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New Lorentz spaces for the restricted weak-type Hardy's inequalities

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Abstract

Associated to the class of restricted-weak type weights for the Hardy operator R_p , we find a new class of Lorentz spaces for which the normability property holds. This result is analogous to the characterization given by Sawyer for the classical Lorentz spaces. We also show that these new spaces are very natural to study the existence of equivalent norms described in terms of the maximal function.

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1 Introduction

During the last decade, the characterization of the normability of all sorts of weighted Lorentz spaces in terms of either weak-type or strong inequalities for the Hardy operator, has been completely settled down (see [AM], [Sa], [CGS], and [So]). One of the main results of this paper is to consider the class of weights R_p introduced by Neugebauer ([Ne]), which characterizes the restricted weak-type boundedness of the Hardy operator, and show that there exists a new satisfactory Lorentz space, in the sense that the normability condition is described in terms of R_p (thus completing the picture of equivalences between Banach Lorentz-type spaces and weighted inequalities). We will also prove that these new spaces are useful to consider the problem of finding equivalent norms in the classical Lorentz spaces depending upon the maximal function (the case $p = 1$ was an open problem, which we now solve in its full generality).

In section 2 we introduce the spaces $\Gamma_\alpha^q(w)$, study the embeddings with respect to both $\Lambda^q(w)$ and $\Gamma^q(w)$, and characterize the existence of an equivalent norm. In section 3 we prove that all Banach Lorentz spaces $\Lambda^1(w)$ admit a norm depending on the maximal function, and give an explicit formula for it (see Theorem 3.1). This study leads us to investigate how the space $\Lambda^1(w)$ fits among the range of Banach spaces $\Gamma^{1,p}(w)$, which arise in a natural way in the theory. As a consequence we find that, on this scale, there are not intermediate weighted inequalities for the Hardy operator, that is, as soon as one gets a better estimate than the weak-type inequality, we obtain the best possible estimate, namely the strong-type (1,1) inequality, (see Theorem 3.4). We finally give some examples and applications to show that we can have all sort of different situations for the embeddings $\Gamma^1(w) \subset \Lambda^1(w) \subset \Gamma^{1,\infty}(w)$.

In what follows, we will use the notation decreasing (increasing) with the meaning nonincreasing (resp. nondecreasing). We will say that a function is positive whenever it is nonnegative. A weight is a positive measurable function, locally integrable on \mathbb{R}^+ . Given a weight w , we denote by $W(t) = \int_0^t w(s) ds$. f^* denotes the decreasing rearrangement of the function f . $L_{\text{dec}}^p(w)$ is the cone of positive and decreasing functions in $L^p(w)$ (similarly for $L_{\text{dec}}^{p,\infty}(w)$). Constants such as C may have different values from one occurrence to the next, but they will always be irrelevant for the arguments used. Two positive quantities A and B , are said to be equivalent ($A \approx B$) if there exists a constant $C > 1$ (independent of the essential parameters defining A and B) such that $C^{-1}A \leq B \leq CA$. For other standard notations we refer the reader to [BS].

2 The spaces $\Gamma_\alpha^q(w)$

We begin by introducing the definitions of the spaces we are going to study, and give some well-known results which will be useful in our proofs.

Definition 2.1 Let $0 < p < \infty$ and w be a weight. Then we define the weighted Lorentz space

$$\Lambda^p(w) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^+; \|f\|_{\Lambda^p(w)} < \infty\},$$

where,

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty (f^*(t))^p w(t) dt \right)^{1/p}. \quad (1)$$

Classical examples are obtained when one considers power weights. Thus, if $w(t) = t^{(p/q)-1}$ then $\Lambda^p(w) = L^{q,p}$, which is the Lebesgue space L^p if $p = q$. In general (1) does not define a norm. In fact, Lorentz proved ([Lo]) that (1) is a norm, if and only if $p \geq 1$ and w is a decreasing function. This result was later on improved by Sawyer ([Sa]) by giving a characterization of the normability of $\Lambda^p(w)$, $p > 1$. In order to formulate this result, we need first to introduce some important operators.

Definition 2.2 The Hardy operator is defined as

$$Sf(t) = \frac{1}{t} \int_0^t f(s) ds, \quad t > 0. \quad (2)$$

The maximal function of f is

$$f^{**}(t) = Sf^*(t). \quad (3)$$

The main result about the boundedness of the Hardy operator that we need, is the following theorem due to Ariño and Muckenhoupt:

Theorem 2.3 For $0 < p < \infty$, the following conditions are equivalent:

- (i) $S : L_{\text{dec}}^p(w) \longrightarrow L^p(w)$,
- (ii) $w \in B_p$; i.e., for all $r > 0$,

$$\int_r^\infty \frac{w(s)}{s^p} ds \leq \frac{C}{r^p} \int_0^r w(s) ds. \quad (4)$$

There are also some other Lorentz-type spaces that we are going to consider:

