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Endpoint estimates from restricted rearrangement inequalities*

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Abstract

Let T be a sublinear operator such that $(Tf)^*(t) \leq h(t, \|f\|_1)$ for some positive function $h(t, s)$ and every function f such that $\|f\|_\infty \leq 1$. Then, we show that T can be extended continuously from a logarithmic type space into a weighted weak Lorentz space. This type of result is connected with the theory of restricted weak type extrapolation and extends a recent result of Arias-de-Reyna concerning the pointwise convergence of Fourier series to a much more general context.

1 Introduction

Let S be the Carleson maximal operator,

$$Sf(x) = \sup_n |S_n f(x)|,$$

where $S_n f(x) = (D_n * f)(x)$, being D_n the Dirichlet kernel on $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ and $f \in L^1(\mathbb{T})$. Then, it was proved in [16] that, for every measurable set $E \subset \mathbb{T}$,

$$(S\chi_E)^*(t) \preceq \frac{|E|}{t}\left(1 + \log^+ \frac{t}{|E|}\right). \tag{1}$$

This result can be improved using the following lemma due to Antonov (see [1]).

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Lemma 1.1 **(Antonov)** *Let $S^N f(x) = \sup_{0\leq n\leq N} |S_n f(x)|$. Then, for every $\varepsilon > 0$, every $N \in \mathbb{N}$ and every $0 \leq f(x) \leq 1$, there exists a measurable set F such that $|F| = \|f\|_1$ and $\left\|S^N (f - \chi_F)\right\|_\infty \leq \varepsilon$.*

Using this lemma and the above estimate on characteristic functions we can conclude that

$$(Sf)^*(t) \preceq \frac{\|f\|_1}{t}\left(1 + \log^+ \frac{t}{\|f\|_1}\right), \tag{2}$$

for every $f \in L^1$ such that $\|f\|_\infty \leq 1$.

Antonov's lemma has been extended in [17] to more general operators, namely to any maximal operator of the form

$$Tf(x) = \sup_j |K_j * f(x)|,$$

where $K_j \in L^1$, and therefore, (2) holds for any operator T of the above form such that T satisfies (1). Examples of such operators are given in [17] in the setting of differentiation of integrals and the Halo conjecture.

In particular, (and this is the connection with the weak extrapolation theory, see [11] and [18]) if T is an operator such that, for every $1 < p \leq 2$,

$$(Tf)^*(t)t^{1/p} \leq \frac{1}{(p-1)^m} \|f\|_p,$$

then, for every $f \in L^1$ such that $\|f\|_\infty \leq 1$,

$$(Tf)^*(t) \leq \frac{1}{(p-1)^m} \frac{\|f\|_1^{1/p}}{t^{1/p}},$$

and taking the infimum in p, we conclude that

$$(Tf)^*(t) \preceq \frac{\|f\|_1}{t}\left(1 + \log^+ \frac{t}{\|f\|_1}\right)^m. \tag{3}$$

Our main purpose (see Theorem 3.1) is to show that if T is a sublinear operator satisfying

$$(Tf)^*(t) \preceq h(t, \|f\|_1),$$

for some positive function h and every $\|f\|_\infty \leq 1$, then

$$T : Q_D \longrightarrow M(R)$$

is bounded, where $h(t,s) \le D(s)R(t)$,

$$Q_D = \{f; f = \sum_k e_k f_k,\ \|f_k\|_\infty \le 1, \|f\|_{Q_D} < \infty\},$$

with

$$\begin{aligned} &\|f\|_{Q_D} \\ = \quad &\inf\left\{ \sum_k e_k D(\|f_k\|_1)\Big(1+\log\frac{1}{a_k}\Big);\ \sum_k a_k = 1, a_k \ge 0, f = \sum_k e_k f_k \right\}, \end{aligned}$$

and

$$\|f\|_{M(R)} := \sup_{t>0} \frac{f^*(t)}{R(t)}.$$

In particular, if $D(s) = s\Big(1+\log^+\frac{1}{s}\Big)$ and $T = S^*$, then $Q_D = QA$, where QA is, up to now, the biggest space where the pointwise convergence of the Fourier series is known to hold (see [2]).

Our proof turns out to be very simple and is based in the following basic result (see [7]):

Lemma 1.2 *(***Basic result:***) Let $f = \sum_n f_n$ with $f_n \ge 0$ and let $c_n > 0$ be such that $\sum_n c_n = 1$. Then*

$$f^*(3t) \le \sum_n \left(f_n^*(t) + \frac{1}{t}\int_{c_n t}^t f_n^*(s)ds \right).$$

¿From it, the main result of this paper, which covers as a particular case the result of Arias-de-Reyna, can be immediately obtained.

The point now is that the space Q_D is difficult to handle and, therefore, it is convenient for the applications to find spaces of Logarithmic type L such that $L \subset Q_D$. In [2], it is proved that the space $L\log L\log\log\log L(\mathbb{T}) \subset QA$. We shall extend this result to our general context.

