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EQUALITY-FREE SATURATED MODELS

Saturated models are a powerful tool in model theory. The properties of universality and homogeneity of the saturated models of a theory are useful for proving facts about this theory. They are used in the proof of interpolation and preservation theorems and also as work-spaces. Sometimes we work with models which are saturated only for some sets of formulas, for example, recursively saturated models, in the study of models of arithmetic or atomic compact, in model theory of modules. In this article we introduce the notion of equality-free saturated model, that is, roughly speaking, a model which is saturated for the set of equality-free formulas. Our aim is to understand better the role that identity plays in classical model theory, in particular with regard to this process of saturation.

Given an infinite cardinal κ, we say that a model is equality-free κ-saturated if it satisfies all the 1-types over sets of parameters of power less than κ, with all the formulas in the type that are equality-free. We compare this notion with the usual notion of κ-saturated model. We prove the existence of infinite models $\mathfrak{A}$, which are L^{-}-$|A|^{+}$-saturated. From

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this last fact it follows that L^--saturated models have a different behavior than usual saturated models. Several characterizations of models with this property are given.

We pay special attention to reduced structures. It is said that a structure $\mathfrak{A}$ is *reduced* if there are not different elements in $\mathfrak{A}$ with the same atomic equality-free type over $\mathfrak{A}$. Examples of this type of structures are all linear orders and the random graph. The importance of reduced structures in equality-free logic comes from the fact that any structure is a strict homomorphic image of a reduced structure. Therefore, they satisfy exactly the same equality-free sentences. One interesting result obtained is that, for reduced structures, L^--saturation implies strong homogeneity. The notion of L^--ω-saturated model is considered independently by G. C. Nelson in [19]. In his article this concept is studied in relation to the notion of ω-categorical theory for languages without equality.

The following notation will be used in this work. From now on L will be a similarity type with at least one relation symbol. We denote also by L the set of first-order formulas of type L and by L_0 the set of quantifier-free formulas of L. L^- and L_0^- will be the set of all formulas of L and L_0, respectively, that do not contain the equality symbol. Given L-structures $\mathfrak{A}$ and $\mathfrak{B}$, we write $\mathfrak{A} \equiv^- \mathfrak{B}$ and $\mathfrak{A} \equiv_0^- \mathfrak{B}$ to mean that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same sentences of L^- and L_0^-, respectively. For any L-structure $\mathfrak{A}$ and any set $B \subseteq A$, we denote by $L(B)$ the similarity type obtained from L by adding a new constant symbol for each element of B and we denote by $\mathfrak{A}_B$ the natural expansion of $\mathfrak{A}$ to $L(B)$, where every new constant denotes its corresponding element. For the sake of clarity we use the same symbol for the constant and for the element that is denoted by the constant. $|A|$ denotes the power of the set A.

1. Reduced structures

The present interest for the study of languages without equality has its origin in the works of J. Czelakowski, W. Blok and D. Pigozzi, (see [1], [2], [3], [7]and [8]). They use two main concepts that allow the development of this study, the notion of Leibniz congruence and the notion of relative relation. The notion of Leibniz congruence dates back to 1949 when it was defined by Łoś in the context of Lindenbaum matrices. Leibniz congruences were extensively used by Wójcicki in [23] and by other logicians under the name of the largest matrix congruence (see [9]) . Motivated by their works a general classical model-theoretical study of this logic was carried on in [4], [10], [13] and [14]. Independently, in [18], [20] and [21], G. C. Nelson and O. Neswan introduced the notion of quasi-isomorphism, which is equivalent to the notion of relative relation. Now we present these notions and some basic facts about them without proof.

Definition 1.1. If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures, it is said that an homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is *strict* if for any n-adic relation symbol $R \in L$ and any $a_1, \dots, a_n \in A$,

$$\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{A}} \iff \langle h(a_1), \dots, h(a_n) \rangle \in R^{\mathfrak{B}}.$$

It is a well-known fact that if $h : \mathfrak{A} \to \mathfrak{B}$ is a strict homomorphism onto $\mathfrak{B}$, then $\mathfrak{A} \equiv^- \mathfrak{B}$ and the kernel of h is a strict congruence of $\mathfrak{A}$. Moreover, for any strict congruence θ of $\mathfrak{A}$, the canonical homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}/\theta$ is strict. Given a class K of L-structures, we denote by $\mathbf{H_S}(K)$ the class of all strict homomorphic images of members of K and by $\mathbf{H_S^{-1}}(K)$ the class of all strict homomorphic counter-images of members of K.

Let $\mathfrak{A}$ be an L-structure and B a subset of A. We expand the language adding a new constant symbol for each element of B and also add new variables. Given a cardinal κ, it is said that a set p of formulas of $L^-(B)$ in the variables $\{x_\alpha : \alpha \in \kappa\}$ is an *L^--κ-type over B in $\mathfrak{A}$* if p is consistent with $\mathrm{Th}^-(\mathfrak{A}_B)$. In addition, p is *L^--complete* iff for any formula $\phi \in L^-(B)$ in the variables $\{x_\alpha : \alpha \in \kappa\}$, $\phi \in p$ or $\neg\phi \in p$. Observe that for any set p of formulas of $L^-(B)$, p is consistent with $\mathrm{Th}^-(\mathfrak{A}_B)$ iff p is consistent with $\mathrm{Th}(\mathfrak{A}_B)$. Therefore, a set p of formulas of $L(B)$ is an L^--κ-type over B in $\mathfrak{A}$ iff p is a κ-type over B in $\mathfrak{A}$ and all the formulas of p are equality-free. Given a κ-tuple $\overline{a} = (a_\alpha : \alpha \in \kappa)$ of elements of A, the *equality-free type of $\overline{a}$ over B* in $\mathfrak{A}$, in symbols $\mathrm{tp}^-_{\mathfrak{A}}(\overline{a}/B)$, is the set of all formulas of $L^-(B)$ in the variables $\{x_\alpha : \alpha \in \kappa\}$ satisfied by $\overline{a}$. By $\mathrm{atp}^-_{\mathfrak{A}}(\overline{a}/B)$ we denote the set of atomic formulas of $\mathrm{tp}^-_{\mathfrak{A}}(\overline{a}/B)$ and call it the *equality-free atomic type of $\overline{a}$ over B* in $\mathfrak{A}$.

Definition 1.2. Given an L-structure $\mathfrak{A}$, we define the relation $\Omega(\mathfrak{A})$ on $\mathfrak{A}$ as follows: $\langle a, b \rangle \in \Omega(\mathfrak{A})$ iff $\mathrm{atp}^-_{\mathfrak{A}}(a/A) = \mathrm{atp}^-_{\mathfrak{A}}(b/A)$, for any $a, b \in A$.

$\Omega(\mathfrak{A})$ is the greatest strict congruence relation on $\mathfrak{A}$, that is, every strict congruence relation refines it. It is called the *Leibniz strict congruence of* $\mathfrak{A}$. We say that a structure is *reduced* if there are no different elements with the same equality-free atomic type over the structure, that is, if its

Leibniz strict congruence is the identity. The quotient structure $\mathfrak{A}/\Omega(\mathfrak{A})$ is reduced and is denoted by $\mathfrak{A}^*$. This structure is called *the reduction of* $\mathfrak{A}$. Observe that from the definition it follows that $\mathfrak{A}^* \cong (\mathfrak{A}^*)^*$. Moreover, it is easy to check that the canonical homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}^*$ is strict. It is easy to prove, by induction, that given $a, b \in A$, $\langle a, b\rangle \in \Omega(\mathfrak{A})$ iff $\mathrm{tp}^-_{\mathfrak{A}}(a/A) = \mathrm{tp}^-_{\mathfrak{A}}(b/A)$.