Definition 2.4 Let $0 < p < \infty$, $0 < q \leq \infty$, and w be a weight. Then we define:

(i) The weak-type weighted Lorentz space,

$$\Lambda^{p,\infty}(w) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^+; \|f\|_{\Lambda^{p,\infty}(w)} < \infty\},$$

where,

$$\|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} (f^{**}(t)W^{1/p}(t)). \quad (5)$$

(ii) The Gamma space,

$$\Gamma^{p,q}(w) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^+; \|f\|_{\Gamma^{p,q}(w)} < \infty\},$$

where,

$$\|f\|_{\Gamma^{p,q}(w)} = \left(\int_0^\infty (f^{**}(t))^q (W(t))^{(q/p)-1} w(t) dt \right)^{1/q}, \quad (6)$$

if $q < \infty$, and if $q = \infty$,

$$\|f\|_{\Gamma^{p,\infty}(w)} = \sup_{t>0} (f^{**}(t)W^{1/p}(t)). \quad (7)$$

In case $p = q$ we simply write $\Gamma^p(w)$.

The importance of (3) is the subadditivity property $(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t)$. This immediately implies that (6) defines a (complete) norm if $p \geq 1$ (in (7) this is true for $p > 0$), and one always has that $\Gamma^p(w) \subset \Lambda^p(w) \subset \Lambda^{p,\infty}(w)$ (see [CPSS] for more information on this kind of embeddings). We can now write Sawyer's theorem:

Theorem 2.5 *If $1 < p$, then the following conditions are equivalent:*

- (i) $\Lambda^p(w)$ is a Banach space.
- (ii) $w \in B_p$.
- (iii) $\Lambda^p(w) = \Gamma^p(w)$.

Thus, as a consequence we have that if $\Lambda^p(w)$ is normable, then the equivalent norm is always described in terms of the maximal function (in fact $\|f\|_{\Lambda^p(w)} \approx \|f\|_{\Gamma^p(w)}$). Similarly, the normability of the weak-type spaces $\Lambda^{p,\infty}(w)$ was also characterized in [So]:

Theorem 2.6 *If $0 < p$, then the following conditions are equivalent:*

- (i) $\Lambda^{p,\infty}(w)$ is a Banach space.
- (ii) $w \in B_p$.
- (iii) $\Lambda^{p,\infty}(w) = \Gamma^{p,\infty}(w)$.

Thus, as in Theorem 2.5, we also have in this case that the equivalent norm in $\Lambda^{p,\infty}(w)$ is given in terms of the maximal function. In view of these results, we investigate what happens in the only case that is left, namely $\Lambda^1(w)$. For this endpoint space, we recall the normability characterization proved in [CGS]:

Theorem 2.7 *The following conditions are equivalent:*

- (i) $\Lambda^1(w)$ is a Banach space.
- (ii) $\Lambda^1(w) \subset \Gamma^{1,\infty}(w)$.
- (iii) $\frac{1}{t} \int_0^t w(r) dr \leq \frac{C}{s} \int_0^s w(r) dr$, if $0 < s \leq t$.

Remark 2.8 As a corollary of this theorem we have that if $\Lambda^1(w)$ is normable, there exists a decreasing weight \tilde{w} such that $\Lambda^1(w) = \Lambda^1(\tilde{w})$, and hence, in many cases we can assume without loss of generality, that w is already a decreasing weight. Also, it can be shown that (iii) of the previous theorem is equivalent to the weak-type inequality $S : L_{\text{dec}}^1(w) \rightarrow L^{1,\infty}(w)$, and also that the B_p condition is equivalent to $S : L_{\text{dec}}^p(w) \rightarrow L^{p,\infty}(w)$, if $p > 1$. Hence the above theorems can be summarized as follows:

- $\Lambda^p(w)$ is a Banach space, if and only if $p \geq 1$ and $S : L_{\text{dec}}^p(w) \rightarrow L^{p,\infty}(w)$.
- $\Lambda^{p,\infty}(w)$ is a Banach space, if and only if $S : L_{\text{dec}}^p(w) \rightarrow L^p(w)$.

There exists a third class of inequalities, considered in [Ne], which are referred as R_p , where the restricted weak-type boundedness for the Hardy operator and monotone functions (i.e., $w(\{S\chi_{(0,r)} > \lambda\}) \leq CW(r)/\lambda^p$), is characterized by the condition

$$\frac{1}{s^p} \int_0^s w(x) dx \leq \frac{C}{r^p} \int_0^r w(x) dx, \quad (8)$$

for $0 < r < s < \infty$.

We will show that there exist a suitable class of Lorentz spaces for which (8) is the necessary and sufficient condition for the normability.