Another situation we consider in this work is the following: Let Ω be any domain in $\mathbb{R}^n$ and let $W^{1,p}(\Omega)$ be the classical Sobolev space and set $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

$$\|f\|_{W_0^{1,p}(\Omega)} = \|f\|_p + \|\nabla f\|_p,$$

where ∇f is the gradient of f. Let T be a sublinear operator such that

$$T : W_0^{1,p}(\Omega) \longrightarrow L^{p,\infty}$$

is bounded with constant C_p for every $p \in I \subset [1,\infty)$. Then, for every f such that $\|f\|_\infty + \|\nabla f\|_\infty \leq 1$, it holds that

$$\begin{aligned}(Tf)^*(t)t^{1/p} &\leq C_p \left(\int_0^\infty f^*(t)^p + |\nabla f|^*(t)^p \, dt\right)^{1/p} \Bigg] \\ &\leq C_p \left(\int_0^\infty f^*(t) + |\nabla f|^*(t) \, dt\right)^{1/p}.\end{aligned}$$

Consequently,

$$(Tf)^*(t) \leq \inf_{p\in I} \left(C_p \left(\frac{\|f\|_{W_0^{1,1}(\Omega)}}{t} \right)^{1/p} \right) := h(t, \|f\|_{W_0^{1,1}(\Omega)}).$$

Then we show that the technique developed in Section 2 can also be extended to cover this situation and, in fact, our theory can be presented in the setting of compatible pairs of Banach spaces $\bar{A} = (A_0, A_1)$ using some of the ideas developed in [6]; that is, our operator T will be a sublinear operator acting on elements of the sum space $A_0 + A_1$ and taking values on the set of measurable functions:

$$T : A_0 + A_1 \longrightarrow L^0(\mu).$$

Our first task is to extend the notion of characteristic functions to the setting of pairs. This will be done in Section 3.

As usual, the symbol $f \approx g$ will indicate the existence of a universal positive constant C (independent of all parameters involved) so that $(1/C)f \leq g \leq Cf$, while the symbol $f \preceq g$ means that $f \leq Cg$. $(\mathcal{M}, \mu)$ will be a totally $\sigma-$finite resonant measure space and we shall denote by $L^0(\mu)$ the class of measurable functions that are finite μ a.e., endowed with the topology of the convergence in measure. We write $\|g\|_p$ to denote $\|g\|_{L^p(\mu)}$, $\lambda_g^\mu(y) = \mu\left(\{x \in \mathcal{M} : \ |g(x)| > y\}\right)$ is the distribution function of g with respect to the measure μ and $g_\mu^*(t) = \inf \left\{ s : \lambda_g^\mu(s) \leq t \right\}$ is the decreasing rearrangement (we refer the reader to [3] for further information about distribution functions and decreasing rearrangements).

In what follows we shall omit the indices μ whenever it is clear the measure we are working with.

2 Main results

First of all, given a positive concave function D such that $D(0+) = 0$, we define the space

$$\Lambda(D) = \left\{ f;\ \|f\|_{\Lambda(D)} = \int_0^\infty D(\lambda_f(y))\,dy = \int_0^\infty f^*(s)dD(s) \right\}.$$

Then, we have that the following properties holds:

Lemma 2.1 *Given a positive concave function D such that $D(0+) = 0$, it holds that*

$$\Lambda(D) \subset L^1 + L^\infty,$$

and

$$Q_D \subset \Lambda(D),$$

with continuous embeddings.

Proof: The first embedding is wellknown, since $\min(1, s) \preceq D(s)$ and hence

$$\begin{aligned} \|f\|_{L^1+L^\infty} &= \int_0^1 f^*(s)ds = \int_0^\infty \min(\lambda_f(y), 1)dy \preceq \int_0^\infty D(\lambda_f(y))\,dy \\ &= \|f\|_{\Lambda(D)}. \end{aligned}$$

For the second embedding, let us observe that if $\|f\|_\infty \leq 1$, then

$$\|f\|_{\Lambda(D)} = \int_0^1 D(\lambda_f(y))dy \leq D\bigg(\int_0^1 \lambda_f(y)dy \bigg) = D(\|f\|_1),$$

and hence, if $f = \sum_k e_k f_k$, with $\|f_k\|_\infty \leq 1$, we obtain that

$$\|f\|_{\Lambda(D)}. \leq \sum_k e_k \|f_k\|_{\Lambda(D)} \leq \sum_k e_k D(\|f_k\|_1) \leq \|f\|_{Q_D}. \qquad \square$$

Now we are ready to formulate our first main result:

Theorem 2.1 *Let T be a sublinear operator such that*

$$T : L^1(\mu) + L^\infty(\mu) \longrightarrow L^0(\mu)$$

is bounded, and let us assume that, for every $f \in L^1 \cap L^\infty$ with $\|f\|_\infty \leq 1$,

$$(Tf)^*(t) \leq h(t, \|f\|_1),$$

for some positive function $h : (0,\infty) \times (0,\infty) \to (0,\infty)$ such that for every $t > 0$, the function $h(t, \cdot)$ is increasing and, for every $s > 0$, $t \cdot h(t,s)$ is also an increasing function in the variable t. Then, if D and R are such that

$$h(t,s) \leq D(s)R(t),$$

we have that

$$T : Q_D \longrightarrow M(R)$$

is bounded.

Although no conditions are assumed on R, it is clear that since $t \cdot h(t,s)$ is increasing in the variable t, we can assume without loss of generality that this condition also holds for R.