Examples of reduced structures are all linear orders and the random graph. There are theories without reduced models, for example the theory of an equivalence relation with infinitely many classes all of them infinite because in any model of this theory, any two elements in the same equivalence class have the same equality-free atomic type over the model. Observe also that any theory axiomatized by a set of equality-free sentences has reduced models and non-reduced ones.

Remark 1.3. For any theory T of L, the reduced models of T are those that omit the following set of formulas

$$p_{\mathrm{red}} = \{x \not\approx y\} \cup \left\{\forall \bar{z}[\phi(x, \bar{z}) \leftrightarrow \phi(y, \bar{z})] : \phi \in L^- \text{ atomic}\right\}.$$

It is easy to check that to isolate p_{red} is a sufficient condition, for any theory of L, for having non-reduced models. And in the case that L is countable, by the classical Omitting Types Theorem, if T is a consistent theory of L and p_{red} is non-isolated in T, then some models of T are reduced.

Assume now that L is countable. It is a consequence of the Löwenheim-Skolem-Tarski theorem that if a theory T has a non-reduced infinite model, then it has such models in each infinite power. But the existence of a reduced infinite model of T may not imply the existence of reduced models in each infinite power. It might be interesting to find the Hanf number of the notion of reduced structure, that is to determine the least cardinal κ such that, for any theory T, the existence of a reduced model of T of power κ implies the existence of such models in any infinite power. Some results in this line will be obtained in next section.

Finally, as straightforward corollary of the following proposition, we obtain an interesting result: for any closed theory T in a countable similarity type, if T has reduced models and p_{red} is non-isolated in T, then the theory of the reduced models of T is precisely T.

Proposition 1.4. *Let L be countable and T a consistent closed theory of L. Then p is non-isolated in T iff $T = \mathrm{Th}(\{\mathfrak{A} \models T : \mathfrak{A}$ omits $p\})$.*

Proof. See [5], Proposition 1.3. □

To end this section we introduce the notion of relative correspondence, which turns out to be the equivalent, for equality-free languages, to the notion of isomorphism.

Definition 1.5. Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. A relation $R \subseteq A \times B$ is a relative correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ if $\mathrm{dom}(R) = A$, $\mathrm{rg}(R) = B$ and

(1) for any constant $c \in L, c^{\mathfrak{A}} R c^{\mathfrak{B}}$,

(2) for any n-adic function symbol $f \in L$, any $a_1, \ldots, a_n \in A$ and any $b_1, \ldots, b_n \in B$ such that $a_i R b_i$ for each $i = 1, \ldots, n$,

$$f^{\mathfrak{A}}(a_1, \ldots, a_n) R f^{\mathfrak{B}}(b_1, \ldots, b_n),$$

(3) for any n-adic relation symbol $S \in L$, any $a_1, \ldots, a_n \in A$ and any $b_1, \ldots, b_n \in B$ such that $a_i R b_i$ for each $i = 1, \ldots, n$,

$$\langle a_1, \ldots, a_n \rangle \in S^{\mathfrak{A}} \Longleftrightarrow \langle b_1, \ldots, b_n \rangle \in S^{\mathfrak{B}}.$$

Two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *relatives*, in symbols $\mathfrak{A} \sim \mathfrak{B}$, if there is a relative correspondence between them. The relation of being either a strict homomorphic image or a strict homomorphic counter-image is not in general transitive. Its transitivization is precisely the relative relation, as the following proposition shows. The proof of next proposition can be found in [4].

Proposition 1.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. The following are equivalent:*

i) $\mathfrak{A} \sim \mathfrak{B}$.

ii) *There are $n \in \omega$ and L-structures $\mathfrak{C}_0, \ldots, \mathfrak{C}_n$ such that $\mathfrak{A} = \mathfrak{C}_0$, $\mathfrak{B} = \mathfrak{C}_n$ and for any $i < n$, $\mathfrak{C}_{i+1} \in \mathbf{H_S}(\mathfrak{C}_i)$ or $\mathfrak{C}_{i+1} \in \mathbf{H_S^{-1}}(\mathfrak{C}_i)$.*

iii) $\mathfrak{A}$, $\mathfrak{B} \in \mathbf{H_S}(\mathfrak{C})$, *for some* $\mathfrak{C}$.

iv) $\mathfrak{A}$, $\mathfrak{B} \in \mathbf{H_S^{-1}}(\mathfrak{C})$, *for some* $\mathfrak{C}$.

v) $\mathfrak{A}^* \cong \mathfrak{B}^*$.

vi) *There are enumerations of A and B, $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ respectively, such that the map $h : A^* \to B^*$ defined by: $h([a_i]_{\Omega(\mathfrak{A})}) = [b_i]_{\Omega(\mathfrak{B})}$, for any $i \in I$, is an isomorphism.*

vii) *There are enumerations of* A *and* B, $\overline{a} = (a_i : i \in I)$ *and* $\overline{b} = (b_i : i \in I)$ *respectively, such that* $(\mathfrak{A}, \overline{a}) \equiv_0^- (\mathfrak{B}, \overline{b})$.

viii) *There are enumerations of* A *and* B, $\overline{a} = (a_i : i \in I)$ *and* $\overline{b} = (b_i : i \in I)$ *respectively, such that* $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b})$.

2. Equality-free saturated models

Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures and $r \subseteq A \times B$ a relation. For any formula $\phi \in L(\mathrm{dom}(r))$, $\phi = \phi(\bar{x}, a_1, \ldots, a_n)$, let Σ_ϕ^r be the following set of formulas of $L(\mathrm{rg}(r))$:

$$\{\phi(\bar{x}, b_1, \ldots, b_n) \in L(\mathrm{rg}(r)) : \text{for every } i \in \{1, \ldots, n\}\,,\ \langle a_i, b_i \rangle \in r\}\,,$$

where $\phi(\bar{x}, b_1, \ldots, b_n)$ is obtained from ϕ by substituting b_i for a_i, for every $i \in \{1, \ldots, n\}$. Given a set p of formulas of $L(\mathrm{dom}(r))$, let $p^r = \bigcup_{\phi \in p} \Sigma_\phi^r$. In particular, if $\mathfrak{A}$ is an L-structure, D a subset of A and p a set of formulas of $L^-(D)$, we denote by p^* the set p^r, where $r \subseteq A \times A^*$ is the relation defined by: $r = \left\{\langle d, [d]_{\Omega(\mathfrak{A})} \rangle : d \in D\right\}$. And we denote by D^* the set $\left\{[d]_{\Omega(\mathfrak{A})} : d \in D\right\}$.

Remark 2.1. If $\overline{a} = (a_i : i \in I)$ and $\overline{b} = (b_i : i \in I)$ are sequences of elements of A and B respectively, such that $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b})$, and $r = \{\langle a_i, b_i \rangle : i \in I\}$, then for any set p of formulas of $L^-(\mathrm{dom}(r))$, p is an L^--κ-type over $\mathrm{dom}(r)$ in $\mathfrak{A}$ iff p^r is an L^--κ-type over $\mathrm{rg}(r)$ in $\mathfrak{B}$. Moreover p^r is L^--complete if p is L^--complete.

The notion of elementary substructure can be generalized to equality-free logic in a natural way. By means of elementary substructures we will give a characterization of L^-- complete types.

