A relation between p-adic L-functions and the Tamagawa number conjecture for Hecke characters

Francesc Bars *

Abstract

We prove that the submodule in K-theory which gives the exact value (up to $\mathbb{Z}_{(p)}^*$) of the L-function by the Beilinson regulator map at non-critical values for Hecke characters of imaginary quadratic fields K with $cl(K)=1$(p-local Tamagawa number conjecture) satisfies that the length of its coimage under the local Soulé regulator map is the p-adic valuation of certain special values of p-adic L-functions associated to the Hecke characters. This result yields immediately, up to Jannsen's conjecture, an upper bound for $\#H^2_{et}(\mathcal{O}_K[1/S], V_p(m))$ in terms of the valuation of these p-adic L-functions, where V_p denotes the p-adic realization of a Hecke motive.

1 Introduction

Consider a Hecke character ψ of an imaginary quadratic field K. Then there is an associated complex L-function and, for any prime p, which we impose that it is totally split in K, certain p-adic L-functions associated to the two primes above p. In this paper we find, under technical conditions, a relation between the special values of the complex L-funcions at non-critical points and some special values of the p-adic L-functions.

Deninger [6] defines a pure motive for any Hecke character ψ of an imaginary quadratic field whose L-function is equal to the L-function of the corresponding Hecke character. He also proves the Beilinson conjecture for these motives.
For the p-adic L-functions Geisser proves in [8][9] a p-adic analogue of Beilinson's conjecture, relating the p-adic valuation of some special values of the p-adic L-functions with the length of a coimage module obtained from a map between Iwasawa modules.
The local Tamagawa number conjecture predicts the exact value at integer points of this L-function for the Hecke character ψ (up to $\mathbb{Z}_{(p)}^*$) in terms of the corresponding motive. At non-critical points we can rewrite basically the conjecture as the Beilinson conjecture for these motives and some p-adic control. The study of the local Tamagawa number conjecture for these Hecke characters is made in [2][3].

In this paper we compare the works [8] and [3] and we precise how in the case of Hecke characters the values for the p-adic L-functions appear in Kato's reformulation of the weak Tamagawa number conjecture. We consider the case $cl(K)=1$ for simplicity but we believe that the results should generalize with similar techniques to the case $p\nmid [H:K]$ where H is the Hilbert class field of K. The paper is basically a global-local comparison between Iwasawa modules and étale cohomology groups. Where we work with the subspace coming from the étale realization of the motivic elliptic polylogarithm [11]. A relation in

*Work partially supported by DGI, BHA2000-0180.

the p-adic world between the elliptic polylogarithm and a special value for the p-adic L-function is studied by K. Bannai [1].

I would like to thank Xavier Xarles for his useful comments and suggestions on different versions of this paper. It is a pleasure to thank him and Enric Nart for their encouragement along my work on the Tamagawa number conjecture. Kind thanks also to the referee for his/her comments and for noticing some inaccuracies. I thank Jérôme Scherer and Xavier Xarles for proof-reading my english and for more suggestions.

2 The Tamagawa number conjecture for Hecke characters

Let E be a fixed elliptic curve with CM, which is defined over an imaginary quadratic field K, with CM by $\mathcal{O}_K$, the ring of integers of K. Note that this implies $cl(K) = 1$. Associated to this elliptic curve there is a Hecke character φ of the imaginary quadratic field K with conductor $\mathfrak{f}$, an ideal of $\mathcal{O}_K$ that coincides with the conductor of the elliptic curve E.

Let us introduce the motives that we will use and the result on the weak local Tamagawa number conjecture for them. Consider the category of Chow motives $\mathcal{M}_{\mathbb{Q}}(K)$ over K with morphisms induced by graded correspondences in Chow theory tensored with $\mathbb{Q}$. Then, the motive of the elliptic curve E has a canonical decomposition $h(E)_{\mathbb{Q}} = h^0(E)_{\mathbb{Q}} \oplus h^1(E)_{\mathbb{Q}} \oplus h^2(E)_{\mathbb{Q}}$. The motive $h^1(E)_{\mathbb{Q}}$ has a multiplication by K [5, §1.3]. Let us consider the motive $\otimes^w_{\mathbb{Q}} h^1(E)_{\mathbb{Q}}$, for w a strictly positive integer, which has multiplication by $T_w := \otimes^w_{\mathbb{Q}} K$. Observe that T_w has a decomposition $\prod_\theta T_\theta$ as a product of fields T_θ, where θ runs through the $Aut(\mathbb{C})$-orbits of $\Upsilon^w = Hom(T_w, \mathbb{C})$, where $\Upsilon = Hom(K, \mathbb{C})$. This decomposition defines some idempotents e_θ and gives also a decomposition of the motive and its realizations. Let us fix once and for all an immersion $\lambda : K \to \mathbb{C}$ as in [6, p.135].

