L^p-estimates for Riesz Transforms on Forms in the Poincaré Space

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ABSTRACT. Using hyperbolic form convolution with doubly isometry-invariant kernels, the explicit expression of the inverse of the de Rham laplacian Δ acting on m-forms in the Poincaré space $\mathbb{H}^n$ is found. Also, by means of some estimates for hyperbolic singular integrals, L^p-estimates for the Riesz transforms $\nabla^i \Delta^{-1}$, $i \leq 2$, in a range of p depending on m, n are obtained. Finally, using these, it is shown that Δ defines topological isomorphisms in a scale of Sobolev spaces $H^s_{m,p}(\mathbb{H}^n)$ in case $m \neq (n \pm 1)/2, n/2$.

1. STATEMENT OF RESULTS AND PRELIMINARIES

1.1. The main object of study in this paper is the Hodge-de Rham Laplacian Δ acting on m-forms in the Poincaré hyperbolic space $(\mathbb{H}^n, g)$. The aim is to prove that Δ defines topological isomorphisms in a range $H^s_{m,p}(\mathbb{H}^n)$ of Sobolev spaces of forms defined as follows. For $0 \leq m \leq n$, $1 \leq p < \infty$ and $s \in \mathbb{N}$, the Sobolev space $H^s_{m,p}(\mathbb{H}^n)$ is the completion of the space $\mathcal{D}_m(\mathbb{H}^n)$ of smooth m-forms with compact support with respect to the norm

$$\|\eta\|_{p,s} = \sum_{i=0}^{s} \|\nabla^{(i)}\eta\|_p.$$

Here $\nabla^{(i)}$ means the i-th covariant differential of η, and for a covariant tensor α

$$\|\alpha\|_p = \left(\int_{\mathbb{H}^n} |\alpha(x)|^p \, d\mu(x) \right)^{1/p},$$

$|\alpha|$ being the pointwise norm of α with respect to the metric g and $d\mu$ the volume-invariant measure on $\mathbb{H}^n$ given by g. The space $H^s_{m,p}(\mathbb{H}^n)$ can be alternatively defined in terms of weak derivatives. The main result of this paper is the following theorem.

Theorem A. *Δ is a topological isomorphism from $H^{s+2}_{m,p}(\mathbb{H}^n)$ to $H^s_{m,p}$ for $p \in (p_1, p_2)$ with*

$$p_1 = \frac{2(n-1)}{n-2+|n-2m|}, \qquad \frac{1}{p_1} + \frac{1}{p_2} = 1$$

in case $m \neq (n \pm 1)/2$, $n/2$.

In the exceptional case $m = (n \pm 1)/2$, Δ is one to one but is not a topological isomorphism for any p. For this case we obtain as well some weighted estimates. If $m = n/2$, Δ is known to have a non-trivial kernel. Of course, Sobolev spaces $H^s_{m,p}$ can be considered for non integer s as well, and the same results hold by interpolation.

Notice that the Hodge star operator $*$ establishes an isometry from $H^s_{m,p}(\mathbb{H}^n)$ to $H^s_{n-m,p}(\mathbb{H}^n)$ which commutes with Δ, and this is why the range (p_1, p_2) depends only on $|n-2m|$. Notice too that the range (p_1, p_2) always contains $p = 2$ in the non-critical case $|n-2m| > 1$ and that for functions ($m = 0$), the range of p is $(1, \infty)$ (see comments below). We point out that the range (p_1, p_2) equals $|1/p - \frac{1}{2}| < \sqrt{\mu}/(n-1)$, where μ denotes the greatest lower bound for the spectrum of Δ in $H^0_{m,2}(\mathbb{H}^n)$, whose value ([4]) is $\mu = (n-1-2m)^2/4$ (for $m < n/2$).

For the Sobolev spaces for $p = 2$, $H^s_{m,2}(\mathbb{H}^n)$, another proof of the theorem, based on energy methods and valid for an arbitrary complete hyperbolic manifold, is given in [1]. The motivation for the theorem, as with [1], comes from mathematical physics, where most operators exhibit Δ as their principal part, and results like the above become essential to establish existence and uniqueness theorems.

Our method of proof is simply to construct an explicit inverse L for Δ on $\mathcal{D}_m(\mathbb{H}^n)$ and show that there is a gain of two covariant derivatives

$$\|L\eta\|_{p,s+2} \leq \text{const}\, \|\eta\|_{p,s}.$$

Thus $L\eta$ plays the role of the classical Riesz transform in the Euclidean setting. The most delicate part is of course

$$\|\nabla^{(2)} L\eta\|_p \leq \text{const}\, \|\eta\|_p, \quad p_1 < p < p_2,\ \eta \in \mathcal{D}_m(\mathbb{H}^n).$$

Riesz-type operators such as $\nabla\Delta^{-1/2}$, $\nabla^{(2)}\Delta^{-1}$ have extensively been studied in different contexts, for the case of *functions*. On symmetric spaces, they are bounded in L^p, $1 < p < \infty$ and of weak type $(1,1)$. This was shown in [2] for the first order ones in some spaces, and later extended to all symmetric spaces in [3]. The L^p-boundedness holds as well for higher order Riesz transforms in symmetric spaces, but not generally the weak type $(1, 1)$ estimate. In more general contexts, this has been shown in [6], [7], [8], among others. In case of m-forms, $0 < m < n$, as far as we know, there are much less known results, and is for those that our result is new. In [12], [13] some aspects of harmonic analysis of forms are developed; in particular, the exact expression for the heat kernel is given, and it is very likely that from it one can get as well an explicit expression for Δ^{-1}. Strictly speaking, to prove the result, an exact expression of Δ^{-1} is not needed, it is enough having estimates for the resolvent both local and at infinity. In [8], estimates of this kind are obtained and applied to Sobolev-type inequalities for forms, and they might work for this purpose too.[1] However, we feel that our approach, that we next describe, is more elementary and might be interesting in itself.

The de Rham Laplacian Δ is invariant by all isometries φ of $\mathbb{H}^n$. These form a group that we denote here by $\mathrm{Iso}(\mathbb{H}^n)$. Denoting by $\varphi^*(\eta)(x) = \eta(\varphi(x))$ the pull-back of a form η by φ, this means that Δ and φ^* commute, for all $\varphi \in \mathrm{Iso}(\mathbb{H}^n)$. Therefore the inverse L of Δ should commute too with $\mathrm{Iso}(\mathbb{H}^n)$. Among all isometries of $\mathbb{H}^n$, the *hyperbolic translations* $\mathrm{Tr}(\mathbb{H}^n)$ constitute a (non-commutative) subgroup, in one to one correspondence with $\mathbb{H}^n$ itself. In Section 2 we do some harmonic analysis for forms in $\mathbb{H}^n$ and introduce *hyperbolic convolution of forms* to describe all operators acting on m-forms and commuting with $\mathrm{Tr}(\mathbb{H}^n)$. In a second step (Subsection ***2.2***) we characterize the hyperbolic convolution kernels $k(x, y)$ corresponding to operators commuting with the full group $\mathrm{Iso}(\mathbb{H}^n)$.

Once the general expression of an operator commuting with $\mathrm{Iso}(\mathbb{H}^n)$ has been found, we look for our L among these. This corresponds to L having a kernel $k(x, y)$ which is a *fundamental solution* of Δ in a certain sense, and having the best decay at infinity. This kernel turns out to be unique for $m \neq (n \pm 1)/2$, $n/2$, we call it the *Riesz kernel for m-forms in* $\mathbb{H}^n$, it is found in Subsection ***3.1*** and estimated in Subsection ***3.2***. Section 4 is devoted to the proof of the L^p-estimates. Here we use standard techniques in real analysis (Haussdorf-Young inequalities, Schur's lemma, etc.). For the second-order Riesz transform, to show its boundedness in the specified range (p_1, p_2) needs considering some notion of "hyperbolic singular integral." There exist some references dealing with this, e.g. [9], [11], and giving some criteria for L^p-boundedness that might apply; however, as the singular integral arises locally, we have found it easier and more elementary to treat it with the classical Euclidean Calderón-Zygmund theory as a local model, and patch it in a suitable way to infinity.

1.2. We collect here several notations and known facts about $\mathbb{H}^n$. We will use both the unit ball model $\mathbb{B}^n$ with metric $g = 4(1 - |x|^2)^{-2} \sum_i dx^i\, dx^i$ and

[1] Added in proof. It has been brought to the author's attention by Professor John M. Lee that when $p = 2$, the result in Theorem A is implicit in the work by R. Mazzeo in *Comm. Partial Differential Equations* **16** (1991), 1615–1664, and in *J.Differential Geometry* **28** (1988), 309–339. Also, a similar result appears in J.M. Lee's preprint in `http://www.arxiv.org/math.DG/0105046`.

the half-space model $\mathbb{R}^n_+ = \{x_n > 0\}$ with metric $g = x_n^{-2}\sum_i dx^i\,dx^i$. Both models are connected via the Cayley transform $\psi\colon \mathbb{R}^n_+ \to \mathbb{B}^n$ given in coordinates by

$$y_i = \frac{2x_i}{\sum\limits_i^{n-1} x_i^2 + (x_n+1)^2}, \quad i = 1,\dots,n-1;$$

$$y_n = \frac{\sum\limits_1^{n} x_i^2 - 1}{\sum\limits_1^{n-1} x_i^2 + (x_n+1)^2}.$$

We denote by $e \in \mathbb{H}^n$ the point $(0,0,\dots,1) \in \mathbb{R}^n_+$ or $0 \in \mathbb{B}^n$.

The metric g defines a pointwise inner product $(\alpha,\beta)(x)$ between forms at x, for every $x \in \mathbb{H}^n$, and a volume measure $d\mu$. In the ball model $d\mu$ is written $d\mu(x) = 2^n(1-|x|^2)^{-n}dx^1\cdots dx^n$, and $d\mu(x) = x_n^{-n}dx^1\cdots dx^n$ in the half-space model. We denote by $\langle\ ,\ \rangle$ the pairing between forms that makes $H^s_{m,2}(\mathbb{H}^n)$ a Hilbert space

$$\langle\alpha,\beta\rangle = \int_{\mathbb{H}^n} (\alpha,\beta)(x)\,d\mu(x).$$

We write $|\alpha|$ and $\|\alpha\|$ for the pointwise and global norms, respectively, of the form α. In terms of the Hodge star operator $*$ the inner product can be written too

$$\langle\alpha,\beta\rangle = \int_{\mathbb{H}^n} \alpha\wedge *\beta.$$

The group $\mathrm{Tr}(\mathbb{H}^n)$ of hyperbolic translations is in one to one correspondence $x \mapsto T_x$ with $\mathbb{H}^n$ through the equation $T_x(e) = x$. The equations of $z = T_x y$ are better described in the half-space model by

$$z_i = x_n y_i + x_i,\ i = 1,\dots,n-1; \quad z_n = x_n y_n.$$

It is easily checked that indeed $\mathrm{Tr}(\mathbb{H}^n)$ is a (non-commutative) group. The inverse transformation of T_x will be denoted S_x. Another explicit isometry φ_x mapping e to x, satisfying $\varphi_x^{-1} = \varphi_x$, is given in the ball model by

$$\varphi_x(y) = \frac{(|x|^2-1)y + (|y|^2 - 2xy + 1)x}{|x|^2\,|y|^2 - 2xy + 1}. \tag{1.1}$$

Since the isotropy group of 0 is the orthogonal group $O(n)$, the general expression of $\varphi \in \mathrm{Iso}(\mathbb{H}^n)$ is $\varphi = \varphi_x \circ U$, with $x = \varphi(0)$.