Definition 2.9 Let $0 < p < \infty$, $0 \leq \alpha \leq p$, and w be a weight. Then we define the space

$$\Gamma_\alpha^p(w) = \{f : \mathbb{R}^n \rightarrow \mathbb{R}^+; \|f\|_{\Gamma_\alpha^p(w)} < \infty\},$$

where,

$$\|f\|_{\Gamma_\alpha^p(w)} = \left(\int_0^\infty (f^*(t))^\alpha (f^{**}(t))^{p-\alpha} w(t) dt \right)^{1/p}. \quad (9)$$

Remark 2.10 Observe that $\Gamma_0^p(w) = \Gamma^p(w)$, $\Gamma_p^p(w) = \Lambda^p(w)$, and $\Gamma^p(w) \subset \Gamma_\alpha^p(w) \subset \Gamma_\beta^p(w) \subset \Lambda^p(w)$, if $0 \leq \alpha \leq \beta \leq p$. It is also easy to prove that if $1 \leq p < \infty$, and $0 < \alpha \leq p$, then $\Gamma^p(w) = \Gamma_\alpha^p(w)$, if and only if $w \in B_p$.

We are going to consider first $\alpha = 1$. For this case we need some previous results which are of independent interest.

Lemma 2.11 *If w is a decreasing weight such that $w(\infty) = 0$, then for every $1 \leq q < \infty$, there exists a weight w_q such that,*

$$\left(\int_0^r w(x) dx \right)^q \approx \int_0^r w_q(x) dx + r^q \int_r^\infty w_q(x) \frac{dx}{x^q}. \quad (10)$$

Proof. We are going to assume first that $w \in \mathcal{C}^1(\mathbb{R}^+)$. Then, taking

$$w_q(r) = -r^q \frac{d}{dx} \left(\frac{W^{q-1}w}{x^{q-1}} \right)(r), \quad (11)$$

which defines a positive function, we have that

$$\int_r^\infty \frac{w_q(s)}{s^q} ds = \frac{W^{q-1}(r)w(r)}{r^{q-1}},$$

and, since $0 \leq \lim_{s \rightarrow 0} sW^{q-1}(s)w(s) \leq \lim_{s \rightarrow 0} W^q(s) = 0$, then

$$\int_0^r w_q(s) ds = -rW^{q-1}(r)w(r) + \left(\int_0^r w(s) ds \right)^q,$$

which proves (10). If w is only a continuous function we define the auxiliary function

$$\Phi(t) = \frac{1}{t} \int_t^{2t} W(s) ds = \int_1^2 W(ts) ds.$$

It is clear that $\Phi \in \mathcal{C}^2(\mathbb{R}^+)$, and we only need to show that Φ is concave, $\Phi \approx W$, and $\lim_{t \rightarrow 0} t\Phi'(t) = \lim_{t \rightarrow \infty} \Phi'(t) = 0$:

Take $a, b \in [0, 1]$, $a + b = 1$, and $s, t > 0$. Then

$$a\Phi(s) + b\Phi(t) = \int_1^2 (aW(sr) + bW(tr)) dr \leq \int_1^2 W(asr + btr) dr = \Phi(as + bt),$$

which shows the concavity. Also,

$$W(t) \leq \Phi(t) \leq W(2t) \leq 2W(t).$$

Now, since

$$\Phi'(t) = \frac{2W(2t) - W(t)}{t} - \frac{1}{t^2} \int_t^{2t} W(s) ds,$$

then $\lim_{t \rightarrow 0} t\Phi'(t) = 0$, and $\lim_{t \rightarrow \infty} \Phi'(t) = 0$, since, by hypothesis,

$$\lim_{t \rightarrow \infty} \frac{1}{t^2} \int_t^{2t} W(s) ds = \lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} \approx \lim_{t \rightarrow \infty} \frac{W(t)}{t} = 0.$$

If we now define the function w_q as in (11), changing W by Φ , we have that w_q satisfies (10). For a general weight w , we repeat the previous argument twice. \square

Proposition 2.12 *Let w be a decreasing weight.*

(a) *If $1 < q < \infty$ and v is a weight for which $\Lambda^1(w) \subset \Gamma^q(v) \subset \Gamma^{1,\infty}(w)$, then*

$$\int_0^\infty (f^{**}(x))^q v(x) dx \approx \int_0^\infty f^*(x) (f^{**}(x))^{q-1} W^{q-1}(x) w(x) dx,$$

that is, $\Gamma^q(v) = \Gamma_1^q(W^{q-1}w)$.

(b) *If $w(\infty) = 0$, $1 < q < \infty$ and w_q is as in (10), then $\Lambda^1(w) \subset \Gamma^q(w_q) \subset \Gamma^{1,\infty}(w)$.*

(c) *If $w(\infty) > 0$ then the embeddings $\Lambda^1(w) \subset \Gamma^q(v) \subset \Gamma^{1,\infty}(w)$ never hold.*

Proof. To prove (a) we use the fact that the embeddings $\Lambda^1(w) \subset \Gamma^q(v) \subset \Gamma^{1,\infty}(w)$ are equivalent to the following condition (see [CPSS]):

$$\left(\int_0^r w(x) dx \right)^q \approx \int_0^r v(x) dx + r^q \int_r^\infty v(x) \frac{dx}{x^q}, \quad (12)$$

for every $r > 0$. Hence,

$$V(r) + r^q \int_r^\infty v(x) \frac{dx}{x^q} \leq C \int_0^r W^{q-1}(x) w(x) dx,$$

which is equivalent to (see Theorem 2.2 in [Ne])

$$\int_0^\infty (f^{**}(x))^q v(x) dx \leq C \int_0^\infty f^*(x) (f^{**}(x))^{q-1} W^{q-1}(x) w(x) dx.$$

The other inequality follows similarly using now Theorem 3.2 in [Ne].

