a Law of the Iterated Logarithm in this setting, but some technical difficulties arise because no version of Lemma 4.2.9 in [2] seems to hold in our setting.

Theorem 3.4. *Under the hypothesis of Theorem 3.2, and assuming that* $(N_{\alpha,1,\infty}u)(0) < \infty$, *there exists a constant* $C = C(M,n,\alpha,\beta,\varphi)$ *such that*

$$\limsup_{t\to 0} \frac{(N_\alpha u)(x,t)}{\varphi((S_\beta u)(x,t)) \log\log(\varphi((S_\beta u)(x,t)))} < C$$

at almost every point $x \in \{x \in \mathbb{R}^n : (S_\beta u)(x) = \infty\}$.

Proof. It is enough to show the result for points $x \in Q$, where Q is any sufficiently large cube in $\mathbb{R}^n$. We may assume that there exist $x \in Q$ and $t > 0$ such that $(S_\beta u)(x,t) < \infty$. Given $x \in Q$, consider the set

$$\eta_k(x) = \inf\left\{t > 0 : (S_\beta u)(x,t) < \varphi^{-1}(2^k)\right\}$$

and the domains $\Omega_k(\alpha) = \bigcup \Gamma_\alpha(x, \eta_k(x))$, $\Omega_k(\beta') = \bigcup \Gamma_{\beta'}(x, \eta_k(x))$, where $\alpha < \beta' < \beta$. Observe that $\Omega_k(\alpha)$, $\Omega_k(\beta')$ are Lipschitz domains with Lipschitz constant depending only on α, β, M, φ, but not on k. We will apply the proof of Theorem 3.2 in the domain Ω'_k. The main difficulty is that we only know that

$$S_{\beta'}(P) \lesssim \varphi^{-1}(2^k),$$

at points $P \in \partial\Omega_k(\beta')$ of the form $P = (x, \eta_k(x))$. As before, it is worth mentioning that the analogue of Lemma 4.2.9 in [2] does not seem to hold in our situation. However, observe that for any point $P \in \partial\Omega_k(\beta')$

$$(3.14)\quad \int_{\Gamma_{\beta'}(P)\cap\Omega_k(\alpha)} \left(|\nabla u(w)|^2 + |u(w)\Delta u(w)|\right) |w - P|^{1-n}\, dm(w) \lesssim \varphi^{-1}(2^k).$$

Actually, if P is of the form $P = (x, \eta_k(x))$, we already know it. For general $P = (x_0, t_0) \in \partial\Omega_k(\beta')$ let $Q = (x, \eta_k(x))$ such that P is in the closure of $\Gamma_{\beta'}(Q)$. Since $P = (x_0, t_0) \in \partial\Omega_k(\beta')$, the vertical cone (in the negative direction), $\{(x,t) : |x - x_0| < \beta'(t_0 - t)\}$, does not contain any point of $\Omega_k(\beta')$. Since $\alpha < \beta'$, if $w \in \Gamma_{\beta'}(P) \cap \Omega_k(\alpha)$, $|w - P|$ is comparable to $|w - Q|$. Therefore

$$\int_{\Gamma_{\beta'}(P)\cap\Omega_k(\alpha)} \left(|\nabla u(w)|^2 + |u(w)\Delta u(w)|\right) |w - P|^{1-n}\, dm(w)$$
$$\lesssim \int_{\Gamma_{\beta'}(Q)} \left(|\nabla u(w)|^2 + |u(w)\Delta u(w)|\right) |w - Q|^{1-n}\, dm(w) \lesssim \varphi^{-1}(\lambda)^2.$$

Hence, (3.13) holds.

Observe that, if we would know that $(S_{\beta'} u)(P) \lesssim \varphi^{-1}(2^k)$ for any point $P \in \partial\Omega_k(\beta')$, we could directly apply Theorem 3.2 to obtain

$$(3.15)\qquad |\{x \in Q : (N_\alpha u)(x, \eta_k(x)) > \gamma_k 2^k\}| \le C_1 e^{-C_2 \gamma_k} |Q|.$$

To deduce (3.15) from (3.14), we follow the proof of Theorem 3.2. We apply Green's formula to the functions u^2 and g' in a subdomain $\mathcal{R}'$ of $\Omega_k(\alpha)$. Here g' is Green's function of a convenient subdomain of $\Omega_k(\alpha)$. As in the proof of Theorem 3.2, the key estimate (3.13) follows easily from (3.14). These considerations accomplish the first step in the proof of Theorem 3.2. Successive steps run in the same way as in Theorem 3.2.

Choose $\gamma_k = (2\log k)/c_2$. Then $\sum_k e^{-c_2\gamma_k} < \infty$, and the Borel-Cantelli lemma implies that almost all $x \in Q$ are, at most, in a finite number of the sets $\{x \in Q : N_{\alpha,\eta_k(x),\infty}u(x) > \gamma_k 2^k\}$. So, for almost all $x \in Q$, $N_{\alpha,\eta_k(x),\infty}u(x) \leq \gamma_k 2^k$, eventually, that is, there exists an integer k_0 that may depend on x, such that for all $k \geq k_0$, we have $N_{\alpha,\eta_k(x),\infty}u(x) \leq \gamma_k 2^k$. Consider such points x which in addition satisfy $(S_\beta u)(x) = \infty$. For these points $\eta_k(x) \to 0$ as $k \to \infty$. So, for any $t < \eta_{k_0}(x)$, choosing k such that $\eta_{k+1}(x) < t < \eta_k(x)$, we get

$$\varphi^{-1}(2^k) < (S_{\beta,t,\infty}u)(x) < \varphi^{-1}(2^{k+1})$$

and

$$(N_{\alpha,t,\infty}u)(x) < (N_{\alpha,\eta_{k+1}(x),\infty}u)(x) < \gamma_{k+1}2^{k+1}.$$

Thus, $2^k < \varphi((S_{\beta,t,\infty}u)(x))$ and

$$(N_{\alpha,t,\infty}u)(x) < c\varphi(S_{\beta,t,\infty}u(x)) \log\log \varphi((S_{\beta,t,\infty}u)(x)),$$

as we wanted to prove. □

We end this section applying these techniques to obtain a Fatou-type result.