The hyperbolic (or geodesic) distance between $x, y \in \mathbb{H}^n$ is written $d(x,y)$. We will rather use the *pseudohyperbolic distance* $r = r(x,y)$, related to d by the formula $d(x,y) = 2\,\mathrm{arctanh}\, r(x,y)$. The explicit expression of $r(x,y)^2$ in the $\mathbb{R}^n_+$ model and the $\mathbb{B}^n$ model is respectively

$$r^2 = \frac{|x-y|^2}{|x-y|^2 + 4x_n y_n}, \quad x, y \in \mathbb{R}^n_+, \tag{1.2a}$$

$$r^2 = |\varphi_x(y)|^2 = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2) + |x-y|^2}, \quad x, y \in \mathbb{B}^n. \tag{1.2b}$$

Associated to the group of translations we have the basis of orthonormal translation-invariant vector fields $X_i(x) = (T_x)_*(X_i(e))$, such that $X_i(e) = \partial/\partial x_i$. They satisfy $X_i(u \circ T_x) = (X_i u) \circ T_x$ for every smooth function u. We will denote by $w^i(x)$ the dual basis of X_i, which accordingly is orthonormal and translation invariant too: $T_x^* w^i = w^i$. Their expression in the $\mathbb{R}^n_+$ model is simply

$$X_i(x) = x_n \frac{\partial}{\partial x_i}, \quad w^i(x) = x_n^{-1} dx^i, \quad i = 1, \dots, n.$$

Because of their translation-invariance property, the (X_i, w^i) are more suitable than the (X_i, η^i) defined in the ball model $\mathbb{B}^n$ by

$$Y_i(x) = \frac{(1-|x|^2)}{2} \frac{\partial}{\partial x_i}, \quad \eta^i(x) = 2(1-|x|^2)^{-1}\, dx^i.$$

For an increasing multiindex I of length $|I| = m$ we write $w^I = w^{i_1} \wedge w^{i_2} \wedge \cdots \wedge w^{i_m}$, and similarly dx^I or η^I. The $\{w^I\}_I$ is an orthonormal translation-invariant basis of m-forms.

Recall that the de Rham Laplacian is defined as $\Delta = d\delta + \delta d$, where δ is the adjoint of d with respect to $\langle\ ,\ \rangle$. Although strictly speaking not needed, the following expression of Δ in w^I-coordinates will simplify the analysis at some points. If $\alpha = \sum_I \alpha_I w^I$, a computation shows that in case $n \notin J$

$$(\Delta\alpha)_J = \Delta\alpha_J + 2\sum_{k \in J} X_k \alpha_{Jk} - p(n-p-1)\alpha_J. \tag{1.3}$$

Here Jk means the multiindex obtained replacing k by n. In case $n \in J$,

$$(\Delta\alpha)_J = \Delta\alpha_J - 2\sum_{\ell \notin J} X_\ell \alpha_{\ell J} - (1-p)(p-n)\alpha_J, \tag{1.4}$$

where ℓJ means the multiindex obtained replacing n by ℓ. For a function f

$$\Delta f = -\sum_{i=1}^n X_i^2 f + (n-1) X_n f.$$

In the ball model, with usual coordinates,

$$\Delta f = -\frac{1}{4}(1-|x|^2)^2 \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} + \left(1-\frac{n}{2}\right)(1-|x|^2)\sum x_i \frac{\partial f}{\partial x_i}. \tag{1.5}$$

2. Translation Invariant and Isometry Invariant Operators on Forms

2.1. We are interested in finding the general expression of an operator acting on m-forms, and isometry-invariant. In a first step we consider *translation-invariant operators* acting on m-forms; these are described by what we might call *hyperbolic convolution* as follows. Let $k(x,y)$ be a double m-form in x, y and define

$$(C_k\alpha)(x) = \int_{\mathbb{H}^n} \alpha(y) \wedge *_y k(x,y) = \langle \alpha, k(x,\cdot)\rangle, \quad \alpha \in \mathcal{D}_m(\mathbb{H}^n).$$

If T_z is a translation with inverse S_z

$$\begin{aligned} C_k(T_z^*\alpha)(x) &= \int_{\mathbb{H}^n} (T_z^*\alpha)(y) \wedge *_y k(x,y) \\ &= \int_{\mathbb{H}^n} \alpha(T_z y) \wedge *_y k(x,y) \\ &= \int_{\mathbb{H}^n} \alpha(y) \wedge *_y k(x,S_z y), \\ T_z^*(C_k\alpha)(x) &= C_k\alpha(T_z x) \\ &= \int_{\mathbb{H}^n} \alpha(y) \wedge *_y k(T_z x, y). \end{aligned}$$

Therefore C_k is translation invariant if k is doubly translation invariant in the sense that

$$k(x,y) = k(S_z x, S_z y), \quad \forall\, S_z.$$

Using the translation-invariant basis of m-forms w^I we see that the general expression of k is

$$k(x,y) = \sum_{I,J} k_{I,J}(x,y) w^I(x) \otimes w^J(y),$$

where $k_I(x,y)$ are doubly-invariant functions, that is, of the form $k_{I,J}(x,y) = a_{I,J}(S_y x)$ for some function (or distribution) $a_{I,J}$. If δ_0 denotes the Delta-mass at e and

$$\delta(x,y) = \sum_{I,J} \delta_0(S_y x) w^I(x) \otimes w^J(y),$$

then formally

$$\alpha(x) = \int_{\mathbb{H}^n} \alpha(y) \wedge *_y \delta(x, y).$$

If P is an operator on m-forms commuting with the T_y, S_y, we will thus have

$$P\alpha(x) = \int_{\mathbb{H}^n} \alpha(y) \wedge *_y P_x(\delta(x, y)),$$

and indeed $k(x, y) = P_x(\delta(x, y))$ is formally doubly-invariant. This shows, in loose terms, that the operator C_k of convolution with a doubly translation invariant kernel k gives the general translation-invariant operator acting on m-forms. If

$$k(x, y) = \sum_{I,J} a_{I,J}(S_y x) w^I(x) \otimes w^J(y)$$

and $\alpha(x) = \sum \alpha_I(x) w^I(x)$, then $C_k\alpha$ has in the basis $w^I(x)$ coefficients given by

$$(C_k\alpha)_I(x) = \sum_J \int_{\mathbb{H}^n} a_{I,J}(S_y x)\alpha_J(y)\, d\mu(y).$$

Thus in the basis w^I everything reduces of course to convolution of functions. For a function convolution kernel $a(S_y x)$ and a test function $u \in \mathcal{D}(\mathbb{H}^n)$ we may think of

$$C_a u(x) = \int_{\mathbb{H}^n} u(y) a(S_y x)\, d\mu(y)$$

as an infinite linear combination of inverse translates $a(S_y x)$ of $a(x)$. Since the vector fields X_i commute with translations, it follows that, whenever everything makes sense,

$$X_i(C_a u) = C_{X_i a} u. \tag{2.1}$$

We point out that this convolution is not commutative; $C_a u$ is in general different from $C_u a$. Correspondingly, $X_i C_a u - C_a X_i u$ is in general not zero; in fact one can easily show ([1, Lemma 3.1]) that these commutators are linear combinations of other convolution operators built from $a(S_y x)$.

2.2. Let P be a generic translation-invariant operator acting on m-forms. We have seen in the previous subsection that we can associate to P a doubly-translation invariant kernel $k(x, y)$ so that $P = C_k$. By the same argument as before, P will be isometry invariant if and only if $k(\varphi x, \varphi y) = k(x, y)$ $\forall\, \varphi \in \mathrm{Iso}(\mathbb{H}^n)$, in which case we say that k is *doubly isometry-invariant*. Working in the ball model and since every $\varphi \in \mathrm{Iso}(\mathbb{H}^n)$ is the composition of a translation with some $U \in O(n)$, the additional requirement on the kernel $k(x, y) =$

$\sum a_{I,J}(S_y x) w^I(x) \otimes w^J(y)$ amounts to $k(Ux, U0) = k(x, 0)$, that is,

$$\sum_{I,J} a_{I,J}(Ux) U^* w^I(x) \otimes U^* w^J(0) = \sum_{I,J} a_{I,J}(x) w^I(x) \otimes w^J(0), \quad \forall\, U.$$

Thus we are interested in describing those $k(x, 0)$—which are m-forms at 0 whose coefficients are m-forms in x—that are doubly invariant by all $U \in O(n)$ in the sense above. Once the $k(x, 0)$ having this property are known, $k(x, y) = k(S_y x, 0)$ defines the general doubly isometry invariant m-form. For $m = 0$ the $k(x, 0)$ are simply the radial functions $a(|x|)$, and $a(|S_y x|) = a(|\varphi_y x|)$ is the general doubly isometry invariant function. For $m \neq 0$ their general expression is not so simple. We find it more convenient to use the usual basis dx^I so we look at $k(x, 0)$ in the form

$$k(x, 0) = \sum_{|I|=|J|=m} b_{I,J}(x)\, dx^I \otimes dx^J(0), \tag{2.2}$$

and we must impose $\sum_{I,J} b_{I,J}(Ux) d(Ux)^I \otimes d(Ux)^J(0) = k(x, 0)$, $\forall\, U$. For instance,

$$\gamma(x, 0) = \sum_{i=1}^{n} dx^i \otimes dx^i(0)$$

is easily seen to be doubly $O(n)$-invariant, and so is

$$\gamma_m = \frac{1}{m!} \gamma \wedge \cdots \wedge \gamma = \sum_{|I|=m} dx^I \otimes dx^I(0)$$

(here we use the symbol $\wedge$ to denote as well the exterior product of double forms defined by $(\alpha_1 \otimes \beta_1) \wedge (\alpha_2 \otimes \beta_2) = (\alpha_1 \wedge \alpha_2) \otimes (\beta_1 \wedge \beta_2)$). Another doubly $O(n)$-invariant 1-form is

$$\tau(x, 0) = \Big(\sum_{i=1}^{n} x_i\, dx^i \Big) \otimes \Big(\sum_{i=1}^{n} x_i\, dx^i(0) \Big).$$

Lemma 2.1. *The double forms γ and τ generate all doubly $O(n)$-invariant $k(x, 0)$. More precisely, their general expression in the ball model is*

$$k(x, 0) = \begin{cases} A_1(|x|)\gamma_m + A_2(|x|)\tau \wedge \gamma_{m-1}, & 0 < m < n, \\ A(|x|)\gamma_m, & m = 0, n. \end{cases} \tag{2.3}$$

Proof. First we prove by induction the following statement $S(n)$: if $k(x, 0)$ is a doubly invariant (p, q)-form $\sum_{|I|=p, |J|=q} c_{I,J}\, dx^I \otimes dx^J(0)$ with constant coefficients, then $k \equiv 0$ if $p \neq q$, or k is diagonal, i.e., $k(x,0) = c\sum_{|I|=p} dx^I \otimes dx^I(0) = c\gamma_p$ if $p = q$. Of course $S(1)$ is obvious; assuming $S(n-1)$, let us break $k(x,0)$ in four pieces, depending on whether i_1, $j_1 = 1$ or not:

$$k = \sum_{i_1=j_1=1} c_{I,J}\, dx^I \otimes dx^J(0) + \sum_{i_1=1,\ j_1\neq 1} + \sum_{i_1\neq 1,\ j_1=1} + \sum_{i_1\neq 1,\ j_1\neq 1}$$
$$\stackrel{\text{def}}{=} k_1 + k_2 + k_3 + k_4.$$