Definition 2.2. If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures, $\mathfrak{A}$ is an L^-*-substructure* of $\mathfrak{B}$, in symbols $\mathfrak{A} \preceq^- \mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and for any $\phi(x_1, \ldots, x_n) \in L^-$ and any $a_1, \ldots, a_n \in A$,

$$\mathfrak{A} \models \phi\,[a_1, \ldots, a_n] \Longleftrightarrow \mathfrak{B} \models \phi\,[a_1, \ldots, a_n]\,.$$

If $\mathfrak{A}$ is an L^--substructure of $\mathfrak{B}$, it is said that $\mathfrak{B}$ is an L^-*-extension* of $\mathfrak{A}$. $\mathfrak{A}$ -$^-$ $\mathfrak{B}$ means that $\mathfrak{A}$ is isomorphic to an L^--substructure of $\mathfrak{B}$.

Proposition 2.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. If $\mathfrak{A} \mathbin{\text{-}}^{-} \mathfrak{B}$, then the map $j : \mathfrak{A}^* \to \mathfrak{B}^*$ defined by:*

$$j([a]_{\Omega(\mathfrak{A})}) = [a]_{\Omega(\mathfrak{B})}\,,$$

for any $a \in A$, is an embedding that preserves all the equality-free formulas.

Proof. See [11], Proposition 2.8. □

Proposition 2.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. Then the following are equivalent:*

i) *There is an enumeration of A, $\overline{a} = (a_i : i \in I)$, and a sequence of elements of B, $\overline{b} = (b_i : i \in I)$, such that*

$$(\mathfrak{A}, \overline{a}) \equiv^{-} (\mathfrak{B}, \overline{b}).$$

ii) $\mathfrak{A}^* \mathbin{\text{-}}^{-} \mathfrak{B}^*$.

Proof. See [11], Proposition 2.8. □

Lemma 2.5. *Let $\mathfrak{A}$ be an L-structure, D a subset of A and κ a cardinal. For any set p of formulas of $L^-(D)$ in the variables $\{x_\alpha : \alpha \in \kappa\}$, the following are equivalent:*

i) *p is an L^--complete L^--κ-type over D in $\mathfrak{A}$.*

ii) *There is $\mathfrak{A}'$ such that $D \subseteq A'$ and $\mathfrak{A}'_D \models \mathrm{Th}^-(\mathfrak{A}_D)$ and there is a sequence $\overline{m} = (m_\alpha : \alpha \in \kappa)$ of elements of A' such that $p = \mathrm{tp}^-_{\mathfrak{A}'}(\overline{m}/D)$.*

iii) *There is $\mathfrak{A}'$ such that $\mathfrak{A} \preceq^- \mathfrak{A}'$ and there is a sequence $\overline{m} = (m_\alpha : \alpha \in \kappa)$ of elements of A' such that $p = \mathrm{tp}^-_{\mathfrak{A}'}(\overline{m}/D)$.*

Proof. ii) ⇒ i) and iii) ⇒ i) are clear. i) ⇒ ii) is easy to prove using the fact that p is consistent with $\mathrm{Th}(\mathfrak{A}_D)$ and for i) ⇒ iii) we use the fact that p is consistent with $\mathrm{Th}(\mathfrak{A}_A)$. □

Observe that in ii) of Lemma 2.5, we can take $\mathfrak{A}'$ such that $|A'| \leq \max(|D|, |L|, \aleph_0)$. And in iii) we can take $\mathfrak{A}'$ such that $|A'| \leq \max(|A|, |L|, \aleph_0)$. If we consider reduced structures, we can obtain the following version of Lemma 2.5:

Corollary 2.6. *Let $\mathfrak{A}$ be a reduced L-structure, D a subset of A and κ a cardinal. For any set p of formulas of $L^-(D)$ in the variables $\{x_\alpha : \alpha \in \kappa\}$, the following are equivalent:*

i) *p is an L^--complete L^--κ-type over D in $\mathfrak{A}$.*

ii) *There is a reduced L-structure $\mathfrak{A}'$ such that $\mathfrak{A} \preceq^- \mathfrak{A}'$ and there is a sequence $\overline{m} = (m_\alpha : \alpha \in \kappa)$ of elements of A' such that $p = \mathrm{tp}^-_{\mathfrak{A}'}(\overline{m}/D)$.*

Proof. ii) $\Rightarrow$ i) is clear. i) $\Rightarrow$ ii) Since p is an L^--complete L^--κ-type over D in $\mathfrak{A}$, by Lemma 2.5, there is $\mathfrak{B}$ such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and a sequence $\bar{l} = (l_\alpha : \alpha \in \kappa)$ of elements of B such that $p = \mathrm{tp}^-_{\mathfrak{B}}(\bar{l}/D)$. Then, since $\mathfrak{A}$ is reduced, by Proposition 2.3, the map $j : \mathfrak{A} \to \mathfrak{B}^*$ defined by: $j(a) = [a]_{\Omega(\mathfrak{B})}$, for any $a \in A$, is an embedding that preserves all the equality-free formulas, thus $\mathfrak{A}$ -$^-$ $\mathfrak{B}^*$. If $\overline{k} = ([l_\alpha]_{\Omega(\mathfrak{B})} : \alpha \in \kappa)$, clearly $p^* = \mathrm{tp}^-_{\mathfrak{B}^*}(\overline{k}/D^*)$. With standard arguments we can find an L-structure $\mathfrak{A}'$ such that $\mathfrak{A} \preceq^- \mathfrak{A}'$ and an isomorphism $h : (\mathfrak{A}', a)_{a\in A} \to (\mathfrak{B}^*, [a]_{\Omega(\mathfrak{B})})_{a\in A}$ such that h $^-$ $A = j$. Let $\overline{m} = (h^{-1}([l_\alpha]_{\Omega(\mathfrak{B})}) : \alpha \in \kappa)$. Clearly $p = \mathrm{tp}^-_{\mathfrak{A}'}(\overline{m}/D)$ and since $\mathfrak{B}^*$ is reduced, $\mathfrak{A}'$ is also reduced. □

Now we introduce the main concept of this article: equality-free saturated models.

Definition 2.7. Given an L-structure $\mathfrak{A}$ and a cardinal κ, we say that $\mathfrak{A}$ is *L^--κ-saturated* iff for any $D \subseteq A$ with $|D| < \kappa$, $\mathfrak{A}$ realizes every L^--1-type over D in $\mathfrak{A}$. And we say that $\mathfrak{A}$ is L^--saturated iff $\mathfrak{A}$ is L^--$|A|$-saturated.

Since any L^--κ-type can be extended to an L^--complete L^--κ-type, a model $\mathfrak{A}$ is L^--κ-saturated if for any $D \subseteq A$ with $|D| < \kappa$, $\mathfrak{A}$ realizes every L^--complete L^--1-type over D in $\mathfrak{A}$. Now we show that the relative relation preserves the L^--saturation of models:

Proposition 2.8. *Let $\mathfrak{A}$ be an L-structure and κ a cardinal. Then, $\mathfrak{A}$ is L^--κ-saturated iff $\mathfrak{A}^*$ is L^--κ-saturated.*

Proof. $\Leftarrow$) Assume that $\mathfrak{A}^*$ is L^--κ-saturated. Let E be a subset of A of power less than κ and p an L^--1-type over E in $\mathfrak{A}$. Then, by Remark 2.1, p^* is an L^--1-type over E^* in $\mathfrak{A}^*$. Since $\mathfrak{A}^*$ is L^--κ-saturated and $|E^*| < \kappa$, there is an element $x \in A^*$ that realizes p^*. Let $a \in A$ be a member of the equivalence class x. Clearly a is a realization of p in $\mathfrak{A}$.

