Onsager Machlup functional
for a stochastic evolution equation

by

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Abstract

We compute the Onsager Machlup functional for a stochastic evolution equation with additive noise, using L^2-type techniques. The regularity that has to be imposed to the drift coefficient depends on the covariance of the corresponding stochastic convolution.

Key words and phrases: Onsager Machlup functionals, stochastic evolution equations.

1   Introduction

Let H be a real separable Hilbert space, and consider the following stochastic differential equation on H:

$$\begin{cases} dX(t) & = [AX(t) + F(X(t))]dt + BdW(t), \quad t \in [0,1], \\ X(0) & = x, \end{cases} \tag{1}$$

where $x \in H$, $A : D(A) \subset H \to H$ generates a C_0-semigroup $\{\exp(tA); t \geq 0\}$, F is a Lipschitz function defined on H, B is a non-negative bounded linear operator from H to H, and W is a cylindrical Wiener process in H. It is known that if $\int_0^1 \| \exp(tA)B\|^2_{HS}dt$ is finite, then (1) admits a unique solution $X \in L^2([0,1]; H)$, in a sense that will be specified later (see [3] for further details). In the remainder of the paper, we will also denote by W^A the stochastic convolution of A by W, that is the solution to (1) for $F \equiv 0$ and $x = 0$.

This article proposes to study the limiting behaviour of ratios of the form

$$\gamma_\varepsilon(\phi) \equiv \frac{P\left(\|X - \phi\| \leq \varepsilon\right)}{P\left(\|W^A\| \leq \varepsilon\right)}$$

when ε tends to 0, where ϕ is a deterministic function satisfying some regularity conditions and $\| \cdot \|$ is a suitable norm defined on the functions from $[0,1]$ to H. When $\lim_{\varepsilon \to 0} \gamma_\varepsilon(\phi) = \exp(J_0(\phi))$ for all ϕ in a reasonable class of functions, then the functional J_0 is called the Onsager Machlup functional associated to (1) and $\| \cdot \|$. It is worth noticing that J_0 can easily be interpreted as a generalized likelihood functional in infinite dimension, which makes its calculation an interesting problem.

For usual stochastic differential equations, namely when $H = \mathbb{R}^d$, $A = 0$, $B = \mathrm{Id}$, the problem of computing the Onsager Marchlup functional for (1) has been widely investigated. Ikeda and Watanabe [8] gave a rigorous proof for the case of any $\phi \in C^2([0,1]; \mathbb{R}^d)$ for the norm $\| \cdot \|_\infty$ defined on $C([0,1]; \mathbb{R}^d)$. This result has been enhanced then in two directions: Shepp and Zeitouni proved first in [12] that the function ϕ could be taken in the Cameron-Martin space $W_0^{1,2}([0,1]; \mathbb{R}^d)$, and Capitaine proved in [1] and [2] (basing this last result on some techniques inspired by the computation of the Onsager Machlup functional for diffusions on manifolds, see e.g. [7]) that the norm $\| \cdot \|$ could be taken as any euclidian norm on the functions from $[0,1]$ to $\mathbb{R}^d$ making sense for the solution to (1) and dominating the norm on

2

$L^2([0,1];\mathbb{R}^d)$. It is important to notice that in the case of finite dimensional diffusions, the functional J_0 does not depend on the norm considered.

Our current work fits in a more global project of studying the Onsager Machlup functionals for infinite stochastic systems. The first results in that direction have been obtained by Dembo and Zeitouni [4] for a class of trace class elliptic SPDEs on a bounded domain of $\mathbb{R}^d$, and then by Mayer-Wolf and Zeitouni [10] in the non trace case. We shall use some of their techniques in order to get our main result: if $\|\cdot\|$ is chosen as the Hilbert norm on $L^2([0,1];H)$, then, for ϕ satisfying some suitable hypothesis,

$$J_0(\phi) = -\int_0^1 \frac{1}{2}\left|B^{-1}\left[A\phi(t) + F(\phi(t)) - \dot{\phi}(t)\right]\right|_H^2 dt - \mathrm{Tr}\left(\frac{(PR^* + (PR^*)^*)}{2}\right),$$

where R is a certain bounded linear operator based on the covariance function of W^A and P is an operator depending on ∇F and ϕ.

With respect to the finite dimensional case, some differences can already be stressed in this introduction: first, the generality of the result obtained by Capitaine in [2] seems beyond our hopes at this moment, and it is for instance an open problem for us to know if some computations can be leaded for the norm $\|\cdot\|_\infty$ on $C([0,1];H)$. On the other hand, since an independence of the functional J_0 with respect to the norm $\|\cdot\|$ can also be expected in our SPDE case, we chose to work with the Hilbert norm on $L^2([0,1];H)$ for two main reasons:

1. We are working here with the minimal assumptions on A and B under which equation (1) has a unique solution in $L^2([0,1];H)$, though we will also make the additional assumption that A and B can be diagonalized in the same orthonormal basis of H.

2. We will be able to use the conditional exponential moments results stated in [10], where the conditioning is over an infinite dimensional Gaussian random variable (these results will be recalled at Section 2). However, the fact that we are dealing with an evolution type equation will force us to delve deeper into the different Karhunen decompositions involved in the application of the results of [10]. We shall give some details about these decompositions at Section 2.

Another relevant difference between the finite and infinite dimensional case is that the normalizing factor in $\gamma_\varepsilon(\phi)$ cannot be a function of the norm

3

of the cylindrical Brownian motion. This leads us to the natural choice of a
normalization by $P(\|W^A\| \leq \varepsilon)$, which prives us of the rotational invariance
type properties of the Brownian motion used by Ikeda and Watanabe ([8,
Lemma 8.2]) and by Capitaine ([1, Lemma 4], [2, Lemma 3]) as a fundamental
step towards the computation of the Onsager Machlup functional.

Our paper is organized as follows: in Section 2, we recall some basic
results and fix our notations for the stochastic evolution equation considered.
We recall a basic lemma of Mayer-Wolf and Zeitouni [10, Lemma 2.5] that
we shall use later on, and give the Karhunen expansion of an Ornstein-
Uhlenbeck process in dimension one. In Section 3 we obtain the Onsager
Machlup functional: at Subsection 3.1, we reduce our problem by Girsanov's
transform. Subsection 3.2. is then devoted to some details about the linear
case, that is when F is a linear bounded operator, which will lead us to the
general case after Taylor's expansion, and is of independent interest, since
the conditions given on F in this case will be more explicit, especially when
F can be diagonalized in the same complete orthonormal system than A and
B. At Subsection 3.3, we will deal with the general non linear case.

2 Notations and preliminary results

2.1 Stochastic evolution systems

2.1.1 The operators A and B

Let H be a real separable Hilbert space and $A : D(A) \subset H \to H$ an un-
bounded operator on H. The norm on H will be denoted by $|\cdot|_H$, and the
scalar product by $\langle \cdot, \cdot \rangle$. The L^2 norm in $L^2([0,1]; H)$ will be denoted by $\|\cdot\|_2$.
Let $\mathcal{L}(H)$ the set of bounded linear operators on H. The norm $\|\cdot\|$ will be
the usual operator norm defined on $\mathcal{L}(H)$, that is, $\|T\| = \sup_{x \in H} \frac{|T(x)|_H}{|x|_H}$. We
shall suppose

(H1) The operator A generates a self adjoint C_0-semigroup $\{\exp(tA) ; t \geq 0\}$
of negative type. Moreover, there exists a complete orthonormal system
$\{e_j; j \geq 1\}$ which diagonalizes A. We shall denote by $\{-\alpha_j; j \geq 1\}$ the
corresponding set of eigenvalues and we assume that $\{\alpha_j; j \geq 1\}$ is an
increasing sequence of real numbers such that $\alpha_j > 0$ and $\lim_{j \to \infty} \alpha_j =
\infty$.

