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A useful estimate for the decreasing rearrangement of the sum of functions. *

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Abstract

Given $f = \sum_{n\in\mathbb{N}} f_n$, where $\{f_n\}_n$ are measurable functions, we show that

$$f^*(3t) \leq \sum_n \left(f_n^*(t) + \frac{1}{t}\int_{c_n t}^{t} f_n^*(s)ds \right),$$

where $\{c_n\}_n$ are positive numbers such that $\sum_n c_n = 1$. Several consequences in the setting of weighted Lorentz spaces are also given.

1 Introduction

Let $(\mathcal{N}, \mu)$ be a $\sigma-$finite resonant measure space and let $L^0(\mathcal{N})$ be the class of measurable functions that are finite μ a.e.

Let $g_\mu^*(t) = \inf\left\{ s : \lambda_g^\mu(s) \leq t \right\}$ be the decreasing rearrangement of g, where $\lambda_g^\mu(y) = \mu\left(\{x \in \mathcal{N} : \ |g(x)| > y\}\right)$ is the distribution function of g with respect to the measure μ. Then, it holds (see [2]) that, for every f and g in $L^0(\mathcal{N})$ and every $a \geq 0$ and $b \geq 0$ such that $a + b = 1$,

$$(f+g)_\mu^*(t) \leq f_\mu^*(at) + g_\mu^*(bt),$$

and, in general, the $*$-operator is not subadditive. This estimate can be extended to the case of a countable set of functions $(f_n)_{n\in\mathbb{N}} \subset L^0(\mathcal{N})$ in the natural way; that is, for every sequence of positive elements $(c_n)_n$ such that $\sum_n c_n = 1$, we have that

$$\Big(\sum_n f_n \Big)_\mu^*(t) \leq \sum_n (f_n)_\mu^*(c_n t).$$

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However, on many occasions this estimate is too rough to be useful. Our purpose is to present an estimate for the decreasing rearrangement of $\sum_n f_n$ which allows us to get several useful consequences. In particular, a generalization of the following lemma due to E.M. Stein and N.J. Weiss (see [8]) will be presented.

Lemma 1.1 *Let* $f_n \subset L^0(\mathcal{N})$ *be such that* $\sup_{t>0} t(f_n)^*_\mu(t) \leq 1$ *and let* $(a_n)_n > 0$ *be such that* $\sum_n a_n \log(1/a_n) < \infty$. *Then, if* $f = \sum_n a_n f_n$, *we have that* $\sup_{t>0} t f^*_\mu(t) < \infty$.

As usual, the symbol $f \approx g$ will indicate the existence of a universal positive constant C (independent of all parameters involved) so that $(1/C)f \leq g \leq Cf$, while the symbol $f \preceq g$ means that $f \leq Cg$. In what follows we shall omit the indices μ whenever it is clear the measure we are working with.

2 Main result

To prove our main estimate, we first need some technical lemma. Let $f \in L^0(\mathcal{N})$ be a positive function and let $r > 0$. If

$$A(f, r) = \{x \in \mathcal{N} : f(x) = r\},$$

then $A(f, r_1) \cap A(f, r_2) = \emptyset$ if $r_1 \neq r_2$, and, from the $\sigma-$finite condition, $\mu(A(f, r)) = 0$ except perhaps for at most a countable number of r.

Lemma 2.1 *Let* $f \in L^0(\mathcal{N})$ *be a positive function and* $\varepsilon > 0$. *Then there exists a function* $g \in L^0(\mathcal{N})$ *such that* $\mu(A(g, r)) = 0$ *for each* $r > 0$ *and*