Proof: Let $f \in Q_D$ and let us write $f = \sum_k e_k f_k$ with $\|f_k\|_\infty \leq 1$. Then, by the previous lemma, we have that the convergence of the series is in $L^1 + L^\infty$ and therefore, we can conclude that

$$(Tf)^*(t) \leq \Big(\sum_k e_k T f_k \Big)^*(t).$$

Using now the basic result together with the hypothesis, we obtain that, for every sequence (a_k) of positive numbers such that $\sum_k a_k = 1$,

$$\begin{aligned} (Tf)^*(3t) &\leq \sum_k e_k (Tf_k)^*(t) + \frac{1}{t} \sum_k e_k \int_{a_k t}^t (Tf_k)^*(s)\, ds \\ &\leq \sum_k e_k h(t, \|f_k\|_1) + \frac{1}{t} \sum_k e_k \int_{a_k t}^t h(s, \|f_k\|_1)\, ds. \end{aligned}$$

And, using the properties of the function h, we conclude that

$$(Tf)^*(3t) \leq \sum_k e_k D(\|f_k\|_1) R(3t) + R(3t) \sum_k e_k D(\|f_k\|_1) \log \frac{1}{a_k},$$

and hence,

$$\|Tf\|_{M(R)} = \sup_t \frac{(Tf)^*(t)}{R(t)} \leq \|f\|_{Q_D}. \qquad \square$$

As was mentioned in the introduction, the point now is to analyze the space Q_D to make it useful for the applications. To this end, we have to introduce the following logarithmic spaces:

Definition 2.1 *Let φ be a positive and concave function such that $\varphi(0^+) = 0$.*
(1) The space $L \log|\log L|(\varphi)$ is defined as the set of measurable functions f such that

$$\|f\|_{L\log|\log L|(\varphi)} := \int_0^\infty f^*(s)\,(1 + \log(|\log s| + e))\,d\varphi(s) < \infty. \tag{4}$$

(2) The space $L \log\log L(\varphi)$ is defined as the set of measurable functions f such that

$$\|f\|_{L\log\log L(\varphi)} := \int_0^\infty f^*(s)\left(1 + \log^+\log^+\frac{1}{s}\right)d\varphi(s) < \infty. \tag{5}$$

(3) The space $L \log\log\log L(\varphi)$ is defined as the set of measurable functions f such that

$$\|f\|_{L\log\log\log L(\varphi)} := \int_0^\infty f^*(s)\left(1 + \log^+\log^+\log^+\frac{1}{s}\right)d\varphi(s) < \infty. \tag{6}$$

We also need the two following technical lemmas:

Lemma 2.2 *Let $\Phi(s) = s(1 + \log^+\frac{1}{s})$ and let f be such that $\|f\|_{\Lambda(\varphi)} = 1$, then*

$$\int_0^\infty \Phi\left(f^*(s)\varphi(s)\right)\frac{d\varphi(s)}{\varphi(s)} \simeq \int_0^\infty \Phi\left(s\varphi\left(\lambda_f(s)\right)\right)\frac{ds}{s} \preceq \|f\|_{L\log|\log L|(\varphi)}.$$

Proof: To show the first equivalence, let $H = f^* \circ \varphi^{-1}(s)$. Then, one has that

$$\lambda_H(s) = \varphi(\lambda_f(s))$$

and, by Proposition 4.3 of [19], we have that

$$\int_0^\infty \Phi\left(s\lambda_H(s)\right)\frac{ds}{s} \simeq \int_0^\infty \Phi\left(sH(s)\right)\frac{ds}{s}.$$

A simple change of variable ends the proof of the first part.

For the second part, let us consider the sets

$$E_0 = \left\{s < 1 : \varphi(s)f^*(s) > \left(\log\frac{1}{s} + e\right)^{-2}\right\},$$

and

$$E_1 = \left\{s \geq 1 : \varphi(s)f^*(s) > (\log s + e)^{-2}\right\}.$$

Then, we can write

$$\begin{aligned}
&\int_0^\infty \Phi\left(f^*(s)\varphi(s)\right)\frac{d\varphi(s)}{\varphi(s)}\\
=\ &\left(\int_{E_0}+\int_{(0,1)\setminus E_0}+\int_{E_1}+\int_{(1,\infty)\setminus E_1}\right)\Phi\left(f^*(s)\varphi(s)\right)\frac{d\varphi(s)}{\varphi(s)}\\
=\ &I_1+I_2+I_3+I_4.
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 &= \int_{E_0} f^*(s)\left(1+\log^+\frac{1}{f^*(s)\varphi(s)}\right)d\varphi(s)\\
&\leq \int_0^1 f^*(s)\left(1+2\log\left(\log\frac{1}{s}+e\right)\right)d\varphi(s)\leq 2\,\|f\|_{L\log|\log L|(\varphi))}\,.
\end{aligned}$$

On the other hand, since Φ is increasing, $d\varphi(s)\leq(\varphi(s)/s)ds$ and $1=\|f\|_{\Lambda(\varphi)}\leq\|f\|_{L\log|\log L|(\varphi)}$, we obtain that

$$\begin{aligned}
I_2 &\leq \int_0^1 \frac{\left(1+2\log\left(\log\frac{1}{s}+e\right)\right)}{\left(\log\frac{1}{s}+e\right)^2}\frac{d\varphi(s)}{\varphi(s)}\\
&\leq \int_0^1 \frac{\left(1+2\log\left(\log\frac{1}{s}+e\right)\right)}{s\left(\log\frac{1}{s}+e\right)^2}ds\preceq\|f\|_{L\log|\log L|(\varphi)}\,.
\end{aligned}$$