We shall also consider an operator $B \in \mathcal{L}(H)$ satisfying

4

(H2) The operator B is of non-negative type and is diagonal when expressed in the orthonormal basis $\{e_j; j \geq 1\}$. We shall denote by $\{\beta_j; j \geq 1\}$ the corresponding set of eigenvalues. Furthermore, we shall suppose that

$$\sum_{j=1}^{\infty} \frac{\beta_j^2}{1 + \alpha_j} < \infty.$$

In our case where A and B can be diagonalized in the same complete orthonormal system, notice that the last hypothesis corresponds to the more general one $\int_0^1 \| \exp(tA)B \|_{HS}^2 < \infty$, that can be found in [3] in order to ensure the existence and uniqueness of a solution to (1).

2.1.2 Stochastic evolution equations

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a stochastic basis and $\{W^j(t); t \in [0,1], j \geq 1\}$ a sequence of mutually independent Brownian motions adapted to $\mathcal{F}_t$. The cylindrical Brownian motion on H is defined by the formal series

$$W(t) = \sum_{j=1}^{\infty} W^j(t) e_j, \tag{2}$$

where $\{e_j; j \geq 1\}$ is the complete orthonormal system of H introduced at Subsection 2.1.1. Notice that the series (2) does not converge in H, but for any $h \in H$, $\{\langle W(t), h \rangle; t \in [0,1]\}$ is a linear Brownian motion with covariance $|h|_H^2$ (see [3]). We shall consider the stochastic evolution equation (1), where A and B have been defined at subsection 2.1.1 and $F : H \to H$ is a Lipschitz function (some further hypothesis on F will be made at Subsections 3.2 and 3.3). Equation (1) is only formal, and has to be interpreted in the usual *mild* sense: we will say that $X = \{X(t); t \in [0,1]\}$ is a solution to (1) if it is an H-valued $\mathcal{F}_t$-adapted square integrable process such that

$$X(t) = \exp(tA)x + \int_0^t \exp((t-s)A)F(X(s))ds + \int_0^t \exp((t-s)A)B dW(s), \tag{3}$$

for all $t \in [0,1]$, where the last integral is of Itô's type. The following proposition is then a particular case of [3, Theorem 7.4].

Proposition 2.1 *Suppose that (H1) and (H2) are satisfied. Then there exists a unique solution X to (3), such that $X \in L^2(\Omega \times [0,1]; H)$.*

5

Under our assumptions, X is also a solution to (1) in the *weak* sense : if $Ah \in H$, for all $t \in [0,1]$

$$\langle X(t), h \rangle = \langle x, h \rangle + \int_0^t \langle X(s), Ah \rangle \, ds$$

$$+ \int_0^t \langle F(X(s)), h \rangle \, ds + \langle Bh, W(t) \rangle,$$

P-a.s. In particular, when $x = 0$ and $F = 0$ we have the following integration by parts formula (see [3, Lemma 5.5]): if $m \in W^{1,2}([0,1]; H)$ is such that $Am \in L^2([0,1]; H)$, then, for all $t \in [0,1]$,

$$\left\langle W^A(t), m(t) \right\rangle = \int_0^t \left\langle W^A(s), Am(s) + \dot{m}(s) \right\rangle ds + \int_0^t \left\langle Bm(s), dW(s) \right\rangle.$$

$$(4)$$

2.2 Some conditioned exponential inequalities

We will recall here some basic lemmae that we shall use for the computation of our Onsager Machlup functional. The first one is a crucial, though elementary, inequality that can be found in [8, p. 537].

Lemma 2.2 *For a fixed $n \geq 1$, let $z_1, \ldots, z_n$ be n random variables defined on $(\Omega, \mathcal{F}, P)$ and $\{A_\varepsilon; \varepsilon > 0\}$ a family of sets in $\mathcal{F}$. Suppose that for any $c \in \mathbb{R}$ and any $i = 1, \ldots, n$, we have*

$$\limsup_{\varepsilon \to \infty} E \left[\exp \left(c z_i \right) | A_\varepsilon \right] \leq 1.$$

Then

$$\lim_{\varepsilon \to \infty} E \left[\exp \left(\sum_{i=1}^n z_i \right) \Big| A_\varepsilon \right] = 1.$$

In the sequel of the paper, we shall use some inequalities involving trace class operators. Let us state now the notion of trace that we shall consider. Let K be a separable Hilbert space.

Definition 2.3 *Let $T : K \to K$ be a compact symmetric operator. Let $\{\tau_i; i \geq 1\}$ be the eigenvalues of the operator T. We will say that T is a trace class operator if $\sum_{i=1}^\infty |\tau_i|$ is finite.*

If T is trace class, we define the trace of T, $\mathrm{Tr}(T)$, as $\sum_{i=1}^\infty \langle T e_i, e_i \rangle$ for any basis $\{e_i; i \geq 1\}$. In particular $\mathrm{Tr}(T) = \sum_{i=1}^\infty \tau_i$.

We finish this subsection recalling two technical lemmas from [10]. The first one is a version of [10, Lemma 2.4]. In the second one, we state a slight variation of a fundamental lemma in [10]. We denote here by ℓ^2 the set of sequences of real numbers $\{\eta_i; i \geq 1\}$ such that $\sum_{i \geq 1} \eta_i^2 < \infty$.

Lemma 2.4 *Let $\{z_i; i \geq 1\}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables defined on $(\Omega, \mathcal{F}, P)$, and $\{\eta_i; i \geq 1\}$ and $\{\nu_i; i \geq 1\}$ two ℓ^2 sequences of real numbers. Then*

$$\lim_{\varepsilon \to 0} E\left[\exp\left(\sum_{i=1}^{\infty} z_i \nu_i\right) \Big| \sum_{i=1}^{\infty} \eta_i^2 z_i^2 \leq \varepsilon\right] = 1.$$

Lemma 2.5 *Let $\{z_i; i \geq 1\}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables defined on $(\Omega, \mathcal{F}, P)$, and $\{\eta_i; i \geq 1\}$ a ℓ^2 sequence of real numbers. Let $T : \ell^2 \to \ell^2$ be a Hilbert-Schmidt operator, $\{m_i; i \geq 1\}$ a complete orthonormal system of ℓ^2, and denote $\langle T m_i, m_j \rangle$ by $T_{i,j}$.*

1. *If $\frac{1}{2}(T + T^*)$ is trace class, then*

$$\lim_{\varepsilon \to 0} E\left[\exp\left(\sum_{i,j=1}^{\infty} z_i z_j T_{i,j}\right) \Big| \sum_{i=1}^{\infty} \eta_i^2 z_i^2 \leq \varepsilon\right] = 1,$$

2. *If $\sum_{i=1}^{\infty} T_{i,i} = +\infty$ (resp. $-\infty$), then*

$$\lim_{\varepsilon \to 0} E\left[\exp\left(\sum_{i \neq j} z_i z_j T_{i,j} + \sum_{i=1}^{\infty} (z_i^2 - 1) T_{i,i}\right) \Big| \sum_{i=1}^{\infty} \eta_i^2 z_i^2 \leq \varepsilon\right] = 0$$

(resp. $+\infty$).

Proof: We refer the reader to the proof of [10, Lemma 2.5]. Notice only that if we denote by $\hat{T}$ the operator $\frac{1}{2}(T + T^*)$, then for any $j, i \geq 1$

$$z_i z_j (T_{i,j} + T_{j,i}) = z_i z_j (\hat{T}_{i,j} + \hat{T}_{j,i}),$$

where $\hat{T}_{i,j} = \langle \hat{T} m_i, m_j \rangle$. $\qquad \square$

2.3 The Karhunen-Loève expansion

We compute here a Karhunen-Loève expansion for a class of one-dimensional Ornstein-Uhlenbeck processes that will appear in the decomposition of W^A. The following lemma is presumably fairly standard, but we include it for the sake of completeness.

Lemma 2.6 *Let β a standard Brownian motion and $\lambda > 0$. Then, the process $X = \{X(t) = \int_0^t \exp(-\lambda(t-s))d\beta(s), 0 \le t \le 1\}$ has the following Karhunen-Loève expansion:*

$$X(t) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda^2 + x_k^2}} Y_k \, g_k(t)$$

where, for each $k \ge 1$, x_k is the unique positive solution of the equation $\tan(x) = -\frac{1}{\lambda}x$ in the interval $[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2})$, $\{g_k(t) = A_k \sin(x_k t), k \ge 1\}$ is an orthonormal basis of $L^2([0,1])$ with the normalizing constants A_k satisfying $\sup_k |A_k| \le 2$, and $\{Y_k, k \ge 1\}$ is a family of orthogonal gaussian random variables with mean 0 and variance 1, defined by $Y_k = \sqrt{\lambda^2 + x_k^2} \int_0^1 X(t) \, g_k(t) dt$, for all $k \ge 1$.