(b) follows immediately from Lemma 2.11 and (12).

To finish, if we assume that the embeddings $\Lambda^1(w) \subset \Gamma^q(v) \subset \Gamma^{1,\infty}(w)$ hold, then by (12) and since $w(\infty) = c > 0$, we have that

$$0 < c^q \leq C \liminf_{t \rightarrow \infty} \frac{1}{t^q} \int_0^t v(x) dx,$$

and hence $\int_t^\infty v(s) ds / s^q = \infty$, which is a contradiction. \square

We will also need to recall the following elementary observation.

Lemma 2.13 *If μ is a non-atomic positive measure in $[0, \infty)$ and g is a positive decreasing function, then the mean value function*

$$\frac{1}{\mu(0, t)} \int_0^t g(x) d\mu(x),$$

is decreasing.

We are now ready to prove our main result.

Theorem 2.14 *Let $1 \leq q < \infty$. The following are equivalent:*

- (a) $\Gamma_1^q(w)$ is a Banach space.
- (b) $\Gamma_1^q(w) \subset \Gamma^{q, \infty}(w)$.
- (c) $w \in R_q$.

Proof. If $q = 1$, this result is Theorem 2.7. Assume now that $1 < q < \infty$. If $\|\cdot\|_{q,1}^*$ is a (rearrangement invariant) norm equivalent to $\|\cdot\|_{\Gamma_1^q(w)}$, and since the fundamental function Φ of $(\Gamma_1^q(w), \|\cdot\|_{q,1}^*)$ satisfies that

$$\Phi(t) \approx W^{1/q}(t),$$

then (see [BS]):

$$(\Gamma_1^q(w), \|\cdot\|_{\Gamma_1^q(w)}) = (\Gamma_1^q(w), \|\cdot\|_{q,1}^*) \subset \Gamma^{1, \infty}(d\Phi) = \Gamma^{q, \infty}(w).$$

Now if (b) holds, then by checking this embedding on characteristic functions we find that

$$\sup_{r < s} \frac{W^{1/q}(s)}{s} \leq C \frac{W^{1/q}(r)}{r},$$

and hence $w \in R_q$.

To finish, let us first observe that $w \in R_q$ is equivalent to the condition $w/W^{1/q'} \in R_1$ and hence $\Lambda^1(w/W^{1/q'})$ is Banach (Theorem 2.7). Thus, there exists a decreasing weight \tilde{w} such that $\Lambda^1(\tilde{w}) = \Lambda^1(w/W^{1/q'})$, and

$$\int_0^r \tilde{w}(x) dx \approx \int_0^r \frac{w(x)}{W^{1/q'}(x)} dx \approx \left(\int_0^r w(x) dx \right)^{1/q},$$

and so,

$$\int_0^r \widetilde{W}^{q-1}(x) \tilde{w}(x) dx \approx \left(\int_0^r \tilde{w}(x) dx \right)^q \approx \int_0^r w(x) dx.$$

Thus, by Hardy's lemma (see [BS]),

$$\int_0^\infty f^*(x) (f^{**}(x))^{q-1} \widetilde{W}^{q-1}(x) \tilde{w}(x) dx \approx \int_0^\infty f^*(x) (f^{**}(x))^{q-1} w(x) dx,$$

which proves that $\Gamma_1^q(w) = \Gamma_1^q(\widetilde{W}^{q-1}\tilde{w})$. Therefore it suffices to study when $\Gamma_1^q(W^{q-1}w)$ is Banach, assuming w is decreasing.

If $w(\infty) = 0$ then using Proposition 2.12 we find a weight v such that $\Gamma_1^q(W^{q-1}w) = \Gamma^q(v)$, which is always a Banach space.

If $w(\infty) = a > 0$ and $w(0) = b < \infty$ then $w \approx 1$ and hence

$$\begin{aligned} & \left(\int_0^\infty f^*(x)(f^{**}(x))^{q-1}W^{q-1}(x)w(x) dx \right)^{1/q} \\ & \approx \left(\int_0^\infty f^*(x) \left(\int_0^x f^* \right)^{q-1} dx \right)^{1/q} \approx \int_0^\infty f^*(x) dx, \end{aligned}$$

and $\Gamma_1^q(W^{q-1}w) = L^1$.