Similarly,

$$I_3\leq\int_1^\infty f^*(s)\left(1+2\log\left(\log s+e\right)\right)d\varphi(s)\preceq\|f\|_{L\log|\log L|(\varphi)}\,,$$

and

$$I_4\leq\int_1^\infty\frac{\left(1+2\log\left(\log s+e\right)\right)}{s\left(\log s+e\right)^2}ds\preceq\|f\|_{L\log|\log L|(\varphi)}\,. \qquad \square$$

Lemma 2.3 *([9]) Let w be a positive and measurable function and let φ be a positive and concave function such that $\varphi(0^+)=0$. Then*

$$\int_0^\infty\varphi\left(\lambda_f(s)\right)w(s)ds=\int_0^\infty\left(\int_0^{f^*(s)}w(t)dt\right)d\varphi(s).$$

Theorem 2.2 *Let D be any positive and concave function D such that $D(0^+) = 0$. Then,*
1)
$$L \log|\log L|(D) \subset Q_D.$$

2) If $s \le D(s)$, then
$$L \log\log L(D) \subset Q_D.$$

3) If $s \le D(s)$ and, for every $0 \le s \le 1$, $D(s^2) \preceq sD(s)$, then
$$L \log\log\log L(D) \subset Q_D.$$

Proof: 1) Let $f \in L \log|\log L|(D)$ be such that $\|f\|_{\Lambda(D)} = 1$ and let us write
$$f = \sum_{i\in\mathbb{Z}} 2^{i+1} f_i,$$
where $f_i = \frac{1}{2^{i+1}} f\chi_{\{2^i<|f|\le 2^{i+1}\}}$. Then, for every sequence of positive number $(a_i)_i$ such that $\sum_{i\in\mathbb{Z}} a_i = 1$, we have that
$$\begin{aligned}\|f\|_{Q_D} &\preceq \sum_{i\in\mathbb{Z}} 2^i D(\|f_i\|_1)\left(1+\log\frac{1}{a_i}\right)\\ &\le \sum_{i\in\mathbb{Z}} 2^i D(\lambda_f(2^i))\left(1+\log\frac{1}{a_i}\right).\end{aligned}$$
Taking now
$$a_i = \frac{2^i D(\lambda_f(2^i))}{\sum_i 2^i D\left(\lambda_f(2^i)\right)},$$
we conclude that
$$\|f\|_{Q_D} \preceq \int_0^\infty D(\lambda_f(s))\left(1+\log\frac{1}{sD(\lambda_f(s))}\right) ds,$$
and the result now follows by Lemma 2.2. □
2) Since $s \le D(s)$ we have that $L \log\log L(D) \subseteq \Lambda(D) \subseteq L^1$. Let $f \in L \log\log L(D)$ be such that $\|f\|_{\Lambda(D)} = 1$, and decompose f as
$$f = f\chi_{\{|f|\le 1\}} + \left(\sum_{i\ge 0} 2^{i+1} f_i\right),$$

where $f_i = \frac{1}{2^{i+1}} f\chi_{\{2^i<|f|\leq 2^{i+1}\}}$. Then, for every $(a_i)_i$ such that $\sum_i a_i = 1$,

$$\|f\|_{Q_D} \preceq D(\|f\|_1) + \sum_{i\geq 0} 2^i D(\lambda_f(2^i))\Big(1 + \log\frac{1}{a_i}\Big),$$

and taking $(a_i)_i$ as in 1), we get

$$\|f\|_{Q_D} \preceq 1 + \int_1^\infty D(\lambda_f(s))\left(1 + \log\frac{1}{sD(\lambda_f(s))}\right) ds \preceq 1 + I.$$

To estimate I, it follows, by Lemma 2.2, that

$$I \preceq \int_{\{f^*\geq 1\}} f^*(s)\left(1 + \log\frac{1}{f^*(s)D(s)}\right) dD(s),$$

and since $s\lambda_{f^*}(s) \leq 1$, we get that $\lambda_{f^*}(s) \leq 1$ if $s \geq 1$. Hence, $\{f^* \geq 1\} \subseteq [0, 1]$ and using the same argument than in the proof of Lemma 2.2, it follows that

$$I \preceq \int_0^1 f^*(s)\left(1 + \log\frac{1}{f^*(s)D(s)}\right) dD(s) \preceq \|f\|_{L\log\log L(D)}.$$

3) In this case, we take $f \in L\log\log\log L(D)$ such that $\|f\|_{\Lambda(D)} = 1$ and we write

$$f = f\chi_{\{|f|\leq 2\}} + \sum_{i=0}^\infty 2^{2^{i+1}} f_i,$$

where

$$f_i = \frac{1}{2^{2^{i+1}}} f\chi_{\{2^{2^i}|f|\leq 2^{2^{i+1}}\}}.$$

Then, if $\sum_i a_i = 1$,

$$\begin{aligned}
\|f\|_{Q_D} &\preceq 1 + \sum_{i=0}^\infty 2^{2^{i+1}} D(\|f_i\|_1)\Big(1 + \log\frac{1}{a_i}\Big) \\
&\preceq 1 + \sum_{i=0}^\infty 2^{2^{i+1}} D\Big(\frac{1}{2^{2^{i+1}}}\sum_{j=2^i}^{2^{i+1}-1} 2^j\lambda_f(2^j)\Big)\Big(1 + \log\frac{1}{a_i}\Big),
\end{aligned}$$

and since D is concave,

$$\|f\|_{Q_D} \preceq 1 + \sum_{i=0}^\infty 2^{2^{i+1}} \sum_{j=2^i}^{2^{i+1}-1} D\Big(\frac{2^j}{2^{2^{i+1}}}\lambda_f(2^j)\Big)\Big(1 + \log\frac{1}{a_i}\Big).$$