Proof: Notice that X is a Gaussian process with covariance function

$$
\begin{aligned}
K(t,s) &= \int_0^{t \wedge s} e^{-\lambda(t-u)} e^{-\lambda(s-u)} du \\
&= \frac{1}{2\lambda} \left(e^{-\lambda(t \vee s - t \wedge s)} - e^{-\lambda(t+s)} \right),
\end{aligned}
\tag{5}
$$

$s, t \in [0,1]$. To find the eigenvalues and the eigenvectors of the symmetric operator in $\mathcal{L}(L^2([0,1]))$ associated with the kernel K, we have to solve the equation

$$\int_0^1 K(t,s)g(t)dt = \mu g(s), \quad 0 \le s \le 1, \tag{6}$$

that is for $0 \le s \le 1$

$$\frac{1}{2\lambda} \left(e^{-\lambda s} \int_0^s e^{\lambda t} g(t) dt + e^{\lambda s} \int_s^1 e^{-\lambda t} g(t) dt - e^{-\lambda s} \int_0^1 e^{-\lambda t} g(t) dt \right) = \mu g(s).$$

8

Differentiating twice it is easy to check that g satisfies

$$(\lambda^2\mu - 1)g(s) = \mu g''(s), \quad 0 \le s \le 1, \tag{7}$$

with initial conditions $g(0) = 0$ and $\lambda g(1) = -g'(1)$. Notice that equation (7) clearly implies that $\mu \ne 0$.

Set $\alpha_{\lambda,\mu} = \frac{\mu}{\lambda^2\mu - 1}$. Then $\alpha_{\lambda,\mu}$ is well-defined and strictly negative. Indeed, suppose that $\lambda^2\mu - 1 = 0$. Then $g''(s) = 0$, $0 \le s \le 1$, and the initial conditions imply $g \equiv 0$. Finally, suppose that $\alpha_{\lambda,\mu} > 0$. In this case, the solution of the differential equation (7) is of the form

$$g(s) = c_1 \exp\left(\frac{s}{\sqrt{\alpha_{\lambda,\mu}}}\right) + c_2 \exp\left(-\frac{s}{\sqrt{\alpha_{\lambda,\mu}}}\right),$$

where c_1, c_2 are real constants. Then, the initial conditions $g(0) = 0$ and $\lambda g(1) = -g'(1)$ yield

$$\tanh\left(\frac{1}{\sqrt{\alpha_{\lambda,\mu}}}\right) = -\frac{1}{\lambda}\frac{1}{\sqrt{\alpha_{\lambda,\mu}}}$$

and this equation has no solution.

Consequently, we can assume $\frac{\mu}{\lambda^2\mu - 1} < 0$, and the solution of (7) is of the form

$$g(s) = c_1 \sin\left(\frac{s}{\sqrt{|\alpha_{\lambda,\mu}|}}\right) + c_2 \cos\left(\frac{s}{\sqrt{|\alpha_{\lambda,\mu}|}}\right).$$

The condition $g(0) = 0$ implies $c_2 = 0$, and $\lambda g(1) = -g'(1)$ yields

$$\tan\left(\frac{1}{\sqrt{|\alpha_{\lambda,\mu}|}}\right) = -\frac{1}{\lambda}\frac{1}{\sqrt{|\alpha_{\lambda,\mu}|}}.$$

Set $x = |\alpha_{\lambda,\mu}|^{-1/2}$. The relation $\alpha_{\lambda,\mu} = \frac{\mu}{\lambda^2\mu - 1}$ implies that the eigenvalues of the operator K form a family $\{\mu_n; n \ge 1\}$, where $\mu_n = \frac{1}{\lambda^2 + x_n^2}$ and x_n is the solution of the equation $\tan(x) = -\frac{1}{\lambda}x$ in the interval $[(2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2})$; and the orthonormalized eigenfunctions are of the form $g_n(s) = A_n \sin(x_n s)$, $n \ge 1$. An easy computation shows that $|A_n| \le 2$ for all n.

The classical Karhunen-Loève Theorem (see e.g. [9]) finishes the proof.

$\square$

9

3 Onsager-Machlup functional

In this section we compute the Onsager Machlup functional for our equation, following the usual scheme used for both finite and infinite cases: we apply first the Girsanov transform in order to reduce our problem to the evaluation of a functional of the stochastic convolution W^A, up to some easily controlled correction terms (Subsection 3.1). We are then left with the evaluation of the conditional exponential moments of a stochastic integral with respect to the cylindrical Brownian motion, that can be easily handled when F is a linear operator (Subsection 3.2). The general case for F can be deduced then by Taylor's expansion (Subsection 3.3).

3.1 Application of Girsanov's transform

Set $Q = B^2$ (recall that A and B are supposed to be symmetric). Fix a differentiable function $h : [0, 1] \longrightarrow H$ satisfying

(H3) $\frac{dh}{ds} = \dot{h} \in L^2([0, 1]; H)$, $Ah \in L^2([0, 1]; H)$ and $Bh \in L^2([0, 1]; H)$.

Let ϕ^h be the solution of the infinite dimensional equation

$$\begin{cases} d\phi^h(t) &= A\phi^h(t)dt + Qh(t)dt, \quad t \in [0, 1], \\ \phi^h(0) &= x. \end{cases} \tag{8}$$

We will compute the Onsager Machlup functional on $L^2([0, 1]; H)$ at points of the form ϕ^h. The regularity assumption (H3) is required in order to apply Girsanov's transform, and to control the term T_3 (defined at (10)) by means of an integration by parts argument.

Assume also the following conditions:

(h1) For any $t \in [0, 1]$, $F(X(t)) \in \mathrm{Im}(B)$ a.s. and one of the two following relations holds: for some $\delta > 0$,

$$\sup_{t \in [0,1]} E\left[\exp(\delta|B^{-1}F(X(t))|_H^2)\right] < +\infty,$$

or

$$E\left[\exp\left(\frac{1}{2}\int_0^1 |B^{-1}F(X(t))|_H^2 dt\right)\right] < +\infty,$$

(h2) there exist a positive constant K such that $|B^{-1}F(x) - B^{-1}F(y)|_H \leq K|x-y|_H$ and $|B^{-1}F(x)|_H \leq K(1 + |x|_H)$, for any $x, y \in H$.

Then, using Girsanov's transformation (see [3, Theorem 10.14 and Proposition 10.17]) with $\hat{W}(t) = W(t) + \int_0^t B^{-1}F(X(s))ds$, we obtain for any $\varepsilon > 0$

$$P(\|X - \phi^h\|_2 \leq \varepsilon) = E\Big[\exp\Big(\int_0^1 \langle B^{-1}F(W^A(s) + e^{sA}x), dW(s)\rangle$$

$$-\frac{1}{2}\int_0^1 |B^{-1}F(W^A(s) + e^{sA}x)|_H^2 ds\Big) \mathbf{1}_{\{\|e^{\cdot A}x + W^A - \phi^h\|_2 \leq \varepsilon\}}\Big].$$

Notice that to simplify the notation we have denoted $\hat{W}$ by W. Since $Bh \in L^2([0,1]; H)$ we can apply again Girsanov's transformation, now with $\bar{W}(t) = W(t) - \int_0^t Bh(s)ds$. We get, for any $\varepsilon > 0$,

$$P(\|X - \phi^h\|_2 \leq \varepsilon) = E\Big[\exp\Big(\int_0^1 \langle B^{-1}F(W^A(s) + \phi^h(s)), dW(s)\rangle$$

$$+ \int_0^1 \langle B^{-1}F(W^A(s) + \phi^h(s)), Bh(s)\rangle ds$$

$$- \frac{1}{2}\int_0^1 |B^{-1}F(W^A(s) + \phi^h(s))|_H^2 ds$$

$$- \int_0^1 \langle Bh(s), dW(s)\rangle - \frac{1}{2}\int_0^1 |Bh(s)|_H^2 ds\Big) \mathbf{1}_{\{\|W^A\|_2 \leq \varepsilon\}}\Big].$$