We may write $k_1 = (dx^1 \otimes dx^1(0)) \wedge \widetilde{k_1}$, $k_2 = (dx^1 \otimes 1) \wedge \widetilde{k_2}$, $k_3 = (1 \otimes dx^1(0)) \wedge \widetilde{k_3}$, with $\widetilde{k_1}$, $\widetilde{k_2}$, $\widetilde{k_3}$, k_4 double forms in the $dx^2, \ldots, dx^n, dx^2(0), \ldots, dx^n(0)$ of bidegrees $(p-1, q-1)$, $(p-1, q)$, $(p, q-1)$, and (p,q), respectively. Imposing that k is doubly invariant by U of the type

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & & & \\ 0 & & U_1 & \\ \vdots & & & \\ 0 & & & \end{pmatrix}, \quad U_1 \in O(n-1), \tag{2.4}$$

we see that $\widetilde{k_1}$, $\widetilde{k_2}$, $\widetilde{k_3}$, and k_4 are $O(n-1)$-invariant. We apply the induction hypothesis: if $p = q$, $\widetilde{k_2} = \widetilde{k_3} = 0$, and $\widetilde{k_1}$, k_4 are diagonal, i.e.,

$$k = c_1 \sum_{i_1=1} dx^I \otimes dx^I(0) + c_2 \sum_{i_1\neq 1} dx^I \otimes dx^I(0). \tag{2.5}$$

If we use now $U \in O(n)$ permuting the first two axes, we see that $c_1 = c_2$ and hence k is diagonal, establishing $S(n)$ in case $p = q$. If $|p-q| > 1$, everything is 0. Finally if $|p-q| = 1$, say $p = q+1$, then $\widetilde{k_2}$ is diagonal and all others are zero

$$k = c(dx^1 \otimes 1) \wedge \sum_{|J|=q} dx'^J \otimes dx'^J(0),$$

where $x' = (x_2, \ldots, x_n)$. If we impose the invariance under the permutation of the first two axes as before, it is clear that k must be zero.

Having proved that $S(n)$ holds for all n, let now $k(x,0)$ be as in (2.2), doubly $O(n)$-invariant. Clearly $k(x,0)$ is then determined by its values $k(\vec{r},0)$, where $\vec{r} = (r, 0, 0, \ldots, 0)$. Fixed r, $k(\vec{r},0)$ may be regarded as a double (m,m)-form with constant coefficients, which is invariant by all $U \in O(n)$ fixing $\vec{r}$, that is, of type (2.4). We write now the decomposition of $k(\vec{r},0)$ in terms of $\widetilde{k_1}(r,0)$, $\widetilde{k_2}(r,0)$, $\widetilde{k_3}(r,0)$, and $k_4(r,0)$ as before, and applying $S(n)$ we get (2.5)

$$\begin{aligned}k(\vec{r},0) &= \\ &= c_1(r)\sum_{i_1=1} dx^I(\vec{r}\,)\otimes dx^I(0) + c_2(r)\sum_{i_1\neq 1} dx^I(\vec{r}\,)\otimes dx^I(0)\end{aligned}$$

(if $m = n$, the last term is zero and the first is γ_m), which we write

$$\begin{aligned}&= (c_1(r)-c_2(r))\sum_{i_1=1,\ |I|=m} dx^I(\vec{r}\,)\otimes dx^I(0) + c_2(r)\sum_{|I|=m} dx^I(\vec{r}\,)\otimes dx^I(0)\\ &= (c_1(r)-c_2(r))\, dx^1(\vec{r}\,)\otimes dx^1(0)\wedge \sum_{|I|=m-1} dx^I(\vec{r}\,)\otimes dx^I(0)\\ &\qquad\qquad + c_2(r)\sum_{|I|=m} dx^I(\vec{r}\,)\otimes dx^I(0)\\ &= (c_1(r)-c_2(r))r^{-2}\tau(\vec{r},0)\gamma_{m-1}(\vec{r},0) + c_2(r)\gamma_m(\vec{r},0).\end{aligned}$$

Finally, with fixed x, we choose U such that $Ux = \vec{r}$, $r = |x|$, and use the invariance of k, τ, γ to find (2.3) with $A_1(r) = c_2(r)$, $A_2(r) = r^{-2}(c_1(r) - c_2(r))$. ◻

To find the general expression of a doubly isometry invariant kernel $k(x,y)$ we must translate $k(x,0)$ to an arbitrary point: $k(x,y) = k(S_y x, S_y y)$. We may use any isometry mapping y to 0, for instance we may use φ_y given by (1.1) instead of S_y. We introduce the basic forms α, β, τ, and γ

$$\begin{aligned}\alpha &= \alpha(x,y)\\ &= \sum_i \varphi_y^i(x)\, d\varphi_y^i(x),\end{aligned}$$

$$\begin{aligned}\beta &= \sum_i \varphi_y^i(x)\, d\varphi_y^i(y)\\ &= -\sum_i \varphi_y^i(x)\frac{dy^i}{1-|y|^2},\end{aligned}$$

$$\tau = \alpha\otimes\beta,$$

$$\begin{aligned}\gamma(x,y) &= \sum_{i=1}^n d\varphi_y^i(x)\otimes d\varphi_y^i(y)\\ &= \frac{-1}{1-|y|^2}\sum_{i=1}^n d\varphi_y^i(x)\otimes dy^i = d_x\beta.\end{aligned}$$

The lemma gives part (a) of the following theorem. Part (b) gives other equivalent general expressions, which are intrinsic, that is, independent of the model of $\mathbb{H}^n$ at use.

Theorem 2.2.

(a) *The general expression of an* (m, m)*-form* $k(x, y)$ *doubly isometry-invariant in* $\mathbb{H}^n$*, in the ball model, is*

$$k(x,y) = \begin{cases} A_1(|\varphi_y x|)\gamma_m(x,y) \\ \qquad + A_2(|\varphi_y x|)\tau(x,y) \wedge \gamma_{m-1}(x,y), & 0 < m < n, \\ A(|\varphi_y x|)\gamma_m(x,y), & m = 0, n. \end{cases}$$

(b) *Another equivalent expression for* $0 < m < n$ *is*

$$\begin{aligned} k(x,y) &= B_1(D)(d_x d_y D)^m + B_2(D)(d_x D \otimes d_y D) \wedge (d_x d_y D)^{m-1} \\ &= (C_1(D) d_x d_y D + C_2(D) d_x D \otimes d_y D)^m, \end{aligned}$$

where D *denotes an arbitrary function of the geodesic distance* $d(x, y)$.

(c) *All such* $k(x, y)$ *are symmetric in* x, $y \in \mathbb{H}^n$.

Proof. Part (a) has been already proved. For (b) note first that it is enough to consider *one* function of d: we choose $D = r(x, y)^2$, which in the ball model equals $|\varphi_y(x)|^2$. Then $d_x D = 2\alpha$, and using (1.1), (1.2a) one finds

$$d_y D = 2(1-D)\sum_i \varphi_y^i(x)\frac{d_y^i}{1-|y|^2} = -2(1-D)\beta.$$

This gives $\tau = \alpha \otimes \beta = -\frac{1}{4}(1/(1-D)) d_x D \otimes d_y D$, and

$$d_x d_y D = +2 d_x D \otimes \beta - 2(1-D) d_x \beta = +4\tau - 2(1-D)\gamma.$$

Therefore $(d_x d_y D)^{m-1}$ and $2^{m-1}(1-D)^{m-1}\gamma_{m-1}$ differ in a term containing τ, and so (b) follows. Part (c) is a consequence of (b). ☐

We will need the expression of the generators τ, γ in terms of the invariant basis w^i. We obtain these using formula (1.2a) for $r^2(x, y)$ in the half-space model. First

$$\alpha = \frac{d_x r^2}{2} = \frac{1-r^2}{2(|x-y|^2 + 4x_n y_n)} \times \Big(2\sum_{i=1}^{n-1} x_n(x_i - y_i) w^i(x) + (2x_n(x_n - y_n) - |x-y|^2) w^n(x)\Big),$$

$$\beta = \frac{d_y r^2}{2(r^2-1)} = \frac{-1}{2(|x-y|^2 + 4x_n y_n)} \times \Big(2\sum_{j=1}^{n} y_n(y_j - x_j) w^j(y) + (2y_n(y_n - x_n) - |x-y|^2) w^n(y)\Big).$$

In the following we write $w^{ij} = w^i(x) \otimes w^j(y)$. We have

$$\tau = \alpha \otimes \beta = \frac{1}{4}\frac{1-r^2}{(|x-y|^2+4x_n y_n)^2}\sum_{ij} P_{i,j}(x,y)w^{i,j},$$

where the $P_{ij}(x,y)$ are certain homogeneous polynomials. As we know, everything can be written in terms of $z = S_y x$: for instance

$$1-r^2 = \frac{4x_n y_n}{|x-y|^2+4x_n y_n} = \frac{4z_n}{|z|^2+2z_n+1},$$

and say for $i,\, j < n$

$$\frac{P_{ij}}{(|x-y|^2+4x_n y_n)} = \frac{x_n y_n (x_i-y_i)(x_j-y_j)}{(|x-y|^2+4x_n y_n)^2} = \frac{z_n z_i z_j}{(|z|^2+2z_n+1)^2}.$$

Therefore we may write

$$\tau = \frac{1-r^2}{(|z|^2+2z_n+1)^2}\sum_{i,j} p_{i,j}(z)w^{i,j}. \tag{2.6}$$

For $\gamma = d_x\beta$ we obtain a similar expression

$$\begin{aligned}\frac{4}{1-r^2}\gamma = &\sum_{i,j=1}^{n-1}\left(\delta_{ij} - \frac{2(x_i-y_i)(x_j-y_j)}{|x-y|^2+4x_n y_n}\right)w^{i,j}\\ &+\left(1-\frac{2\sum_{i=1}^{n-1}|x_i-y_i|^2}{|x-y|^2+4x_n y_n}\right)w^{n,n}\\ &+\sum_{i=1}^{n-1}\frac{2(x_i-y_i)(x_n-y_n)}{|x-y|^2+4x_n y_n}(w^{i,n}-w^{n,i}).\end{aligned}$$

Again this can be written

$$\begin{aligned}\gamma &= \frac{1-r^2}{4(|x-y|^2+4x_n y_n)}\sum_{i,j} Q_{ij}(x,y)w^{i,j}\\ &= \frac{1-r^2}{(|z|^2+2z_n+1)}\sum q_{ij}(z)w^{i,j}.\end{aligned} \tag{2.7}$$

Notice that

$$\frac{p_{ij}(z)}{(|z|^2+2z_n+1)^2}=O(1), \qquad \frac{q_{ij}(z)}{(|z|^2+2z_n+1)}=O(1),$$

and hence

$$|\tau(x,y)|=O(1-r^2), \quad |\gamma(x,y)|=O(1-r^2). \tag{2.8}$$

3. Riesz Forms and Riesz Form-potentials in $\mathbb{H}^n$

3.1. Our next objective is now to find an explicit left-inverse L for Δ on $\mathcal{D}_m(\mathbb{H}^n)$. Since Δ is invariant by all isometries, L should be too. By what has been discussed in Section 2, L should have a kernel $k_m(x,y)$,

$$L\eta(x)=\int_{\mathbb{H}^n}\eta(y)\wedge *_y k_m(x,y),$$

doubly invariant by all isometries. Alternatively, notice that if k is *some* kernel such that

$$\eta(x)=\int_{\mathbb{H}^n}\Delta\eta(y)\wedge *_y k(x,y), \quad \eta\in\mathcal{D}_m(\mathbb{H}^n) \tag{3.1}$$

(which formally exists because $\Delta\eta=0$, $\eta\in\mathcal{D}_m(\mathbb{H}^n)$ imply $\eta=0$), then its average over the unitary group $O(n)$ with respect to the normalized left-invariant measure $d\mu(U)$,

$$k_1(x,y)=\int_{O(n)}k_0(Ux,Uy)\,d\mu(U),$$

still satisfies (3.1), and it is doubly invariant by $O(n)$. If φ_x is an isometry mapping x to 0, $k_2(x,y)=k_1(\varphi_x x,\varphi_x y)$ is independent of φ_x, satisfies (3.1), and is doubly invariant by all isometries.