The L-function associated to the motive $e_\theta(\otimes^w h^1(E)_{\mathbb{Q}})$ corresponds to the L-function associated to $\psi_\theta = e_\theta(\otimes^w \varphi) : \mathbb{A}^*_K \to K^*$ ([6, §1.3.1]) a CM-character (for equivalent definitions of Hecke characters over an imaginary quadratic field, see [8, §2.2]), which, with the fixed embedding λ, corresponds to $\varphi^a \overline{\varphi}^b$, where $a, b \geq 0$ are integers such that $w = a + b$. The pair (a, b) is the infinite type for ψ_θ. We note that there are different θ with the same infinite type. Every θ gives two elements of Υ^w, one given by the infinite type $\vartheta \in \theta \cap Hom_K(T_w, \mathbb{C})$ and the other coming from the other embedding.

We introduce the notation

$$M_\theta := e_\theta(\otimes^w h^1(E)),$$

which is considered as an integral Chow motive (we can consider e_θ as an idempotent integer with $e_\theta(\otimes^w \mathcal{O}_K) \cong \mathcal{O}_K$ an $\otimes^w \mathcal{O}_K \cong \prod_{e_\theta} \mathcal{O}_K$ because $cl(K) = 1$, see [9, lemma 5.1] or [8, p.57]), that is an element in the category $\mathcal{M}(K)$ constructed like $\mathcal{M}_{\mathbb{Q}}(K)$ but without tensoring the correspondences by $\mathbb{Q}$. Observe that M_θ has multiplication by $\mathcal{O}_K \cong e_\theta(\otimes^w \mathcal{O}_K) =: \mathcal{O}_\theta$. Since we want to consider the values at positive integers of the L-function associated to the above motive in the non-critical strip band, we take integers $l \geq 0$ and we will study the Tamagawa number conjecture for the motive

$$M_\theta(w+l+1).$$

This conjecture predicts the value of the L-function for M_θ at $w+l+1$. Via the functional equation proved by Hecke for the Hecke characters, the study of these values is equivalent to the study of the value of the L-function at $-l$ since M_θ has weight w. We will formulate the Tamagawa number conjecture and our result on Hecke characters using this functional equation instead of the original formulation made by Bloch and Kato [4]. In this reformulation, the Tamagawa number conjecture predicts the cardinal of the coimage for the Beilinson regulator map as well as for the Soulé regulator map at every prime number p, of a lattice in a K-theory group associated to the motive. Let us introduce all these objects. The K-theory group corresponding to our motive $M_\theta(w+l+1)$ is

$$H_{\mathcal{M}} := K_{2(w+l)-w+1}(M_\theta)^{(w+l+1)} \otimes \mathbb{Q}$$

where the K-groups are the Quillen K-groups and the superscript indicates the Adam's filtration on them. Let S be a finite set of primes of K, which contains the primes above p, and such that the p-adic realization of the motive, $H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+l+1))$, is unramified outside S (in our situation S contains the finite primes of K which divide $p\mathfrak{f}_\theta$, where $\mathfrak{f}_\theta$ is the conductor of ψ_θ). Denote for simplicity

$$M_{\theta\mathbb{Z}_p}(w+l+1) := H^w_{\acute{e}t}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+l+1)),$$

and

$$M_{\theta\mathbb{Q}_p}(w+l+1) := H^w_{\acute{e}t}(M_\theta \times_K \overline{K}, \mathbb{Q}_p(w+l+1)).$$

We impose that $w-2(w+l+1) \leq -3$. We have a Beilinson regulator map,

$$r_{\mathcal{D}} : H_{\mathcal{M}} \otimes \mathbb{R} \to H^w(M_{\theta\mathbb{C}}, \mathbb{Q}(w+l)) \otimes \mathbb{R},$$

where the cohomology group on the right is the Betti realization for our motive, which coincides with $e_\theta(\otimes^w_{\mathbb{Q}} H^1(E(\mathbb{C}), \mathbb{Q}(1))(l)$. We have inside this $\mathbb{Q}$-vector space a $\mathbb{Z}$-lattice given by $H_{h,\mathbb{Z}} := e_\theta(\otimes^w_{\mathbb{Z}} H^1_B(E(\mathbb{C}), \mathbb{Z}(1)))(l)$, which is an $e_\theta(\otimes^w_{\mathbb{Z}} \mathcal{O}_K) \cong \mathcal{O}_K$-module of rank 1.