$\Rightarrow$) Assume that $\mathfrak{A}$ is L^--κ-saturated. Let E be a subset of A^* of power less than κ and p an L^--1-type over E in $\mathfrak{A}^*$. We choose for any equivalence class $e \in E$ a representative $a_e \in e$. Let $D = \{a_e : e \in E\}$. For any formula $\phi \in p$, $\phi = \phi(x, e_1, \ldots, e_n)$, let ϕ' be the formula of $L(D)$ obtained from ϕ by substituting a_{e_i} for e_i, for $i = 1, \ldots, n$. Let $q = \{\phi' : \phi \in p\}$, clearly $q^* = p$ and, by Remark 2.1, q is an L^--1-type over D in $\mathfrak{A}$. Since $\mathfrak{A}$ is L^--κ-saturated and $|D| < \kappa$, there is an element $a \in A$ that realizes q. Then $[a]_{\Omega(\mathfrak{A})}$ is clearly a realization of p in $\mathfrak{A}^*$. □

Corollary 2.9. *Let κ be a cardinal and $\mathfrak{A}$ and $\mathfrak{B}$ L-structures such that $\mathfrak{A} \sim \mathfrak{B}$. Then, $\mathfrak{A}$ is L^--κ-saturated iff $\mathfrak{B}$ is L^--κ-saturated.*

Proof. Assume that $\mathfrak{A}$ is L^--κ-saturated. By Proposition 2.8, $\mathfrak{A}^*$ is L^--κ-saturated. Since $\mathfrak{A} \sim \mathfrak{B}$, by Proposition 1.6, $\mathfrak{A}^* \cong \mathfrak{B}^*$. Therefore, $\mathfrak{B}^*$ is L^--κ-saturated, and again by Proposition 2.8, $\mathfrak{B}$ is L^--κ-saturated. The other direction is analogous. □

Let us see now that, for reduced structures and finite and relational similarity types, the two concepts coincide:

Proposition 2.10. *Let L be a finite and relational similarity type and $\mathfrak{A}$ a reduced L-structure. Then, $\mathfrak{A}$ is L^--saturated iff $\mathfrak{A}$ is saturated.*

Proof. $\Leftarrow$) is clear. $\Rightarrow$) Suppose that $\mathfrak{A}$ is L^--saturated. Let D be a subset of A with $|D| < |A|$ and p an 1-type over D in $\mathfrak{A}$. Since L is finite and relational and $\mathfrak{A}$ is reduced, there is a finite set Γ of formulas of the form $\forall \overline{z}\, [\phi(x, \overline{z}) \leftrightarrow \phi(y, \overline{z})]$, where $\phi \in L^-$ is atomic, such that, if $\psi(x, y)$ is the conjunction of all the formulas in Γ, then $\mathfrak{A} \models \forall x \forall y\, [x \approx y \leftrightarrow \psi(x, y)]$. For any formula $\phi \in L(D)$, let $\phi' \in L^-(D)$ be the formula obtained from ϕ by replacing each appearance of a formula of the form $t_1 \approx t_2$ by an appearance of $\psi(t_1, t_2)$. Let $p' = \{\phi' : \phi \in p\}$. It is easy to check that p' is an L^--1-type over D in $\mathfrak{A}$. Since $\mathfrak{A}$ is L^--saturated, there is a realization of p', $a \in A$. Clearly, a is also a realization of p. □

Now, we present some results on the existence of L^--saturated models. First, we introduce the notion of L^--complete theory. We say that a theory T is L^-*-complete* iff for any sentence $\sigma \in L^-$, $T \models \sigma$ or $T \models \neg\sigma$.

Proposition 2.11. *If T is L^--complete and $\lambda \geq \max(|L|, \aleph_0)$, then T has a L^--λ-saturated model of power $\leq 2^\lambda$.*

Proof. Extend T to a complete theory $T'$and apply the analogous result for logic with equality. □

We can obtain a better result in stable theories. We say that a theory T is L^{-}-λ-*stable* if for any model $\mathfrak{A}$ of T, for any $X \subseteq A$ with $|X| \leq \lambda$, there are $\leq \lambda$ L^{-}-complete L^{-}-1-types over X in $\mathfrak{A}$. It is said that T is L^{-}-*stable* if for some λ, T is L^{-}-λ-*stable.*

Proposition 2.12. *If T is L^{-}-complete, λ is a regular cardinal $\lambda \geq \max(|L|, \aleph_0)$ and T is L^{-}-λ-*stable*, then T has a L^{-}-λ-saturated model of power λ.*

Proof. We build an elementary chain (in the usual sense of logic with equality) $(\mathfrak{A}_\alpha : \alpha \in \lambda)$, of models of T of power λ. Let $\mathfrak{A}_0$ be any model of T. Assume that we have chosen $\mathfrak{A}_\alpha$, by L^{-}-λ-stability, there are at most λ L^{-}-complete L^{-}-1-types over $\mathfrak{A}_\alpha$. Let $\mathfrak{A}_{\alpha+1}$ be an elementary extension of $\mathfrak{A}_\alpha$ which realizes all these types. At limit ordinals, take unions. Let $\mathfrak{A} = \bigcup_{\alpha\in\lambda} \mathfrak{A}_\alpha$, since λ is regular, it is easy to check that $\mathfrak{A}$ is a L^{-}-λ-saturated model of T of power λ. □

In stability theory, several improvements can be obtained in reference to singular cardinals. The notion of L^{-}-*stable* is not studied in this work. In view of Propositions 2.18 and 2.19, perhaps, it could be of some interest to compare this notion with the usual one. The following existence theorem for L^{-}-ω-saturated models is stated by G. C. Nelson in [19].

Theorem 2.13. *Let T be L^{-}-complete. Then T has a L^{-}-ω-saturated model iff for each $n \in \omega$, T has only countably many L^{-}-types in n variables.*

We have also the corresponding theorems with set-theoretically assumptions, for example, for any λ strongly inaccessible, every theory with $\lambda \geq \max(|L|, \aleph_0)$ has a L^{-}-λ-saturated model of power λ. Now we state some facts without proof. The proof of these statements is analogous to the proof in the case of logic with equality.

Fact 2.14. (1) *Given a cardinal κ, if $\mathfrak{A}$ is L^{-}-κ-saturated, then for any $D \subseteq A$ with $|D| < \kappa$, $\mathfrak{A}$ realizes every L^{-}-κ-type over D in $\mathfrak{A}$.*

(2) *Given a cardinal κ, any model has an L^{-}-extension that is L^{-}-κ-saturated.*

(3) *Any finite model is L^{-}-κ-saturated, for any cardinal κ.*

(4) *Any two L^{-}-equivalent L^{-}-saturated models of the same power are relatives.*

Observe that, from Fact (4), it follows that two reduced models of the same power that are L^--equivalent and L^--saturated are isomorphic, because any two reduced models which are relatives are isomorphic. This is not true in general. Let us see now that the converse of Fact (3) is not true, because there is a property of L^--saturation that usual saturation does not share: there are L-structures $\mathfrak{A}$ that are L^--$|A|^+$-saturated. Let us see an example:

Example 2.15. Let $L = \{P_n : n \in \omega\}$, where for any $n \in \omega$, P_n is a unary relation symbol and T be the theory of the infinite independent properties, that is, the set of consequences of the following set of sentences:

$$\exists x (P_{i_0} x \wedge \ldots \wedge P_{i_n} x \wedge \neg P_{j_0} x \wedge \ldots \wedge \neg P_{j_k} x),$$

for any distinct $i_0, \ldots, i_n, j_0, \ldots, j_k \in \omega$. Let $\mathfrak{A} = (P(\omega), P_n^{\mathfrak{A}})_{n\in\omega}$, where for any $X \in P(\omega)$, $X \in P_n^{\mathfrak{A}}$ iff $n \in X$.