Now, using $D(s)/s$ decreases, and that $2^i \le j < 2^{i+1}$, we obtain that

$$2^{2^{i+1}} D\left(\frac{2^j}{2^{2^{i+1}}}\lambda_f(2^j)\right) \le \left(2^j\right)^2 D\left(\frac{\lambda_f(2^j)}{2^j}\right).$$

Now we take $a_i = 6/(\pi^2(i+1)^2)$, and hence

$$\begin{aligned} \|f\|_{Q_D} &\preceq 1 + \sum_{i=0}^{\infty} \sum_{j=2^i}^{2^{i+1}-1} \left(2^j\right)^2 D\left(\frac{\lambda_f(2^j)}{2^j}\right)(1+\log(i+1)) \\ &\preceq \sum_{i=0}^{\infty} \sum_{j=2^i}^{2^{i+1}-1} (2^j)^2 D\left(\frac{\lambda_f(2^j)}{2^j}\right)(1+\log^+\log^+\log^+ 2^j) \\ &\preceq \int_1^\infty sD\left(\frac{\lambda_f(s)}{s}\right)(1+\log^+\log^+\log^+ s)\, ds. \end{aligned}$$

Since $s\lambda_f(s) \le 1$, we get that

$$\frac{s}{\lambda_f(s)} \le \left(\frac{1}{\lambda_f(s)}\right)^2$$

and since $sD(1/s)$ increases

$$sD\left(\frac{\lambda_f(s)}{s}\right) \le \frac{1}{\lambda_f(s)} D\left((\lambda_f(s))^2\right).$$

Moreover, since $\lambda_f(s) \le s\lambda_f(s) \le 1$, if $s \ge 1$, by condition (ii)

$$\frac{1}{\lambda_f(s)} D\left((\lambda_f(s))^2\right) \preceq D\left(\lambda_f(s)\right).$$

Using this estimate and Lemma 2.3 we get

$$\begin{aligned} I &\preceq \int_1^\infty D\left(\lambda_f(s)\right)\left(1+\log^+\log^+\log^+ s\right)\, ds \\ &\le \int_0^\infty D\left(\lambda_f(s)\right)\left(1+\log^+\log^+\log^+ s\right)\, ds \\ &= \int_0^\infty \left(\int_0^{f^*(s)} \left(1+\log^+\log^+\log^+ t\right)\, dt\right) dD(s). \end{aligned}$$

Now, since $\left(1+\log^+\log^+\log^+ t\right)$ is increasing and $sf^*(s) \le 1$

$$\begin{aligned} I &\le \int_0^\infty f^*(s)\left(1+\log^+\log^+\log^+ f^*(s)\right)\, dD(s) \\ &\le \int_0^\infty f^*(s)\left(1+\log^+\log^+\log^+ \frac{1}{s}\right)\, dD(s). \qquad \square \end{aligned}$$

3 Extension to arbitrary compatible pairs

Let $\bar{A} = (A_0, A_1)$ be a **compatible pair** of Banach spaces, that is, we assume that there is a topological vector space $\mathcal{U}$ such that $A_i \subset \mathcal{U}$, $i = 0, 1$, continuously. In what follows we drop the terms "compatible" and "Banach" and refer to a compatible Banach pair simply as a "pair".

The Peetre $K-$**functional** (see [3], [4] and [5]) associated with a pair $\bar{A}$ is defined, for each $a \in A_0 + A_1$ and $t > 0$, by

$$K(a,t) = K(a,t;\bar{A}) = \inf\left\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} :\ a = a_0 + a_1,\ a_i \in A_i\right\}.$$

It is easy to see that $K(t,a)$ is a nonnegative and concave function of $t > 0$, (and thus also continuous). Therefore

$$K(a,t;\bar{A}) = K(a,0^+;\bar{A}) + \int_0^t k(a,s;\bar{A})\,ds,$$

where the $k-$**functional**, $k(a,s;\bar{A}) = k(a,s)$, is a uniquely defined, nonnegative, decreasing and right-continuous function of $s > 0$.

In order to find the analogue of the set $\{f \in L^1; \|f\|_\infty \leq 1\}$ in the setting of pairs, let us recall that the Gagliardo completion $\tilde{A}_0$ and $\tilde{A}_1$ of a pair $\bar{A}$ is defined by $\|a\|_{\tilde{A}_0} = \sup_t K(t,a;\bar{A})$ and $\|a\|_{\tilde{A}_1} = \sup_t K(t,a;\bar{A})/t$ (see [3]).

Definition 3.1 *Given a pair $\bar{A}$, we say that a is a characteristic element of $\bar{A}$ if $a \in \tilde{A}_0 \cap \tilde{A}_1$ and $\|a\|_{\tilde{A}_1} \leq 1$.*

The collection of characteristic elements of a pair $\bar{A}$ will be denoted by $C(\bar{A})$.

The following lemma was proved in [6] and it is fundamental for our purpose.