Theorem 3.5. *Under the hypotheses of Theorem 3.2 and assuming that* $(N_{\alpha,1,\infty}u)(0) < \infty$, *for almost all* $x \in \{x \in \mathbb{R}^n : (S_\beta u)(x) < \infty\}$, *one has*

(i) $(N_\alpha u)(x) < \infty$.

(ii) *The function* u *has a finite non-tangential limit, that is,* $\lim_{w\to(x,\phi(x))} u(w)$ *exists, where the limit is taken when* $w \in \Gamma_\alpha(x)$, $w \to (x,\phi(x))$, $0 < \alpha < 1/M$.

Proof. It is quite easy to deduce (i) from Theorem 3.2 and a standard point of density argument, therefore we omit its proof. Assume now that (ii) does not hold. Since $(N_\alpha u)(x)$ is finite a.e. $x \in \{x : S_\beta u(x) < \infty\}$, there must exist $\delta > 0$ and a set

$$E \subset \{x \in \mathbb{R}^n : (N_\alpha u)(x) < \infty,\ (S_\beta u)(x) < \infty\},$$

with $|E| > 0$, such that for any $x \in E$,

$$\overline{\lim}\, u(w) - \underline{\lim}\, u(w) > \delta, \tag{3.16}$$

where the limits are taken when w tends to $(x,\phi(x))$, $w \in \Gamma_\alpha(x)$. Choose $\varepsilon > 0$. Then a point of density argument provides a dyadic cube $Q \subset \mathbb{R}^n$, and a set $F \subset Q$ such that $|F|/|Q| > \frac{3}{4}$, and for all $x \in F$, $(S_{\beta,0,\ell(Q)})(x) < \varepsilon$ and (3.16) holds. Consider the Lipschitz domains

$$\Omega' = \bigcup_{x\in F} \Gamma_\alpha(x) \subset \Omega = \bigcup_{x \in F} \Gamma_\beta(x).$$

As before, we only know that the area function of u is bounded (by $\subset \varepsilon$) at points $P \in \partial\Omega$ of the form $P = x \in F$. If we knew that, for any point in $\partial\Omega$, then, applying Lemma 3.1 to $\Omega \cap \widehat{\Omega}$, we would get a contradiction, since $\varphi(0) = 0$ and the constant ε can be taken arbitrarily small. As before, to overcome this difficulty we will take profit of the two different angles $\alpha < \beta$. Arguing as in the proof of Theorem 3.2, one obtains that for any $P \in \partial\Omega$,

$$\int_{\Gamma_\beta(P)\cap\Omega'\cap\widehat{Q}} \left(|\nabla u(w)|^2 + |u(w)\Delta u(w)|\right) |w-P|^{1-n}\, dm(w) \lesssim \varepsilon.$$

Now, one can apply the proof of Lemma 3.1 to obtain a contradiction. □

4. CONTROL OF THE AREA FUNCTION BY THE NON-TANGENTIAL MAXIMAL FUNCTION

Let $f \geq 0$ be a measurable function in Ω. If $0 < \varepsilon < 1/\sqrt{1+M^2}$, we introduce another function $f^\sharp_\varepsilon$ by setting

$$f^\sharp_\varepsilon(z,y) = \fint_{B_\varepsilon(z,y)} f,$$

where $B_\varepsilon(z,y) = B((z,y), \varepsilon(y-\phi(z)))$ for $(z,y) \in \Omega$. Note that, since $\varepsilon < 1/\sqrt{1+M^2}$, $B_\varepsilon(z,y) \subset \Omega$ if $(z,y) \in \Omega$. The following two technical results are elementary.

Proposition 4.1. *Let* $0 < \alpha < 1/M$, $0 < \varepsilon < 1/\sqrt{1+M^2}$.

(1) *If* $(z,y) \in \Gamma_\alpha(x)$, *then:*
$$y - \phi(z) \simeq y - \phi(x),$$
with comparison constants depending on M *and* α.

(2) *If* $(z',y') \in B_\varepsilon(z,y)$ *and* $(z,y) \in \Gamma_\alpha(x)$, *then*
 (i) $y' - \phi(x) \simeq y - \phi(x)$,
 (ii) $y' - \phi(z') \simeq y - \phi(z)$,
 (iii) $|y - y'| \lesssim y - \phi(x)$,
 (iv) $|z' - x| \lesssim y - \phi(x)$,

with comparison constants depending on M, α *and* ε.

The following lemma relates the truncated area functions of f, $f^\sharp_\varepsilon$ defined in Section 2.

Lemma 4.2. *Let* $0 < \alpha < 1/M$, *and* $h > 0$. *Then there exists* $\varepsilon_0 = \varepsilon_0(\alpha, M)$ *such that if* $0 < \varepsilon < \varepsilon_0$, *one can find* $\alpha_\pm$ *and* $h_\pm$ *such that*

(i) $C^{-1}\mathcal{A}_{\alpha_-,0,h_-}(f_\varepsilon^\sharp) \le \mathcal{A}_{\alpha,0,h}(f) \le C\mathcal{A}_{\alpha_+,0,h_+}(f_\varepsilon^\sharp)$,

(ii) $C^{-1}\mathcal{A}_{\alpha,h_+,\infty}(f_\varepsilon^\sharp) \le \mathcal{A}_{\alpha,h,\infty} \le C\mathcal{A}_{\alpha_+,h_-,\infty}(f_\varepsilon^\sharp)$,

where $\alpha_- < \alpha < \alpha_+$, $h_- < h < h_+$, *and* C *are all positive constants depending on* M, α *and* ε.

In particular, $C^{-1}\mathcal{A}_{\alpha_-}(f_\varepsilon^\sharp) \le \mathcal{A}_\alpha(f) \le C\mathcal{A}_{\alpha_+}(f_\varepsilon^\sharp)$.