Clearly $\mathfrak{A}$ is reduced. We show that $\mathfrak{A}$ is L^--$|A|^+$-saturated. Let D be any subset of A and p an L^--complete L^--1-type over D in $\mathfrak{A}$. Since $\mathfrak{A}$ is reduced, by Corollary 2.6, there are a reduced L-structure $\mathfrak{A}'$ and an element $a \in A'$ such that $\mathfrak{A} \preceq^- \mathfrak{A}'$ and $p = \mathrm{tp}^-_{\mathfrak{A}'}(a/D)$. Consider now the set $p_0 = \mathrm{atp}^-_{\mathfrak{A}'}(a/A')$. All the formulas of p_0 are of the form $P_n x$, for some $n \in \omega$. Let $Y = \{n \in \omega : P_n x \in p_0\}$. Observe that $Y \in A$ and $\mathrm{atp}^-_{\mathfrak{A}'}(Y/A') = \mathrm{atp}^-_{\mathfrak{A}'}(a/A')$. Therefore, since $\mathfrak{A}'$ is reduced, $Y = a$. Hence, p is realized in $\mathfrak{A}$. We conclude that $\mathfrak{A}$ is L^--$|A|^+$-saturated. However, $\mathfrak{A}$ is not saturated because the following 2-type is not realized in A

$$p = \{x \not\approx y\} \cup \{P_n x \leftrightarrow P_n y : n \in \omega\}.$$

Now we give a characterization of L-structures $\mathfrak{A}$ that are L^--$|A|^+$-saturated. We obtain the following result: $\mathfrak{A}$ is L^--$|A|^+$-saturated iff any L^--extension of $\mathfrak{A}$ is relative of $\mathfrak{A}$. Later we will see that, if $\mathfrak{A}$ is L^--$|A|^+$-saturated, intuitively speaking, $\mathfrak{A}^*$ is the greatest reduced model of $\mathrm{Th}^-(\mathfrak{A})$.

Lemma 2.16. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. Suppose that $\mathfrak{A} \preceq^- \mathfrak{B}$, $a \in A$ and $b \in B$. If $\mathrm{tp}^-_{\mathfrak{B}}(b/A) = \mathrm{tp}^-_{\mathfrak{B}}(a/A)$, then $\mathrm{atp}^-_{\mathfrak{B}}(b/B) = \mathrm{atp}^-_{\mathfrak{B}}(a/B)$.*

Proof. Suppose that $\mathrm{tp}^-_{\mathfrak{B}}(b/A) = \mathrm{tp}^-_{\mathfrak{B}}(a/A)$. Then, since for any equality-free atomic formula $\phi(x, \bar{z}) \in L, \forall \bar{z}(\phi(a, \bar{z}) \leftrightarrow \phi(y, \bar{z})) \in \mathrm{tp}^-_{\mathfrak{B}}(a/A)$, we have for any equality-free atomic formula $\phi(x, \bar{z}) \in L, \forall \bar{z}(\phi(a, \bar{z}) \leftrightarrow \phi(y, \bar{z})) \in \mathrm{tp}^-_{\mathfrak{B}}(b/A)$, and thus, $\mathfrak{B} \models \forall \bar{z}(\phi(x, \bar{z}) \leftrightarrow \phi(y, \bar{z}))\,[a, b]$. Therefore, $\mathrm{atp}^-_{\mathfrak{B}}(b/B) = \mathrm{atp}^-_{\mathfrak{B}}(a/B)$. □

Lemma 2.17. *Let $\mathfrak{A}$ be an L-structure. If $\mathfrak{A}$ is L^--$|A|^+$-saturated, then for any $\mathfrak{B}$ such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and any $b, b' \in B$, the following are equivalent:*

i) $\mathrm{atp}^-_{\mathfrak{B}}(b/A) = \mathrm{atp}^-_{\mathfrak{B}}(b'/A)$.

ii) $\mathrm{tp}^-_{\mathfrak{B}}(b/A) = \mathrm{tp}^-_{\mathfrak{B}}(b'/A)$.

iii) $\mathrm{tp}^-_{\mathfrak{B}}(b/B) = \mathrm{tp}^-_{\mathfrak{B}}(b'/B)$.

Proof. iii)$\Rightarrow$ i) is clear. i)$\Rightarrow$ ii) Suppose that $\mathrm{atp}^-_{\mathfrak{B}}(b/A) = \mathrm{atp}^-_{\mathfrak{B}}(b'/A)$. Let $p = \mathrm{tp}^-_{\mathfrak{B}}(b/A)$ and $p' = \mathrm{tp}^-_{\mathfrak{B}}(b'/A)$. Since $\mathfrak{A}$ is L^--$|A|^+$-saturated and $\mathfrak{A} \preceq^- \mathfrak{B}$, there are $a, a' \in A$ such that a is a realization of p and a' is a realization of p'. Therefore,

$$\mathrm{atp}^-_{\mathfrak{B}}(a/A) = \mathrm{atp}^-_{\mathfrak{B}}(b/A) = \mathrm{atp}^-_{\mathfrak{B}}(b'/A) = \mathrm{atp}^-_{\mathfrak{B}}(a'/A),$$

and since $\mathfrak{A} \preceq^- \mathfrak{B}$, $\mathrm{atp}^-_{\mathfrak{A}}(a/A) = \mathrm{atp}^-_{\mathfrak{A}}(a'/A)$. Thus, $\mathrm{tp}^-_{\mathfrak{A}}(a/A) = \mathrm{tp}^-_{\mathfrak{A}}(a'/A)$ and again since $\mathfrak{A} \preceq^- \mathfrak{B}$, $\mathrm{tp}^-_{\mathfrak{B}}(a/A) = \mathrm{tp}^-_{\mathfrak{B}}(a'/A)$. Consequently,

$$\mathrm{tp}^-_{\mathfrak{B}}(b/A) = \mathrm{tp}^-_{\mathfrak{B}}(a/A) = \mathrm{tp}^-_{\mathfrak{B}}(a'/A) = \mathrm{tp}^-_{\mathfrak{B}}(b'/A).$$

ii) $\Rightarrow$ iii) Suppose that $p = \mathrm{tp}^-_{\mathfrak{B}}(b/A) = \mathrm{tp}^-_{\mathfrak{B}}(b'/A)$. Since $\mathfrak{A}$ is L^--$|A|^+$-saturated and $\mathfrak{A} \preceq^- \mathfrak{B}$, there is $a \in A$ such that a is a realization of p. Therefore, $\mathrm{tp}^-_{\mathfrak{B}}(b/A) = \mathrm{tp}^-_{\mathfrak{B}}(a/A) = \mathrm{tp}^-_{\mathfrak{B}}(b'/A)$. Then, by Lemma 2.16, $\mathrm{atp}^-_{\mathfrak{B}}(b/B) = \mathrm{atp}^-_{\mathfrak{B}}(a/B) = \mathrm{atp}^-_{\mathfrak{B}}(b'/B)$ and thus $\mathrm{tp}^-_{\mathfrak{B}}(b/B) = \mathrm{tp}^-_{\mathfrak{B}}(b'/B)$. □

Proposition 2.18. *Let $\mathfrak{A}$ be an L-structure. The following are equivalent:*

i) *$\mathfrak{A}$ is L^--$|A|^+$-saturated.*

ii) *For any $\mathfrak{B}$ such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and any $b \in B$ there is $a \in A$ such that*

$$\mathrm{atp}^-_{\mathfrak{B}}(b/B) = \mathrm{atp}^-_{\mathfrak{B}}(a/B).$$

iii) *For any $\mathfrak{B}$ such that $\mathfrak{A} \preceq^- \mathfrak{B}$, $\mathfrak{B} \sim \mathfrak{A}$.*

iv) *For any $\mathfrak{B}$ such that $\mathfrak{A} \preceq \mathfrak{B}$, $\mathfrak{B} \sim \mathfrak{A}$.*