If $w(\infty) = a > 0$ and $w(0) = \infty$, assuming without loss of generality that $a = 1$, we define $\bar{w} = w - 1$, and $\bar{W}(t) = \int_0^t \bar{w}(x) dx$. Then

$$\begin{aligned} & \left(\int_0^\infty f^*(x)(f^{**}(x))^{q-1}W^{q-1}(x)w(x) dx \right)^{1/q} \\ & \approx \left(\int_0^\infty f^*(x)(f^{**}(x))^{q-1}[\bar{W}^{q-1}(x)\bar{w}(x) + x^{q-1}\bar{w}(x) + \bar{W}^{q-1}(x)] dx \right)^{1/q} \\ & + \left(\int_0^\infty f^*(x)(f^{**}(x))^{q-1}x^{q-1} dx \right)^{1/q}, \end{aligned}$$

and hence,

$$\Gamma_1^q(W^{q-1}w) = \Gamma_1^q(u) \cap L^1,$$

where $u(x) = \bar{W}^{q-1}(x)\bar{w}(x) + x^{q-1}\bar{w}(x) + \bar{W}^{q-1}(x)$. Thus we only need to show that $\Gamma_1^q(u)$ is Banach. We first of all observe that the function $u(t)/t^{q-1}$ is decreasing, and hence by Lemma 2.13 we obtain that $t^{-q} \int_0^t u(x) dx$ is also a decreasing function. Thus, if we define $V(r) = (\int_0^r u(x) dx)^{1/q}$, we have that V is a positive increasing function such that $V(r)/r$ is decreasing, and hence there exists a decreasing weight ϕ such that $V(r) \approx \Phi(r) = \int_0^r \phi(x) dx$. Therefore, since

$$\int_0^r V^{q-1}(x)v(x) dx \approx V^q(r) \approx \Phi^q(r) \approx \int_0^r \Phi^{q-1}(x)\phi(x) dx,$$

we conclude that

$$\Gamma_1^q(u) = \Gamma_1^q(V^{q-1}v) = \Gamma_1^q(\Phi^{q-1}\phi),$$

and by the previous argument, if we show that $\phi(\infty) = 0$ we are done. But, since ϕ is decreasing, then

$$\phi(\infty) = \lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} \leq C \lim_{t \rightarrow \infty} \frac{V(t)}{t} = C \lim_{t \rightarrow \infty} \left(\frac{\int_0^t u(x) dx}{t^q} \right)^{1/q}$$

$$\begin{aligned}
&\leq C \lim_{t \rightarrow \infty} \left(\frac{\int_0^t [\overline{W}^{q-1}(x) \overline{w}(x) + x^{q-1} \overline{w}(x) + \overline{W}^{q-1}(x)] dx}{t^q} \right)^{1/q} \\
&\leq C \lim_{t \rightarrow \infty} \left(\frac{\overline{W}^q(t)}{t^q} + \frac{\overline{W}(t)}{t} + \frac{\overline{W}^{q-1}(t)}{t^{q-1}} \right)^{1/q} = 0,
\end{aligned}$$

since $\overline{w}(\infty) = 0$. \square

Remark 2.15 From the proof of the previous result, it is easy to show that if $\Gamma_\alpha^q(w)$ is a Banach space, then we always have that $w \in R_q$.

We consider next the normability characterization for the case $1 < \alpha \leq q$. We observe that now, we obtain the same condition as for the $\Lambda^q(w)$ spaces.

Theorem 2.16 *Let $1 < \alpha \leq q$. Then, $\Gamma_\alpha^q(w)$ is a Banach space, if and only if, $w \in B_q$.*

Proof. If $w \in B_q$ then by Theorem 2.5 we have that $\Lambda^q(w) = \Gamma_\alpha^q(w) = \Gamma^q(w)$. Conversely, if $\Gamma_\alpha^q(w)$ is a Banach space, then it is a rearrangement invariant Banach function space, and hence (see [BS]) $\Gamma_\alpha^q(w) \subset \Gamma^{q,\infty}(w)$. In particular, if for fixed $0 < a < \infty$ and $s > 1$, we choose the function

$$f^*(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq a, \\ \frac{a}{x} & \text{if } a < x \leq sa, \\ x & \text{if } sa < x, \end{cases}$$

then this embedding gives

$$f^{**}(x) W^{1/q}(x) \leq C \left(\int_0^\infty (f^*(t))^\alpha (f^{**}(t))^{q-\alpha} w(t) dt \right)^{1/q},$$

which for $x = as$ is equal to

$$\left(\frac{1 + \log s}{s} \right)^q W(as) \leq C \left(\int_0^a w(x) dx + \int_a^{as} \left(\frac{a}{x} \right)^q \left(1 + \log \frac{x}{a} \right)^{q-\alpha} w(x) dx \right). \quad (13)$$

On the other hand, if we choose the function $f^* = \chi_{[0,s]}$ we easily obtain that $w \in R_q$, and hence (Lemma 7.1 in [Ne]),

$$\int_a^{as} \left(\frac{a}{x} \right)^q w(x) dx \leq C(1 + \log s) \int_0^a w(x) dx.$$

If we combine this with (13), we obtain,

$$\left(\frac{1 + \log s}{s} \right)^q W(as) \leq C(1 + \log s)^{q-\alpha+1} \int_0^a w(x) dx,$$

i.e.,

$$\frac{W(as)}{(as)^q} \leq C(1 + \log s)^{1-\alpha} \frac{W(a)}{a^q}.$$

Finally, taking s big enough so that $C(1 + \log s)^{1-\alpha} < 1$ (which we can do since $\alpha > 1$), and using Lemma 6.3 in [Ne], we obtain that $w \in B_q$. \square

We can also give a characterization for the case $q = 1$ and $0 < \alpha < 1$.