Anyway, we look for a doubly isometry-invariant kernel k_m for which (3.1) holds, and then consider the operator L defined by k_m as above. Taking for granted by now that this operator L is well defined on $\mathcal{D}_m(\mathbb{H}^n)$ and maps $\mathcal{D}_m(\mathbb{H}^n)$ into locally integrable m-forms, notice that (3.1) and the symmetry of k_m together imply that L is a right-inverse too, that is, $\Delta L\alpha=\alpha$ for $\alpha\in\mathcal{D}_m(\mathbb{H}^n)$ in the weak sense:

$$\begin{aligned}\langle\Delta L\alpha,\eta\rangle=\langle L\alpha,\Delta\eta\rangle&=\int_x L\alpha(x)\wedge *\Delta\eta(x)\\&=\int_x\Big\{\int_y\alpha(y)\wedge *_y k_m(x,y)\Big\}\wedge *\Delta\eta(x)\\&=\int_y\alpha(y)\wedge *_y\Big\{\int_x k_m(x,y)\wedge *\Delta\eta(x)\Big\}=\langle\alpha,\eta\rangle.\end{aligned}$$

We work in the ball model. By Theorem 2.2, $k_m(x,y)$ is of type

$$k_m(x,y) = \begin{cases} A(|\varphi_x y|)\gamma_m, & m = 0, n, \\ A_1(|\varphi_x y)\gamma_m + A_2(|\varphi_x y|)\tau \wedge \gamma_{m-1}, & 0 < m < n, \end{cases}$$

where $\gamma = \sum_i d\varphi_x^i(x) \otimes d\varphi_x^i(y), \tau = \alpha \otimes \beta$ with $\alpha = \sum_i \varphi_x^i(y)\, d\varphi_x^i(x)$, $\beta = \sum_i \varphi_x^i(y)\, d\varphi_x^i(y)$ (notice that we are exchanging x, y, using (c) in Theorem 2.2). Condition (3.1) implies $\Delta_y k_m(x,y) = 0$ in $y \neq x$ (while $\Delta L w = w$ implies $\Delta_x k_m(x,y) = 0$ in $x \neq y$). In fact, (3.1) amounts to requiring $\Delta_y k_m(x,y) = \delta_x$ in a sense to be described below.

3.2. In a first step we look for conditions on the A_1, A_2, so that $\Delta_y k_m(x,y) = 0$ in $y \neq x$. A lengthy computation will show that the general harmonic k_m depends on four parameters. By the invariance of k_m, we may assume $x = 0$, in which case, writing $r = |y|$,

$$k_m(x,y) = A(r)\gamma_m, \quad m = 0, n,$$
$$k_m(0,y) = A_1(r)\gamma_m + A_2(r)\tau \wedge \gamma_{m-1},$$

with $\gamma = \sum dx^i(0) \otimes dy^i$, $\tau = \alpha \otimes \beta$, $\alpha = \sum y^i\, dx^i(0)$, $\beta = r\, dr$. Since $*_x *_y k_m(x,y)$ is again doubly invariant, it must have an analogous expression with m replaced by $n - m$. Indeed, it is easily checked that

$$*_x *_y \gamma_m = \frac{m!}{(n-m)!}(1-r^2)^{2m-n}\gamma_{n-m},$$

$$*_x *_y (\tau \wedge \gamma_{m-1}) = (m-1)!(1-r^2)^{2m-n}\left(r^2\frac{\gamma_{n-m}}{(n-m)!} - \frac{\tau \wedge \gamma_{n-m-1}}{(n-m-1)!}\right),$$

whence

$$*_x *_y k_m(x,y) = \frac{m!}{(n-m)!}(1-r^2)^{2m-n}\gamma_{n-m}, \quad \text{for } m = 0, n,$$

and

$$*_x *_y k_m(0,y) = \frac{(m-1)!(1-r^2)^{2m-n}}{(n-m)!}[(mA_1 + r^2 A_2)\gamma_{n-m} - (n-m)A_2\tau \wedge \gamma_{n-m-1}], \quad \text{for } 0 < m < n. \tag{3.2}$$

Moreover, since $*$ commutes with Δ, it is natural to require as well that $*_x *_y k_m = k_{n-m}$, that is, we may assume from now on that $0 \le m \le n/2$. For $m = 0$, using (1.5) we find

$$\Delta(A(r)) = \frac{1}{4}(1-r^2)[-1(1-r^2)A'' + ((3-n)r + r^{-1}(1-n))A'],$$

from which it follows that $A'(r) = c_0(1-r^2)^{n-2}r^{1-n}$ and

$$A(r) = c_1 - c_0 \int_r^1 (1-s^2)^{n-2}s^{1-n}\,ds.$$

We start now computing $\Delta_y k_m(0,y)$ for $0 < m \le n/2$, using that on m-forms Δ equals $(-1)^{m+1}(*d*d + (-1)^n d*d*)$. The double form $\Delta_y k_m(x,y)$ is also doubly invariant, and therefore it must have the same expression as k_m with A_1, A_2 replaced by other functions B_1, B_2 to be found. In the computations we will use besides (3.2) the equations

$$\begin{aligned}
&d_y\alpha = \gamma,\\
&d_y(\tau \wedge \gamma_{m-1}) = -r\,dr \wedge \gamma_m = -\beta \wedge \gamma_m,\\
&*_x *_y dr \wedge \gamma_m = (-1)^m \frac{m!}{(n-m-1)!}(1-r^2)^{2m+2-n}r^{-1}\alpha \wedge \gamma_{n-m-1},
\end{aligned}$$

which are easily checked as well. First, $d_y k_m(0,y) = (A_1' - rA_2)\,dr \wedge \gamma_m$, so by the equations above

$$\begin{aligned}
(3.3)\quad & *_x *_y d_y k_m(0,y)\\
&= (-1)^m \frac{m!}{(n-m-1)|!}(1-r^2)^{2m+2-n}(A_1' - rA_2)r^{-1}\alpha \wedge \gamma_{n-m-1}\\
&\overset{\text{def}}{=} \frac{(-1)^m m!}{(n-m-1)!}A_3\alpha \wedge \gamma_{n-m-1},
\end{aligned}$$

$$*_x d_y *_y d_y k_m(0,y) = \frac{(-1)^m m!}{(n-m-1)!}(A_3\gamma_{n-m} + A_3' r^{-1}\tau \wedge \gamma_{n-m-1})$$

$$\begin{aligned}
*_y d_y *_y d_y k_m(0,y) &= (-1)^{m(n-m-1)} *_y *_x (A_3\gamma_{n-m} + A_3' r^{-1}\tau \wedge \gamma_{n-m-1})\\
&= (-1)^{m(n-m-1)} \frac{m!}{(n-m-1)!}(1-r^2)^{n-2m}\\
&\quad \times \left(A_3 \frac{(n-m)!}{m!}\gamma_m + A_3' r \frac{(n-m-1)!}{m!}\gamma_m \right.\\
&\qquad \left. - A_3' r^{-1}\frac{(n-m-1)!}{(m-1)!}\tau \wedge \gamma_{m-1}\right)\\
&= (-1)^{m(n-m+1)}(1-r^2)^{n-2m}\\
&\quad \times [((n-m)A_3 + A_3' r)\gamma_m - mA_3' r^{-1}\tau \wedge \gamma_{m-1}].
\end{aligned}$$

By analogous computation, applying d_y to (3.2)

$$*_x d_y *_y k_m(0,y) = \frac{(m-1)!}{(n-m)!}\Big[[(mA_1 + r^2A_2)(1-r^2)^{n-2m}]' + (n-m)rA_2(1-r^2)^{n-2m}\Big]\,dr\wedge \gamma_{n-m},$$

$$\begin{aligned}(3.4)\quad *_y d_y *_y k_m(0,y) &= \\ &= (-1)^{(m+1)(n-m)}(1-r^2)^{2m+2-n}r^{-1} \\ &= \Big[[(mA_1+r^2A_2)(1-r^2)^{n-2m}]' + (n-m)rA_2(1-r^2)^{n-2m}\Big]\alpha\wedge\gamma_{m-1} \\ &\stackrel{\text{def}}{=} (-1)^{(m+1)(n-m)}A_4\alpha\wedge\gamma_{m-1},\end{aligned}$$

$$d_y *_y d_y *_y k_m(0,y) = (-1)^{(n-m)(m+1)}(A_4' r^{-1}\tau\wedge\gamma_{m-1} + A_4\gamma_m).$$

It follows finally that $\Delta = (-1)^{nm+1}(*d*d + (-1)^n d*d*)$ on k_m equals

$$\Delta_y k_m(0,y) = B_1\gamma_m + B_2\tau\wedge\gamma_{m-1},$$

with

$$\begin{aligned}B_1 &= -A_4 - (1-r^2)^{n-2m}((n-m)A_3 + A_3' r), \\ B_2 &= -A_4 r^{-1} + m(1-r^2)^{n-2m}A_3' r^{-1}.\end{aligned}$$

Therefore, $\Delta_y k(0,y) = 0$ is equivalent to the system $B_1 = 0$, $B_2 = 0$. It easily follows from this that A_3 satisfies the equation

$$r(1-r^2)A_3'' + [(n+1) - r^2(3n+1-4m)]A_3' - 2(n-2m)(n-m)rA_3 = 0.$$

Replacing in the equation $B_1 = 0$, A_4 by its expression in terms of A_1 and A_2, and then A_2 by its expression in terms of A_1 and A_3, we find that A_1 satisfies the inhomogeneous equation

$$r(1-r^2)A_1'' + [(n+1) + (n-1-4m)r^2]A_1' + 2m(n-2m)rA_1 = 2rA_3(1+r^2)(1-r^2)^{n-2m-2}.$$

The change of variables $A_1(r) = G(x)$, $A_3(r) = H(x)$, $x = r^2$, transforms these into the hypergeometric equations

$$(3.5)\quad x(1-x)H''(x) + \left[\frac{n}{2} + 1 - \left(\frac{3}{2}n + 1 - 2m\right)x\right]H'(x) - \left(\frac{n}{2} - m\right)(n-m)H = 0,$$

$$x(1-x)G''(x)+\left[\frac{n}{2}+1-\left(2m+1-\frac{n}{2}\right)x\right]G'(x)-m\left(m-\frac{n}{2}\right)G = \frac{1}{2}(1+x)(1-x)^{n-2m-2}H(x) \overset{\text{def}}{=} f(x). \tag{3.6}$$

This system is equivalent to $\Delta_y k_m(x,y) = 0$ in $y \neq x$, whence the general doubly-invariant k_m harmonic in $y \neq x$ depends on four parameters. Note that for $m = n/2$, the homogeneous equations are the same and can be solved explicitly: the general solution is $H = as^{-n/2} + b$ and

$$G(x) = cx^{-n/2} + d + \frac{1}{2}\int_{1/2}^{x} t^{-n/2-1}\left\{\int_0^t s^{n/2}(1+s)(1-s)^{-3}(as^{-n/2}+b)\,ds\right\}dt. \tag{3.7}$$