Proof. We will only prove (i), the proof of (ii) being similar. We start with the right hand side. If $\delta = \delta(\varepsilon, M)$ is conveniently chosen, then, by Proposition 4.1 and Fubini's Theorem:

$$\int_{\Gamma_{\alpha,0,h}(x)} f(z,y)(y-\phi(x))^{1-n}\,dz\,dy$$
$$\le C\int_{\Gamma_{\alpha,0,h}(x)}\int_{\mathbb{R}^{n+1}_+} f(z,y)(y'-\phi(x))^{1-n}\frac{1}{|B_\delta(z,y)|} \times \chi_{B_\delta(z,y)}(z',y')\,dz'\,dy'\,dz\,dy$$
$$\le C\int_{\Gamma_{\alpha_+,0,h_+}(x)}\int_{\mathbb{R}^{n+1}_+}(y'-\phi(x))^{1-n}\frac{\chi_{B_\varepsilon(z',y')}(z,y)}{|B_\varepsilon(z',y')|}f(z,y)\,dz\,dy\,dz'\,dy'$$
$$\le C\int_{\Gamma_{\alpha_+,0,h_+}(x)}(y'-\phi(x))^{1-n}⨍_{B_\varepsilon(z',y')} f(z,y)\,dz\,dy\,dz'\,dy'$$
$$= C\int_{\Gamma_{\alpha_+,0,h_+}(x)}(y'-\phi(x))^{1-n}f_\varepsilon^\sharp(z',y')\,dz'\,dy',$$

where C is some positive constant depending only on M, α, ε. For the left-hand side, we also get:

$$\int_{\Gamma_{\alpha_-,0,h_-}(x)} f_\varepsilon^\sharp(z,y)(y-\phi(x))^{1-n}\,dz\,dy$$
$$\le \int_{\Gamma_{\alpha_-,0,h_-}(x)}\int_{B_\varepsilon(z,y)} f(z',y')\frac{(y-\phi(x))^{1-n}}{|B_\varepsilon(z,y)|}\,dz'\,dy'\,dz\,dy$$
$$\le C\int_{\Gamma_{\alpha,0,h}(x)} f(z',y')(y'-\phi(x))^{1-n}\,dz'\,dy'. \qquad \square$$

Now, if $f \ge 0$ is measurable in Ω, $Q \subset \mathbb{R}^n$ is a cube of side length ℓ and $x \in Q$, consider the truncated cones $\Gamma_{\alpha,0,\ell}(x)$, $\Gamma_{\alpha,\ell,\infty}(x)$ and the corresponding area functions $(\mathcal{A}_{\alpha,0,\ell}f)(x)$, $(\mathcal{A}_{\alpha,\ell,\infty}f)(x)$ associated to f. The following two results were proved in the harmonic setting by Bañuelos and Moore [3].

Lemma 4.3. *Assume that $f \geq 0$ is measurable in Ω and satisfies the uniform estimate*

$$f(z,y) \leq \frac{A}{(y-\phi(z))^2}$$

for some $A > 0$ and any $(z,y) \in \Omega$. Then if $Q \subset \mathbb{R}^n$ is a cube of side length ℓ, $0 < \alpha < 1/M$ and $\mathcal{A}_{\alpha,\ell,\infty}(f)$ is as above, we have

$$|\mathcal{A}^2_{\alpha,\ell,\infty}(f)(x_1) - \mathcal{A}^2_{\alpha,\ell,\infty}(f)(x_2)| \leq C$$

for any x_1, $x_2 \in Q$, where C is a constant depending only on M, α, A, n.

Proof. Take $\ell' = 2M\sqrt{n}\ell$. An elementary commutation shows that

$$\mathcal{A}^2_{\alpha,\ell',\infty}(f)(x) - \mathcal{A}^2_{\alpha,\ell,\infty}(f)(x) \leq C = C(M,n,\alpha,A)$$

for any $x \in \mathbb{R}^n$. Therefore, it is enough to prove the lemma with ℓ' instead of ℓ. Now,

$$\begin{aligned}
&\left|\mathcal{A}_{\alpha,\ell',\infty}(f)(x_1) - \mathcal{A}^2_{\alpha,\ell',\infty^2}(f)(x_2)\right| \\
&\quad\leq \int_{\Gamma_{\alpha,\ell',\infty}(x_1)} f(z,y)|(y-\varphi(x_1))^{1-n} - (y-\varphi(x_2))^{1-n}|\,dz\,dy \\
&\qquad + \int_{\Gamma_{\alpha,\ell',\infty}(x_1)\Delta\Gamma_{\alpha,\ell',\infty}} f(z,y)|y-\varphi(x_2)|^{1-n}\,dz\,dy = \text{(I)} + \text{(II)}.
\end{aligned}$$

We claim that (I) and (II) are bounded by some constant $C = C(M,n,\alpha,A)$. To estimate (I), note that, from the choice of ℓ' it follows,

$$\frac{1}{2} \leq \frac{y-\varphi(x_2)}{y-\varphi(x_1)} \leq \frac{3}{2}$$

whenever $(z,y) \in \Gamma_{\alpha,\ell',\infty}(x_1)$. Then

$$\left|(y-\varphi(x_1))^{1-n} - (y-\varphi(x_2))^{1-n}\right| \leq C(M,n)\ell\frac{1}{(y-\varphi(x_1))^n},$$

so, by Proposition 4.1, (1):

$$\begin{aligned}
\text{(I)} &\leq \int_{\Gamma_{\alpha,\ell',\infty}(x_1)} \frac{A}{(y-\varphi(z))^2}\,\frac{C\ell}{(y-\varphi(x_1))^n}\,dz\,dy \\
&\leq \ell C(M,n,\alpha,A)\int_{\Gamma_{\alpha,\ell',\infty}(x_1)} \frac{dz\,dy}{(y-\varphi(x_1))^{n+2}} \leq C(M,n,\alpha,A).
\end{aligned}$$

To estimate (II), note that

$$\sigma_n \left\{ z \in \mathbb{R}^n : (z, y) \in \Gamma_{\alpha,\ell',\infty}(x_1) \Delta \Gamma_{\alpha,\ell',\infty}(x_2) \right\} \leq C(M, n, \alpha) \ell (y - \varphi(x_2))^{n-1}$$

where σ_n is the Lebesgue measure in $\mathbb{R}^n$. As in [3], this last inequality, together with Fubini's Theorem, gives (II) $\leq C(M, n, \alpha, A)$. □

Now, for a cube $Q \subset \mathbb{R}^n$, centered at $x_0 \in \mathbb{R}^n$ and of side length ℓ, denote by Q^* the cube also centered at x_0 with side length 4ℓ, and let g, ω be Green's function and the harmonic measure in $\widehat{Q^*}$, with respect to $p_{Q^*} = (x_0, \phi(x_0) + \ell/2)$. We follow the notation introduced in Section 2, that is, for $E \subset Q^*$, $\omega^*(E) = \omega(G_E, p_{Q^*}, \widehat{Q^*})$, where $G_E = \{(x, \phi(x)) : x \in E\}$.