Theorem 2.17 *Let w be a weight and $0 < \alpha < 1$. Then, $\Gamma_\alpha^1(w)$ is a Banach space, if and only if, $w \in R_1$ and $\Gamma_\alpha^1(w) = \Lambda^1(w)$.*

Proof. If $\Gamma_\alpha^1(w)$ is a Banach space, then by Remark 2.15 we have that $w \in R_1$. Also, since the fundamental function of $\Gamma_\alpha^1(w)$ is like W , then $\Lambda^1(w) \subset \Gamma_\alpha^1(w)$, and the equality follows. The converse result is trivial by Theorem 2.7. \square

Remark 2.18 Since $\Gamma_0^1(w)$ is always a Banach space, and $\Gamma_1^1(w)$ is Banach, if and only if $w \in R_1$, it would a priori be natural to expect that the normability characterization for the space $\Gamma_\alpha^1(w)$, $0 < \alpha < 1$, should be weaker than R_1 , contrary to what we have just proved. In fact, it is easy to show that in general $w \in R_1$ does not imply $\Gamma_\alpha^1(w) = \Lambda^1(w)$. For example, with $w = 1$, and f^* a bounded function, $f^*(t) = t^{-1} \log^{-1/\alpha}(t)$ for t big enough, one can prove that $f \in \Lambda^1(w) \setminus \Gamma_\alpha^1(w)$.

3 Equivalent norms and embeddings

We consider in this section the existence of a norm in the space $\Lambda^1(w)$, which depends on the maximal function f^{**} . This question has a positive answer in all the other cases of normability: for the case $\Lambda^q(w)$, $q > 1$, see Theorem 2.5, and for $\Lambda^{q,\infty}(w)$ see Theorem 2.6. Observe that Theorem 2.7 does not give any information on whether there exists such an equivalent norm in $\Lambda^1(w)$ (i.e., if it is a Gamma space). Since for w decreasing, we have that $\Gamma^1(w) \subset \Lambda^1(w) \subset \Gamma^{1,\infty}(w)$, and the endpoint spaces are normable in terms of the maximal function, we ask ourselves when can we get equality on each case:

- (i) $\Lambda^1(w) = \Gamma^1(w)$, if and only if $w \in B_1$ (Theorem 2.3).
- (ii) $\Lambda^1(w) = \Gamma^{1,\infty}(w)$, if and only if, $w \in L^\infty$, and either $w(\infty) > 0$ or, $w(\infty) = 0$ and $w \in L^1$ (see [CPSS]).

It is now clear that there exist weights w which do not satisfy either (i) or (ii). Thus, for the corresponding spaces $\Lambda^1(w)$, the problem of normability with respect to f^{**} was an open question. The following result gives a positive answer in all cases.

Theorem 3.1 *Let w be a decreasing weight. Then:*

- (i) *There exists a weight v such that $\Lambda^1(w) = \Gamma^1(v)$, if and only if $w(\infty) = 0$.*
- (ii) *$\Lambda^1(w) = L^1$, if and only if $w(\infty) > 0$ and $w \in L^\infty$.*
- (iii) *There exists a weight v such that $\Lambda^1(w) = \Gamma^1(v) \cap L^1$, $L^1 \not\subset \Gamma^1(v)$, and $\Gamma^1(v) \not\subset L^1$, if and only if $w(\infty) > 0$ and $w \notin L^\infty$.*

Proof. Let us first prove (i). $\Lambda^1(w) = \Gamma^1(v)$ is equivalent to the condition (see [CPSS]),

$$\frac{1}{r} \int_0^r w(s) ds \approx \frac{1}{r} \int_0^r v(s) ds + \int_r^\infty \frac{v(s)}{s} ds = S(S^*v)(r), \quad (14)$$

where S^* is the adjoint operator of S :

$$S^*f(r) = \int_r^\infty \frac{f(s)}{s} ds.$$

Suppose that $w(\infty) = \rho > 0$. Since $S(S^*v)$ is a decreasing function then, using (14), we obtain the existence of the limit

$$\lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r v(s) ds = \alpha \geq 0.$$

Since $S(S^*v) = S^*(Sv)$, then we find that necessarily $\alpha = 0$. But on the other hand,

$$\rho = \lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r w(s) ds \leq C \lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r v(s) ds,$$

and we get the contradiction that $\alpha \geq \rho/C > 0$. Thus, $\rho = 0$.