For $m < n/2$, a fundamental family for the equation (3.5) is given by

$$u_1(x) = x^{-n/2}F\left(-m, \frac{n}{2}-m, 1-\frac{n}{2}, x\right),$$
$$u_2(x) = F\left(\frac{n}{2}-m, n-m, \frac{n}{2}+1, x\right).$$

The hypergeometric function in u_1 is a polynomial in x of degree m with positive coefficients, $1+x$ if $m = 1$. A fundamental family for the equation (3.6) is given by

$$u_3(x) = x^{-n/2}F\left(m-n, m-\frac{n}{2}, 1-\frac{n}{2}, x\right) = x^{-n/2}(1-x)^{n+1-2m}F\left(\frac{n}{2}+1-m, 1-m, 1-\frac{n}{2}, x\right),$$
$$u_4(x) = F\left(m, m-\frac{n}{2}, 1+\frac{n}{2}, x\right).$$

The hypergeometric function in u_3 is a polynomial of degree $m-1$ with positive coefficients (see [5] for all these facts). The wronskian $w(x)$ for this second equation is, by Liouville's formula,

$$W(x) = W(x_0)\exp -\int_{x_0}^{x}\frac{\frac{n}{2}+1-\left(2m+1-\frac{n}{2}\right)t}{t(1-t)}\,dt = c_{mn}x^{-n/2-1}(1-x^{n-2m}).$$

It follows from this that the parametrization for G is given by

$$G(x) = c(x)u_3(x) + d(x)u_4(x), \tag{3.8}$$

where $c(x)$, $d(x)$ satisfy, with $H(x) = au_1(x) + bu_2(x)$,

$$\begin{aligned} c'(x) &= \frac{u_4(x)f(x)}{x(1-x)W(x)} \\ &= \frac{1}{2}c_{mn}^{-1}H(x)(1+x)x^{n/2}(1-x)^{-3}u_4(x), \\ d'(x) &= -\frac{u_3(x)f(x)}{x(1-x)W(x)} \\ &= -\frac{1}{2}c_{mn}^{-1}H(x)(1+x)x^{n/2}(1-x)^{-3}u_3(x). \end{aligned}$$

Once $A_1(r) = G(r^2)$ and $A_3(r) = H(r^2)$ are known, the kernel $k_m(x,y)$ is completely known, because by the definition of A_3 in (3.3),

$$A_2(r) = -(1-r^2)^{n-2m-2}A_3(r) + r^{-1}A_1'(r) = -(1-x)^{n-2m-2}H(x) + 2G'(x).$$

The choice $a = 0$, $c(0) = 0$ ($a = c = 0$ in the parametrization (3.7) for $m = n/2$) gives all doubly invariant $k_m(x,y)$ which are *globally* harmonic, with no singularity, and they are therefore spanned by the forms corresponding to the choice $G = u_4$ and to the choice $a = 0$, $b = 1$, $c(0) = 0$, $d(0) = 0$,

$$\begin{aligned} G(x) = &\left\{\int_0^x (1+t)(1-t)^{-3}t^{n/2}u_2(t)u_4(t)\,dt\right\}u_3(x) \\ &- \left\{\int_0^x (1+t)(1-t)^{-3}t^{n/2}u_2(t)u_3(t)\,dt\right\}u_4(x). \end{aligned}$$

As a particular case, note that for $m = n/2$, γ_m is harmonic in $\mathbb{H}^{2m}$, and it is the simplest example of a non-zero harmonic m-form in $L^2(\mathbb{H}^{2m})$.

3.3. Besides being harmonic in $y \neq x$, the singularity at $y = x$ must be such that (3.1) holds. Again, we may assume $x = 0$; we check this property using *second's Green identity*, whose version for general forms we recall now.

The operator δ being the adjoint of d, one has, for a smooth domain $\bar{\Omega} \subset \mathbb{B}^n$ and α, β smooth forms on $\bar{\Omega}$ with $\deg\alpha = \deg\beta - 1$,

$$\int_{\partial\Omega} \alpha \wedge *\beta = \int_\Omega d\alpha \wedge *\beta - \int_\Omega \alpha \wedge *\delta\beta.$$

Given two m-forms η, ω, applying this with $\alpha = \delta\eta$, $\beta = \omega$, next with $\alpha = \omega$, $\beta = d\eta$ and subtracting, one gets *the first Green's identity for m-forms*

$$\int_{\partial\Omega} (\delta\eta \wedge *\omega - \omega \wedge *d\eta) = \int_\Omega (\Delta\eta \wedge *\omega - \delta\eta \wedge *\delta\omega - d\eta \wedge *d\omega).$$

Permuting ω, η and subtracting again gives *the second Green's identity*

$$\int_{\partial\Omega}(\delta\eta\wedge *\,\omega-\omega\wedge *\,d\eta-\delta\omega\wedge *\,\eta+\eta\wedge *\,d\omega)=\int_{\Omega}(\Delta\eta\wedge *\,\omega-\Delta\omega\wedge *\,\eta).$$

We apply this to $\Omega = B(0,R) - B(0,\varepsilon)$ $0 < \varepsilon < R < 1$, $\eta \in \mathcal{D}_m(\mathbb{H}^n)$ and our $k_m(0,y)$ to get

$$\text{(3.9)}\quad \int_{|y|\geq\varepsilon}\Delta\eta\wedge *_y\, k_m(0,y) = \int_{|y|=\varepsilon}(k_m\wedge *\,d\eta+\delta_y k_m\wedge *\,\eta-\delta\eta\wedge *_y\, k_m-\eta\wedge *\,dk_m).$$

In case $m = 0$, the terms in δk_m, $\delta\eta$ are of course zero; to get a term in $\eta(0)$ on the right when $\varepsilon \to 0$, we need dk_m of the order of ε^{1-n} and k_m of the order of ε^{2-n} in $|y| = \varepsilon$. That makes k_m locally integrable too, and (3.1) is obtained letting $\varepsilon \to 0$. This means that, for $m = 0$, k is unique and is given by the well-known Green's function

$$\text{(3.10)}\qquad A(r) = c_n\int_r^1 (1-s^2)^{n-2}s^{1-n}\,ds,$$

for an appropriate choice of c_n. In case $m > 0$, again we need $|k_m(0,y)| = o(r^{1-n})$ as $r \to 0$, so that the first and third terms on the right have limit 0 as $\varepsilon \to 0$; then k_m is integrable in y, and the integral on the left converges to $\int \Delta\eta\wedge *\,k_m$. Using the expression for $*\,dk_m$ in (3.3), we find

$$\int_{|y|=\varepsilon}\eta\wedge *\,d_y k_m = \frac{(-1)^{m(n-m+1)}n!}{(n-m-1)!}A_3(\varepsilon)\,*_x\int_{|y|=\varepsilon}\eta\wedge\alpha\wedge\gamma_{n-m-1}.$$

By Stoke's theorem, and since $\alpha = O(r)$, the last integral equals

$$(-1)^m\int_{|y|<\varepsilon}\eta\wedge\gamma_{n-m}+O(\varepsilon).$$

If $A_3(\varepsilon) = a_0\varepsilon^{-n}+\cdots$, we see that

$$\lim_{\varepsilon}\int_{|y|=\varepsilon}\eta\wedge *\,d_y k_m = c_n(n-m)\,m!\,a_0\eta(0).$$

Using (3.4) for $\delta k_m = (-1)^{n(m+1)+1} *\,d\,*$, and proceeding in the same way,

$$\int_{|y|=\varepsilon}\delta_y k_m\wedge *\,\eta = -A_4(\varepsilon)\int_{|y|=\varepsilon}\alpha\wedge\gamma_{m-1}\wedge *\,\eta$$
$$= -A_4(\varepsilon)\int_{|y|<\varepsilon}(\gamma_m\wedge *\,\eta+O(\varepsilon)).$$

But by the equation $B_1 = 0$, $A_4(\varepsilon) = -(1-\varepsilon^2)^{n-2m}((n-m)A_3(\varepsilon)+\varepsilon A_3'(\varepsilon)) = a_0 m\varepsilon^{-n} + O(\varepsilon^{1-n})$, and hence the limit of the above expression is $-c_n m! a_0 m\eta(0)$. Altogether, we conclude that if $A_3(\varepsilon) = a_0\varepsilon^{-n} + O(\varepsilon^{1-n})$ and $k_m(0,y) = o(r^{1-n})$, one has

$$\int \Delta\eta \wedge *_y k_m(0,y) = -c_n n m! a_0 \eta(0),$$

so (3.1) will hold for an appropriate choice of a_0. Taking into account the definition of A_3 in (3.3) and that $|k_m| \simeq |A_1| + r^2|A_2|$, we see from (3.7) that if $m = n/2$, this is accomplished by the choice $c = 0$, $a = a_0$; then $G(x) \sim \log x$, $A_1(r) \sim \log r$, $A_1'(r) = O(1/r)$, $A_2(r) = O(r^2)$ if $n = 2$; if $n > 2$, $A_1(r) \sim r^{2-n}$ and $A_2 = O(r^{-n})$. For $0 < m < n/2$, in terms of the functions H, G introduced before, this translates to $H(x) \sim a_0 x^{-n/2}$, $G(x) \sim x^{1-n/2}$. Now look at the general expression of H, G in (3.8). The condition $H(x) \sim c_0 x^{-n/2}$ fixes $a = a_0$; then near $x = 0$, $c'(x)$ is bounded and $d'(x)$ behaves like $x^{-n/2}$. Since $u_4(x)$ is bounded, the term $d(x)u_4(x)$ behaves like $x^{1-n/2}$. So, we must normalize $c(x)$ by $c(0) = 0$, so that $c(x) = O(x)$, and the other term $c(x)u_3(x)$ will behave like $x^{1-n/2}$.

In conclusion, all this discussion shows that the doubly invariant kernels $k_m(x,y)$ satisfying (3.1) constitute a *two parameter family* described by $H = a_0 u_1(x) + b u_2(x)$, $c(0) = 0$. The two parameters are b and the constant of integration for $d(x)$ in (3.8). Equivalently, they are obtained by adding to the form corresponding to $H = a_0 u_1(x)$, $c(0) = 0$, and say $d(\frac{1}{2}) = 0$ the general globally smooth one described before.

3.4. In order to produce the best estimates, in a sense we need to choose the best of the kernels k_m. Naturally enough, we choose the k_m having the best behaviour at infinity, $x = 1$, that is, so that G, H have the best decrease in size as $x \to 1$. In case $m = n/2$, where we already have the normalization $c = 0$, $a = a_0$, the choice $b = -a$ gives the best growth $H(x) = O(1-x)$ and $G(x) = O(\log(1-x))$.