$$f(x) \leq g(x) \leq (1 + \varepsilon) f(x) \quad a.e.\ x \in \mathcal{N}.$$

Proof: If $\mu(A(f, r)) = 0$ for each $r > 0$, then $f = g$, otherwise there exists a finite or countable sequence $\{r_n\}_n$ such that $\mu(A(f, r_n)) > 0$. Let $h : [0, \infty) \to [0, \infty)$ be defined by $h(t) = \varepsilon e^{-t}$, which is continuous, strictly decreasing and $\lim_{t\to\infty} h(t) = 0$. Hence there is a measure-preserving transformation $\sigma : (\mathcal{N}, \mu) \to ([0, \infty), m)$ such that $H = h \circ \sigma \in L^0(\mathcal{N})$ where m is the Lebesgue measure and $H^*_\mu(t) = h(t)$ [see [2], Corollary 7.6]. Moreover $\mu(A(H, r)) = 0$ for each $r > 0$, since H^*_μ and λ_H are inverse to each other, and H^*_μ is continuous and strictly decreasing.

We define g in the following way

$$g(x) = \begin{cases} f(x) & x \in \mathcal{N} \setminus \cup_n A(f, r_n) \\ (1 + H(x)) \sum_n r_n \chi_{A(f,r_n)}(x) & x \in \cup_n A(f, r_n). \end{cases}$$

Then

$$f(x) \leq g(x) \leq (1 + \varepsilon) f(x)$$

since $0 < H \leq \varepsilon$ and $f = r_n$ on $A(f, r_n)$. Also, $\mu\left(A(g, r)\right) = 0$ for each $r > 0$, since given $r > 0$

$$\begin{aligned} A(g, r) &= \{x \in \mathcal{N} \setminus \cup_n A(f, r) : f(x) = r\} \\ & \bigcup \left(\cup_n \{x \in A(f, r_n) : r_n (1 + H(x)) = r\}\right). \end{aligned}$$

Obviously

$$\mu\left(\{x \in \mathcal{N} \setminus \cup_n A(f, r_n) : f(x) = r\}\right) = 0$$

and

$$\begin{aligned} \mu(\{x \in A(f, r_n) : r_n (1 + H(x)) = r\}) &\leq \mu\left(\left\{x \in \mathcal{N} : H(x) = \frac{r - r_n}{r_n}\right\}\right) \\ &= \mu\left(A\left(H, \frac{r - r_n}{r_n}\right)\right) = 0. \ \square \end{aligned}$$

Theorem 2.1 *Let* $f = \sum_n f_n$ *with* $f_n \geq 0$ *and let* $c_n > 0$ *be such that* $\sum_n c_n = 1$. *Then*

$$f^*(3t) \leq \sum_n \left(f_n^*(t) + \frac{1}{t} \int_{c_n t}^{t} f_n^*(s) ds\right).$$

Proof: Given $\varepsilon > 0$, by our previous Lemma, we can replace each function f_n by $\tilde{f}_n$ such that $f_n \leq \tilde{f}_n \leq (1 + \varepsilon) f_n$, with $\mu\left(A(\tilde{f}_n, r)\right) = 0$ for each $r > 0$. For a fixed $t > 0$, let us write

$$\begin{aligned} \tilde{f}_n &= \tilde{f}_n^1 + \tilde{f}_n^2 + \tilde{f}_n^3 \\ &= \tilde{f}_n \chi_{\left\{x \in \mathcal{N} \,:\, \tilde{f}_n(x) > \tilde{f}_n^*(c_n t)\right\}} + \tilde{f}_n \chi_{\left\{x \in \mathcal{N} \,:\, \tilde{f}_n^*(t) < \tilde{f}_n(x) \leq \tilde{f}_n^*(c_n t)\right\}} \\ &\quad + \tilde{f}_n \chi_{\left\{x \in \mathcal{N} \,:\, \tilde{f}_n(x) \leq \tilde{f}_n^*(t)\right\}}. \end{aligned}$$

Since $\mu\left(\left\{x \in \mathcal{N} : \tilde{f}_n(x) = \tilde{f}_n^*(c_n t)\right\}\right) = 0$,

$$\tilde{f}_n^2 = \tilde{f}_n \chi_{\left\{x \in \mathcal{N} \,:\, \tilde{f}_n^*(t) < \tilde{f}_n(x) < \tilde{f}_n^*(c_n t)\right\}}.$$