Then

$$\gamma_\varepsilon(\phi^h) = \frac{P(\|X - \phi^h\|_2 \leq \varepsilon)}{P(\|W^A\|_2 \leq \varepsilon)}$$

can be written as

$$\exp\Big(-\frac{1}{2}\int_0^1 |B^{-1}F(\phi^h(s)) - Bh(s)|_H^2 ds\Big) E\Big[\exp\Big(\sum_{i=1}^3 T_i\Big) \Big| \|W^A\|_2 \leq \varepsilon\Big],$$

$$(9)$$

with

$$T_1 := \int_0^1 \langle B^{-1}F(W^A(s) + \phi^h(s)), dW(s)\rangle,$$

11

$$T_2 \quad := \quad \int_0^1 \langle B^{-1}F(W^A(s) + \phi^h(s)) - B^{-1}F(\phi^h(s)), Bh(s)\rangle ds$$

$$-\frac{1}{2}\int_0^1 \left(|B^{-1}F(W^A(s) + \phi^h(s))|_H^2 - |B^{-1}F(\phi^h(s))|_H^2\right) ds$$

and

$$T_3 := -\int_0^1 \langle Bh(s), dW(s)\rangle. \tag{10}$$

On the set $\{\|W^A\|_2 \le \varepsilon\}$, using (h2), we clearly have that $|T_2| \le C\varepsilon$. Moreover, since Bh is deterministic, by relation (4), we obtain $T_3 := T_{3(a)} + T_{3(b)}$ with

$$T_{3(a)} \quad = \quad \int_0^1 \left\langle Ah(s) + \dot{h}(s), W^A(s)\right\rangle ds,$$

$$T_{3(b)} \quad = \quad -\left\langle h(1), W^A(1)\right\rangle.$$

Assumption (H3) trivially gives $|T_{3(a)}| \le C\varepsilon$ on the set $\{\|W^A\|_2 \le \varepsilon\}$.

On the other hand, assuming hypothesis (H1) and (H2), by Lemma 2.6 and decomposition (2)

$$W^A(t) \quad = \quad \sum_{j=1}^\infty \left(\int_0^t \beta_j\, e^{-\alpha_j(t-s)} dW^j(s)\right) e_j$$

$$= \quad \sum_{j=1}^\infty \sum_{k=1}^\infty \mu_{k,j} Y_{k,j} (g_{k,j} \otimes e_j)(t), \tag{11}$$

where $\mu_{k,j} = \frac{\beta_j}{\sqrt{\alpha_j^2 + x_{k,j}^2}}$, and $x_{k,j}$, $Y_{k,j}$ and $g_{k,j}$ are the x_k, Y_k and g_k defined in Lemma 2.6 when $\lambda = \alpha_j$. Moreover, $\{Y_{k,j}, k \ge 1, j \ge 1\}$ is a family of independent centered random variables with variance 1.

Given $f \in L^2([0,1])$ and $e \in H$, we denote by $f \otimes e$ the function of $L^2([0,1]; H)$ such that $(f \otimes e)(s) = f(s)e$. Notice then that $\{g_{k,j} \otimes e_j, k \ge 1, j \ge 1\}$ is an orthonormal basis of $L^2([0,1]; H)$ such that for any $j, k \ge 1$: $\mathrm{Cov}(\langle W^A, g_{k,j} \otimes e_j\rangle_{L^2([0,1];H)}) = \mu_{k,j}^2$. Thus,

$$\|W^A\|_2^2 = \sum_{j=1}^\infty \sum_{k=1}^\infty \mu_{k,j}^2 Y_{k,j}^2,$$

12

with

$$\sum_{j,k=1}^{\infty} \mu_{k,j}^2 = \sum_{j,k=1}^{\infty} \frac{\beta_j^2}{\alpha_j^2 + x_{k,j}^2} < \sum_{j=1}^{\infty} \frac{\beta_j^2}{\alpha_j^2} \int_1^{\infty} \frac{1}{1 + \frac{\pi^2}{4\alpha_j^2}x^2} dx \leq C \sum_{j=1}^{\infty} \frac{\beta_j^2}{\alpha_j} < +\infty.$$

Then, by (11)

$$T_{3(b)} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} r_{k,j} Y_{k,j}$$

with $r_{k,j} := -\mu_{k,j} g_{k,j}(1)\langle h(1), e_j \rangle$. Since

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} r_{k,j}^2 \leq 4|h(1)|_H^2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu_{k,j}^2 < \infty,$$

by Lemma 2.4 we get

$$\lim_{\varepsilon \to 0} E\left[\exp\left(cT_{3(b)} \right) \Big| \|W^A\|_2 \leq \varepsilon \right] = 1,$$

for any $c \in \mathbb{R}$. Hence, using Lemma 2.2, the only point remaining to determine $\lim_{\varepsilon \downarrow 0} \gamma_\varepsilon(\phi^h)$ is the study of the term $E[\exp(T_1)\big|\|W^A\|_2 \leq \varepsilon]$.

3.2   The linear case

In this subsection we discuss the case of a linear function F. We obtain an Onsager Machlup functional and we study carefully the particular case where F is diagonal in the same basis than the operators A and B.

The main theorem of this part states as follows:

Theorem 3.1 *Assume that (H1) and (H2) are satisfied, h is a function satisfying (H3), ϕ^h is defined by (8) and $F \in \mathcal{L}(H)$ is such that $\hat{P} = B^{-1}F$ is a bounded operator. Denote by P and R the linear operators defined on $L^2([0,1]; H)$ such that $P(h \otimes e_j) = h \otimes \hat{P}(e_j)$ and $R(h \otimes e_j) = R_j(h) \otimes e_j$ with*

$$(R_j h)(s) := \int_s^1 \beta_j e^{-\alpha_j(t-s)} h(t) dt.$$

Then, for any $j, k \geq 1$,

$$\langle (R^*R)(g_{k,j} \otimes e_j), g_{k,j} \otimes e_j \rangle_{L^2([0,1];H)} = \mathrm{Cov}(\langle W^A, g_{k,j} \otimes e_j \rangle_{L^2([0,1];H)})$$

and

13

(i) if $\frac{1}{2}(PR^* + (PR^*)^*)$ is trace class, then

$$\lim_{\varepsilon \downarrow 0} \gamma_\varepsilon(\phi^h) \;=\; \exp\Big(-\int_0^1 \frac{1}{2}\Big|B^{-1}\big[(A+F)\phi^h(t) - \dot{\phi}^h(t)\big]\Big|_H^2 dt$$

$$-Tr\Big(\frac{1}{2}(PR^* + (PR^*)^*)\Big)\Big),$$

(ii) if $\sum_{j,k}\langle PR^*(g_{k,j} \otimes e_j), g_{k,j} \otimes e_j\rangle = +\infty$ (resp. $-\infty$), then

$$\lim_{\varepsilon \downarrow 0} \gamma_\varepsilon(\phi^h) = 0$$

(resp. $+\infty$).

Proof:

Step 1: Reduction to a stochastic integral involving W^A and W.

Since $\hat{P}$ is a bounded operator, condition (h2) is clearly satisfied. On the other hand, using that F is a linear operator, we get

$$E\left[\exp\left(\frac{1}{2}\int_0^1 |B^{-1}F(X(t))|_H^2 dt\right)\right]$$

$$\leq \exp\left(\frac{C}{2}\int_0^1 (1 + \|e^{t(A+F)}\|^2 |x|_H^2) dt\right) E\left[\exp\left(\frac{C}{2}\int_0^1 |W^{A+F}(t)|_H^2 dt\right)\right].$$

Following the proof of [3, Theorem 10.20] we check that (h1) is also satisfied, using the fact that $(A + F)$ generates a C_0-semigroup such that $\int_0^1 \|e^{t(A+F)}B\|_{HS}^2 ds$ is finite whenever $F \in \mathcal{L}(H)$ (see e.g. Goldberg [6, Chapter 5.1]). Hence, it is sufficient to study $\lim_{\varepsilon \downarrow 0} E[\exp(T_1)\,\big|\,\|W^A\|_2 \leq \varepsilon]$.