The hypergeometric function u_3 behaves like $(1-x)^{n+1-2m}$ near $x = 1$, while $u_4(x) = F(m, m-n/2, 1+n/2, x)$ is bounded because $1 + n/2 - m - (m - n/2) = 1 + n - 2m > 0$. Similarly, u_1 is bounded near $x = 1$; for $u_2(x) = F(n/2-m, n-m, n/2+1, x)$ we have $n/2+1-(n/2-m)-(n-m) = 2m+1-n$ and hence it behaves like $(1-x)^{2m+1-n}$ if $2m < n-1$, and like $\log(1-x)$ if $2m = n-1$. We use equations (3.8)

$$c(x) = c_{m,n}\int_0^x H(t)(1+t)t^{n/2}(1-t)^{-3}u_4(t)\,dt,$$

$$d(x) = -c_{m,n}\int_{1/2}^x H(t)(1+t)t^{n/2}(1-t)^{-3}u_3(t)\,dt + d_0.$$

If $b \neq 0$, then $H(t) = a_0 u_1(t) + b u_2(t)$ behaves like $(1-t)^{2m+1-n}$ if $2m < n-1$, and like $\log(1-t)$ if $2m = n-1$, resulting in $c(x) = O(1-x)^{2m-n-1}$, $d(x) = O(\log(1-x))$ if $2m < n-1$, and $c(x) = O((1-x)^{-2}\log(1-x))$, $d(x) = O((1-x)^{-1}\log(1-x))$ if $2m = n-1$. So if $b \neq 0$, one has $G(x) = O(\log(1-x))$ if $2m < n-1$ and $G(x) = O((1-x)^{-1}\log(1-x))$ if $2m = n-1$. If $b = 0$, then H is bounded, giving $c(x) = O((1-x)^{-2})$ and $d(x) = O(1)$ for $2m < n-1$, $d(x) = O(\log(1-x))$ for $2m = n-1$. In case $2m < n-1$, however, we can choose the constant d_0 so that $d(1) = 0$, and then $d(x) = O(1-x)^{n-2m-1}$. This choice gives $G(x) = O(1-x)^{n-2m-1}$ for $2m < n-1$. For $2m = n-1$, no choice of d_0 can improve the bound $G(x) = O(\log(1-x))$.

It remains to estimate the growth of $A_2(r)$ near $r = 1$. Recall that the definition (3.3) of A_3 translates to $A_2(r) = 2G'(x) - (1-x)^{n-2m-2}H(x)$. Both terms grow like $(1-x)^{n-2m-2}$, but a cancellation occurs. The functions u_1, u_3 are C^∞ at 1 and have developments

$$
\begin{aligned}
u_3(x) &= A(1-x)^{n+1-2m} + O(1-x)^{n+2-2m},\\
u_3'(x) &= -A(n+1-2m)(1-x)^{n-2m} + O(1-x)^{n+1-2m},\\
H(x) &= a_0 u_1(x) = B + O(1-x).
\end{aligned}
$$

In $u_4(x) = F(m, m-n/2, 1+n/2, x)$, $1+n/2-m-(m-n/2) = n+1-2m \geq 2$, whence u_4 has a finite derivative at 1 and a development

$$
u_4(x) = C + D(1-x) + O(1-x)^{1+\varepsilon} \quad \forall\, \varepsilon < 1, \quad u_4'(x) = O(1).
$$

Then $W(x) = u_3' u_4 - u_3 u_4' = CA(2m-n-1)(1-x)^{n-2m} + \cdots$, and so the constant c_{mn} in (3.8) is $CA(2m-n-1)$. Then from (3.8)

$$
\begin{aligned}
c'(x) &= \frac{B(1-x)^{-3}}{A(2m-n-1)} + O(1-x)^{-2},\\
d'(x) &= -\frac{B(1-x)^{n-2m-2}}{C(2m-n-1)} + O(1-x)^{n-2m-1},
\end{aligned}
$$

which gives

$$
\begin{aligned}
c(x) &= \frac{1}{2}\frac{B}{2(2m-n-1)}(1-x)^{-2} + O(1-x)^{-1},\\
d(x) &= \begin{cases} O(1-x)^{n-2m-1}, & 2m < n-1,\\ O(\log(1-x)), & 2m = n-1. \end{cases}
\end{aligned}
$$

But $G' = c(x)u_3'(x) + d(x)u_4'(x)$; the second term $d(x)u_4'(x)$ satisfies the required bound, while the first $c(x)u_3'(x)$ has a development

$$c(x)u_3'(x) = -\frac{1}{2}\frac{B}{A(2m-n-1)}A(n+1-2m)(1-x)^{n-2m-2} + O(1-x)^{n-2m-1}$$
$$= \frac{B}{2}(1-x)^{n-2m-2} + O(1-x)^{n-2m-1}.$$

As $(1-x)^{n-2m-2}H(x) = B(1-x)^{n-2m-2} + O(1-x)^{n-2m-1}$, the bound for A_2 follows for $2m \le n-1$.

However, for $m = n/2$, this no longer holds. Indeed, from (3.7), where $c = 0$, $a = a_0$, $b = -a$,

$$2G'(x) = x^{-n/2-1}\int_0^x s^{n/2}(1+s)(1-s)^{-3}a(s^{-n/2}-1)\,ds$$

has development

$$2G'(x) = na(1-x)^{-1} + O(\log(1-x)),$$

while

$$(1-x)^{-2}a(x^{-n/2}-1) = \frac{n}{2}a(1-x)^{-1} + \cdots .$$

We point out that all this can be obtained, in loose terms, working directly with the hypergeometric equations relating G, H,

$$x(1-x)G''(x) + \left[\frac{n}{2} + 1 - \left(2m+1-\frac{n}{2}\right)x\right]G'(x) - m\left(m-\frac{n}{2}\right)G = \frac{1}{2}(1+x)(1-x)^{n-2m-2}H(x),$$

and using asymptotic developments. If $H(x) = h_0 + h_1(1-x) + \cdots$ and $G(x) = g_j(1-x)^j + \cdots$, identifying the lower order terms in both sides gives,

$$g_j j(j-1-n+2m)(1-x)^{j-1} = h_0(1-x)^{n-2m-2}.$$

When $H \equiv 0$, one must have either $j = 0$ (corresponding to u_4) or $j = n-2m+1$ (corresponding to u_3). For the inhomogeneous equation, if $j \neq 0$, $j \neq n+1+2m$ (that is, G contains no contribution from u_3, u_4), one finds $j = n-2m-1$ if $2m < n-1$ and $g_j j = -h_0/2$. Then $2G'(x) = h_0(1-x)^{n-2m-2} + \cdots$, $(1-x)^{n-2m-2}H(x) = h_0(1-x)^{n-2m-2}$, showing cancellation. An analogous argument works if $2m = n-1$, but not for $2m = n$.

We summarize the results in this and the previous subsections in the following theorem.

Theorem 3.1. *For $|n-2m|>1$, there is a unique doubly invariant kernel*

$$k_m(x,y)=\begin{cases}A_1(|\varphi_x y|)\gamma_m + A_2(|\varphi_x y|)\tau\wedge\gamma_{m-1}, & m\neq 0,\\ A(|\varphi_x y|)\gamma_m, & m=0,n,\end{cases}$$

for which (3.1) holds, and satisfying moreover

$$|A_i(r)| = O(1-r^2)^{|n-2m|-1}, \quad \text{as } r\to 1.$$

For $m=(n\pm 1)/2$, there is a one-parameter family of such kernels satisfying

$$|A_i(r)| = O(\log(1-r^2)).$$

For $m=n/2$, there is a one-parameter family of such kernels satisfying

$$|A_i(r)| = O(1-r^2)^{-1}.$$

In all cases $A_1(r)\sim r^{2-n}$, $A_2(r)\sim r^{-n}$ as $r\to 0$.

For $|n-2m|>1$, we call $k_m(x,y)$ the *Riesz kernel for m-forms* in $\mathbb{H}^n$, and

$$L\eta(x)=\int_{\mathbb{H}^n}\eta(y)\wedge *_y k_m(x,y)$$

the Riesz potential of η, whenever this is defined. From (2.8) we see that

$$|k_m(x,y)| = O(1-r^2)^{n-m-1}. \tag{3.11}$$

With the notations used before, the function $A_3(r)=H(r^2)$ is bounded with bounded derivatives near $r=1$. Then (3.3) and symmetry imply

$$|d_x k_m(x,y)|,\ |d_y k_m(x,y)| = O(1-r^2)^{n-m-1} \tag{3.12}$$

too. The growth of A_3 also implies $A_4=O(1-r^2)^{n-2m}$ because $B_1\equiv 0$, and then (3.4) gives as well

$$|\delta_x k_m(x,y)|,\ |\delta_y k_m(x,y)| = O(1-r^2)^{n-m-1}. \tag{3.13}$$

By construction, one has $L\Delta\eta=\eta$ for $\eta\in\mathcal{D}_m(\mathbb{H}^n)$. We will need the following generalization of this fact.

Proposition 3.2. *If η is a smooth form in $\mathbb{H}^n$ such that*

$$|\eta(y)|,\ |\nabla\eta(y)| = o(1-|y|^2)^m, \quad y\in\mathbb{B}^n,$$

then $L\Delta\eta=\eta$.

Proof. In (3.9) we would get an extra term

$$\int_{|y|=R} (k_m \wedge * d\eta + \delta k_m \wedge * \eta - \delta\eta \wedge * k_m - \eta \wedge * dk_m).$$

Estimates (3.11), (3.12) and (3.13) imply that, with x fixed and $|y| = R \nearrow 1$,

$$|k_m|,\ |\delta k_m|,\ |dk_m| = O(1 - R^2)^{n-m-1}.$$

Inserting $|\eta(y)|, |\nabla\eta(y)| = o(1 - |y|^2)^m$ we see that this extra term vanishes as $R \nearrow 1$. □

4. Proof of the Main Theorem

4.1. Once the Riesz form $k_m(x, y)$ has been found, our aim is now to prove that the corresponding convolution

$$L_m\eta(x) = \int_{\mathbb{H}^n} \eta(y) \wedge *_y k_m(x, y)$$

satisfies

$$\|L_m\eta\|_{p,s+2} \le c\|\eta\|_{p,s}, \tag{4.1}$$

for $m \ne (n \pm 1)/2$, $n/2$, and p in the range $p_1(m) = (n-1)/(n-1-m) < p < (n-1)/m = p_2(m)$, and for a compactly supported m-form η (recall that we are assuming without loss of generality that $m \le n/2$). Since these are dense in the Sobolev spaces and we already know that $\Delta L_m\eta = L_m\Delta\eta = \eta$, this will prove the theorem for $m \ne (n \pm 1)/2$, $n/2$. The case $m = (n \pm 1)/2$ will be commented later.

We work in the translation invariant basis w^I. Taking into account formulas (2.6) and (2.7) for γ, τ, the Riesz form is written in the $\mathbb{R}^n_+$ model

$$k_m(x, y) = \sum_{|I|=|J|=m} a_{I,J}(S_y x) w^I(x) \otimes w^J(y),$$

where each coefficient $a_{I,J}$ has an expression, with $z = S_y x$,

$$a_{I,J}(z) = \Psi_{I,J}(r) \frac{p_{I,J}(z)}{(|z|^2 + 2z_n + 1)^{2m}},$$

$$r^2 = \frac{1 + |z|^2 - 2z_n}{1 + |z|^2 + 2z_n} = \frac{|x - y|^2}{|x - y|^2 + 4x_n y_n}.$$

Here $p_{I,J}(z)$ is a certain polynomial in $z_1, \ldots, z_n$, $\Psi_{I,J}$ is C^∞ in $(0, 1)$ with $\Psi_{I,J}(r) \sim c_0 r^{2-n}$ as $r \searrow 0$, $\Psi_{I,J}(r) = O(1 - r^2)^{n-m-1}$ as $r \nearrow 1$. The term $q_{I,J}(z) = p_{I,J}(z)/(|z|^2 + 2z_n + 1)^{2m}$ is bounded.