Then

$$f \leq \tilde{f}^1 + \tilde{f}^2 + \tilde{f}^3 := \sum_n \tilde{f}_n^1 + \sum_n \tilde{f}_n^2 + \sum_n \tilde{f}_n^3.$$

Now, since

$$\mu\left(\left\{x \in \mathcal{N} : \tilde{f}_n^1(x) > 0)\right\}\right) \leq \mu\left(\left\{x \in \mathcal{N} : \tilde{f}_n(x) > \tilde{f}_n^*(c_n t)\right\}\right) \leq c_n t,$$

it follows that $\left(\tilde{f}_n^1\right)^*(c_n t) = 0$, and therefore,

$$0 \leq \left(\tilde{f}^1\right)^*(t) \leq \sum_n \left(\tilde{f}_n^1\right)^*(c_n t) = 0.$$

On the other hand,

$$\begin{aligned} \left(\tilde{f}^2\right)^*(t) &\leq \frac{1}{t}\left\|\tilde{f}^2\right\|_1 \leq \frac{1}{t}\sum_n \int_{\left\{x\in\mathcal{N}\,:\,\tilde{f}_n^*(t)<\tilde{f}_n(x)<\tilde{f}_n^*(c_n t)\right\}} \tilde{f}_n(s)ds \\ &= \frac{1}{t}\sum_n \int_{c_n t}^t \tilde{f}_n^*(s)ds \end{aligned}$$

and

$$\left(\tilde{f}^3\right)^*(t) \leq \sum_n \left(\tilde{f}_n^3\right)^*(c_n t) \leq \sum_n \tilde{f}_n^*(t).$$

Since $\tilde{f}_n \leq (1+\varepsilon) f_n$, we have that

$$\begin{aligned} f^*(3t) &\leq \left(\tilde{f}^1\right)^*(t) + \left(\tilde{f}^2\right)^*(t) + \left(\tilde{f}^3\right)^*(t) \\ &\leq \sum_n \tilde{f}_n^*(t) + \frac{1}{t}\sum_n \int_{c_n t}^t \tilde{f}_n^*(s)ds \\ &\leq (1+\varepsilon)\sum_n \left(f_n^*(t) + \frac{1}{t}\int_{c_n t}^t f_n^*(s)ds\right), \end{aligned}$$

and letting ε tends to zero, we are done. □

3 Consequence and applications

Given a positive locally Lebesgue integrable function on $(0, \infty)$ (i.e., a weight), we recall that the weak type version of the Lorentz space $\Lambda^p(w)$, introduced in [6], are the spaces $\Lambda^{p,\infty}(w)$ defined as the set of μ-measurable functions such that

$$\|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} W^{1/p}(t) f^*(t) < \infty,$$

where $W(t) = \int_0^t w$ and $0 < p < \infty$. In order to have that these spaces are quasi-normed, it is known that W needs to satisfy the so-called Δ_2-condition; that is $W(2t) \leq CW(t)$ for every t (see [3]). This assumption will be assumed throughout the rest of the paper.

In general, these spaces are not normable and their normability has been studied in several papers (see [4] and [7]). Consequently, if $(f_n)_n \subset \Lambda^{p,\infty}(w)$ are such that $\|f_n\|_{\Lambda^{p,\infty}(w)} \leq 1$ and $(a_n)_n$ satisfies that $\sum_n a_n < \infty$, the function $\sum_n a_n f_n$ is not, in general, in $\Lambda^{p,\infty}(w)$. However, the following result holds:

Theorem 3.1 *Let $(f_n)_n$ be such that $\|f_n\|_{\Lambda^{p,\infty}(w)} \leq 1$ and $(a_n)_n$ such that $\sum_n a_n < \infty$. If there exists a sequence $(c_n)_n \geq 0$ such that $\sum_n c_n = 1$ and*

$$a_W := \sup_{t>0} \frac{W^{1/p}(t)}{t} \sum_n \left[a_n \int_{c_n t}^t \frac{1}{W^{1/p}(s)}\, ds \right] < \infty, \tag{1}$$

then the function $\sum_n a_n f_n \in \Lambda^{p,\infty}(w)$.