The following lemma is a standard consequence of Fubini's Theorem, Theorem 2.1 (c) and the elementary estimates in Proposition 4.1.

Lemma 4.4. *For any $f \geq 0$ measurable in Ω, one has*

$$\int_Q (\mathcal{A}^2_{\alpha,0,\ell} f)(x)\, d\omega^*(x) \leq C \int_{\widehat{Q^*}} f(z, y) g(z, y)\, dz\, dy,$$

where C is a constant depending on α, M and n.

In what follows, we will be interested in the class of functions $u \in C^2(\Omega)$ that satisfy

$$(**) \qquad |u \Delta u| \leq \theta |\nabla u|^2 \quad \text{in } \Omega,$$

for some θ, $0 < \theta < 1$. As mentioned in the introduction, if $0 \leq t \leq s \leq +\infty$, we will define

$$(A_{\alpha,t,s} u)(x) = \left(\int_{\Gamma_{\alpha,t,s}(x)} |\nabla u(z, y)|^2 (y - \phi(x))^{1-n}\, dz\, dy \right)^{1/2}.$$

The following lemma is a Caccioppoli inequality for this class. This is where our assumption $0 < \theta < 1$ gets used.

Lemma 4.5. *Suppose that $u \in C^2(\Omega)$ satisfies $(**)$ for some θ, $0 < \theta < 1$. Then, for each $(x, y) \in \Omega$ and any ε, $0 < \varepsilon < 1/4\sqrt{1 + M^2}$, one has*

$$\int_B |\nabla u|^2 \leq \frac{C}{r^2} \int_{2B} u^2,$$

where $r = \varepsilon(y - \phi(x))$, $B = B((x, y), r)$, $2B = B((x, y), 2r)$ and C only depends on n, θ.

Proof. The proof goes as in the usual Caccioppoli inequality. Fix B and let $\varphi \in C_0^\infty(2B)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on B and $\|\Delta\varphi\|_\infty \le Cr^{-2}$, where C only depends on n. Then by Green's formula applied to u^2, φ in $2B$

$$\int_{2B} \varphi \Delta u^2 = \int_{2B} u^2 \Delta\varphi.$$

So

$$\begin{aligned} 2(1-\theta)\int_B |\nabla u|^2 &\le 2(1-\theta)\int_{2B} \varphi |\nabla u|^2 \\ &\le 2\int_{2B} \varphi(|\nabla u|^2 + u\Delta u) \\ &= \int_{2B} u^2 \Delta\varphi \le \frac{C}{r^2}\int_{2B} u^2. \end{aligned}$$

□

Corollary 4.6. *Suppose that $u \in C^2(\Omega)$ satisfies $(**)$ and $|u| \le 1$ in Ω. Then, for any cube $Q \subset \mathbb{R}^n$ of side length ℓ,*

$$\int_Q (\mathcal{A}^2_{\alpha,0,\ell} u)(x)\, d\omega^*(x) \le C$$

where C depends on M, n, α and θ, the constant in $(**)$.

Proof. We use the notation of Lemma 4.4. By Green's Theorem,

$$\int_{\widehat{Q^*}} g\Delta u^2 = \int_{\partial\widehat{Q^*}} (u^2 - u^2(p_{Q^*}))\, dw^* \le 2.$$

Since $\Delta u^2 = 2(|\nabla u|^2 + u\Delta u) \ge 2(1-\theta)|\nabla u|^2$, it follows:

$$\int_{\widehat{Q^*}} |\nabla u|^2 g \le \frac{1}{1-\theta}$$

and the conclusion follows from Lemma 4.4, applied to $f \equiv |\nabla u|^2$. □

If $u \in C^2(\Omega)$, $f = |\nabla u|^2$ and $0 < \varepsilon < 1/\sqrt{1+M^2}$, we remind that $f_\varepsilon^\sharp$ denotes the density introduced at the beginning of this section.

Theorem 4.7. *Let $0 < \alpha < \beta < 1/M$. Suppose that $u \in C^2(\Omega)$ satisfies $(**)$ with constant θ, that $|u| \le 1$ in Ω, and that $A_{\beta,1,\infty}u(x_0) < \infty$ for some $x_0 \in \mathbb{R}^n$. Then*

$$(\mathcal{A}_\alpha f_\varepsilon^\sharp)^2 \in \mathrm{BMO}(\mathbb{R}^n)$$

for some appropriate choice of $\varepsilon = \varepsilon(\alpha, \beta, M)$, and its BMO*-norm only depends on M, n, α, β, θ.*

Proof. Choose $\varepsilon > 0$ so that $\alpha \le \beta_-$ with the notation of Lemma 4.2. If Q is any cube in $\mathbb{R}^n$, centered at x_0, with side length ℓ, it follows by Lemma 4.3 that $(\mathcal{A}_{\alpha,\ell,\infty} f_\varepsilon^\sharp)(x) < \infty$ for any $x \in Q$. On the other hand, $(A_{\beta,0,\ell} u)(x) < \infty$ for a.e. (ω^*) $x \in Q$, by Corollary 4.6, so $(\mathcal{A}_{\alpha,0,\ell} f_\varepsilon^\sharp)(x) < \infty$ for a.e. $(w^*) x \in Q$. Since Q is arbitrary, it follows that $(\mathcal{A}_\alpha f_\varepsilon^\sharp)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$ (recall that the harmonic measure and the surface measure are mutually absolutely continuous on the boundary of any Lipschitz domain). Now as in [3] we show that $(\mathcal{A}_\alpha f_\varepsilon^\sharp)^2 \in \mathrm{BMO}(\mathbb{R}^n)$. Indeed, let Q be a cube in $\mathbb{R}^n$ of side length ℓ. Corollary 4.6 gives

$$\omega^* \{x \in Q : (\mathcal{A}_{\alpha,0,\ell} f_\varepsilon^\sharp)^2(x) > \lambda\} \le C \frac{\omega^*(Q)}{\lambda}.$$