Proof. iii) $\Rightarrow$ iv) is clear. i) $\Rightarrow$ ii) Let $\mathfrak{B}$ be such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and $b \in B$, consider $p = \mathrm{tp}^-_{\mathfrak{B}}(b/A)$. By i), since $\mathfrak{A} \preceq^- \mathfrak{B}$ there is an element $a \in A$ such that a is a realization of p. Therefore, by Lemma 2.16, $\mathrm{atp}^-_{\mathfrak{B}}(b/B) =$

$\text{atp}^-_{\mathfrak{B}}(a/B)$. ii) $\Rightarrow$ iii) Let $\mathfrak{B}$ be such that $\mathfrak{A} \preceq^- \mathfrak{B}$. By ii), for any $b \in B - A$, we can choose $a_b \in A$ such that $\text{atp}^-_{\mathfrak{B}}(b/B) = \text{atp}^-_{\mathfrak{B}}(a_b/B)$. For any $b \in A$, let $a_b = b$. Then $(a_b : b \in B)$ and $(b : b \in B)$ are enumerations of A and B respectively, such that $(\mathfrak{A}, a_b)_{b\in B} \equiv^-_0 (\mathfrak{B}, b)_{b\in B}$. By Proposition 1.6, $\mathfrak{B} \sim \mathfrak{A}$. iv) $\Rightarrow$ i) Let $\mathfrak{B}$ be an L-structure such that $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathfrak{B}$ is $|A|^+$-saturated. Then $\mathfrak{B}$ is L^--$|A|^+$-saturated. By iv), $\mathfrak{B} \sim \mathfrak{A}$ and by Proposition 2.8, $\mathfrak{A}$ is L^--$|A|^+$-saturated. □

Now we give another characterization of L^--$|A|^+$-saturated models:

Proposition 2.19. *Let $\mathfrak{A}$ be an L-structure. The following are equivalent:*

i) *$\mathfrak{A}$ is L^--$|A|^+$-saturated.*

ii) *$\mathfrak{A}$ is L^--ω-saturated and for any $\mathfrak{B}$ such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and any $b \in B$ there is a finite $E \subseteq A$ such that for any $c \in B$,*

$$\text{tp}^-_{\mathfrak{B}}(b/E) = \text{tp}^-_{\mathfrak{B}}(c/E) \quad \textit{iff} \quad \text{atp}^-_{\mathfrak{B}}(b/B) = \text{atp}^-_{\mathfrak{B}}(c/B).$$

iii) *There is an infinite cardinal $\kappa \leq |A|$ such that $\mathfrak{A}$ is L^--κ-saturated and for any $\mathfrak{B}$ such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and any $b \in B$ there is $E \subseteq A$ with $|E| < \kappa$ such that for any $c \in B$,*

$$\text{tp}^-_{\mathfrak{B}}(b/E) = \text{tp}^-_{\mathfrak{B}}(c/E) \quad \textit{iff} \quad \text{atp}^-_{\mathfrak{B}}(b/B) = \text{atp}^-_{\mathfrak{B}}(c/B).$$

Proof. ii) $\Rightarrow$ iii) is clear. i) $\Rightarrow$ ii) If $\mathfrak{A}$ is L^--$|A|^+$-saturated, then $\mathfrak{A}$ is L^--ω-saturated. Let $\mathfrak{B}$ be such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and $b \in B$. Since $\mathfrak{A}$ is L^--$|A|^+$-saturated, by Proposition 2.18, there is $a \in A$ such that $\text{atp}^-_{\mathfrak{B}}(b/B) = \text{atp}^-_{\mathfrak{B}}(a/B)$ and therefore, $\text{tp}^-_{\mathfrak{B}}(b/B) = \text{tp}^-_{\mathfrak{B}}(a/B)$. Let $E = \{a\}$ and suppose that $c \in B$. If $\text{tp}^-_{\mathfrak{B}}(b/E) = \text{tp}^-_{\mathfrak{B}}(c/E)$, then for any equality-free atomic formula $\phi(x, \bar{z}) \in L$, since $\forall \bar{z}(\phi(a, \bar{z}) \leftrightarrow \phi(y, \bar{z})) \in \text{tp}^-_{\mathfrak{B}}(b/E)$ we have that $\forall \bar{z}(\phi(a, \bar{z}) \leftrightarrow \phi(y, \bar{z})) \in \text{tp}^-_{\mathfrak{B}}(c/E)$, and then, $\mathfrak{B} \models \forall \bar{z}(\phi(x, \bar{z}) \leftrightarrow \phi(y, \bar{z}))\,[a, c]$. Consequently,

$$\text{atp}^-_{\mathfrak{B}}(c/B) = \text{atp}^-_{\mathfrak{B}}(a/B) = \text{atp}^-_{\mathfrak{B}}(b/B).$$

Conversely, if $\text{atp}^-_{\mathfrak{B}}(b/B) = \text{atp}^-_{\mathfrak{B}}(c/B)$, then $\text{tp}^-_{\mathfrak{B}}(b/B) = \text{tp}^-_{\mathfrak{B}}(c/B)$ and thus $\text{tp}^-_{\mathfrak{B}}(b/E) = \text{tp}^-_{\mathfrak{B}}(c/E)$.

iii) $\Rightarrow$ i) Let p be an L^--1-type over $D \subseteq A$ in $\mathfrak{A}$. Assume that p is L^--complete. By Lemma 2.5 there is $\mathfrak{B}$ such that $\mathfrak{A} \preceq^- \mathfrak{B}$ and $b \in B$ such that $p = \mathrm{tp}^{-}_{\mathfrak{B}}(b/D)$. By iii), there is $E \subseteq A$ with $|E| < \kappa \leq |A|$ such that for any $c \in B$,

$$\mathrm{tp}^{-}_{\mathfrak{B}}(b/E) = \mathrm{tp}^{-}_{\mathfrak{B}}(c/E) \quad \text{iff} \quad \mathrm{atp}^{-}_{\mathfrak{B}}(b/B) = \mathrm{atp}^{-}_{\mathfrak{B}}(c/B).$$

Let $q = \mathrm{tp}^{-}_{\mathfrak{B}}(b/E)$. Since $\mathfrak{A}$ is L^{-}-κ-saturated and $\mathfrak{A} \preceq^{-} \mathfrak{B}$, there is an element $a \in A$ such that a is a realization of q. Therefore, $\mathrm{tp}^{-}_{\mathfrak{B}}(b/E) = \mathrm{tp}^{-}_{\mathfrak{B}}(a/E)$, and then, by assumption, $\mathrm{atp}^{-}_{\mathfrak{B}}(b/B) = \mathrm{atp}^{-}_{\mathfrak{B}}(a/B)$, consequently, $\mathrm{tp}^{-}_{\mathfrak{B}}(b/D) = \mathrm{tp}^{-}_{\mathfrak{B}}(a/D)$ and a is a realization of p. Then, we can conclude that $\mathfrak{A}$ is L^{-}-$|A|^{+}$-saturated. □

Corollary 2.20. *Let L be a similarity type such that the arity of all the symbols in L is ≤ 1. Then, any L^{-}-ω-saturated structure $\mathfrak{A}$ is L^{-}-$|A|^{+}$-saturated.*

Proof. Observe that, in case that the arity of all the symbols in L is ≤ 1, for any L-structure $\mathfrak{B}$, the following holds: $\mathrm{atp}^{-}_{\mathfrak{B}}(b/\emptyset) = \mathrm{atp}^{-}_{\mathfrak{B}}(c/\emptyset)$ iff $\mathrm{atp}^{-}_{\mathfrak{B}}(b/B) = \mathrm{atp}^{-}_{\mathfrak{B}}(c/B)$, for any $b, c \in B$. Consequently, by Proposition 2.19, for any L-structure $\mathfrak{A}$, if $\mathfrak{A}$ is L^{-}-ω-saturated, then $\mathfrak{A}$ is L^{-}-$|A|^{+}$-saturated. □

We end this section with some examples. First we see another example of an structure $\mathfrak{A}$ that is L^{-}-$|A|^{+}$-saturated, using Proposition 2.19.