We have also, for every prime number p, the Soulé regulator map:

$$r_p : H_{\mathcal{M}} \otimes \mathbb{Q}_p \to H^1_{\acute{e}t}(\mathcal{O}_K[1/S], M_{\theta\mathbb{Q}_p}(w+l+1)).$$

The L-function associated to the motive M_θ is defined by

$$L_S(M_\theta, s) = L_S(M_{\theta\mathbb{Q}_p}, s) := \prod_{\mathfrak{p}\notin S} P_{\mathfrak{p}}(M_{\theta\mathbb{Q}_p}, s) \;\text{ for } Re(s) >> 0,$$

where $M_{\theta\mathbb{Q}_p} = H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Q}_p)$ and the local Euler factors $P_{\mathfrak{p}}(M_{\theta\mathbb{Q}_p}, s)$ are given by

$$P_{\mathfrak{p}}(M_{\theta\mathbb{Q}_p}, s) := det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{p}} N\mathfrak{p}^{-s} | M^{I_{\mathfrak{p}}}_{\theta\mathbb{Q}_p}) = (1 - \psi_\theta(\mathfrak{p}) N\mathfrak{p}^{-s})(1 - \overline{\psi_\theta}(\mathfrak{p}) N\mathfrak{p}^{-s})$$

where Fr means the geometric Frobenius and $I_{\mathfrak{p}}$ is the inertia group at $\mathfrak{p}$ [3, Lemma 2.5].

After all these preliminaries we can formulate the local weak Tamagawa number conjecture at p for odd primes for our motives:

Conjecture 2.1. *Let us fix a prime number* $p \neq 2$ *and let* S *be the finite primes of* K *which divide* $p\mathfrak{f}_\theta$*, where* $\mathfrak{f}_\theta$ *is the conductor of* ψ_θ *with infinite type* (a, b)*,* $a, b \geq 0$ *integers with* $w = a + b \geq 1$*. Let* $l \geq 0$ *be an integer. Then, there is a subspace* $H_{\mathcal{M}}^{constr}$ *in* $H_{\mathcal{M}}$ *such that*

1. *The maps* $r_{\mathcal{D}}$ *and* r_p *restricted to* $H_{\mathcal{M}}^{constr}$ *are isomorphisms and the group* $H^2(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w + l + 1))$ *is finite.*
2. $dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=-l} L_S(H^w(M_\theta, \mathbb{Q}_p), s)$=*2.*
3. *Let* $\eta \in det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$ *be a* $\mathbb{Z}$*-basis. There is an element* $\xi \in det_{\mathbb{Q}} H_{\mathcal{M}}^{constr}$ *such that*
$$r_{\mathcal{D}}(\xi) = (\lim_{s \to -l} s^{-2} L_S(H^w(M_\theta, \mathbb{Q}_p), s))\eta.$$
4. *The element* $r_p(\xi)$ *is a basis of the* $\mathbb{Z}_p$*-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w + l + 1)))^{-1}$$
$$\subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Q}_p}(w + l + 1))[-1]).$$

Remark 2.2. The above conjecture when we take $H_{\mathcal{M}}^{constr} = H_{\mathcal{M}}$ corresponds to the local Tamagawa number conjecture.

Theorem 2.3 (Theorem 5.13 [3]). *Let be* p *a prime* $\neq 2, 3$ *and* $p > N_{K/\mathbb{Q}}\mathfrak{f}$*. Consider* l *a strictly positive integer. Suppose that* ψ_θ *has infinite type* (a, b) *with* a, b *non-negative integers, such that* $a \not\equiv b \ mod |\mathcal{O}_K^*|$ *and* $w = a + b \geq 1$ *verifies* $-w - 2l \leq -3$*. Suppose furthermore that* $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ *is injective. Write* Δ *for the Galois group* $Gal(K(E[p])/K)$*, and consider the* Δ*-representation*

$$Hom_{\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p}(M_{\theta\mathbb{Z}_p}(w + l), \mathcal{O}_K \otimes \mathbb{Z}_p).$$

Finally, suppose that this representation is irreducible, that it is not the cyclotomic character, and that it satisfies the hypothesis of [12, theorem 4.1] on the main Iwasawa conjecture (these conditions are satisfied if p splits). Then, the above conjecture 2.1 is true up to the finiteness for $H^2(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))$ *and the injectivity on the Soulé regulator map.*

Let us write down explicitly the construction of $H_{\mathcal{M}}^{constr}$ in theorem 2.3, and briefly indicate a sketch of the proof. The elements in K-theory for these motives were first constructed by Deninger in [5]. Denote by $\Gamma = \Omega\mathcal{O}_K$ the lattice of the elliptic curve, and let $f \in \mathcal{O}_K$ be a generator for $\mathfrak{f}_\theta$. Then we have that $\Omega f^{-1} \in \mathfrak{f}_\theta^{-1}\Gamma$, and hence (Ωf^{-1}) gives a divisor in $\mathbb{Z}[E[\mathfrak{f}_\theta] \setminus 0]$ defined over $K(E[\mathfrak{f}_\theta]) \subset K(E[\mathfrak{f}]) = K(\mathfrak{f})$ since $\mathfrak{f}_\theta$ divides the conductor $\mathfrak{f}$ of the elliptic curve E. We obtain next a divisor defined over K by taking the norm:

$$\beta_\theta := N_{K(E[\mathfrak{f}])/K}((\Omega f^{-1})).$$

Deninger constructs a map from the divisors to $H_{\mathcal{M}}$ as a composition of an Eisenstein symbol map with a projection map (see [6, §2]),

$$\mathcal{D}en^{w,l,\theta} : \mathbb{Z}[E[\mathfrak{f}_\theta] \setminus 0] \to K_{2(w+l)-w+1}(M_\theta)^{(w+l+1)} \otimes \mathbb{Q}.$$