Since we are in the linear case, we can write $T_1 := T_{1(a)} + T_{1(b)}$ with

$$T_{1(a)} \;=\; \int_0^1 \langle B^{-1}F(W^A(s)), dW(s)\rangle,$$

$$T_{1(b)} \;=\; \int_0^1 \langle B^{-1}F(\phi^h(s)), dW(s)\rangle.$$

Since the function $B^{-1}F$ is a bounded linear operator, $T_{1(b)}$ can be studied likewise the term T_3, using the integration by parts formula (4).

14

The study of the term $T_{1(a)}$ will follow the ideas presented by Mayer-Wolf and Zeitouni [10].

Step 2: Expression of $T_{1(a)}$ in terms of Skorokhod integrals.

Consider now $f_{k,j} := R_j g_{k,j}$, $j \geq 1, k \geq 1$. Then, for any $j \geq 1$, $\{h_{k,j} = \frac{1}{\mu_{k,j}} f_{k,j}, \; k \geq 1\}$ is an orthonormal basis of $L^2([0,1])$. Indeed, the $h_{k,j}$ form an orthogonal family since, by (5) and (6),

$$
\begin{aligned}
\langle f_{m,j}, f_{n,j} \rangle_{L^2[0,1]} &= \beta_j^2 \int_0^1 \Big(\int_s^1 e^{-\alpha_j(t-s)} g_{m,j}(t) dt \Big) \Big(\int_s^1 e^{-\alpha_j(u-s)} g_{n,j}(u) du \Big) ds \\
&= \beta_j^2 \int_0^1 \int_0^1 K_j(t,u) g_{m,j}(t) g_{n,j}(u) dt du \\
&= \mu_{m,j}^2 \langle g_{m,j}, g_{n,j} \rangle_{L^2[0,1]},
\end{aligned} \tag{12}
$$

for any $m, n \geq 1$, where K_j denotes the covariance function defined at (5) with $\lambda = \alpha_j$. Thus, in order to prove that it is a basis, it is sufficient to show that, if $h \in L^2([0,1])$ satisfies $\langle f_{k,j}, h \rangle_{L^2[0,1]} = 0$ for all $k \geq 1$, then $h \equiv 0$. But this follows easily from the fact that if for all $k \geq 1$,

$$
\begin{aligned}
0 = \langle f_{k,j}, h \rangle_{L^2[0,1]} &= \int_0^1 \Big(\int_s^1 \beta_j \, e^{-\alpha_j(t-s)} g_{k,j}(t) dt \Big) h(s) ds \\
&= \beta_j \langle g_{k,j}, \varphi^h \rangle_{L^2[0,1]},
\end{aligned}
$$

then $\varphi^h \equiv 0$ with $\varphi^h(t) = \int_0^t e^{-\alpha_j(t-s)} h(s) ds$, and of course $h \equiv 0$.

Furthermore,

$$
\begin{aligned}
Y_{k,j} &= \sqrt{\alpha_j^2 + x_{k,j}^2} \int_0^1 \Big(\int_0^t e^{-\alpha_j(t-s)} dW^j(s) \Big) g_{k,j}(t) dt \\
&= \int_0^1 \Big(\sqrt{\alpha_j^2 + x_{k,j}^2} \int_s^1 e^{-\alpha_j(t-s)} g_{k,j}(t) dt \Big) dW^j(s) \\
&= I_j(h_{k,j}),
\end{aligned}
$$

where $I_j(l)$ denotes the Wiener integral of l with respect to W^j, that is

15

$I_j(l) = \int_0^1 l(s) dW^j(s)$. Then

$$T_{1(a)} = \int_0^1 \langle \hat{P}(W^A(s)), dW(s) \rangle$$

$$= \sum_{i,j,k=1}^{\infty} \mu_{k,j} \int_0^1 Y_{k,j} \langle \hat{P}((g_{k,j} \otimes e_j)(s)), e_i \rangle dW^i(s)$$

$$= \sum_{i,j,k=1}^{\infty} \mu_{k,j} \langle \hat{P}e_j, e_i \rangle \int_0^1 Y_{k,j}\, g_{k,j}(s) dW^i(s). \tag{13}$$

It is worth noticing that the random variables $Y_{k,j}$ are $\mathcal{F}_1$-measurable. Hence, some of the stochastic integrals appearing in (13) are anticipating. When they are of this kind, we have taken them in the Skorokhod sense, and we switch from Itô's integrals to Skorokhod's ones using the fact that they coincide on the set L_a^2 of square integrable adapted processes (see for instance [11] for an account on Skorokhod's integrals). Moreover, using [11, Equation (1.45)], observe that when $j = i$

$$\int_0^1 Y_{k,j} g_{k,j}(s) dW^j(s) = Y_{k,j} \sum_{m=1}^{\infty} \langle g_{k,j}, h_{m,j} \rangle_{L^2[0,1]} I_j(h_{m,j})$$

$$- \langle h_{k,j}, g_{k,j} \rangle_{L^2[0,1]},$$

and when $j \neq i$

$$\int_0^1 Y_{k,j} g_{k,j}(s) dW^i(s) = Y_{k,j} \sum_{m=1}^{\infty} \langle g_{k,j}, h_{m,i} \rangle_{L^2[0,1]} I_i(h_{m,i}).$$

Using the fact that $h_{k,j} = \mu_{k,j}^{-1} R_j g_{k,j}$ and $Y_{k,j} = I_j(h_{k,j})$, we can write (13) in the following way:

$$T_{1(a)} = \sum_{(k,j) \neq (m,i)} \frac{\mu_{k,j}}{\mu_{m,i}} Y_{k,j} Y_{m,i} \langle \hat{P}e_j, e_i \rangle \langle g_{k,j}, R_i g_{m,i} \rangle_{L^2[0,1]}$$

$$+ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (Y_{k,j}^2 - 1) \langle \hat{P}e_j, e_j \rangle \langle g_{k,j}, R_j g_{k,j} \rangle_{L^2[0,1]}. \tag{14}$$

Step 3: Expression of $T_{1(a)}$ in terms of P and R^*.

16

Define now the operator $T : \ell^2_{\mathbb{N}^2} \to \ell^2_{\mathbb{N}^2}$ by

$$T_{(k,j),(m,i)} = \frac{\mu_{k,j}}{\mu_{m,i}} \langle \hat{P} e_j, e_i \rangle \langle g_{k,j}, R_i g_{m,i} \rangle_{L^2[0,1]}, \quad (k,j),(m,i) \in \mathbb{N}^2.$$

Then

$$T_{1(a)} = \sum_{(k,j) \neq (m,i)} T_{(k,j),(m,i)} Y_{k,j} Y_{m,i} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} T_{(k,j),(k,j)} (Y_{k,j}^2 - 1). \qquad (15)$$

Let Λ be the linear operator such that $\Lambda(g_{k,j} \otimes e_j) = \mu_{k,j}(g_{k,j} \otimes e_j)$. Observe that, for any $j, k \geq 1$,

$$
\begin{aligned}
(R^* R)(g_{k,j} \otimes e_j) &= \sum_{n=1}^{\infty} \langle (R_j^* R_j) g_{k,j}, g_{n,j} \rangle_{L^2[0,1]} (g_{n,j} \otimes e_j) \\
&= \sum_{n=1}^{\infty} \langle R_j g_{k,j}, R_j g_{n,j} \rangle_{L^2[0,1]} (g_{n,j} \otimes e_j),
\end{aligned}
$$

and by (12) this last quantity equals $\mu_{k,j}^2 (g_{k,j} \otimes e_j)$.