The converse is a direct consequence of (14) and Lemma 2.11.

The proof of (ii) is trivial, since the condition says that $w \approx 1$. Let us finally consider (iii). Observe that $\Lambda^1(w) \subset L^1$ is equivalent to $w(\infty) > 0$, and hence, if we define $u(t) = w(t) - w(\infty)$, using (i) we find a weight v such that $\Lambda^1(u) = \Gamma^1(v)$. Thus, $\Lambda^1(w) = \Gamma^1(v) \cap L^1$. Also, if $L^1 \subset \Gamma^1(v)$, then $\Lambda^1(w) = L^1$ which contradicts the fact that $w \notin L^\infty$. Similarly, if $\Gamma^1(v) \subset L^1$ then (see [CPSS])

$$t \leq C \left(V(t) + t \int_t^\infty \frac{v(s)}{s} ds \right),$$

and hence, $v(\infty) > 0$, which implies $\Gamma^1(v) = \{0\}$. To finish, we only need to observe that if $\Lambda^1(w) = \Gamma^1(v) \cap L^1$ then, as we have mentioned before, $w(\infty) > 0$ and, if $w \in L^\infty$ then $w \approx 1$, which says that $\Lambda^1(w) = L^1$, contradicting the hypothesis. \square

Remark 3.2 (i) Since $\|f\|_{L^1} = \sup_{t>0}(tf^{**}(t))$, it is now clear that Theorem 3.1 shows that in $\Lambda^1(w)$ there always exists an equivalent norm depending on the maximal function.

(ii) For the weight $w = \chi_{(0,1)}$, one can show that the function obtained by the argument used in the previous proof is $v(t) = t^{-2}(\log(4t)\chi_{(1/4,1/2)}(t) - \log t\chi_{(1/2,1)}(t))$. In fact, it is easy to prove that in order for (14) to hold, it suffices that v is a bounded function with compact support, which vanishes on a neighborhood of 0.

We consider now other kind of embeddings for $\Lambda^1(w)$.

Proposition 3.3 *Let w be a decreasing weight. Then the following are equivalent:*

- (i) *There exist $1 < q < \infty$ and a weight v such that $\Lambda^1(w) = \Gamma^q(v)$.*
- (ii) *For every $1 < q < \infty$, $\Lambda^1(w) = \Gamma_1^q(w_q)$, where w_q is as in (10).*
- (iii) *$\Lambda^1(w) = \Gamma^{1,\infty}(w)$, $w(\infty) = 0$ and $w \in L^1$.*

Proof. It is easy to see that it suffices to show that (i) implies (iii). By Proposition 2.12 we have that $\Gamma^q(v) = \Gamma_1^q(W^{q-1}w)$, and hence

$$\begin{aligned} \int_0^\infty f^*(x)w(x) dx &\approx \left(\int_0^\infty f^*(x)(f^{**}(x))^{q-1}W^{q-1}(x)w(x) dx \right)^{1/q} \\ &\leq \left(\left(\int_0^\infty f^*(x)w(x) dx \right) \sup_{t>0} (f^{**}(t)W(t))^{q-1} \right)^{1/q}, \end{aligned}$$

that is,

$$\int_0^\infty f^*(x)w(x) dx \leq \sup_{t>0} (f^{**}(t)W(t)),$$

which shows that $\Gamma^{1,\infty}(w) \subset \Lambda^1(w)$. That $w(\infty) = 0$, is also a consequence of Proposition 2.12, and the fact that $w \in L^1$ follows from [CPSS]. \square

Another kind of natural embeddings follow if we observe that on the one hand, if w is decreasing, $\Gamma^1(w) \subset \Lambda^1(w) \subset \Gamma^{1,\infty}(w)$, and on the other $\Gamma^1(w) \subset \Gamma^{1,q}(w) \subset \Gamma^{1,\infty}(w)$, for $1 < q < \infty$ (see Definition 2.4 (ii)). Thus, since $\Gamma^{1,q}(w)$ is a Banach space, we consider the question of whether $\Lambda^1(w) = \Gamma^{1,q}(w)$, for some $1 < q < \infty$. The answer is rather surprising and says that this only happens in the trivial case $w \in B_1$, i.e., when $\Lambda^1(w) = \Gamma^1(w)$.

Theorem 3.4 *Let w be a decreasing weight, and $1 < q < \infty$. Then, the following are equivalent:*

- (i) $\Lambda^1(w) \subset \Gamma^{1,q}(w)$.
- (ii) $S : L^1_{\text{dec}}(w) \longrightarrow L^{1,q}(w)$.
- (iii) $w \in B_1$.