Lemma 3.1 *Given an element $a \in A_0 + A_1$ such that $K(a,0^+;\bar{A}) = 0$, there exist a constant γ (depending only on $\bar{A}$) and a collection of characteristic elements $(a_i)_{i\in\mathbb{Z}}$ such that*

$$a = \gamma \sum_{i\in\mathbb{Z}} 2^i a_i \ (\textit{convergence in } A_0 + A_1),$$

and

$$\|a_i\|_{\tilde{A}_0} \leq \lambda_{k(a,\cdot)}(2^i).$$

We say that $a = \gamma \sum_{i=-\infty}^{\infty} 2^i a_i$ is a dyadic decomposition of a.

Definition 3.2 *([6]) Given a pair $\bar{A} = (A_0, A_1)$ and a concave function φ, the minimal Lorentz space, $\Lambda(\varphi; \bar{A})$, is the set of elements $a \in A_0 + A_1$ such that $K(a, 0^+; \bar{A}) = 0$ and*

$$\|a\|_{\Lambda(\varphi;\bar{A})} = \int_0^\infty k(a, s; \bar{A})\, d\varphi(s) < \infty.$$

If $\bar{A}$ is the classical pair $\big(L^1(\nu), L^\infty(\nu)\big)$, then $k(a, s) = f^*(s)$ and hence $\Lambda(\varphi; \bar{A}) = \Lambda(\varphi)$ is the classical Lorentz spaces defined in the previous section.

Definition 3.3 *Given a pair $\bar{A}$, and a quasi-Banach lattice $B \subset \Lambda(\varphi)$, we define $B(\varphi; \bar{A})$ as*

$$B(\varphi; \bar{A}) = \Big\{ a \in \Lambda(\varphi; \bar{A}); \|a\|_{B(\varphi;\bar{A})} := \|k(a, \cdot)\|_B < \infty \Big\}. \tag{7}$$

Remark 3.1 *Obviously,*

$$L \log |\log L|(\varphi; \bar{A}) \subset L \log\log L(\varphi; \bar{A}) \subset L \log\log\log L(\varphi; \bar{A}),$$

and the above embeddings are, in general, strict. However, if $\bar{A}$ is an ordered pair, that is $A_1 \subset A_0$) then $k(a, t) = 0$ if $t > 1$, and hence $L \log |\log L|(\varphi; \bar{A}) = L \log\log L(\varphi; \bar{A})$.

Definition 3.4 *Let $h : (0, \infty) \times (0, \infty) \to (0, \infty)$ be such that for every $t > 0$, the function $h(t, \cdot)$ is increasing and, for every $s > 0$, $t \cdot h(t, s)$ is also an increasing function in the variable t. We say that a sublinear continuous operator*

$$T : A_0 + A_1 \longrightarrow L^0(\mathcal{N}),$$

satisfies a restricted $h-$ rearrangement inequality if, for every $t > 0$ and every characteristic element a of $\bar{A}$,

$$(Ta)^*(t) \le h(t, \|a\|_{\tilde{A}_0}). \tag{8}$$

Examples:
1) If $\bar{A} = \big(L^1(\nu), L^\infty(\nu)\big)$, then $C(\bar{A}) = \{f \in L^1; \|f\|_\infty \le 1\}$, and hence, any sublinear operator satisfying (3), satisfies the condition assumed in the previous section.
2) Let Ω be any domain in $\mathbb{R}^n$ and let $W^{1,p}(\Omega)$ be the classical Sobolev space

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_p + \|\nabla f\|_p,$$

where ∇f is the gradient of f. Set $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Then it is known, (see [10]), that if $\bar{A} = (W_0^{1,1}(\Omega), W_0^{1,\infty}(\Omega))$,

$$K(t, f; \bar{A}) \approx t\Big(f^{**}(t) + |\nabla f|^{**}(t)\Big),$$

and therefore $C(\bar{A}) = \{f \in W_0^{1,1}(\Omega); \|f\|_\infty + \|\nabla f\|_\infty \leq 1\}$. Hence, if T is a sublinear operator such that

$$T : W_0^{1,p}(\Omega) \longrightarrow L^{p,\infty}$$

is bounded with constant C_p for every $p \in I \subset [1, \infty)$, then,

$$(Tf)^*(t) \leq \inf_{p \in I} \left(C_p \left(\frac{\|f\|_{W_0^{1,1}(\Omega)}}{t} \right)^{1/p} \right) := h(t, \|f\|_{W_0^{1,1}(\Omega)}).$$

3) Let us now consider, for example, the pair $\bar{A} = (\Lambda^1(w), L^\infty)$, where $\Lambda^1(w)$ is the weighted Lorentz space introduced by Lorentz in [13] and defined by

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p w(t)\, dt \right)^{1/p} < \infty.$$

Let us recall that the weak type version of these spaces are defined by

$$\|f\|_{\Lambda^{p,\infty}(W)} = \sup_{t>0} f^*(t) W^{1/p}(t) < \infty.$$

Consider a sublinear operator T such that, for some weights w and W,

$$T : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(W),$$

with constant less than or equal to C_p. Then, since it is known that $K(t, f; \bar{A}) = \int_0^t (f^*)^*_w (s)\, ds$, we can conclude that

$$C(\bar{A}) = \left\{ f \in \Lambda^1(w) : \|f\|_\infty \leq 1 \right\},$$

and therefore, for every characteristic element,

$$(Tf)^*(t) \leq \inf_p \left(C_p \left(\frac{\|f\|_{\Lambda^1(w)}}{W(t)} \right)^{1/p} \right) := h(t, \|f\|_{\Lambda^1(w)}).$$