Then, we define $\mathcal{R}_\theta$ by

$$(-1)^{l-1}\frac{(2l+w)!P_p(\overline{\psi_\theta},-l)^{-1}\Phi(\mathfrak{f})}{2^{l-1}N_{K/\mathbb{Q}}\mathfrak{f}_\theta^l\psi_\theta(f)\Phi(\mathfrak{f}_\theta)}\mathcal{D}en^{w,l,\theta}(\beta_\theta)\mathcal{O}_K,$$

where $\Phi(\mathfrak{m}) := |(\mathcal{O}_K/\mathfrak{m})^*|$ and P_p is the product of the Euler factor of $\overline{\psi_\theta}$ at primes over p (see [3, remark 2.6] for the non-vanishing of these Euler factors). We take $H^{constr}_{\mathcal{M}}$ equal to $\mathcal{R}_\theta \otimes \mathbb{Q}$.

For the proof of the theorem 2.3 we need to study the image of $H^{constr}_{\mathcal{M}}$ with respect to the Beilinson and Soulé regulator maps. In the case of the Beilinson regulator map this study is basically made by Deninger in [6]. We compare the Soulé regulator map with a map from an Iwasawa module to a global Galois cohomology group,

$$(Soul)_p : (\overline{\mathcal{C}}_\infty \otimes (e_\theta \otimes^w T_pE)(l))_{\mathcal{G}} \to H^1(\mathcal{O}_K[1/S], (e_\theta \otimes^w T_pE)(l+1)),$$

where T_pE means the Tate module of the elliptic curve E at p, $\overline{\mathcal{C}}_\infty$ are the elliptic units and $\mathcal{G}$ is the Galois group $Gal(K_\infty/K)$ (since $K_\infty = \cup_{n\in\mathbb{N}}K(E[p^n])$). Let us introduce here the definition of this elliptic units for our latter interest.

Definition 2.4. *Let C_n be the subgroup of units generated over $\mathbb{Z}[Gal(K(E[p^n]))]$ by*

$$\prod_{\sigma\in Gal(K(\mathfrak{f})/K)} \theta_{\mathfrak{a}}(t_{\mathfrak{f}}^\sigma + h_n)$$

where $\mathfrak{a}$ runs through all ideals prime to $6p\mathfrak{f}$, $t_{\mathfrak{f}}$ is a primitive $\mathfrak{f}$-torsion point and h_n is a primitive p^n-torsion point, and $\theta_{\mathfrak{a}}$ is the classical theta function (see [7, II]). Define the group of elliptic units of $K_n := K(E[p^n])$ as

$$\mathcal{C}_n := \mu_\infty(K_n)C_n.$$

If $\mathcal{U}_n^{\mathfrak{p}}$ is the group of local units at K_n congruent to 1 for all place v of K_n over $\mathfrak{p}$ (called local principal units), denote by $\overline{\mathcal{C}}_n$ the closure of $\mathcal{C}_n \cap \mathcal{U}_n^{\mathfrak{p}}$ in $\mathcal{U}_n^{\mathfrak{p}}$. Considering the limit with respect to the norm maps we obtain:

$$\overline{\mathcal{C}}_\infty := \varprojlim \overline{\mathcal{C}}_n.$$

Remark 2.5. The definition above coincides with the one in [2] but it differs from the one in [3]. Theorem 2.3 however remains valid with this definition as well, using a similar argument as in remark 4.3 [3].

The main Iwasawa conjecture proved by Rubin [12] and the specialization of the elliptic polylogarithm sheaf proved by Kings [11] allows to compare the image of the map e_p with the image of $H^{constr}_{\mathcal{M}}$ under the Soulé regulator map. This concludes the proof of the local weak Tamagawa number conjecture.

3 Relation with Geisser's p-adic analogue of Beilinson's conjectures

We want to relate theorem 2.3 with the results of Geisser (see [8] or [9]). We impose once and for all that p splits in K, with $p = \mathfrak{p}\mathfrak{p}^*$, $\mathfrak{p} \neq \mathfrak{p}^*$. Observe that $M_\theta \otimes \mathbb{Z}_p$ has multiplication by $\mathcal{O}_\theta \otimes \mathbb{Z}_p$ and, as p splits, it decomposes in two idempotents. Denote by e_{Ω_1} and e_{Ω_2} the two idempotents which give the following decomposition

$$M_\theta \otimes \mathbb{Z}_p = M_{\Omega_1} \oplus M_{\Omega_2} \in \mathcal{M}_{\mathbb{Z}_p}(K)$$

in the category of Chow motives with coefficients in $\mathbb{Z}_p$. We have also a direct sum decomposition of the map $(Soul)_p = (Soul)_{\Omega_1} \oplus (Soul)_{\Omega_2}$,