Since ω^* satisfies the A_∞-condition of Theorem 2.1, we deduce

$$|\{x \in Q : (\mathcal{A}_{\alpha,0,\ell} f_\varepsilon^\sharp)^2(x) > \lambda\}| \le C \frac{|Q|}{\lambda^b},$$

for some $b > 0$. On the other hand, Lemma 4.5 gives the necessary estimates for the hypothesis of Lemma 4.3, with the function $f^\sharp$ where $f = |\nabla u|^2$. Actually, the fact that the expressions in Lemma 4.5 are averages is the reason to introduce $\sharp$ functions. Hence, applying Lemma 4.3,

$$|(\mathcal{A}_{\alpha,\ell,\infty} f_\varepsilon^\sharp)^2(x) - (\mathcal{A}_{\alpha,\ell,\infty} f_\varepsilon^\sharp)^2(y)| \le C_1$$

for any $x, y \in Q$. Hence, if x_Q is the center of Q, one has

$$\left| \left\{ x \in Q : |(\mathcal{A}_\alpha f_\varepsilon^\sharp)^2(x) - (\mathcal{A}_{\alpha,\ell\infty} f_\varepsilon^\sharp)^2(x_Q))| > \lambda \right\} \right| \le \frac{C|Q|}{(\lambda - C_1)^b}$$

and one deduces that $(\mathcal{A}_\alpha f_\varepsilon^\sharp)^2 \in \mathrm{BMO}(\mathbb{R}^n)$. $\square$

Now, the John-Nirenberg inequality gives the following result.

Corollary 4.8. *Under the hypothesis of Theorem 4.7, for any cube $Q \subset \mathbb{R}^n$ there exists a constant a_Q such that*

$$\left| \left\{ x \in Q : |(\mathcal{A}_\alpha f_\varepsilon^\sharp)^2(x) - a_Q| > t \right\} \right| \le C_1 e^{-C_2 t} |Q|,$$

for every $t > 0$, where C_1, C_2 depend on M, n, α, β, θ. In particular,

$$\left| \left\{ x \in Q : (\mathcal{A}_\alpha f_\varepsilon^\sharp)^2(x) > t \right\} \right| \le C_1 e^{-C_2 t} |Q|,$$

provided $t > \sqrt{2a_Q}$. Moreover,

$$\left| \left\{ x \in Q : (\mathcal{A}_\alpha f_\varepsilon^\sharp)(x) > 2t \right\} \right| \le C_1 \exp(-C_2 t^2) \left| \left\{ x \in Q : (\mathcal{A}_\alpha f_\varepsilon^\sharp)(x) > t \right\} \right|.$$

Constructing suitable Lipschitz domains, we will show that Theorem 4.7 leads to good-λ inequalities relating the size of the area function and the non-tangential maximal function.

The following theorem is a weak version of the corresponding result for harmonic functions, which is Theorem 4 in [3], but it is enough for the applications that follow. We will use the notation $\Gamma_\alpha^\Omega(x)$, A_α^Ω, N_α^Ω whenever we want to emphasize that we take cones with vertex at $\partial\Omega$.

Theorem 4.9. *Let* $0 < \alpha < \beta < 1/M$. *Assume* $u \in C^2(\Omega)$ *satisfies* $(**)$ *with constant* θ, $0 < \theta < 1$. *Then there exist constants* $C_1, C_2 > 0$, $0 < C_3 < 1$ *depending only on* M, n, α, β, θ *such that for any* t, $\lambda > 0$:

$$\begin{aligned} &|\{x \in \mathbb{R}^n : (A_\alpha u)(x) \geq t\lambda, (N_\beta u)(x) \leq \lambda\}| \\ &\qquad\qquad \leq C_1 e^{-C_2 t^2} \, |\{x \in \mathbb{R}^n : (A_\beta u)(x) > C_3 t\lambda\}| \,. \end{aligned}$$

Proof. Suppose that $|\{x \in \mathbb{R}^n : (A_\beta u)(x) > C_3 t\lambda\}| < \infty$. Since condition $(**)$ also holds, with the same constant θ, if we replace u by u/λ, we can assume that $\lambda = 1$. Define $E = \{x \in \mathbb{R}^n : (N_\beta^\Omega u)(x) \geq 1\}$ and $\Omega' = \bigcup_{x \in E^c} \Gamma_\beta^\Omega(x)$. Then it is easy to see that there is a Lipschitz function $\psi : \mathbb{R}^n \to \mathbb{R}$, with Lipschitz constant at most $1/\beta$ such that $\Omega' = \{(x, y) : x \in \mathbb{R}^n,\ y > \psi(x)\} \subset \Omega$. Note that $|u| \leq 1$ in Ω'.

Fix $\varepsilon > 0$, depending only on α, β such that $\alpha_+ \leq (\beta_-)_- = \gamma$ (with the notation of Lemma 4.2). Then, by Lemma 4.2, Corollary 4.8, and the facts that $\mathcal{A}_\beta^{\Omega'} \geq \mathcal{A}_\beta^\Omega$ and $(\mathcal{A}_{\alpha_+}^{\Omega'} f_\varepsilon^\sharp)^2 \in \mathrm{BMO}(\mathbb{R}^n)$, we have:

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^n : (A_\alpha^\Omega u)(x) \geq t,\ (N_\beta^\Omega u)(x) \leq 1\right\}\right| \\ &\quad = \left|\left\{x \in \mathbb{R}^n \setminus E : (A_\alpha^\Omega u)(x) \geq t\right\}\right| \\ &\quad \leq \left|\left\{x \in \mathbb{R}^n : (A_\alpha^{\Omega'} u)(x) \geq t\right\}\right| \\ &\quad \leq \left|\left\{x \in \mathbb{R}^n : (A_{\alpha_+}^{\Omega'} f_\varepsilon^\sharp)(x) \geq \frac{1}{C} t\right\}\right| \\ &\quad \leq C_1 \exp\{-C_2 t^2\} \left|\left\{x \in \mathbb{R}^n : (\mathcal{A}_{\alpha_+}^{\Omega'} f_\varepsilon^\sharp)(x) \geq \frac{1}{2C} t\right\}\right| \\ &\quad \leq C_1 \exp\{-C_2 t^2\} \left|\left\{x \in \mathbb{R}^n : (\mathcal{A}_\gamma^{\Omega'} f_\varepsilon^\sharp)(x) \geq \frac{1}{2C} t\right\}\right| \\ &\quad \leq C_1 \exp\{-C_2 t^2\} \left|\left\{x \in \mathbb{R}^n : (A_{\beta_-}^{\Omega'} u)(x) \geq \frac{1}{2C^2} t\right\}\right| \\ &\quad \leq C_1 \exp\{-C_2 t^2\} \left|\left\{x \in \mathbb{R}^n : (A_\beta^\Omega u)(x) \geq \frac{1}{2C^2} t\right\}\right|, \end{aligned}$$