Example 2.21. Let $L = \{R_n : n \in \omega\}$, where for any $n \in \omega$, R_n is a binary relation symbol. Consider the L-structure $\mathfrak{A} = ({}^{\omega}2, R^{\mathfrak{A}}_n)_{n \in \omega}$, where for any $n \in \omega$, $R^{\mathfrak{A}}_n$ is the equivalence relation defined by: $\langle f, g \rangle \in R^{\mathfrak{A}}_n$ iff $f \upharpoonright n = g \upharpoonright n$, for any $f, g \in {}^{\omega}2$.

We see that $\mathfrak{A}$ is L^{-}-$|A|^{+}$-saturated. By Proposition 2.18, it is enough to show that for any L-structure $\mathfrak{B}$ such that $\mathfrak{A} \preceq^{-} \mathfrak{B}$ and any $b \in B$, there is $g \in {}^{\omega}2$ such that $\mathrm{atp}^{-}_{\mathfrak{B}}(b/B) = \mathrm{atp}^{-}_{\mathfrak{B}}(g/B)$. Suppose that $\mathfrak{A} \preceq^{-} \mathfrak{B}$ and $b \in B$. Since for any $n \in \omega$ there are exactly 2^n equivalence classes in the partition by the relation $R^{\mathfrak{B}}_n$, for any $n \in \omega$ there is $f_n \in {}^{\omega}2$ such that $\langle f_n, b \rangle \in R^{\mathfrak{B}}_n$. Then, consider the function $g \in {}^{\omega}2$ defined as follows: for any $n \in \omega$, $g(n) = f_n(n)$. Clearly, for any $n \in \omega$, $\langle g, b \rangle \in R^{\mathfrak{B}}_n$ and therefore, $\mathrm{atp}^{-}_{\mathfrak{B}}(b/B) = \mathrm{atp}^{-}_{\mathfrak{B}}(g/B)$.

Now we exhibit a theory without this kind of structures.

Example 2.22. Let L be as in Example 2.21 and $\mathfrak{B}$ be the L-structure $({}^{\omega}\omega, R^{\mathfrak{B}}_n)_{n \in \omega}$, where for any $n \in \omega$, $R^{\mathfrak{B}}_n$ is the equivalence relation defined by: $\langle f, g \rangle \in R^{\mathfrak{B}}_n$ iff $f \upharpoonright n = g \upharpoonright n$, for any $f, g \in {}^{\omega}\omega$.

There is no model $\mathfrak{A}$ of the Th($\mathfrak{B}$) which is L^--$|A|^+$-saturated. Assume that $\mathfrak{A}$ is a model of Th($\mathfrak{B}$) and let X be the set of all equivalence classes of $R_1^{\mathfrak{A}}$. For any $x \in X$ we choose a representative $a_x \in A$. The following L^--1-type over $\{a_x : x \in X\}$ in $\mathfrak{A}$, $p = \{\neg R_1 y a_x : x \in X\}$, is not realized in $\mathfrak{A}$.

¿From the universality properties of L^--saturated models we can deduce that if $\mathfrak{A}$ is a L^--$|A|^+$-saturated model of T, then $\mathfrak{A}^*$ is the greatest reduced model of $\text{Th}^-(\mathfrak{A})$.

Proposition 2.23. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. Then the following are equivalent:*

i) *There is an enumeration of A, $\overline{a} = (a_i : i \in I)$, and a sequence of elements of B, $\overline{b} = (b_i : i \in I)$, such that*

$$(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b}).$$

ii) $\mathfrak{A}^* -^- \mathfrak{B}^*$.

Proof. See [11], Proposition 2.8. □

Proposition 2.24. *Let $\mathfrak{A}$ be an L-structure and κ a cardinal. If $\mathfrak{A}$ is L^--κ-saturated, then for any $\mathfrak{B} \models \text{Th}^-(\mathfrak{A})$ with $|B| \leq \kappa$, $\mathfrak{B}^* -^- \mathfrak{A}^*$.*

Proof. Assume that $\mathfrak{A}$ is L^--κ-saturated and let $\mathfrak{B}$ be an L-structure such that $\mathfrak{B} \models \text{Th}^-(\mathfrak{A})$ and $|B| = \lambda \leq \kappa$. Let $\overline{b} = (b_\alpha : \alpha \in \lambda)$ be an enumeration of B without repetitions and $p = \text{tp}^-_{\mathfrak{B}}(\overline{b}/\emptyset)$. Since $\mathfrak{B} \models \text{Th}^-(\mathfrak{A})$, p is an L^--λ-type over $\emptyset$ in $\mathfrak{A}$. Therefore, since $\lambda \leq \kappa$ and $\mathfrak{A}$ is L^--κ-saturated, p is realized in $\mathfrak{A}$. Let $\overline{a} = (a_\alpha : \alpha \in \lambda)$ be a realization of p in $\mathfrak{A}$. Then, $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{B}, \overline{b})$. Therefore, by Proposition 2.4, $\mathfrak{B}^* -^- \mathfrak{A}^*$. □

Corollary 2.25. *Let $\mathfrak{A}$ be an L^--$|A|^+$-saturated model, then for any $\mathfrak{B} \models \text{Th}^-(\mathfrak{A})$, $\mathfrak{B}^* -^- \mathfrak{A}^*$.*

Proof. By Proposition 2.24. □

In [17] Morley proved that the Hanf number for the class of countable stable theories was $\beth_{\omega_1}$. This bound allows us to obtain the following existence result for models which have this phenomenon of supersaturation.

Proposition 2.26. *Let T be a stable countable theory. If $\mathfrak{A}$ is a reduced L^--$|A|^+$-saturated model of T, then $|A| < \beth_{\omega_1}$.*

Proof. By Corollary 2.25, for any $\mathfrak{B} \models T$, $\mathfrak{B}^* \text{-}^- \mathfrak{A}$. □

Let us consider the class of stable countable theories axiomatized by a set of equality-free sentences. From next proposition follows that the Hanf number for the notion of reduced structure for this class it is the least cardinal κ such that any supersaturated reduced model of a theory of the class is of power less than κ.

Proposition 2.27. *If T is a theory axiomatized by a set of equality-free sentences, then, there is a reduced model $\mathfrak{A}$ of T which is L^--$|A|^+$-saturated of power λ if and only if there is a reduced model of power λ and any model of T of power bigger than λ is not reduced.*

Proof. The direction $\Rightarrow$) is clear by Corollary 2.25. $\Leftarrow$) Let $\mathfrak{A}$ be model of T γ-saturated, for some $\gamma > \lambda$. Then $\mathfrak{A}^*$ is a reduced model of T, L^--γ-saturated and, by assumption, of power $\leq \lambda$. Therefore $\mathfrak{A}^*$ is L^--$|A|^+$-saturated and by Corollary 2.25, for any reduced model $\mathfrak{B}$ of T, $\mathfrak{B} \text{-}^- \mathfrak{A}^*$. Since we have assumed that there is a reduced model $\mathfrak{B}$ of power λ, $\mathfrak{A}^*$ must be of power λ. □

In the previous propositions several improvements can be obtained using the results of Shelah and Hrushovski in [16], for superstable theories. Finally we introduce the notion of strong L^--homogeneity and compare this notion with the usual notion of strong homogeneity.