$$(\overline{\mathcal{C}}_\infty \otimes M_{\Omega_1 \mathbb{Z}_p}(w+l))_{\mathcal{G}} \oplus (\overline{\mathcal{C}}_\infty \otimes M_{\Omega_2 \mathbb{Z}_p}(w+l))_{\mathcal{G}} \to$$
$$H^1(K, M_{\Omega_1 \mathbb{Z}_p}(w+l+1)) \oplus H^1(K, M_{\Omega_2 \mathbb{Z}_p}(w+l+1))$$

where $M_{\Omega_i \mathbb{Z}_p}$ is the p-adic lattice corresponding to $e_{\Omega_i}(\otimes^w H^1_{et}(\overline{M}, \mathbb{Z}_p))$.

Remark 3.1. The identification

$$H^1(K, M_{\theta\mathbb{Z}_p}(w+l+1)) = H^1(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)),$$

comes easily from the localization exact sequence.

Observe that $\psi_\theta \otimes \mathbb{Z}_p = \psi_{\Omega_1} \oplus \psi_{\Omega_2}$ and they satisfy $\overline{\psi_{\Omega_1}} = \psi_{\Omega_2}$. If (a_θ, b_θ) denotes the infinity type of ψ_θ, and we suppose that Ω_1 comes from $\mathfrak{p}$ and Ω_2 from $\mathfrak{p}^*$ by using the identification $\mathcal{O}_\theta = \mathcal{O}_K$, then the infinity types for ψ_{Ω_1} and ψ_{Ω_2} are (a_θ, b_θ) and (b_θ, a_θ) respectively. Here the infinity types come from morphisms to $\mathbb{C}_p$, (see [9] for more details). Let us denote by $\iota : H^1(K, \ \ ) \to H^1(K_\mathfrak{p}, \ \ )$ the restriction map.

Theorem 3.2 (Geisser, theorem 9.1 [9]). *Suppose that $p > 3w + 2l + w + 1$, θ is an idempotent with infinite type (a, b) with $a, b \geq 0$, $w = a+b \geq 1$, $a+l > 0$, $b+l > 0$ and p is a prime which splits in K. Then, the length as an $\mathcal{O}_{\Omega_i} \cong \mathbb{Z}_p$-module of the coimage of the Geisser elliptic units $\overline{\mathcal{C}}^{Rob}$ (defined below) via the map $\iota \circ e_{\Omega_i}$ is equal to the p-adic valuation of the p-adic L-function*

$$G(\psi_{\Omega_i}\kappa^l, u_1^{-a_i-1} - 1, u_2^{-b_i-1} - 1)$$

where (a_i, b_i) is the infinity type for ψ_{Ω_i}, κ is the cyclotomic character for $\mathcal{G}$ and G is the $(\psi_{\Omega_i}\kappa^l)^{-1}$-component of the two variable p-adic L-function (see [9, p. 227] for an explicit definition or see below).

We define now the Geisser elliptic units; and prove that they coincide with the elliptic units introduced in 2.4. The Geisser elliptic units are a modification of the elliptic units one can find in the book of de Shalit [7]. We follow [7, III §1]. Consider the ideal of K given by $\mathfrak{g} := \mathfrak{f}\mathfrak{p}^{*n}$. We define $C'_{n,m}$ as the group generated by the primitive Robert units of conductor $\mathfrak{g}\mathfrak{p}^m$. Then $C'_{n,m}$ is generated by $\theta_\mathfrak{a}(t_\mathfrak{f} + \mathfrak{h}_{n,m})$ where $\mathfrak{h}_{n,m}$ is a point of $\mathfrak{p}^{*n}\mathfrak{p}^m$-torsion, with $(\mathfrak{a}, 6\mathfrak{g}\mathfrak{p}) = 1$. If μ_∞ denotes the roots of unity in $K(\mathfrak{g}\mathfrak{p}^m)$, consider the group $C_{n,m} := C'_{n,m}\mu_\infty(K(\mathfrak{g}\mathfrak{p}^m))$. Define next $\overline{C}_{\mathfrak{g}\mathfrak{p}^m} = \overline{C}_{\mathfrak{f}\mathfrak{p}^{*n}\mathfrak{p}^m}$ as the closure of $C_{n,m} \cap \mathcal{U}^\mathfrak{p}_{\mathfrak{g}\mathfrak{p}^m}$ in $\mathcal{U}^\mathfrak{p}_{\mathfrak{g}\mathfrak{p}^m}$ where $\mathcal{U}^\mathfrak{p}_{\mathfrak{g}\mathfrak{p}^m}$ are the local principal units at $K(\mathfrak{g}\mathfrak{p}^m)$ for the places over $\mathfrak{p}$. Define also $\overline{C}_{\mathfrak{f}(\mathfrak{p}^*)^n} := \varprojlim_m \overline{C}_{\mathfrak{f}\mathfrak{p}^{*n}\mathfrak{p}^m}$, and

$$\overline{\mathcal{C}}(\mathfrak{f}) := \varprojlim_n \overline{C}_{\mathfrak{f}\mathfrak{p}^{*n}},$$

where these limits are defined with respect to the norm maps.