Hence $\Lambda^2 = R^* R$. Define $U = R\Lambda^{-1} = \frac{R}{|R|}$, which is clearly a bounded operator defined in $L^2([0,1]; H)$. We get

$$
\begin{aligned}
T_{(k,j),(m,i)} &= \langle P\Lambda(g_{k,j} \otimes e_j), R\Lambda^{-1}(g_{m,i} \otimes e_i) \rangle_{L^2([0,1];H)} \\
&= \langle U^* P R^* U(g_{k,j} \otimes e_j), (g_{m,i} \otimes e_i) \rangle_{L^2([0,1];H)}. \qquad (16)
\end{aligned}
$$

Observe now that for any $f, h \in L^2([0,1])$, $i, j \geq 1$,

$$\langle R^*(h \otimes e_j), f \otimes e_i \rangle_{L^2([0,1];H)} = \langle h, R_i(f) \rangle_{L^2[0,1]} \langle e_j, e_i \rangle$$

$$= \langle h, R_j(f) \rangle_{L^2[0,1]} \langle e_j, e_i \rangle = \langle R_j^*(h) \otimes e_j, f \otimes e_i \rangle_{L^2([0,1];H)}.$$

So, we get

$$T_{(k,j),(k,j)} = \langle \hat{P} e_j, e_j \rangle \langle g_{k,j}, R_j g_{k,j} \rangle_{L^2[0,1]}$$

$$= \langle P R^*(g_{k,j} \otimes e_j), g_{k,j} \otimes e_j \rangle_{L^2([0,1];H)}, \qquad (17)$$

and clearly, if $\frac{1}{2}(P R^* + (P R^*)^*)$ is a trace class operator, $\sum_{k,j} T_{(k,j),(k,j)} = \mathrm{Tr}(\frac{1}{2}(P R^* + (P R^*)^*))$.

Step 4: Application of Lemma 2.5.

Using Lemma 2.2 and by equation (15), in order to see part (i) it is sufficient to show that

$$E\left[\exp\left(c\sum_{i,k,j,m}Y_{k,j}Y_{m,i}T_{(k,j),(m,i)}\right)\Big|\|W^A\|_2\leq\varepsilon\right]\longrightarrow 1$$

when ε goes to 0, for any $c\in\mathbb{R}$.

Since by assumption $\frac{1}{2}(PR^*+(PR^*)^*)$ is a trace class operator, by (9), the decomposition of W^A given in (11), (16) and applying part 1) in Lemma 2.5 we finish easily the proof of part (i).

To prove (ii), we should proceed with the same computations. From (17) and part 2) of Lemma 2.5 we obtain easily the desired result. □

We finish this subsection with an important Corollary where we study the case where F is a trace class operator. We also examine the diagonal case.

Corollary 3.2 *Assume that (H1) and (H2) are satisfied, h is a function satisfying (H3), ϕ^h is defined by (8) and F is a trace class operator such that $\hat{P}=B^{-1}F$ is a bounded operator. Then, if $\frac{1}{2}(PR^*+(PR^*)^*)$ is trace class*

$$\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon(\phi^h)=\exp\left(-\int_0^1\frac{1}{2}\left|B^{-1}\left[(A+F)\phi^h(t)-\dot{\phi}^h(t)\right]\right|_H^2 dt-\frac{1}{2}Tr(F)\right).$$

Proof. Using Theorem 3.1 it is enough to prove that if F is a trace class operator then $\mathrm{Tr}(\frac{1}{2}(PR^*+(PR^*)^*))=\frac{1}{2}\mathrm{Tr}(F)$.

We have to study the eigenvalues of the operator $\frac{1}{2}(PR^*+(PR^*)^*)$. Denote by $\hat{R}_j$ the operator $\beta_j^{-1}R_j$. Set $V_j\equiv\frac{1}{2}(\hat{R}_j^*+\hat{R}_j)$. First, we can check trivially that for all $j\geq 1$, V_j is a linear operator on $L^2([0,1])$ given by a kernel denoted by $\hat{K}_j$: indeed, for any $h\in L^2([0,1])$

$$[V_j(h)](s)=\int_0^1\hat{K}_j(s,t)h(t)dt,\quad s\in[0,1],$$

with $\hat{K}_j:[0,1]^2\to[0,1]$ defined by $\hat{K}_j(s,t)=\frac{1}{2}e^{-\alpha_j|s-t|}$.

Let us consider now an operator V on $L^2([0,1])$ given by a kernel $\hat{K}(s,t)=\frac{1}{2}e^{-\lambda|s-t|}$, an let us show that V is a non-negative. Indeed, if $\hat{R}$ is the Volterra operator on $L^2([0,1])$ defined by $\hat{R}h=g$ and

$$g(t)=\int_0^t e^{-\lambda(t-s)}h(s)ds,\quad t\in[0,1],$$

18

then it is readily seen that

$$Vh = \frac{1}{2} \langle h, f \rangle_{L^2([0,1])} f + \lambda \hat{R}^* \hat{R} h,$$

where $f(t) = e^{-\lambda(1-t)}$ for all $t \in [0,1]$. Therefore,

$$\langle Vh, h \rangle_{L^2([0,1])} = \frac{1}{2} \langle h, f \rangle^2_{L^2([0,1])} + \langle Rh, Rh \rangle^2_{L^2([0,1])},$$

which shows the positivity.

Thus, V is a positive Hilbert-Schmidt operator given by a continuous kernel on $[0,1]^2$. It is then well-known [5, Proposition 10.1] that

$$\sum_{k=0}^{\infty} \nu_k = \int_0^1 \hat{K}(t,t) dt = \frac{1}{2}.$$

Notice that this value does not depend on λ.

On the other hand, for any $h \in L^2([0,1]), j \geq 1$ we have

$$\frac{1}{2}(PR^* + (PR^*)^*)(h \otimes e_j)(s)$$

$$= \frac{1}{2} \left\{ P \left(\left(\int_0^s e^{-\alpha_j(s-u)} h(u) du \right) \beta_j e_j \right) + R \left[h \otimes F^*(\beta_j^{-1} e_j) \right] (s) \right\}$$

$$= \frac{1}{2} \left\{ \left(\int_0^s e^{-\alpha_j(s-u)} h(u) du \right) \beta_j (B^{-1} F) e_j \right.$$

$$\left. + \sum_{i=1}^{\infty} \beta_i \beta_j^{-1} \langle F^* e_j, e_i \rangle \left(\int_s^1 e^{-\alpha_i(u-s)} h(u) du \right) e_i \right\}.$$

Moreover, observe also that if we denote by $\{\hat{f}_{k,j}, k \geq 1\}$ the $L^2([0,1])$ orthonormal basis that diagonalizes V_j, then $\{\hat{f}_{k,j} \otimes e_j; j \geq 1, k \geq 1\}$ is a orthonormal basis of $L^2([0,1]; H)$. So, since

$$\langle \beta_j(B^{-1} F) e_j, e_j \rangle = \langle (B^{-1} F) e_j, B e_j \rangle = \langle F e_j, e_j \rangle,$$

19

we get

$$\mathrm{Tr}\left(\frac{1}{2}(PR^* + (PR^*)^*)\right)$$

$$= \sum_{j,k=1}^{\infty} \left\langle \frac{1}{2}(PR^* + (PR^*)^*)(\hat{f}_{k,j} \otimes e_j), \hat{f}_{k,j} \otimes e_j \right\rangle_{L^2([0,1];H)}$$

$$= \frac{1}{2}\sum_{j,k=1}^{\infty} \left\langle \hat{R}_j^* \hat{f}_{k,j}, \hat{f}_{k,j} \right\rangle_{L^2[0,1]} \langle Fe_j, e_j \rangle + \left\langle \hat{R}_j \hat{f}_{k,j}, \hat{f}_{k,j} \right\rangle_{L^2[0,1]} \langle F^* e_j, e_j \rangle$$

$$= \sum_{j,k=1}^{\infty} \langle Fe_j, e_j \rangle \left\langle V_j \hat{f}_{k,j}, \hat{f}_{k,j} \right\rangle_{L^2[0,1]}$$

$$= \frac{1}{2}\sum_{j=1}^{\infty} \langle Fe_j, e_j \rangle = \frac{1}{2}\mathrm{Tr}(F).$$

