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Equality of some classical Lorentz spaces

Santiago Boza*and Joaquim Martín†

Abstract

We prove that although the class of M_p-weights of Muckenhoupt is strictly smaller than the class of B_p-weights of Ariño and Muckenhoupt, both classes produce the same classical Lorentz spaces. An analogous result is obtained for other classes of weights.

1 Introduction

Let w be a positive and Lebesgue measurable function on $(0,\infty)$ (briefly a weight).

For $1 \leq p < \infty$, $L^p(w)$ is the class of Lebesgue measurable functions f defined on the interval $(0,\infty)$ such that

$$\|f\|_{L^p(w)} := \left(\int_0^\infty |f(s)|^p \, w(s) ds \right)^{1/p} < \infty.$$

Let us also consider the Hardy operator P defined by

$$Pf(t) := \frac{1}{t} \int_0^t f(x) dx.$$

The study of the boundedness of P on $L^p(w)$ has been considered by several authors (see [5] and the references quoted therein). Their results ensure that P is bounded on $L^p(w)$ if, and only if, $w \in M_p$, where M_p-weights are described by the estimate,

$$\left(\int_r^\infty \frac{w(x)}{x^p} dx \right)^{1/p} \left(\int_0^r w(x)^{1-p'} dx \right)^{1/p'} \leq C, \quad \text{if} \ \ 1 < p < \infty,$$

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where, as usual, $p' = p/(p-1)$. And

$$\int_r^\infty \frac{w(x)}{x} dx \le C w(r), \quad \text{if} \quad p = 1.$$

It is well known that the Hardy operator is closely related to the Hardy-Littlewood maximal function M since (see [2])

$$(Mf)^*(t) \approx Pf^*(t), \tag{1}$$

where f^* is the nonincreasing rearrangement of f and, as usual, the symbol $f \approx g$ will indicate the existence of a universal positive constant C (independent of all parameters involved) so that $(1/C)f \le g \le Cf$. Constants such as C may change from one occurrence to the next.

In 1990 Ariño and Muckenhoupt (see [1]) consider the problem of characterize the weights w so that

$$M : \Lambda^p(w) \to \Lambda^p(w)$$

is bounded, where

$$\Lambda^p(w) = \left\{ f : \|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(x)^p w(x) dx \right)^{1/p} < \infty \right\}$$

is the classical Lorentz space (see [4]).

Ariño and Muckenhoupt's result states that

$$M : \Lambda^p(w) \to \Lambda^p(w) \text{ is bounded } \Leftrightarrow w \in B_p$$

where B_p-weights can be define for $0 < p < \infty$ and are described by the estimate,

$$\int_r^\infty \left(\frac{r}{x} \right)^p w(x) dx \le C \int_0^r w(x) dx. \tag{2}$$

Using (1), it is obvious that if we define

$$L^p(w)^{dec} := \{ f \in L^p(w) : f \text{ is decreasing} \}$$

then

$$M : \Lambda^p(w) \to \Lambda^p(w) \Leftrightarrow P : L^p(w)^{dec} \to L^p(w)^{dec},$$

and from this, it follows that

$$M_p \subseteq B_p.$$

In fact M_p is strictly smaller than B_p (see [1]).

Similarly (see [5]), the adjoint Hardy operator

$$Qf(t) := \int_t^\infty f(x)\frac{dx}{x}$$

is bounded on $L^p(w)$ if, and only if, $w \in M^p$, that is

$$\left(\int_0^r w(x)dx\right)^{1/p}\left(\int_r^\infty \frac{w(x)^{1-p'}}{x^{p'}}dx\right)^{1/p'} \leq C, \quad \text{if} \ \ 1 < p < \infty,$$

and

$$Pw \leq Cw, \quad \text{if} \ \ p = 1.$$

The boundedness of Q on $L^p(w)^{dec}$ was considered by Neugebauer in 1992 (see [6], Theorem 3.3), and he proved that

$$Q : L^p(w)^{dec} \to L^p(w)^{dec} \Leftrightarrow w \in B^*_\infty,$$

where $B^*_\infty-$weights are defined by the condition

$$\int_0^r Pw(x)dx \leq C\int_0^r w(x)dx. \tag{3}$$

($B^*_\infty-$weights are also related with the boundedness of the Hilbert transform on $\Lambda^p(w)$ (see [6], Theorem 4.4)). Again it is easy to see that M^p is strictly smaller than B^*_∞.

We shall prove in this paper that although $M_p \neq B_p$ and $M^p \neq B^*_\infty$ both classes produce the same classical Lorentz spaces (we also refer to [3] and [8] as works related to this topic).