Proof. That (i) and (ii) are equivalent is easy. Also, if $w \in B_1$, then (see Theorem 2.3) $S : L^1_{\text{dec}}(w) \longrightarrow L^1(w) \subset L^{1,q}(w)$, which proves (ii). Therefore it suffices to prove that (ii) implies (iii). A simple calculation shows that if w satisfies (ii), then $W^{q-1}w \in B_q$. Hence there exists $p < q$ such that $W^{q-1}w \in B_p$ (see [AM]), that is,

$$\int_r^\infty \frac{W^q(x)}{x^{p+1}} dx \leq C \frac{W^q(r)}{r^p},$$

(this is the equivalent characterization of B_p proved in [So]). Thus, for $s > 1$,

$$\frac{W^q(sr)}{W^q(r)} \leq Cs^p,$$

and hence, if $s = 2^k$, $k = 1, 2, \dots$,

$$\sup_{r>0} \left(\frac{W(2^k r)}{W(r)} \right)^{1/k} \leq C^{1/k} 2^{p/q},$$

and using Theorems 4.2 and 4.5 in [CGS] we prove that $w \in B_1$. \square

A similar result is obtained if we study the equality between the spaces $\Gamma^{1,q}(w)$ and $\Gamma^q(w_q)$.

Proposition 3.5 *Let w be a decreasing weight. Then, $w \in B_1$, if and only if, $\Gamma^{1,q}(w) = \Gamma^q(w_q)$, for every $1 < q < \infty$.*

Proof. If $w \in B_1$ then $w(\infty) = 0$, and by Proposition 2.12 we can find w_q such that $\Lambda^1(w) \subset \Gamma^q(w_q) \subset \Gamma^{1,\infty}(w)$. Since $\Lambda^1(w) \subset \Gamma^{1,q}(w) \subset \Gamma^{1,\infty}(w)$ also holds (Theorem 3.4), and $\Gamma^{1,q}(w) = \Gamma^q(W^{q-1}w)$, then using again Proposition 2.12 we have that $\Gamma^{1,q}(w) = \Gamma^q(w_q)$. Conversely,

$$\begin{aligned} \int_0^\infty (f^{**}(x))^q W^{q-1}(x) w(x) dx &\approx \int_0^\infty f^*(x) (f^{**}(x))^{q-1} W^{q-1}(x) w(x) dx \\ &\leq \left(\int_0^\infty (f^*(x))^q W^{q-1}(x) w(x) dx \right)^{1/q} \left(\int_0^\infty (f^{**}(x))^q W^{q-1}(x) w(x) dx \right)^{1/q'}, \end{aligned}$$

and hence $\Lambda^q(W^{q-1}w) = \Gamma^{1,q}(w)$, which is a Banach space. Thus $W^{q-1}w \in B_q$, which is equivalent to $w \in B_1$ (see, e.g., [CM]). \square

Remark 3.6 If w is a decreasing weight such that $w(\infty) = 0$, then condition (10) tells us that $\Gamma^{1,q}(w) \subset \Gamma^q(w_q)$ (see [CPSS]). Hence, as a consequence of Proposition 3.5, we have that under those hypotheses on the weight, $w \in R_1 \setminus B_1$, if and only if, $\Gamma^{1,q}(w) \subsetneq \Gamma^q(w_q)$.

To finish, we give some examples which show how the space $\Lambda^1(w)$ fits among the range of Banach spaces $\Gamma^{1,p}(w)$.

Examples 3.7 (i) Contrary to what happens in the case $\Lambda^1(w) \subset \Gamma^{1,q}(w)$ (see Theorem 3.4), the converse embedding can hold true without having that necessarily $\Lambda^1(w) = \Gamma^{1,\infty}(w)$. For example, if $w(\infty) > 0$ and $w(0) = \infty$, then for $1 \leq q < \infty$,

$$\{0\} = \Gamma^{1,q}(w) \subsetneq \Lambda^1(w) \subsetneq \Gamma^{1,\infty}(w).$$

(ii) Another example can be obtained if we chose $w(t) = (1 - \log t)\chi_{(0,1)}(t)$. In this case, if $1 \leq q < \infty$,

$$\{0\} \neq \Gamma^{1,q}(w) \subsetneq \Lambda^1(w) \subsetneq \Gamma^{1,\infty}(w).$$

(iii) If $w(t) = \log t(\log t + 2)\chi_{(0,e^{-2})}$, then, for $1 < p < q < \infty$,

$$\Gamma^1(w) \subsetneq \Gamma^{1,p}(w) \subsetneq \Gamma^{1,q}(w) \subsetneq \Gamma^{1,\infty}(w),$$

$\Gamma^1(w) \subsetneq \Lambda^1(w) \subsetneq \Gamma^{1,\infty}(w)$ and $\Lambda^1(w) \not\subset \Gamma^{1,q}(w)$.

(iv) If $w(t) = t^{-\alpha}\chi_{(0,1)}(t)$, $-1 < \alpha < 0$, then, for $1 < p < q < \infty$,

$$\Gamma^1(w) = \Lambda^1(w) \subsetneq \Gamma^{1,p}(w) \subsetneq \Gamma^{1,q}(w) \subsetneq \Gamma^{1,\infty}(w).$$

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