Let us now define the space

$$Q_D(\bar{A}) = \Big\{ a = \sum_k e_k a_k; \|a_k\|_{\bar{A}_1} \leq 1, \|a\|_{Q_D(\bar{A})} < \infty \Big\},$$

where

$$\|a\|_{Q_D(\bar{A})} = \inf\left\{\sum_k e_k D(\|a_k\|_{\bar{A}_0})\left(1+\log\frac{1}{a_k}\right);\ \sum_k a_k = 1, a_k \geq 0, a = \sum_k e_k a_k\right\}.$$

Then, we have the following extension of Theorem 2.1:

Theorem 3.1 *Let $T : A_0 + A_1 \to L^0(\mathcal{N})$ be a sublinear operator satisfying a restricted $h-$rearrangement inequality. Then, if D and R are two positive functions such that D is concave, $D(0^+) = 0$ and*

$$h(t,s) \leq D(s)R(t), \tag{9}$$

we have that

$$T : Q_D(\bar{A}) \longrightarrow M(R)$$

is bounded.

Proof: Given $a \in \Lambda(D;\bar{A})$ such that $\|a\|_{\Lambda(D;\bar{A})} = 1$, we can decompose a as in Lemma 3.1

$$a = \gamma \sum_{i\in Z} 2^i a_i.$$

Then, if $a_N = \gamma\sum_{i=-N}^{N} 2^i a_i$, we have that $Ta_N \to Ta$ in measure, and therefore, (see [12] p. 67)

$$(Ta_N)^*(t) \to (Ta)^*(t) \quad \text{a.e. } t > 0. \tag{10}$$

By the sublinearity of T we get that

$$(Ta_N)^*(t) \leq \gamma\left(\sum_{i=-N}^{N} 2^i |Ta_i|\right)^*(t) \leq \gamma\left(\sum_{i=-\infty}^{\infty} 2^i |Ta_i|\right)^*(t),$$

and hence

$$(Ta)^*(t) \leq \gamma\left(\sum_{i=-\infty}^{\infty} 2^i |Ta_i|\right)^*(t) \quad \text{a.e. } t > 0.$$

The proof now follows as in Theorem 2.1. □

We also have and analogue to Theorem 2.2:

Theorem 3.2 *Let D be any positive and concave function D such that $D(0^+) = 0$ and $D(\infty) = \infty$. Then,*
1)

$$L \log |\log L|(D; \bar{A}) \subset Q_D(\bar{A}).$$

2) If $s \leq D(s)$, then

$$L \log \log L(D; \bar{A}) \subset Q_D(\bar{A}).$$

3) $s \leq D(s)$ and, for every $0 \leq s \leq 1$, $D(s^2) \preceq sD(s)$, then

$$L \log \log \log L(D; \bar{A}) \subset Q_D(\bar{A}).$$

Proof: 1) In this case, given $a \in \Lambda(D; \bar{A})$ such that $\|a\|_{\Lambda(D;\bar{A})} = 1$, we decompose a as in Lemma 3.1

$$a = \gamma \sum_{i \in Z} 2^i a_i,$$

and continue as in the proof of Theorem 2.2, 1).
2) Since $s \leq D(s)$ we have that $L \log \log L(D; \bar{A}) \subseteq \Lambda(D; \bar{A}) \subseteq \tilde{A}_0$. Let $a \in \Lambda(D; \bar{A})$ such that $\|a\|_{\Lambda(D;\bar{A})} = 1$, and decompose a as

$$a = \gamma \Big(\sum_{i<0} 2^i a_i + \sum_{i \geq 0} 2^i a_i \Big) = \gamma \Big(a^0 + \sum_{i \geq 0} 2^i a_i \Big).$$

Then since $a^0 \in C(\bar{A})$, and $\|a_i\|_{\tilde{A}_0} \leq \lambda_{k(a,\cdot)}(2^i)$, we have that

$$\begin{aligned} \|a\|_{Q_D(\bar{A})} &\preceq D\left(\|a^0\|_{\tilde{A}_0}\right) + \int_1^\infty D(\lambda_{k(a,\cdot)}(s)) \left(1 + \log \frac{1}{sD(\lambda_{k(a,\cdot)}(s))}\right) ds \\ &= I_1 + I_2. \end{aligned}$$

Obviously

$$I_1 \preceq D\left(\|a\|_{\tilde{A}_0}\right) \leq D(\|a\|_{\Lambda(D;\bar{A})}) = D(1) \leq D(1)\, \|a\|_{L \log \log L(D;\bar{A})},$$

and to estimate I_2, we follow as in the proof of Theorem 2.2, 2).

3) In this case, given $a \in L\log\log\log L(D;\bar{A})$ such that $\|a\|_{\Lambda(D;\bar{A})} = 1$, let $a = \gamma\sum_{i\in\mathbb{Z}} 2^i a_i$ be a dyadic decomposition. Then, if, for every $k \in \mathbb{N}$, $d_k = \sum_{i=2^k}^{2^{k+1}-1} 2^i$, we obtain that

$$a = \sum_{i=-\infty}^{0} 2^i a_i + \sum_{k=0}^{\infty} d_k \left(\frac{1}{d_k} \sum_{i=2^k}^{2^{k+1}-1} 2^i a_i \right) = a^0 + \sum_{k=0}^{\infty} d_k A_k,$$

where, it is immediate to see that both a^0 and A_k are characteristic elements. Then, for every $\sum_k c_k = 1$,

$$\begin{aligned} \|a\|_{Q_D(\bar{A})} &\leq D(\|a\|_{\tilde{A}_0}) + \sum_{k=0}^{\infty} d_k D(\|A_k\|_{\tilde{A}_0}) \Big(1 + \log \frac{1}{c_k} \Big) \\ &= \Big(D(\|a\|_{\tilde{A}_0}) + I \Big). \end{aligned}$$