where C is the constant in Lemma 4.2. □

By means of the change of variables, $(z,t) \to (z, t+\phi(x))$, which transforms $\mathbb{R}^{n+1}_+$ into Ω, and the fact that $y - \phi(x) \approx y - \phi(z)$ whenever $(z,y) \in \Gamma^{\Omega}_{\alpha}(x)$, one can show that for any $f \geq 0$ measurable in Ω,

$$\int_{\Gamma^{\Omega}_{\alpha}(x)} f(z,y)(y-\phi(x))^{1-n}\,dz\,dy \approx \int_{\Gamma^{\Omega}_{\alpha}(x)} f(z,y)(y-\phi(z))^{1-n}\,dz\,dy$$
$$= \int_{|z-x|<\alpha t} h(z,t)t^{1-n}\,dt\,dz,$$

where $h(z,t) = f(z, t+\phi(z))$. This shows that we can reduce ourselves to the general setting considered in [10]. In particular, from Proposition 4 there, adapted to our situation, we get $\|\mathcal{A}_{\alpha}u\|_p \simeq \|\mathcal{A}_{\beta}u\|_p$ whenever $0 < \alpha < \beta < 1/M$, where the comparison constant only depends on α, β, p. This observation, together with a well-known standard argument, shows that the weak form of Theorem 4.9 is enough to get the usual comparison of the L^p-norms of the area function and the non-tangential maximal functions, as follows.

Theorem 4.10. *Let* $0 < \alpha < \beta < 1/M$, *and* $u \in C^2(\Omega)$ *satisfying* $(**)$ *with constant* θ, $0 < \theta < 1$. *Then, for* $0 < p < \infty$ *there exists* $C = C(p,\alpha,\beta,n,M,\theta)$ *such that*

$$\|A_{\alpha}u\|_{L^p(\mathbb{R}^n)} \leq C\|N_{\beta}u\|_{L^p(\mathbb{R}^n)},$$

whenever $\|A_{\alpha}u\|_{L^p(\mathbb{R}^n)} < \infty$.

Theorem 4.11. *Let* u, α, β *be as in Theorem 4.9. Assume that there is* $x_0 \in \mathbb{R}^n$ *and* $t_0 > 0$ *such that* $(A_{\beta,t_0,\infty}u)(x_0) < \infty$. *Then there are positive constants* C_1, C_2, C_3 *depending on* M, n, α, β, θ *such that if* $Q \subset \mathbb{R}^n$ *is any cube centered at* x_0, *there is* $a_Q > 0$ *such that, if* $\lambda > 0$, $t > \sqrt{2C_3 a_Q}$, *then*

$$|\{x \in Q : (A_{\alpha}u)(x) > t\lambda,\ (N_{\beta}u)(x) \leq \lambda\}| \leq C_1 e^{-C_2t^2}|Q|.$$

Proof. Assume $\lambda = 1$, as above. Since $(A_{\beta,t_0,\infty}u)(x_0) < \infty$, then for any cube Q of side ℓ centered at x_0 we have $(\mathcal{A}_{\beta,\ell,\infty}f^{\#}_{\varepsilon})(x_0) < \infty$, by Lemma 4.2 and some appropriate choice of ε. (Here $f^{\#}_{\varepsilon}$ is the density associated to $f = |\nabla u|^2$, as above). Therefore, $(\mathcal{A}^{\Omega'}_{\alpha_+}f^{\#}_{\varepsilon})^2 \in \mathrm{BMO}(\mathbb{R}^n)$ by Theorem 4.7 where, as before, let $E = \{x \in \mathbb{R}^n : N^{\Omega}_{\beta}u(x) \geq 1\}$ and $\Omega' = \bigcup_{x\in E^C}\Gamma^{\Omega}_{\beta}(x)$. Then, if $\varepsilon > 0$ is chosen sufficiently small, one has

$$\begin{aligned}
&|\{x \in Q : (A_{\alpha}u)(x) > t,\ (N_{\beta}u)(x) \leq 1|\}| \\
&\leq \left|\left\{x \in E^C \cap Q : (A^{\Omega}_{\alpha}u)(x) > t\right|\}\right| \\
&\leq \left|\left\{x \in Q : (A^{\Omega'}_{\alpha}u)(x) > t\right|\}\right| \\
&\leq \left|\left\{x \in Q : (\mathcal{A}^{\Omega'}_{\alpha_+}f^{\#}_{\varepsilon})(x) > \frac{t}{c}\right\}\right| \leq C_1 e^{-C_2t^2}|Q|
\end{aligned}$$

provided $(t/c)^2 \geq 2a_Q$, where a_Q is as in Corollary 4.8. □

As a consequence of Theorem 4.11 and Theorem 3.5, we get the corresponding Calderón-type result in this context.

Theorem 4.12. *Let* $0 < \alpha < 1/M$, *and* $u \in C^2(\Omega)$, *satisfying* $(**)$. *Then,* u *has finite non-tangential limit almost everywhere on the set*

$$\{x \in \mathbb{R}^n : (N_\alpha u)(x) < \infty\}.$$

As before, the good-λ inequality leads to a Law of the Iterated Logarithm.

Theorem 4.13. *Let* $u \in C^2(\Omega)$, *satisfying* $(**)$ *with a constant* θ. *Suppose that there exists* $x_0 \in \mathbb{R}^n$, $0 < \beta < 1/M$ *and* $y_0 > 0$ *such that* $(A_{\beta,y_0,\infty}u)(x_0) < \infty$. *Then, for each* α, $0 < \alpha < \beta$*:*

$$\varlimsup_{t\to 0} \frac{(A_{\alpha,y,\infty}u)(x)}{(N_{\beta,y,\infty}u)(x)\sqrt{\log\log(N_{\beta,y,\infty}u)(x)}} \leq C$$

a.e. $x \in \{x \in \mathbb{R}^n : (N_\beta u)(x) = \infty\}$, *where* $C = C(M, n, \alpha, \beta, \theta)$.