Definition 2.28. Given an L-structure $\mathfrak{A}$ and a cardinal κ, we say that $\mathfrak{A}$ is *L^--κ-homogeneous* iff for any two sequences $\overline{a} = (a_i : i \in I)$ and $\overline{a}' = (a'_i : i \in I)$ of elements of A such that $|I| < \kappa$ and $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{A}, \overline{a}')$, it happens that for any $d \in A$ there is $d' \in A$ such that $(\mathfrak{A}, \overline{a}, d) \equiv^- (\mathfrak{A}, \overline{a}', d')$. We say that $\mathfrak{A}$ is *L^--homogeneous* if it is L^--$|A|$-homogeneous.

Definition 2.29. Given an L-structure $\mathfrak{A}$ and a cardinal κ, we say that $\mathfrak{A}$ is *strong-ly L^--κ-homogeneous* iff for any two sequences $\overline{a} = (a_i : i \in I)$ and $\overline{a}' = (a'_i : i \in I)$ of elements of A such that $|I| < \kappa$ and $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{A}, \overline{a}')$, there are enumerations $\overline{d} = (d_j : j \in J)$ and $\overline{d}' = (d'_j : j \in J)$ of A such that $(\mathfrak{A}, \overline{d}) \equiv^- (\mathfrak{A}, \overline{d}')$ and $\overline{a} \subseteq \overline{d}$ and $\overline{a}' \subseteq \overline{d}'$. We say that $\mathfrak{A}$ is *strongly L^--homogeneous* if it is strongly L^--$|A|$-homogeneous.

By usual arguments we obtain the following fact:

Remark 2.30. Let $\mathfrak{A}$ be an L-structure. $\mathfrak{A}$ is L^--homogeneous iff $\mathfrak{A}$ is strongly L^--homogeneous.

Proposition 2.31. *Let $\mathfrak{A}$ be an L-structure. If $\mathfrak{A}$ is L^--saturated, then $\mathfrak{A}$ is strongly L^--homogeneous.*

Proof. Suppose that $\overline{a} = (a_i : i \in I)$ and $\overline{a}' = (a'_i : i \in I)$ are sequences of elements of A such that $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{A}, \overline{a}')$ and $|I| < |A|$. Let $r = \{\langle a_i, a'_i\rangle : i \in I\}$. Given an element $d \in A$, consider the type $p = \mathrm{tp}^-_{\mathfrak{A}}(d/\mathrm{dom}(r))$. Since $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{A}, \overline{a}')$, by Remark 2.1, p^r is an L^--1-type over $\mathrm{rg}(r)$ in $\mathfrak{A}$. Since $\mathfrak{A}$ is L^--saturated, there is a realization $d' \in A$ of p^r. Then, $(\mathfrak{A}, \overline{a}, d) \equiv^- (\mathfrak{A}, \overline{a}', d')$. Thus, $\mathfrak{A}$ is L^--homogeneous and by the previous remark, $\mathfrak{A}$ is strongly L^--homogeneous. □

Proposition 2.32. *Let $\mathfrak{A}$ be a reduced L-structure. If $\mathfrak{A}$ is strongly L^--homoge-neous, then $\mathfrak{A}$ is strongly homogeneous.*

Proof. Suppose that $\mathfrak{A}$ is reduced and strongly L^--homogeneous and $\overline{a} = (a_i : i \in I)$ and $\overline{a}' = (a'_i : i \in I)$ are two sequences of elements of A such that $|I| < |A|$ and $(\mathfrak{A}, \overline{a}) \equiv (\mathfrak{A}, \overline{a}')$. Then, $(\mathfrak{A}, \overline{a}) \equiv^- (\mathfrak{A}, \overline{a}')$, and since $\mathfrak{A}$ is strongly L^--homogeneous, there are enumerations $\overline{d} = (d_j : j \in J)$ and $\overline{d}' = (d'_j : j \in J)$ of A, such that $(\mathfrak{A}, \overline{d}) \equiv^- (\mathfrak{A}, \overline{d}')$ and $\overline{a} \subseteq \overline{d}$ and $\overline{a}' \subseteq \overline{d}'$. But since $\mathfrak{A}$ is reduced, by Proposition 1.6, there is an automorphism $f : \mathfrak{A} \to \mathfrak{A}$ such that, for any $j \in J$, $f(d_j) = d'_j$. Therefore, $(\mathfrak{A}, \overline{d}) \equiv (\mathfrak{A}, \overline{d}')$. Consequently, $\mathfrak{A}$ is strongly homogeneous. □

Observe that, in Proposition 2.32 we can not delete the restriction that $\mathfrak{A}$ is reduced.

Example 2.33. Let $L = \{E\}$, where E is a binary relation symbol and $\mathfrak{A} = (\omega_1 + \omega, E^{\mathfrak{A}})$, where $E^{\mathfrak{A}}$ is the equivalence relation defined by: $\langle \alpha, \beta\rangle \in E^{\mathfrak{A}}$ iff either ($\alpha \leq \omega_1$ and $\beta \leq \omega_1$) or ($\alpha > \omega_1$ and $\beta > \omega_1$), for any $\alpha, \beta \in \omega_1 + \omega$.

It is easy to check that $\mathfrak{A}$ is non-reduced and that $\mathfrak{A}^*$ is finite. Then, $\mathfrak{A}^*$ is L^--saturated and consequently, $\mathfrak{A}$ is L^--saturated. But it is not strongly homogeneous: take $\alpha \leq \omega_1$ and $\beta > \omega_1$ and I the set of all finite partial isomorphisms p such that $p(\alpha) = \beta$. It is easy to see that $I : (\mathfrak{A}, \alpha) \cong_p (\mathfrak{A}, \beta)$. Therefore, $(\mathfrak{A}, \alpha) \equiv (\mathfrak{A}, \beta)$. But, since the equivalence classes of α and β are of different power, there is no automorphism $h : \mathfrak{A} \to \mathfrak{A}$ such that $h(\alpha) = \beta$.

There are counterexamples that show that the converse of Proposition 2.32 is not true.

Example 2.34. Take $L = \{P, E, f\}$, where P is a unary relation symbol, E a binary relation symbol and f a unary function symbol. Let $\mathfrak{A} = (A, P^{\mathfrak{A}}, E^{\mathfrak{A}}, f^{\mathfrak{A}})$, where $A = M \cup M' \cup \{b\}$, $M = \{a_n : n \in \omega\}$, $M' = \{a'_n : n \in \omega\}$, $b \notin M \cup M'$ and $M \cap M' = \emptyset$. Let $P^{\mathfrak{A}} = \{a_0, a'_0\}$ and for any $n \in \omega$, $f^{\mathfrak{A}}(a_{n+1}) = a_n$, $f^{\mathfrak{A}}(a_0) = a_0$, $f^{\mathfrak{A}}(a'_{n+1}) = a'_n$, $f^{\mathfrak{A}}(a'_0) = a'_0$ and $f^{\mathfrak{A}}(b) = b$. Finally, let

$$E^{\mathfrak{A}} = \big[(M' \cup \{b\}) \times (M' \cup \{b\})\big] \cup [M \times M] .$$

Clearly $\mathfrak{A}$ is reduced. Observe that $\mathfrak{A}$ is strongly homogeneous: suppose that $\overline{a} = (a_i : i \in I)$ and $\overline{a}' = (a'_i : i \in I)$ are sequences of elements of A such that $(\mathfrak{A}, \overline{a}) \equiv (\mathfrak{A}, \overline{a}')$. It is easy to check that, in this case, for any $i \in I$, $a_i = a'_i$, therefore the identity is the desired automorphism. But using back-and forth systems for equality-free logic it can be shown that $\mathfrak{A}$ is not strongly L^--homogeneous, for the details see ([10]).

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