The proof is now completed. □

Example 3.3 *Consider the case of an operator F which is diagonal when expressed in the orthonormal basis $\{e_j; j \geq 1\}$. We denote by $\{p_j; j \geq 1\}$ the corresponding set of eigenvalues. Assume also the other hypothesis of Theorem 3.1. Then:*

(i') *the operator $\frac{1}{2}(PR^* + (PR^*)^*)$ is trace class if and only if F is, and in this case, $Tr(\frac{1}{2}(PR^* + (PR^*)^*)) = \frac{1}{2}Tr(F)$.*

(ii') *if $\sum_{j\geq 1} p_j = +\infty$, then*

$$\lim_{\varepsilon\downarrow 0} \gamma_\varepsilon(\phi) = 0,$$

(iii') *if $\sum_{j\geq 1} p_j = -\infty$, then*

$$\lim_{\varepsilon\downarrow 0} \gamma_\varepsilon(\phi) = +\infty.$$

Remark 3.4 *1. Under situations (ii') and (iii'), choosing $h = 0$, we find that the limiting behaviour of $P(\|W^A\|_2 \leq \varepsilon)$ and $P(\|W^{A+F}\|_2 \leq \varepsilon)$. when $\varepsilon \to 0$ cannot be compared.*

2. Since we need $B^{-1}F$ to be a bounded operator, we will suppose that $\sup_j \frac{|p_j|}{\beta_j} < \infty$.

20

3. *The conditions (i'), (ii') and (iii') only involve the operator F, and none of the operators A and B.*

Proof. Since now $PR^*(h \otimes e_j) = p_j(\hat{R}_j^*(h) \otimes e_j)$ and $(PR^*)^*(h \otimes e_j) = p_j(\hat{R}_j(h) \otimes e_j)$, it is easily seen that $\frac{1}{2}(PR^* + (PR^*)^*)$ is of trace class iff $\zeta \equiv \sum_{i,j \geq 1} |p_j v_{i,j}| < \infty$, where the family $\{v_{i,j}; i \geq 1\}$ denotes the set of eigenvalues of the operator V_j. As we have seen in Corollary 3.2, for any $j \geq 1$, $\sum_{i \geq 1} |v_{i,j}| = \sum_{i \geq 1} v_{i,j} = \frac{1}{2}$. It is now easily deduced that PR^* is trace class if and only if $\sum_{j \geq 1} |p_j| < \infty$.

Assume now that the operator $\frac{PR^* + (PR^*)^*}{2}$ is not trace class. Observe first that

$$\langle PR^*(g_{k,j} \otimes e_j), g_{k,j} \otimes e_j \rangle_{L^2([0,1];H)} = p_j \langle \hat{R}_j g_{k,j}, g_{k,j} \rangle_{L^2[0,1]}.$$

We can compute easily

$$
\begin{aligned}
\hat{R}_j g_{k,j}(s) &= \int_s^1 e^{-\alpha_j(t-s)} g_{k,j}(t) dt \\
&= \frac{A_{k,j} e^{\alpha_j s}}{\alpha_j^2 + x_{k,j}^2} \Big(e^{-\alpha_j s} \big(x_{k,j} \cos(x_{k,j} s) + \alpha_j \sin(x_{k,j} s) \big) \\
&\quad - e^{-\alpha_j} \big(x_{k,j} \cos(x_{k,j}) + \alpha_j \sin(x_{k,j}) \big) \Big) \\
&= \frac{A_{k,j}}{\alpha_j^2 + x_{k,j}^2} \Big(x_{k,j} \cos(x_{k,j} s) + \alpha_j \sin(x_{k,j} s) \Big),
\end{aligned}
$$

using $x_{k,j} \cos(x_{k,j}) = -\alpha_j \sin(x_{k,j})$. Since $\int_0^1 A_{k,j}^2 \sin^2(x_{k,j} s) ds = 1$, we then obtain

$$
\begin{aligned}
\langle \hat{R}_j g_{k,j}, g_{k,j} \rangle_{L^2[0,1]} &= \int_0^1 \frac{A_{k,j}^2}{\alpha_j^2 + x_{k,j}^2} \Big(x_{k,j} \cos(x_{k,j} s) \sin(x_{k,j} s) + \alpha_j \sin^2(x_{k,j} s) \Big) ds \\
&= \frac{1}{\alpha_j^2 + x_{k,j}^2} \Big(\frac{A_{k,j}^2}{2} \sin^2(x_{k,j}) + \alpha_j \Big).
\end{aligned}
$$

Furthermore, we have that

$$\frac{C_1}{\alpha_j} \leq \sum_{k=1}^{\infty} \frac{1}{\alpha_j^2 + x_{k,j}^2} \leq \frac{C_2}{\alpha_j},$$

for some positive constants C_1 and C_2 not depending on j, comparing the above sum with some integrals of the function $\varphi(y) = \frac{1}{1+y^2}$.

21

Since $\sup_{k,j} |A_{k,j}| \leq 2$, and

$$\sum_{j,k=1}^{\infty} \langle PR^*(g_{k,j} \otimes e_j), g_{k,j} \otimes e_j \rangle_{L^2([0,1];H)} = \sum_{j,k=1}^{\infty} \frac{p_j}{\alpha_j^2 + x_{k,j}^2} \Big(\frac{A_{k,j}^2}{2} \sin^2(x_{k,j}) + \alpha_j \Big),$$

the proofs of (ii') and (iii') in our example are now straightforward.

$\square$

3.3 The general case

In this subsection we deal with the case of a general function F, by means of a linearization procedure around $F(\phi_t)$ at each $t \in [0,1]$, which is usual in Onsager Machlup type results (see e.g. [8]). Let us introduce first some notation: given a differentiable function $S : H \to H$ and $x \in H$ we denote by $D_x S \in \mathcal{L}(H)$ the derivative operator of S at x, and for any $\xi \in L^2([0,1];H)$ we define by $D_\xi S$ the operator defined on $L^2([0,1];H)$ by $((D_\xi S)(\psi))(s) := (D_{\xi(s)} S)(\psi(s))$ for any $\psi \in L^2([0,1];H)$.

The main theorem is the following:

Theorem 3.5 *Assume that (H1) and (H2) are satisfied, h is a function verifying (H3), ϕ^h is defined by (8) and $F : H \to H$ is a Lipschitz continuous function such that $\hat{P} = B^{-1}F$ is $\mathcal{C}_b^2$. Let R be the linear operator defined in Theorem 3.1 and denote by $P : L^2([0,1];H) \to L^2([0,1];H)$ the operator given by*

$$(P(u \otimes l))(s) = u(s)(D_{\phi^h(s)}\hat{P})(l), \quad u \in L^2([0,1]), l \in H.$$

Assume finally that $\frac{1}{2}(PR^* + (PR^*)^*)$ is trace class and there exists a $r > 0$ and a deterministic trace class operator $\hat{T} : L^2([0,1];H) \to L^2([0,1];H)$ such that, for any ξ satisfying $\|\phi^h - \xi\|_2 \leq r$,

$$|\langle (D_\xi \hat{P} - D_{\phi^h}\hat{P})R^*\psi, \psi \rangle_{L^2([0,1];H)}| \leq \langle \hat{T}\psi, \psi \rangle_{L^2([0,1];H)}, \tag{18}$$

for any $\psi \in L^2([0,1];H)$.

Then

$$\lim_{\varepsilon \downarrow 0} \gamma_\varepsilon(\phi) = \exp\Big(-\frac{1}{2}\int_0^1 \Big|B^{-1}\Big[A\phi^h(t) + F(\phi^h(t)) - \dot{\phi}^h(t)\Big]\Big|_H^2 dt$$
$$-Tr\Big(\frac{1}{2}(PR^* + (PR^*)^*)\Big)\Big).$$

22

Proof: Since (h1) and (h2) are satisfied, like in the proof of Theorem 3.1 it is enough to study $\lim_{\varepsilon \downarrow 0} E\big[\exp(T_1)\big|\|W^A\|_2 \leq \varepsilon\big]$.