5. A Sufficient Condition

This section is devoted to proving the following result.

Proposition 5.1. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^{n+1}_+$. *Let* $u \in C^2(\Omega)$ *be such that there exists a constant* $C > 0$ *for which*

$$|u(w)\Delta u(w)| \leq C|\nabla u(w)|^2,$$

for all $w \in \Omega$. *Then* u *satisfies condition* $(*)$ *for* $\varphi(t) = At$, *where* $A = A(C)$ *is a constant depending on* C.

The result still holds under more general assumptions on the domain Ω. Observe that the hypotheses in Proposition 5.1 imply that u^{2k} is subharmonic if k is sufficiently large. Hence, Proposition 5.1 easily follows from the following result.

Proposition 5.2. *Let* B *be a ball in* $\mathbb{R}^n$ *and* $u \in C^2(B)$. *Assume that* u^{2k} *is a subharmonic function in* B, *for some positive integer* k. *Then,*

$$\operatorname{osc}\left(u, \frac{1}{10}B\right) \leq C r_B \left(\fint_B |\nabla u|^2\right)^{1/2},$$

where $C = C(n, k)$ *is a constant.*

Let us fix the notation. Given a ball B_0 contained in $B \subset \mathbb{R}^{n+1}_+$, and a function u defined in B, r_{B_0} will denote the euclidean radius of B_0 and $u_{B_0} = (1/|B_0|)\int_{B_0} u = \fint_{B_0} u$. Fix $0 < \eta < 1$, then we also define $A = r_{B_0}\left(\fint_{\widetilde{B}_0} |\nabla u|^2\right)^{1/2}$, where $\widetilde{B_0} = (1+\eta)B_0 \subset B$. Before proving Proposition 5.2, we need a lemma.

Lemma 5.3. *Assume u^2 is a subharmonic function in the ball $B \subset \mathbb{R}^n$ and let B_0 be another ball such that $\widetilde{B_0} = (1+\eta)B_0 \subset B$. Then, for any $w \in B_0$, one has*

$$|u^2(w) - (u_{B_0})^2| \le C(|u_{B_0}| + A)A,$$

where $C = C(n, \eta)$ is a constant.

Proof. To simplify notation we set $B = B_0$ and keep in mind that a certain duplicate of B is contained in the domain where u^2 is subharmonic. The lemma will be a consequence of several estimates.

(i). $|u_{\widetilde{B}} - u_B| \lesssim A$.
One has

$$|u_{\widetilde{B}} - u_B| \le ⨍_B |u - u_{\widetilde{B}}| \lesssim ⨍_{\widetilde{B}} |u - u_{\widetilde{B}}| \lesssim r_{\widetilde{B}} ⨍_{\widetilde{B}} |\nabla u| \lesssim r_{\widetilde{B}} \Big(⨍_{\widetilde{B}} |\nabla u|^2 \Big)^{1/2},$$

where the last two inequalities follow from Poincaré's and Hölder's inequalities.

(ii). $\Big(⨍_{\widetilde{B}} |u - u_B|^2 \Big)^{1/2} \lesssim A$.
To show (ii) we apply Poincaré's inequality again and (i):

$$⨍_{\widetilde{B}} |u - u_B|^2 \le 2\Big(⨍_{\widetilde{B}} |u - u_{\widetilde{B}}|^2 \Big) + 2|u_{\widetilde{B}} - u_B|^2 \lesssim r_{\widetilde{B}}^2 ⨍_{\widetilde{B}} |\nabla u|^2 + A^2 \simeq A^2.$$

(iii). $\Big(⨍_{\widetilde{B}} u^2 \Big) \lesssim |u_B|^2 + A^2$.
We simply write

$$⨍_{\widetilde{B}} u^2 \le 2\Big(⨍_{\widetilde{B}} (u - u_B)^2 \Big) + 2(u_B)^2$$

and apply (ii).

Let us now estimate $|u^2(w) - (u_B)^2|$. Since u^2 is subharmonic, for any $w \in B$, we have

$$u^2(w) - (u_B)^2 \le ⨍_{B(w)} u^2 - (u_B)^2,$$

where $B(w) \subset \widetilde{B}$ is a ball centered at w of radius comparable to the radius of B. Hence

$$\begin{aligned} |u^2(w) - (u_B)^2| &\lesssim ⨍_{\widetilde{B}} |u^2 - (u_B)^2| \\ &\le \Big(⨍_{\widetilde{B}} (u - u_B)^2 \Big)^{1/2} \Big(⨍_{\widetilde{B}} (u + u_B)^2 \Big)^{1/2} = \text{(I)(II)}. \end{aligned}$$

By (ii), I $\lesssim A$. Also by (iii)

$$(\mathrm{II})^2 \le 2\left(\fint_{\widetilde{B}} u^2\right) + 2(u_B^2) \lesssim (u_B)^2 + A^2 \le (|u_B| + A)^2. \qquad \square$$

Proof of Proposition 5.2. Assume first that u^2 is subharmonic in B. To simplify notation we rename $\frac{1}{10}B$ to be B and keep in mind that a certain duplicate of B is contained in the domain where u^2 is subharmonic. Then, by the previous lemma, for any $w \in B$, one has

$$|u^2(w) - (u_B)^2| \lesssim (|u_B| + A)A. \tag{5.1}$$

Suppose $u_B > 0$, otherwise we would apply the same argument to the function $(-u)$. Let $k_0 = \operatorname{osc}_B(u)$. Then either $(u_B + k_0/2) = u(w)$ for some $w \in B$ or $(u_B - k_0/2) = u(w)$ for some $w \in B$. Our purpose is to show that $k_0 \lesssim A$. In the first case, by (5.1) we get

$$\left|\left(u_B + \frac{k_0}{2}\right)^2 - (u_B)^2\right| \lesssim (|u_B| + A)A,$$

and therefore $k_0 \lesssim A$ as we wanted to show. The second case is harder, and we will need to consider several subcases. By (5.1)

$$\left|\left(u_B - \frac{k_0}{2}\right)^2 - (u_B)^2\right| = k_0\left|u_B - \frac{k_0}{4}\right| \lesssim (|u_B| + A)A. \tag{5.2}$$