The Geisser elliptic units over K are finally defined by taking the norm map from $K(\mathfrak{f})$ to K of $\overline{\mathcal{C}}(\mathfrak{f})$. We will denote these elliptic units by $\overline{\mathcal{C}}^{Rob}$.

Lemma 3.3. *The Geisser elliptic units coincide with the elliptic units 2.4.*

Proof. First of all, notice that

$$N_{K(\mathfrak{f})/K}\varprojlim_{n,m} \cong \varprojlim_{n,m} N_{K(\mathfrak{f})/K},$$

where the projective limit (with respect to the norm maps) on the left is taken over $K(\mathfrak{f}\mathfrak{p}^m(\mathfrak{p}^*)^n)$, and on the right over $K(E[\mathfrak{p}^m\mathfrak{p}^{*n}])$. This equality comes from the fact that $K(E[\mathfrak{g}]) = K(\mathfrak{g})$ if $\mathfrak{f}$ divides $\mathfrak{g}$.

From the above we obtain

$$N_{K(\mathfrak{f})/K}\varprojlim_{n}\varprojlim_{m}\mathcal{U}^{\mathfrak{p}}_{\mathfrak{f}(\mathfrak{p}^*)^n\mathfrak{p}^m} = \varprojlim_{n}\varprojlim_{m}N_{K(\mathfrak{f})/K}\mathcal{U}^{\mathfrak{p}}_{\mathfrak{f}(\mathfrak{p}^*)^n\mathfrak{p}^m} = \varprojlim_{n}N_{K(\mathfrak{f})/K}\mathcal{U}^{\mathfrak{p}}_{\mathfrak{f}p^n} = \varprojlim_{n}\mathcal{U}^{n}_{\mathfrak{p}};$$

the second equality holds because the projectives limits coming from the bicomplex $U_{n,m} := N_{K(\mathfrak{f})/K}\mathcal{U}_{\mathfrak{f}\mathfrak{p}^{*m}\mathfrak{p}^n}$ coincide, and as $K(\mathfrak{f})/K$ is unramified at p we obtain the last equality. We compare now the definitions of elliptic units. For similar reasons we have,

$$\overline{\mathcal{C}}^{Rob} = \varprojlim_{n}\varprojlim_{m}N_{K(\mathfrak{f})/K}\overline{C}_{\mathfrak{f}(\mathfrak{p}^*)^n\mathfrak{p}^m} = \varprojlim_{n}N_{K(\mathfrak{f})/K}\overline{C}_{\mathfrak{f}\mathfrak{p}^n\mathfrak{p}^{*n}}.$$

Let us observe that $N_{K(\mathfrak{f})/K}C_{n,n}$ coincides with $\mathcal{C}_n$ possibly up to roots of unity. Thus as $K(\mathfrak{f})/K$ in unramified at primes above p, $\overline{\mathcal{C}}_n$ and $N_{K(\mathfrak{f})/K}\overline{C}_{\mathfrak{f}(\mathfrak{p}^*)^n\mathfrak{p}^n}$ could differ only up to roots of unity. Since we take intersection with the principal units at places over $\mathfrak{p}$, where the norm map of $K(\mathfrak{f})/K$ is surjective, both definitions must coincide. □

We recall now some facts on p-adic L-functions in our situation. Let us denote by $\mathbb{D}$ the ring of integers of the maximal unramified extension of $K_{\mathfrak{p}}$. All characters of finite groups of order prime to p have values in $\mathbb{D}$.

Observe that the Galois group $\Gamma := Gal(K_\infty/K(E[p]))$ is isomorphic to a product $\Gamma_1\times\Gamma_2$ of two copies of $\mathbb{Z}_p$, Γ_1 being the Galois group $Gal(K(E[\mathfrak{p}^\infty])/K(E[\mathfrak{p}]))$ and Γ_2 the analogue for $\mathfrak{p}^*$. Let γ_i be a generator of Γ_i, and let κ_i be the character of Γ_i giving the action on the torsion points of the elliptic curve. Denote by u_i the image of γ_i in $\mathbb{Z}_p$.