A Taylor's formula for Hilbert space valued functions gives us

$$\hat{P}(W^A(s) + \phi^h(s)) = \hat{P}(\phi^h(s)) + (D_{\phi^h(s)}\hat{P})(W^A(s))$$

$$+ \Big(\int_0^1 \big(D_{\phi^h(s) + \lambda W^A(s)}\hat{P} - D_{\phi^h(s)}\hat{P}\big)d\lambda\Big)(W^A(s)).$$

Then $T_1 := T_{1(c)} + T_{1(d)} + T_{1(e)}$ with

$$T_{1(c)} = \int_0^1 \langle \hat{P}(\phi^h(s)), dW(s)\rangle,$$

$$T_{1(d)} = \int_0^1 \langle (D_{\phi^h(s)}\hat{P})(W^A(s)), dW(s)\rangle,$$

$$T_{1(e)} = \int_0^1 \Big\langle \Big(\int_0^1 \big(D_{\phi^h(s) + \lambda W^A(s)}\hat{P} - D_{\phi^h(s)}\hat{P}\big)d\lambda\Big)(W^A(s)), dW(s)\Big\rangle,$$

Since $B^{-1}F$ is $\mathcal{C}_b^2$, we can deal $T_{1(c)}$ using the same arguments as for T_3.

The term $T_{1(d)}$ can be handled in much the same way as $T_{1(a)}$ in the proof of Theorem 3.1, the only difference being in the analysis of $T_{1(d)}$ (i.e. the dependence on s of the operator $D_{\phi^h(s)}\hat{P}$), but notice that the structure of the proof is still valid. Similarly to (14), $T_{1(d)}$ can be written

$$T_{1(d)} = \sum_{(k,j)\neq(m,i)} \frac{\mu_{k,j}}{\mu_{m,i}} Y_{k,j} Y_{m,i} \langle M_{k,j}^i, R_i g_{m,i}\rangle_{L^2[0,1]}$$

$$+ \sum_{j=1}^\infty \sum_{k=1}^\infty (Y_{k,j}^2 - 1)\langle M_{k,j}^j, R_j g_{m,j}\rangle_{L^2[0,1]},$$

with $M_{k,j}^i(s) := \langle (P(g_{k,j} \otimes e_j))(s), e_i\rangle$. Since

$$\langle M_{k,j}^i, R_i g_{m,i}\rangle_{L^2[0,1]} = \langle P(g_{k,j} \otimes e_j), R_i g_{m,i} \otimes e_i\rangle_{L^2([0,1];H)},$$

proceeding as in Theorem 3.1 we can then obtain

$$E\Big[\exp\big(T_{1(d)}\big)\Big|\|W^A\|_2 \leq \varepsilon\Big] \tag{19}$$

$$= \exp\Big(-\operatorname{Tr}(\frac{1}{2}(PR^* + (PR^*)^*))\Big)E\Big[\exp\big(T_{1(d,2)}\big)\Big|\|W^A\|_2 \leq \varepsilon\Big],$$

23

with

$$T_{1(d,2)} = \sum_{(k,j),(m,i)} Y_{k,j} Y_{m,i} \langle (U^* P R^* U)(g_{k,j} \otimes e_j), g_{m,i} \otimes e_i \rangle_{L^2([0,1];H)}$$

where U is defined in the proof of Theorem 3.1 and

$$\lim_{\varepsilon \to 0} E \Big[\exp \big(T_{1(d,2)} \big) \big| \|W^A\|_2 \le \varepsilon \Big] = 1.$$

Finally we have to study $T_{1(e)}$. We will follow the method used in [10, Theorem 4.1]. Set $\varphi^\lambda = \phi^h + \lambda W^A$. Just like in the case of $T_{1(d)}$, we get

$$\int_0^1 \Big\langle \int_0^1 \big((D_{\varphi^\lambda(s)} \hat{P} - D_{\phi^h(s)} \hat{P}) d\lambda \big) (W^A(s)), dW(s) \Big\rangle$$

$$= \int_0^1 \sum_{(k,j),(m,i)} Y_{k,j} Y_{m,i} T^\lambda_{(k,j),(m,i)} d\lambda - \int_0^1 \sum_{j,k} T^\lambda_{(k,j),(k,j)} d\lambda,$$

where

$$T^\lambda_{(k,j),(m,i)} = \langle (U^*(D_{\varphi^\lambda} \hat{P} - D_{\phi^h} \hat{P}) R^* U)(g_{k,j} \otimes e_j), g_{m,i} \otimes e_i \rangle_{L^2([0,1];H)}.$$

Since $\hat{P}$ is $\mathcal{C}_b^2$, we clearly have

$$|(D_{\varphi^\lambda(s)} \hat{P} - D_{\phi^h(s)} \hat{P})(y)| \le C |\varphi^\lambda(s) - \phi^h(s)|_H |y|_H,$$

for any $y \in H$. So, fixed $\lambda, (k,j), (m,i)$ $\lim_{\varepsilon \to 0} \mathbf{1}_{\{\|W^A\|_2 \le \varepsilon\}} T^\lambda_{(k,j),(m,i)} = 0$. Because of (18), by a dominated convergence argument we can prove that

$$\lim_{\varepsilon \to 0} \mathbf{1}_{\{\|W^A\|_2 \le \varepsilon\}} \int_0^1 \sum_{j=1}^\infty \sum_{k=1}^\infty T^\lambda_{(k,j),(k,j)} d\lambda = 0.$$

Then, by Lemma 2.2, we shall have established the Theorem if we prove that, for all $c \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} E \Big[\exp \Big(c \int_0^1 \sum_{(k,j),(m,i)} Y_{k,j} Y_{m,i} T^\lambda_{(k,j),(m,i)} d\lambda \Big) \Big| \|W^A\|_2 \le \varepsilon \Big] \le 1. \qquad (20)$$

But, by assumption (18) (see[10, Theorem 4.1]),

$$E \Big[\exp \Big(c \int_0^1 \sum_{(k,j),(m,i)} Y_{k,j} Y_{m,i} T^\lambda_{(k,j),(m,i)} d\lambda \Big) \Big| \|W^A\|_2 \le \varepsilon \Big]$$

24

$$\leq E\Big[\exp\Big(|c|\sum_{(k,j),(m,i)} Y_{k,j}Y_{m,i}\hat{T}_{(k,j),(m,i)}\Big)\Big|\,\|W^A\|_2 \leq \varepsilon\Big],$$

where

$$\hat{T}_{(k,j),(m,i)} = \big\langle\hat{T}(g_{k,j}\otimes e_j), g_{m,i}\otimes e_i\big\rangle_{L^2([0,1];H)}.$$

Therefore (20) follows from part 1) of Lemma 2.5. □

Example 3.6 *Suppose $H = L^2([0,1])$ with Dirichlet boundary conditions, $A = \nabla$, $B = Id$, and $F : H \to H$ is given by the Nemitsky operator $F(x) = f(Kx)$, where $K \in \mathcal{L}(H)$ is a trace class operator and $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}_b^2$ function. Then the conditions of Theorem 3.5 are satisfied.*

Proof: We shall concentrate on condition (18), the other ones being easy to verify. Under the assumptions of our example, $\hat{P} = F$, and the operator R^* is given by $[R^*(u \otimes e_j)] = R_j^* u \otimes e_j$ for all $u \in L^2([0,1])$ and $j \geq 1$, with

$$[R_j^* u](s) = \int_0^s e^{-\alpha_j(s-t)} u(t)dt, \quad t \in [0,1].$$

Furthermore,

$$|\langle(D_\xi F - D_{\phi^h} F)R^*(u \otimes e_j), (u \otimes e_j)\rangle_{L^2([0,1];H)}|$$
$$= |\langle[f'(K\xi) - f'(K\phi^h)](R_j u \otimes Ke_j), (u \otimes e_j)\rangle_{L^2([0,1];H)}|$$
$$\leq 2\|f'\|_\infty\langle\hat{T}(u \otimes e_j), (u \otimes e_j)\rangle_{L^2([0,1];H)},$$

with $\hat{T}(u \otimes e_j) = R_j u \otimes Ke_j$. Since K is trace class, it can be shown, like in example 3.3, that condition (18) is then fullfilled.

□

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