If $\eta = \sum_I \eta_I(y) w^I(y)$, the coefficient $(L\eta)_I(x)$ of $L\eta$ in the basis w^I is a finite linear combination of hyperbolic convolutions

$$(L\eta)_I(x) = \sum_J \int_{\mathbb{H}^n} \Psi_{I,J}(r) q_{I,J}(z) \eta_J(y)\, d\mu(y).$$

By ellipticity of Δ, $L\eta$ is a smooth form. Moreover, since η has compact support, we see from (1.2a) and (3.11), (3.12), (3.13) that, in the ball model,

$$|L\eta(x)|,\ |d(L\eta)(x)|,\ |\delta(L\eta)(x)| = O(1-|x|^2)^{n-m-1},$$

which amounts to

$$|(L\eta)_I(x)|,\ |X_i(L\eta)_I(x)| = O(1-|x|^2)^{n-m-1}. \tag{4.2}$$

We claim that for second-order derivatives we have too

$$\begin{aligned} |X_j X_i (L\eta)_I(x)| &= O(1-|x|^2)^{n-m-1}, \quad \text{i.e.,} \\ |\nabla^{(2)}(L\eta)(x)| &= O(1-|x|^2)^{n-m-1}. \end{aligned} \tag{4.3}$$

Notice that since we already know that $\Delta L\eta = \eta$, from the expression of Δ in the basis w^I given in (1.3)–(1.5) it follows that it is enough to show that for $j < n$. We will see below (equation (4.7) and invariance of the X_i) that each of the functions $a(z) = \Psi_{I,J}(r) q_{I,J}(z)$ satisfies

$$|X_j X_i a(z)| = O(1-r^2)^{n-m-1},$$

from which (4.3) follows as before. In fact, the discussion that follows will show that $|\nabla^{(k)} L\eta(x)| = O(1-|x|^2)^{n-m-1}$, $\forall\, k$.

We continue the proof of (4.1). We claim first that it is enough to prove (4.1) for $s = 0$. For a smooth form $\eta = \sum \eta_I w^I$, let $X_i\eta$ denote here the m-form $X_i\eta = \sum X_i \eta_I w^I$. It is clear from formulas (1.3)–(1.5) and the commutation properties,

$$[X_i, X_j] = 0,\ i, j < n, \quad [X_n, X_i] = X_i,\ i < n,$$

that for each i there is an operator P_i of order two in the $X_1, \dots, X_n$ such that

$$X_i \Delta\eta - \Delta(X_i\eta) = P_i(X)\eta.$$

Applying this to $L\eta$, which is smooth by the ellipticity of Δ, we get

$$(X_i - \Delta X_i L)\eta = P_i(X) L\eta.$$

But $X_i L\eta$ satisfies, by (4.2) and (4.3)

$$|X_i L\eta(x)|,\ |d(X_i L\eta)(x)|,\ |\delta(X_i L\eta)(x)| = O(1-|x|^2)^{n-m-1},$$

and hence by Proposition 3.2, $L\Delta = \mathrm{Id}$ on it. We conclude that for all $\eta \in \mathcal{D}_m(\mathbb{H}^n)$

$$(LX_i - X_i L)\eta = LP_i(X)L\eta.$$

Assume that (4.1) has been proved up to s, so that by density it holds for $\alpha \in H^s_{m,p}(\mathbb{H}^n)$ too, and let γ be a multiindex of length $|\gamma| \le s$. For $i = 1, \dots, n$ and $\eta \in \mathcal{D}_m(\mathbb{H}^n)$,

$$X^\gamma X_i L\eta = X^\gamma L X_i \eta - X^\gamma LP_i(X)L\eta,$$

so using twice the induction hypothesis

$$\begin{aligned}\|X^\gamma X_i L\eta\|_p &\le \text{const}\,(\|X_i\eta\|_{p,s} + \|P_i(X)L\eta\|_{p,s}) \\ &\le \text{const}\,(\|\eta\|_{p,s+1} + \|\eta\|_{p,s}),\end{aligned}$$

proving (4.1) for $s+1$. Proving (4.1) for $s > 0$ means proving

$$\|(L\eta)_I\|_p,\ \|X_i(L\eta)_I\|_p,\ \|X_jX_i(L\eta)_I\|_p \le \text{const}\,\|\eta\|_p.$$

As before, using that we already know that $\Delta L\eta = \eta$, we see that for the second-order derivatives we may assume $j < n$. In the following we delete the indexes I, J and denote by $a(z) = \psi(r)Q(z)$ a convolution kernel with ψ, Q as above, and proceed to prove that the convolution

$$(C_a\alpha)(z) = \int_{\mathbb{H}^n} a(S_y x)\alpha(y)\,d\mu(y)$$

satisfies

$$\|C_a\alpha\|_p,\ \|X_i(C_a\alpha)\|_p,\ \|X_jX_iC_a(\alpha)\|_p \le \text{const}\,\|\alpha\|_p, \qquad p_1 \le p \le p_2, \tag{4.4}$$

where in the last case we may assume that $j < n$. The fields X_i are invariant, and therefore $X_iC_a\alpha$, $X_jX_iC_a\alpha$ are obtained, respectively, by convolution with Z_ia, Z_jZ_ia (by (2.1)). Recall that

$$\psi(r) = O(1-r^2)^{n-m-1} = O\left(\frac{4z_n}{1+|z|^2+2z_n}\right)^{n-m-1} \qquad \text{as } r \nearrow 1,$$

and

$$\psi(r) \sim r^{2-n}, \qquad \text{as } r \searrow 0.$$

In order to estimate Z_ia, Z_iZ_ja, we collect first some auxiliary estimates. We claim that

$$\begin{aligned} &|Z_iQ| \le \text{const}, && |Z_iZ_jQ| \le \text{const}, \\ &|Z_ir| \le \text{const}\,(1-r^2), && |Z_iZ_jr| \le \text{const}\,r^{-1}(1-r^2).\end{aligned} \tag{4.5}$$

The first two are routinely checked, for instance, when differentiating the denominator in Q,

$$\begin{aligned}\left| Z_i \frac{1}{(1+|z|^2+2z_n)^{2m}} \right| &= \left| \frac{4mz_n z_1}{(1+|z|^2+2z_n)^{2m+1}} \right| \\ &\leq \frac{\text{const}}{(1+|z|^2+2z_n)^{2m}} \qquad (i<n),\end{aligned}$$

so that the term $p_{I,J}(z)Z_i[(1+|z|^2+2z_n)^{-2m}]$ will still be bounded. All other terms can be treated similarly. Differentiating $1-r^2 = 4z_n/(1+|z|^2+2z_n)$, we get

$$\begin{aligned}
Z_i r &= \frac{1-r^2}{2} \frac{z_i z_n}{r(1+|z|^2+2z_n)}, \\
Z_n r &= -\frac{1-r^2}{2} \frac{1}{r} \frac{1+|z|^2-2z_n^2}{1+|z|^2+2z_n}, \\
Z_j Z_i r &= \frac{1-r^2}{2r} \left\{ \frac{\delta_{ij} z_n^2}{1+|z|^2+2z_n} - \frac{1+5r^2}{2r^2} \frac{z_i z_j z_n^2}{(1+|z|^2+2z_n)^2} \right\}, \quad i,j<n \\
Z_j Z_n r &= \frac{1-r^2}{r} \left\{ -\frac{2z_n^2 z_j(1+z_n)}{(1+|z|^2+2z_n)^2)} + \frac{(1+r^2)}{4r^2} \frac{z_j z_n(1+|z|^2-2z_n)}{(1+|z|^2+2z_n)^2} \right\}, \\
&\qquad j<n.
\end{aligned}$$

These imply (4.5) because

$$\begin{aligned}|z_i z_n|,\ 1+|z|^2-2z_n^2 &\leq (1+|z|^2-2z_n)^{1/2}(1+|z|+2z_n)^{1/2} \\ &= r(1+|z|^2+2z_n).\end{aligned}$$

Now

$$\text{(4.6a)} \qquad Z_i a(z) = \psi'(r) Z_i r Q(z) + \psi(r)(Z_i Q)(z),$$

$$\begin{aligned}\text{(4.6b)} \quad Z_j Z_i a(z) &= \psi''(r)(Z_i r)(Z_j r) Q(z) + \psi'(r)(Z_j Z_i r) Q \psi'(r) Z_i r Z_j Q \\ &\quad + \psi'(r) Z_j r Z_i Q + \psi(r) Z_j Z_i Q.\end{aligned}$$

The estimates (4.5) imply

$$\text{(4.7)} \quad \begin{aligned}&|a(z)|, \quad |Z_i a(z)|, \quad |Z_j Z_i a(z)| = O(1-r^2)^{n-m-1} \quad \text{as } r \nearrow 1, \\ &|a(z)| = O(r^{2-n}), \quad |Z_i a(z)| = O(r^{1-n}), \quad |Z_j Z_i a(z)| = O(r^{-n}).\end{aligned}$$

We will call a convolution kernel $b(z)$ *m-admissible* if $|b(z)| = O(r^{1-n})$ as $r \searrow 0$ and, moreover, $|b(z)| = O(1-r^2)^{n-m-1}$ as $r \nearrow 1$. We will prove later (Theorem 4.2) that a hyperbolic convolution with m-admissible kernels defines a bounded operator in $L^p(\mathbb{H}^n)$ for the range $(p_1(m), p_2(m))$, as specified in the statement of the main result. From the estimates (4.6) we see that a and $Z_i a$ are m-admissible kernels, and so (4.4) will be proved for them. As $|Z_j Z_i a(z)| = O(r^{-n})$ has the critical non-integrable singularity at $r = 0$, $Z_j Z_i a(z)$ is not an m-admissible kernel. Notice however from (4.6), (4.7) that the last three terms $\psi'(r) Z_i r Z_j Q$, $\psi'(r) Z_j r Z_i Q$, $\psi(r) Z_j Z_i Q$ are indeed m-admissible. Moreover, the estimate $|Z_i Q| \le \text{const}$ implies that Q is Lipschitz with respect to the hyperbolic metric, in particular

$$Q(z) = Q(e) + O\left(\log \frac{1+r}{1-r}\right) = Q(e) + O(r),$$

for small r. This means that replacing Q by $Q - Q(e)$ in the first two terms leads to an m-admissible kernel again. All this leaves us with the kernel

$$\psi''(r) Z_i r Z_j r + \psi'(r) Z_j Z_i r, \quad j < n.$$

If $\psi(r) = c_0 r^{2-n} + \cdots$, write $\phi(r) = c_0 r^{2-n}(1-r^2)^{n-m-1}$; then the above differs from

$$\phi''(r) Z_i r Z_j r + \phi'(r) Z_j Z_i r$$

in an m-admissible kernel. By the same reason, we may replace $\phi''(r)$, $\phi'(r)$ respectively by $(r^{2-n})''(1-r^2)^{n-m-1}$, $(r^{2-n})'(1-r^2)^{n-m-1}$, that is to say we must deal with the convolution kernel

$$(1-r^2)^{n-m-1} Z_j Z_i (r^{2-n}). \tag{4.8}$$

We introduce a class of singular hyperbolic convolution kernels to deal with the later. For this purpose it is more convenient to work in the ball model, so now b is defined in $\mathbb{B}^n$, and $r = |z|$. We replace the integrable singularity r^{1-n} by a typical Calderón-Zygmund singularity (see e.g. [14]). Thus, we will call b a *m-Calderón-Zygmund singular kernel* if it has the form

$$b(z) = \Omega(w) r^{-n} (1-r^2)^{n-m-1}, \quad z = rw,\ w \in S^{n-1},$$

where Ω is say a Lipschitz function on S^{n-1} satisfying the cancellation condition

$$\int_{S^{n-1}} \Omega(w)\, d\sigma(w) = O. \tag{4.9}$$

In Theorem 4.2 below we prove that m-Calderón-Zygmund singular kernels define bounded operators in the same range of p. With the following proposition, applied to $\phi_2(z) = |z|^{2-n}$, this will end the proof of the main result. The proposition is the analogue of the well-known statement that for ϕ smooth and homogeneous of degree $1-n$ in $\mathbb{R}^n$, $\partial\phi/\partial x_i$ defines a Calderón-Zygmund kernel; it is homogeneous of degree $-n$, and the cancellation condition (4.9) is automatically satisfied, because

$$\int_{r_1<|x|<r_2} \frac{\partial\phi}{\partial x_i}\, dV(x) = \left(\int_{|x|=r_2} - \int_{|x|=r_1}\right)\phi(x)\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = 0.$$