We recall the connections between measures and power series. We have the isomorphism

$$\Lambda(\Gamma_1 \times \Gamma_2, \mathbb{D}) \cong \mathbb{D}[[T_1, T_2]],$$

mapping a measure μ in $\Lambda(\Gamma, \mathbb{D})$ to the power series

$$G(T_1, T_2) := \int_\Gamma (1 + T_1)^\alpha (1 + T_2)^\beta d\mu(\alpha, \beta).$$

In particular $G(u_1^a - 1, u_2^b - 1)$ is equal to

$$\int (u_1^a)^\alpha (u_2^b)^\beta d\mu(\alpha, \beta) = \int \kappa_1^a \kappa_2^b d\mu.$$

Let μ be a measure in $\Lambda(\mathcal{G}, \mathbb{D})$ for $\mathcal{G} = Gal(K(E[p^\infty])/K)$, and let χ be a character of $Gal(K(E[p])/K)$. We denote the power series associated to the χ-component of μ by $G(\chi^{-1}, T_1, T_2)$. Then we obtain that

$$G(\chi^{-1}, u_1^a - 1, u_2^b - 1) = \int_{\mathcal{G}} \kappa_1^a \kappa_2^b \chi\mu = \int_{\Gamma_1\times\Gamma_2} \kappa_1^a \kappa_2^b d\chi(\mu).$$

We denote $G(\chi\kappa_1^a\kappa_2^b, T_1, T_2)$ by $G(\chi, T_1, T_2)$ for a character $\chi\kappa_1^a\kappa_2^b$ of $\mathcal{G}$ (in particular for the characters $\psi_{\Omega_i}\kappa^l$, see [9, §4 and §9])

By the interpolation theorem [7, theorem 4.14], $G(\chi, u_1^a - 1, u_2^b - 1)$ is a p-adic interpolation of $L(\chi\kappa_1^{b-a}, -b) = L(\chi\kappa_1^{-a}\kappa_2^{-b}, 0)$, at least for $0 \leq -b \leq a$.

Proposition 3.4. *With the hypotheses of theorems 2.3 and 3.2, the length of the coimage of $\iota \circ r_p(\mathcal{R}_\theta)$ in $H^1(K_\mathfrak{p}, M_{\theta\mathbb{Z}_p}(w+l+1))$ is equal to the p-adic valuation of*

$$G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1} - 1, u_2^{-b_\theta-1} - 1)G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1} - 1, u_2^{-a_\theta-1} - 1).$$

Proof. It is easy to check any element in the module $(\overline{\mathcal{C}}_\infty \otimes M_{\Omega_i}(w+l))_{\mathcal{G}}$ is an elliptic unit. This is because M_{Ω_i} is a free $\mathbb{Z}_p$-module of rank 1 endowed with a Galois action ([8, proposition 2.4.6]) and the coinvariants come from the representation $Hom_{\mathbb{Z}_p}(M_{\Omega_i}(w+l), \mathbb{Z}_p)$. Therefore

$$e_{\Omega_i}(\overline{\mathcal{C}}_\infty) = e_{\Omega_i}((\overline{\mathcal{C}}_\infty \otimes M_{\Omega_i}(w+l))_{\mathcal{G}}).$$

The comparison map between the image of e_p and of r_p, see [3, corollary 5.9], yields

$$e_p((\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_{\mathcal{G}}) = r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p).$$

Hence, we obtain the following equality

$$\iota \circ e_p(\overline{\mathcal{C}}_\infty) = \iota \circ r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p).$$

The lenght of the coimage of $e_{\Omega_i}(\overline{\mathcal{C}}_\infty)$ is known factor by factor: it is equal to $G(\psi_{\Omega_i}\kappa^l, u_1^{-a_i-1} - 1, u_2^{-b_i-1} - 1)$. The direct decomposition of e_p and theorem 3.2 permit then to conclude. $\square$

Remark 3.5. In the context of the Tamagawa number conjecture with $w = 1$ for simplicity, (i.e. in the case of an elliptic curve with CM $\mathcal{O}_K$ defined over K), the conjecture follows from controlling the image of $\mathcal{R}_\theta$ by the Beilinson regulator (which gives us the value of the L-function associated to the elliptic curve at $-l$ (up to $\mathbb{Z}_{(p)}^*$)) and the coimage of $r_p(\mathcal{R}_\theta)$ in $H^1(K, T_pE(l+1))$ [11]. We prove above that the coimage of $\iota \circ r_p(\mathcal{R}_\theta)$ is related with some values of the p-adic L-functions associated naturally to E.

Corollary 3.6. † *Under the technical conditions of the above proposition we have the following inclusion of $\mathbb{Z}_p$-modules in $\mathbb{Q}_p$,*

$$det_{\mathbb{Z}_p} H^2_{et}(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)) \subseteq$$

$$p^{-v_p(G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1}-1, u_2^{-b_\theta-1}-1)G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1}-1, u_2^{-a_\theta-1}-1))}\mathbb{Z}_p.$$

Moreover, if the Jannsen conjecture is true for these motives, (see [2, Appendix B] for the formulation and for some positive answers), an upper bound for $\#H^2_{et}(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))$ is given by,

$$p^{v_p(G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1}-1, u_2^{-b_\theta-1}-1)G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1}-1, u_2^{-a_\theta-1}-1))}.$$

†We use the determinant formulae as in [10].