Proof: Let $f = \sum_n a_n f_n$. Then, using Theorem 2.1 and the fact that W satisfies the Δ_2-condition, we obtain that

$$\begin{aligned} \|f\|_{\Lambda^{p,\infty}(w)} &= \sup_{t>0} f^*(t) W^{1/p}(t) \approx \sup_{t>0} f^*(3t) W^{1/p}(t) \\ &\leq \sup_{t>0} \left(\sum_n a_n f_n^*(t) W^{1/p}(t) + \frac{W^{1/p}(t)}{t} \sum_n \left[a_n \int_{c_n t}^t f_n^*(s)\, ds \right] \right) \\ &\preceq \sup_{t>0} \left(1 + \frac{W^{1/p}(t)}{t} \sum_n \left[a_n \int_{c_n t}^t \frac{1}{W^{1/p}(s)}\, ds \right] \right) = 1 + a_W, \end{aligned}$$

from which the result follows. □

Remark 3.1 *If w belongs to the Ariño and Muckenhoupt class of weights B_p (see [1]), it was proved in [7] that*

$$\int_0^t \frac{1}{W^{1/p}(s)}\, ds \preceq \frac{t}{W^{1/p}(t)},$$

and, therefore, (1) holds, for every (c_n).

From this we can give a new proof of the following fact (see [7])

Corollary 3.1 *If $w \in B_p$, then $\Lambda^{p,\infty}(w)$ is normable.*

Proof: Let $N \in \mathbb{N}$ and let $f_n \in \Lambda^{p,\infty}(w)$ for every $n = 1, \cdots N$. Set $f = \sum_{n=1}^{N} f_n$. Then, using Theorem 3.1 and the above remark, we obtain that there exists a positive constant A independent of N such that

$$\|f\|_{\Lambda^{p,\infty}(w)} \leq A \sum_n \|f_n\|_{\Lambda^{p,\infty}(w)}.$$

From this, it follows that the quantity

$$\|f\| := \inf \left\{ \sum_{\text{finite}} \|f_n\|_{\Lambda^{p,\infty}(w)}; \ \sum_{\text{finite}} f_n = f \right\},$$

is a norm equivalent to $\|f\|_{\Lambda^{p,\infty}(w)}$. Therefore $\Lambda^{p,\infty}(w)$ is normable. $\square$

For our next result, which is a natural extension of E.M. Stein and N.J. Weiss's Lemma ([8]), let us recall that an increasing function W is said to be quasi-concave if $W(s)/s$ is a decreasing function.

Corollary 3.2 *Let w be a weight such that $W^{1/p}$ is quasi-concave and let $(f_n)_n \subset \Lambda^{p,\infty}(w)$ be such that $\|f_n\|_{\Lambda^{p,\infty}(w)} \leq 1$. Then if $(a_n)_n > 0$ satisfy that $\sum_n a_n \log(1/a_n) < \infty$, we have that $\sum_n a_n f_n \in \Lambda^{p,\infty}(w)$.*

Proof: The proof follows from the fact that, by the quasi-concavity of $W^{1/p}$,

$$\int_{c_n t}^{t} \frac{1}{W^{1/p}(s)}\, ds \leq \frac{t}{W^{1/p}(t)} \log \frac{1}{c_n},$$

and therefore (1) holds for every (c_n) such that $\sum_n a_n \log \frac{1}{c_n} < \infty$. Taking $c_n = a_n / \sum_i a_i$, we are done. $\square$

This corollary shows that for every weight w such that $W^{1/p}$ is quasi-concave, the space $\Lambda^{p,\infty}(w)$ is logconvex in the terminology of [5].

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