Suppose first that $u_B > k_0/2$. Then $|u_B - k_0/4| \ge u_B/2$ and (5.2) implies

$$k_0 u_B \lesssim (u_B + A)A,$$

and therefore $k_0 \lesssim A$. Assume next that $u_B \le k_0/8$. Then $|u_B - k_0/4| > k_0/8$, and by (5.2)

$$k_0^2 \lesssim (|u_B| + A)A \le \left(\frac{k_0}{8} + A\right)A.$$

So we deduce $k_0 \lesssim A$. Finally, if $k_0/8 \le u_B \le k_0/2$, we will apply (5.1) to some point w such that $u(w) = u_B - k_0/8$. Then (5.2) becomes in this case

$$\left|\left(u_B - \frac{k_0}{8}\right)^2 - u_B^2\right| = k_0\left|\frac{k_0}{16} - u_B\right| \lesssim (|u_B| + A)A,$$

and the previous argument holds as well.

To prove the proposition in the general case, we proceed in a similar way. Assume that u^{2k} is subharmonic. The first step is to show an inequality similar to (5.1). As before, since u^{2k} is subharmonic, for any $\omega \in B$ we have

$$u^{2k}(\omega) - (u_B)^{2k} \leq \fint_{B(\omega)} u^{2k} - (u_B)^{2k},$$

where $B(\omega) \subset \widetilde{B}$ is a ball centered at ω of radius comparable to the radius of B. By estimate (ii) in the proof of Lemma 5.3, the expansion $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \cdots + b^{m-1})$, and Hölder's inequality, we get for any $w \in B$

$$\begin{aligned} |u^{2k}(w) - (u_B)^{2k}| &\leq \fint_{\widetilde{B}} |u^{2k} - (u_B)^{2k}| \\ &\leq \Big(\fint_{\widetilde{B}} |u - u_B|^2\Big)^{1/2} \Big(\fint_{\widetilde{B}} \Big(\sum_{j=1}^{2k} |u|^{2k-j}|u_B|^{j-1}\Big)^2\Big)^{1/2} \\ &\lesssim A \sum_{j=1}^{2k} |u_B|^{j-1} \Big(\fint_{\widetilde{B}} |u|^{2(2k-j)}\Big)^{1/2}. \end{aligned}$$

Since u^{2k} is subharmonic,

$$\sup_{\widetilde{B}} |u| \leq \Big(\fint_{2\widetilde{B}} u^{2k}\Big)^{1/2k},$$

and thus

$$\Big(\fint_{\widetilde{B}} u^{4k}\Big)^{1/4k} \leq \Big(\fint_{2\widetilde{B}} u^{2k}\Big)^{1/2k}.$$

As a consequence of a result of Iwaniec and Nolder ([15], see also [4, Lemma 1.4]), this reverse Hölder inequality improves to

$$\Big(\fint_{\widetilde{B}} u^{4k}\Big)^{1/4k} \lesssim \Big(\fint_{4\widetilde{B}} |u|\Big).$$

Using this and Hölder's inequality, we get that

$$\Big(\fint_{\widetilde{B}} |u|^{2(2k-j)}\Big)^{1/2} \leq C_{k,j} \Big(\fint_{4\widetilde{B}} |u|\Big)^{2k-j}.$$

Arguing as in Lemma 5.3, we get

$$\begin{aligned} \fint_{2\widetilde{B}} |u| &\leq \Big(\fint_{2\widetilde{B}} |u - u_{2\widetilde{B}}|\Big) + |u_{2\widetilde{B}} - u_B| + |u_B| \\ &\leq r_B \Big(\fint_{2\widetilde{B}} |\nabla u|^2\Big)^{1/2} + A + |u_B|. \end{aligned}$$

Thus

$$
\begin{aligned}
(5.3) \qquad |u^{2k}(w) - (u_B)^{2k}| &\lesssim A \sum_{j=1}^{2k} |u_B|^{j-1}(A + |u_B|)^{2k-j} \\
&\lesssim A(|u_B| + A + |u_B|)^{2k-1} \\
&\lesssim A(|u_B| + A)^{2k-1}.
\end{aligned}
$$

We can proceed now as in the case where u^2 was subharmonic. So, we suppose as before that $u_B > 0$, and we let $k_0 = \mathrm{osc}_B(u)$. Then either $(u_B + k_0/2) = u(w)$ for some $w \in B$ or $(u_B - k_0/2) = u(w)$ for some $w \in B$. We will use the following elementary estimate:

$$x^{2k} - y^{2k} = (x-y)(x^{2k-1} + x^{2k-2}y + \cdots + y^{2k-1}) \gtrsim (x-y)(x+y)^{2k-1}.$$

So, in the first case by (5.3), we get

$$\left(u_B + \frac{k_0}{2}\right)^{2k} - (u_B)^{2k} \lesssim A(u_B + A)^{2k-1},$$

and by the observation above we can conclude

$$k_0(u_B + k_0)^{2k-1} \lesssim A(u_B + A)^{2k-1},$$

which implies $k_0 \lesssim A$. In the second case, we will consider subcases as before. Assume first that $u_B > k_0$, then $u_B - k_0/2 \geq u_B/2$ and by (5.3) and the previous observation

$$k_0 u_B^{2k-1} \leq u_B^{2k} - \left(u_B - \frac{k_0}{2}\right)^{2k} \lesssim A(u_B + A)^{2k-1}$$

and therefore $k_0 \lesssim A$. Next, assume $u_B < k_0/8$. Then $k_0/2 - u_B > 3k_0/8$ and (5.3) gives

$$k_0^{2k} \lesssim \left(\frac{k_0}{2} - u_B\right)^{2k} - u_B^{2k} \leq A(u_B + A)^{2k-1},$$

which implies $k_0 \lesssim A$. Finally assume $k_0/8 < u_B < k_0$. Then we apply the same argument to a point $w \in B$ such that $u(w) = u_B - k_0/16$ and we get

$$\left| \left(u_B - \frac{k_0}{16}\right)^2 - u_B^2 \right| \lesssim A(u_B + A)^{2k-1}.$$

Since $u_B > k_0/8$, we obtain as before $k_0 \lesssim A$. ◻

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