Proof. Set $H_p^i := H_{et}^i(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))$. We know that $det_{\mathbb{Z}_p} H_p^2 \cong det_{\mathbb{Z}_p}(H_p^1/r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p))$, and, by the above proposition, we have,

$$det_{\mathbb{Z}_p}(H^1(K_{\mathfrak{p}}, M_{\theta\mathbb{Z}_p}(w+l+1))/\iota(r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p))) \subseteq$$

$$p^{-v_p(G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1}-1, u_2^{-b_\theta-1}-1)G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1}-1, u_2^{-a_\theta-1}-1))}\mathbb{Z}_p.$$

Let us consider the injective map

$$\bar{i} : H_p^1/(r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p) + Ker(\iota)) \to (H^1(K_{\mathfrak{p}}, M_{\theta\mathbb{Z}_p}(w+l+1))/\iota(r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p)))$$

and the short exact sequence,

$$0 \to \frac{r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p) + Ker(\iota)}{r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p)} \to \frac{H_p^1}{r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p)} \to \frac{H_p^1}{r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p) + ker(\iota)} \to 0.$$

From the determinant property for short exact sequences follows that

$$det_{\mathbb{Z}_p}(\frac{H_p^1}{r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p)}) \subseteq det_{\mathbb{Z}_p}(H_p^1/(r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p) + Ker(\iota)))$$

$$\subseteq det_{\mathbb{Z}_p}(H^1(K_{\mathfrak{p}}, M_{\theta\mathbb{Z}_p}(w+l+1))/\iota(r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p))).$$

For the first inclusion we use that $ker(\iota)$ has no torsion of H_p^1 and that by assumption $\iota(r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p)) \neq 0$. To obtain the second inclusion one checks that $det_{\mathbb{Z}_p}(coker(\bar{\iota})) = p^{-\alpha}\mathbb{Z}_p$ with $\alpha \geq 0$.
Let us recall that if A is finite and $det_{\mathbb{Z}_p} A = p^{-\delta}\mathbb{Z}_p$ then $\#A = p^\delta$, thus we obtain an upper bound for H_p^2 when it is finite. □

Corollary 3.7. *Denote by $\mathcal{P}ol_{\mathbb{Q}_p}$ the elliptic polylogarithm sheaf (see [11, §3.2]). Let β_θ be as in §2. We have ([3, theorem 5.2])*

$$r_p(Den^{w,l,\theta}(\beta_\theta)) = -N_{K/\mathbb{Q}}\mathfrak{f}_\theta^{2(w+2l)}\mathcal{K}_{\mathcal{M}}(\beta_\theta^*\mathcal{P}ol_{\mathbb{Q}_p})^{w+2l}$$

in $H_{et}^1(\mathcal{O}_S, Sym^{w+2l}T_pE \otimes \mathbb{Q}_p)$, where $\mathcal{K}_{\mathcal{M}}$ is the projector map ([6, §2.8]). Then under the technical condition of the proposition 3.4 the length of the coimage of the $\mathcal{O}_K \otimes \mathbb{Z}_p$-module generated by the specialization of the polylogarithm sheaf, $\iota(\mathcal{K}_{\mathcal{M}}(\beta_\theta^*\mathcal{P}ol_{\mathbb{Q}_p})^{w+2l}\mathcal{O}_K \otimes \mathbb{Z}_p)$, is the p-adic valuation of

$$(P_p(\overline{\psi_\theta, -l})((2l+w)!)^{-1})^2 G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1}-1, u_2^{-b_\theta-1}-1)G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1}-1, u_2^{-a_\theta-1}-1).$$

Proof. We only need to remark that the difference between the p-adic valuation of ξ_θ and $Den^{w,l,\theta}(\beta_\theta)$ is the factor $P_p(\overline{\psi_\theta, -l})/(2l+w)!$ (see the definition of ξ_θ given after theorem 2.3). The length of the coimage of $\iota(\mathcal{K}_{\mathcal{M}}(\beta_\theta^*\mathcal{P}ol_{\mathbb{Q}_p})^{w+2l}\mathcal{O}_K \otimes \mathbb{Z}_p)$ in $H^1(K_{\mathfrak{p}}, M_{\theta\mathbb{Z}_p}(w+l+1))$, is (under the conditions of proposition 3.4) the length of the coimage of $\frac{P_p(\overline{\psi}_\theta)}{(2l+w)!}\iota(r_p(\mathcal{R}_\theta \otimes \mathbb{Z}_p))$ in $H^1(K_{\mathfrak{p}}, M_{\theta\mathbb{Z}_p}(w+l+1))$. This cohomology group decomposes into two pieces, $H^1(K_{\mathfrak{p}}, M_{\Omega_i}\mathbb{Z}_p)(w+l+1))$ with $i = 1, 2$. From the proof of [8, theorem 3.3.1], we see that the length of the coimage in each piece is

$$v_p(P_p(\overline{\psi_\theta, -l})((2l+w)!)^{-1}G(\psi_{\Omega_i}\kappa^l, u_1^{-a_i-1}-1, u_2^{-b_i-1}-1),$$

obtaining the result. □

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Francesc Bars Cortina
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
Catalunya
Spain
e-mail: francesc@mat.uab.es