Since $a_i \in C(\bar{A})$ and D is subadditive, we have that

$$d_k D(\|A_k\|_{\tilde{A}_0}) \leq d_k \sum_{i=2^k}^{2^{k+1}-1} D\left(\frac{2^i}{d_k} \|a_i\|_{\tilde{A}_0} \right) \leq d_k \sum_{i=2^k}^{2^{k+1}-1} D\left(\frac{2^i}{d_k} \lambda_{k(a,\cdot)}(2^i) \right),$$

and the proof now follows as in Theorem 2.2, 3). □

4 Applications

(I) Let T be a sublinear operator satisfying a restricted h-rearrangement inequality, where

$$h(t,s) = \frac{s}{t}\left(1 + \log^+ \frac{t}{s}\right)^m$$

with $m > 0$, as it happens with the examples we have mentioned in the introduction. Then,

$$h(t,s) \leq \frac{s}{t}\left(1 + \log^+ \frac{t}{s}\right)^m \leq s\left(1 + \log^+ \frac{1}{s}\right)^m \frac{1}{t}(1 + \log^+ t)^m,$$

and we can take $D(s) = s\left(1 + \log^+ \frac{1}{s}\right)^m$ and $R(t) = \frac{1}{t}(1 + \log^+ t)^m$ in our Theorems 3.1 and 3.2 to conclude the following result.

Theorem 4.1 *If $T : A_0+A_1 \to L^0(\mathcal{N})$ satisfies a restricted h-rearrangement inequality with $h(t,s) = \frac{s}{t}\left(1+\log^+ \frac{t}{s}\right)^m$, T can be extended continuously*

$$T : Q_D(\bar{A}) \longrightarrow M(R),$$

where $D(t) = t\left(1+\log^+ \frac{1}{t}\right)^m$, and $R(t) = \frac{1}{t}(1+\log^+ t)^m$.

In particular

$$T : L\log\log\log L(D;\bar{A}) \longrightarrow M(R),$$

is bounded.

If $\bar{A} = (L^1(\mathbb{T}), L^\infty(\mathbb{T}))$ and T is the Carleson maximal operator S^*, then $D(s) = s\Big(1+\log\frac{1}{s}\Big)$ and $R(t) = t$. In this particular case, the above result has been recently obtained by Arias-de-Reyna in [2]. Also, for such function D, it is very easy to see that

$$L\log\log\log L(D;\bar{A}) = L\log L\log\log\log L(\mathbb{T}),$$

and the boundedness

$$S^* : L\log L\log\log\log L(\mathbb{T}) \longrightarrow L^{1,\infty},$$

was obtained previously by Antonov in [1], and for other more general operators, as mentioned in the introduction, in [17].

(II) If $\bar{A} = (W_0^{1,1}(\Omega), W_0^{1,\infty}(\Omega))$ where Ω has finite measure, and

$$T : W_0^{1,p}(\Omega) \longrightarrow L^{p,\infty}$$

is bounded with constant say $1/(p-1)$, then, applying Theorem 4.1, we obtain that

$$T : W_0(\Omega) \longrightarrow L^{1,\infty}$$

is bounded, where $W_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W(\Omega)$ with

$$W(\Omega) = \{f;\ |f| + |\nabla f| \in L\log L\log\log\log L\}.$$

(III) Our third application deals with the theory of weighted Lorentz spaces. Let w_0 and W_1 be weights in $(0,\infty)$ and let T be a sublinear operator such that

$$T : \Lambda^1(w_0) + L^\infty \longrightarrow L^0(\mathbb{R}^n)$$

is continuous and, for every $p > 2$,

$$T : \Lambda^p(w_0) \longrightarrow \Lambda^{p,\infty}(W_1)$$

is bounded with constant p (see, [14], [15], [8] to find examples of operators T satisfying the above condition); that is

$$(Tf)^*(t) W_1(t)^{1/p} \leq p \left(\int_0^\infty f^*(s) w_0(s)\, ds \right)^{1/p}.$$

Now, if we take $\bar{A} = (\Lambda^1(w_0), L^\infty)$, we have that $\tilde{A}_0 = \Lambda^1(w_0)$ and hence, it follows, taking the infimum in $p > 2$, that

$$(Tf)^*(t) \preceq h(W_1(t), \|a\|_{\tilde{A}_0}),$$

where $h(t,s) = \inf_{p>2} p(s/t)^{1/p} \approx (s/t)^{1/2}(1+\log^+(s/t))$. Therefore, we can deduced the following result.

Theorem 4.2 *Let T be a sublinear operator as above. Then, T can be extended continuously*

$$T : L \log |\log L|(D; \bar{A}) \rightarrow M(R),$$

where $D(s) = s^{1/2}(1 + \log^+ s)$, and $R(t) = W_1(t)^{-1/2} \left(1 + \log^+ \frac{1}{W_1(t)}\right)^{-1}$.

Open Question: When is is true that the space $L \log\log\log\log L(D; A) \subset Q_D(\bar{A})$ or in general, $L \log^{(m)} L(D; A) \subset Q_D(\bar{A})$?.

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