Proposition 4.1. *If ϕ_1, ϕ_2 are homogeneous functions of degree $1-n$, $2-n$ respectively, the kernels $(1-r^2)^{n-m-1}Z_i\phi_1$, $(1-r^2)^{n-m-1}Z_jZ_i\phi_2$ are sum of $(m-1)$-admissible and $(m-1)$-Calderón-Zygmund singular kernels.*

Proof. We replace the Z_j by $Y_j = (1-r^2)\partial/\partial z_j$; we have

$$\begin{aligned} Y_i\phi_1 &= (1-r^2)\frac{\partial\phi_1}{\partial z_i}, \\ Y_i\phi_2 &= (1-r^2)\frac{\partial\phi_2}{\partial z_i} = (1-r^2)O(r^{1-n}), \\ Y_jY_i\phi_2 &= (1-r^2)\frac{\partial^2\phi_2}{\partial z_i\partial z_j} - 2(1-r^2)z_j\frac{\partial\phi_2}{\partial z_i} \\ &= (1-r^2)\frac{\partial^2\phi_2}{\partial z_i\partial z_j} + (1-r^2)O(r^{2-n}), \end{aligned}$$

so in all cases we get an extra factor $(1-r^2)$. Besides, $\partial\phi_1/\partial z_i$ and $\partial^2\phi_2/\partial z_i\partial z_j$ are, as noted before, homogeneous of degree $-n$, and satisfy the cancellation condition (4.9). ◻

4.2. It remains to prove the following result.

Theorem 4.2. *Both m-admissible and m-Calderón-Zygmund kernels define, by hyperbolic convolution, bounded operators in $L^p(\mathbb{H}^n)$ for*

$$\frac{n-1}{n-1-m} < p < \frac{n-1}{m}, \quad 0 \le m < \frac{n-1}{2}.$$

We will make use of the following well-known Schur's lemma for boundedness in L^p of an integral operator with positive kernel.

Lemma 4.3. *If $K(x,y)$ is a positive kernel in a measure space X and $1 < p < \infty$, the operator $Kf(x) = \int_X K(x,y)f(y)\,d\mu(y)$ is bounded in $L^p(\mu)$ if and only* *if there exists $h \geq 0$ such that*

$$\int_X K(x,y)h(y)^q \, d\mu(y) = O(h(x)^q), \quad x \in X, \tag{4.10}$$

$$\int_X K(x,y)h(x)^p \, d\mu(x) = O(h(y)^p), \quad y \in Y. \tag{4.11}$$

Here q is the conjugate exponent of p, $1/p + 1/q = 1$. If h can be taken $\equiv 1$, that is,

$$\sup_x \int_X K(x,y) \, d\mu(y), \ \sup_y \int_X K(x,y) \, d\mu(x) < +\infty,$$

then K is bounded in $L^p(\mu)$ for all p, $1 \leq p \leq \infty$.

Proof. Let us prove Theorem 4.2. If b is m-admissible, then $b = b_1 + b_2$ with $b_1(z) = O(r^{1-n})$ for $r \leq \frac{1}{2}$, $b_1(z) = 0$ for $r > \frac{1}{2}$, and $b_2(z) = O(1-r^2)^{n-m-1}$ for all r. We apply to b_1 the second criterion in Lemma 4.3, working in the ball model (recall that $|S_y X| = |\varphi_y x|$ is symmetric in x, y).

$$\begin{aligned}
\int_X b_1(S_y x) \, d\mu(x), \ \int_X b_1(S_y x) \, d\mu(y) &\leq c \int_{|S_y x| \leq 1/2} |S_y x|^{1-n} \, d\mu(x) \\
&= c \int_{|z| \leq 1/2} |z|^{1-n} \, d\mu(z) \\
&= \text{const} \int_0^{1/2} \frac{dr}{(1-r^2)^n} < +\infty.
\end{aligned}$$

We apply to $(1-r^2)^{n-m-1}$ the criteria of the first part on Lemma 4.3, working this time for convenience in the half-space model, where the kernel is written

$$\begin{aligned}
K(x,y) &= (1-r^2)^{n-m-1} \\
&= \left(\frac{4z_n}{1 + |z|^2 + 2z_n} \right)^{n-m-1} \\
&= \left(\frac{4x_n y_n}{|x-y|^2 + 4x_n y_n} \right)^{n-m-1}.
\end{aligned}$$

We test $h(y) = y_n^\alpha$ in (4.10) for an exponent α to be chosen, so we need

$$\int_{y_n > 0} \frac{y_n^{-m-1+\alpha q} \, dy}{(|x-y|^2 + 4x_n y_n)^{n-m-1}} = O(x_n^{\alpha q + m + 1 - n}).$$

We write $|x-y|^2+4x_ny_n = |x'-y'|^2+(x_n+y_n)^2$, where $x' = (x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1}$, and analogously for y', and integrate first in y'. One has for $2m < n-1$

$$\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|x'-y'|^2+(x_n+y_n)^2)^{n-m-1}} = c\int_0^\infty \frac{s^{n-2}}{(s^2+(x_n+y_n)^2)^{n-m-1}}$$
$$= O((x_n+y_n)^{2m+1-n}),$$

and so the above becomes

$$\int_0^\infty \frac{y_n^{\alpha q-m-1}\,dy_n}{(x_n+y_n)^{n-1-2m}} = O(x_n^{\alpha q+m+1-n}).$$

By homogeneity ($y_n = x_n t$) this reduces to

$$\int_0^\infty \frac{t^{\alpha q-m-1}}{(1+t)^{n-1-2m}} = O(1),$$

which holds whenever $m < \alpha q < n-1-m$. By symmetry, for (4.11) we need as well $m < \alpha p < n-1-m$. Therefore, a choice of α is possible whenever $m\max(1/p,1/q) < (n-1-n)\min(1/p,1/q)$, and this gives the range

$$\frac{n-1}{n-1-m} < p < \frac{n-1}{m}.$$

Consider now a m-Calderón-Zygmund kernel $b(z) = \Omega(w)r^{-n}(1-r^2)^{n-m-1}$. Since $|S_y x| = |\varphi_y x|$, we may replace $z = S_y x$ by $z = \varphi_x y$. Using (1.1) this is given by

$$z = \frac{(x-y)(1-|x|^2)+x|x-y|^2}{A},$$

where we use the notation $A = (1-|x|^2)(1-|y|^2)+|x-y|^2$; note that

$$(1-|x|^2),\ (1-|y|^2) \lesssim A^{1/2}.$$

Also recall that $r = |z|$ and $Ar^2 = |x-y|^2$. Hence we can write

$$\frac{z}{r} - \frac{x-y}{|x-y|} = \frac{x-y}{|x-y|}\left(\frac{1-|x|^2}{\sqrt{A}} - 1\right) + x\cdot r.$$

But

$$\frac{1-|x|^2}{\sqrt{A}} - 1 = \frac{(1-|x|^2)^2 - A}{\sqrt{A}((1-|x|^2)+\sqrt{A})}$$
$$= \frac{(1-|x|^2)O(|x-y|)+O(|x-y|^2)}{A}$$

is $O(r)$. Therefore, modulo an m-admissible kernel, we may replace $\Omega(w)$ by $\Omega((x-y)/(|x-y|))$. This leaves us with the kernel

$$
\begin{aligned}
K &= (1-r^2)^{n-m-1}\Omega\left(\frac{x-y}{|x-y|}\right) r^{-n} \\
&= (1-r^2)^{n-m-1}|x-y|^{-n}\Omega\left(\frac{x-y}{|x-y|}\right) A^{n/2}(x,y).
\end{aligned}
$$

Fix p, $1<p<\infty$. Write

$$
A^{n/2}(x,y) = (1-|x|^2)^{n/p}(1-|y|^2)^{n/q} + O(|x-y|A^{(n-1)/2}).
$$

Since $|x-y|^{1-n}A^{(n-1)/2} = r^{1-n}$, the kernel K differs from

$$
(1-r^2)^{n-m-1}|x-y|^{-n}\Omega\left(\frac{x-y}{|x-y|}\right)(1-|x|^2)^{n/p}(1-|y|^2)^{n/q}
$$

in a m-admissible kernel, so we keep this one. We write it as the sum of

$$
|x-y|^{-n}\Omega\left(\frac{x-y}{|x-y|}\right)(1-|x|^2)^{n/p}(1-|y|^2)^{n/q} = K_1(x,y)
$$

and another $K_2(x,y)$, which we estimate by

$$
\begin{aligned}
|K_2(x,y)| &= O(r^2|x-y|^{-n}(1-|x|^2)^{n/p}(1-|y|^2)^{n/q}) \\
&= O(r^{2-n}(1-|x|^2)^{n/p}(1-|y|^2)^{n/q}A^{-n/2}).
\end{aligned}
$$

Write K_Ω for the (euclidean) Calderón-Zygmund convolution operator with kernel $|x-y|^{-n}\Omega((x-y)/(|x-y|))$, which as it is well-known, satisfies an $L^p(dV)$-estimate. Notice that

$$
K_1 f(x) = (1-|x|^2)^{n/p}K_\Omega(f(1-|y|^2)^{-n/p})
$$

and therefore, using the L^p-boundedness of K_Ω

$$
\begin{aligned}
\int_{\mathbb{B}^n} |K_1 f(x)|^p\,d\mu(x) &= \int_{\mathbb{B}^n} |K_\Omega(f(1-|y|^2)^{-n/p})|^p\,dV(x) \\
&\leq \int_{\mathbb{B}^n} |f(x)|^p\,d\mu(y).
\end{aligned}
$$

For K_2, we can ignore the integrable singularity r^{2-n} and arguing as we just did with K_1, we need to show that the integral operator

$$
K_3 f(x) = \int_{|y|\leq 1} \frac{1}{(1-|x|+|x-y|)^n} f(y)\,dV(y)
$$

satisfies $L^p(dV)$-estimates for all p, $1<p<\infty$. To see this, just check that the criteria in Lemma 4.3 holds, with $h(x) = (1-|x|^2)^{-1/(pq)}$. ◻

Notice that in case $m = 0$ a m-Calderón-Zygmund kernel defines a bounded operator in all $L^p(\mathbb{H}^n)$, $1 < p < \infty$: this is the right analogue of the euclidian kernels, because $(1 - r^2)^{n-1}$ is the typical growth at infinity of a weak $L^1(d\mu)$ function in $\mathbb{H}^n$.

4.3. Finally we make some comments, with no proofs, on the critical case $m = (n-1)/2$ in the main theorem.

In this case, the m-admissible and m-Calderón-Zygmund operators appearing in $X_j X_i C_a u$, etc. have $(1-r^2)^{(n-1)/2}\log(1/(1-r^2))$ instead of $(1-r^2)^{n-m-1} = (1-r^2)^{(n-1)/2}$ as a factor. One can then prove that for $\beta > 0$ and $2 \le p < 2 + 2\beta/(n-1)$,

$$\|L_p \eta\|_{p,2} \le \text{const} \int_{\mathbb{B}^n} |\eta|^p (1 - |y|^2)^{-\beta}\, d\mu(y).$$

The L^p-estimates do not hold in this case for any p, because they do not hold for $p = 2$ and Δ is self-adjoint.

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