Our main result is the following:

Theorem 1.1 *Let w be a weight and $p \geq 1$. Then*

1. *If $w \in B_p$ there exists $\widetilde{w} \in M_p$ such that $\Lambda^p(w) = \Lambda^p(\widetilde{w})$.*
2. *If $w \in B^*_\infty$ there exists $\widetilde{w} \in M^p$ such that $\Lambda^p(w) = \Lambda^p(\widetilde{w})$.*
3. *If $w \in B_p \cap B^*_\infty$ there exists $\widetilde{w} \in M_p \cap M^p$ such that $\Lambda^p(w) = \Lambda^p(\widetilde{w})$.*

Before proving this theorem, we will introduce the following operators. For $q > 0$, let us define

$$Q_q f(t) := t^{q-1}\int_t^\infty f(x)\,\frac{dx}{x^q},$$

also for $0 \leq \lambda < 1$, let us consider the operators

$$P_\lambda f(t) := \frac{1}{t^{1-\lambda}} \int_0^t \frac{f(x)}{x^\lambda} \, dx.$$

We observe that $P_0 = P$ and also $Q_1 = Q$. The following fact concerning this operators, which can be easily proved using Fubini's theorem, will be useful

$$P_\lambda \circ Q_q = Q_q \circ P_\lambda = \frac{1}{q-\lambda}(P_\lambda + Q_q).$$

We also recall here that for a weight $w \in B_p$, $w \in B_{p-\epsilon}$ for every $0 < \epsilon < \epsilon_0$, where ϵ_0 depends only on p and also on the constant C that appears in condition (2) (see [1], or [7] for a simpler proof of this result).

Finally the following simple inequality will be used repeatedly (see [2]).

Lemma 1.1 *(Hardy's Lemma) Let us assume that w_0, w_1 are two weights such that $\int_0^r w_0(x)\, dx \approx \int_0^r w_1(x)\, dx$, for every $r > 0$, then, for any decreasing function f*

$$\int_0^\infty f(x) w_0(x) dx \approx \int_0^\infty f(x) w_1(x) dx.$$

2 Proof of Theorem 1.1

PART 1.

For $w \in B_p$, let us take $\epsilon > 0$ such that $w \in B_{p-\epsilon}$. Define

$$\widetilde{w}(x) = x^{p-1-\epsilon} \int_x^\infty \frac{w(s)}{s^{p-\epsilon}} \, ds = Q_{p-\epsilon} w(x).$$

Then by use of the definition and Fubini's theorem we have

$$\int_t^\infty \frac{\widetilde{w}(x)}{x^p} \, dx = \int_t^\infty \frac{w(s)}{s^{p-\epsilon}} \left(\int_t^s x^{-1-\epsilon} \, dx \right) ds \leq C t^{-\epsilon} \int_t^\infty \frac{w(s)}{s^{p-\epsilon}} \, ds. \quad (4)$$

In the case $p = 1$, this last inequality means that $Q\widetilde{w} \leq C\widetilde{w}$ which is condition M_1 for the weight $\widetilde{w}$. For $p > 1$, we also have

$$\begin{aligned} \int_0^t \widetilde{w}(x)^{1-p'} \, dx &\leq \int_0^t x^{-1+(\epsilon/(p-1))} \left(\int_t^\infty \frac{w(s)}{s^{p-\epsilon}} \, ds \right)^{-1/(p-1)} dx \quad (5) \\ &= C t^{\epsilon p'/p} \left(\int_t^\infty \frac{w(s)}{s^{p-\epsilon}} \, ds \right)^{-p'/p}. \end{aligned}$$

Using inequalities (4) and (5) shows immediately that $\widetilde{w} \in M_p$. By use of Fubini's theorem and then the fact that $w \in B_{p-\epsilon}$ it follows that

$$P\widetilde{w} = (P \circ Q_{p-\epsilon})w \approx Pw + Q_{p-\epsilon}w \approx Pw,$$

So then, Lemma 1.1 implies $\Lambda^p(w) = \Lambda^p(\widetilde{w})$.

PART 2.

If $w \in B^*_\infty$, there exists $C > 0$ such that $(P \circ P)\, w \leq CPw$. Let us take $\epsilon > 0$ such that $\epsilon C < 1$, and let us define $\widetilde{w}$ as

$$\widetilde{w}(s) = w(s) + P_\epsilon w(s).$$

A standard argument (see for example [2], page 152) shows that with this choice of ϵ we can represent $P_\epsilon w$ as the sum of the following series

$$P_\epsilon w(s) = \sum_{n=1}^{\infty} \epsilon^{n-1} P^{(n)} w(s),$$

where $P^{(n)} = P \circ \overset{(n)}{\cdots} \circ P$.

Obviously $P^{(n)}w \leq C^{n-1}Pw$, $n \geq 2$, which implies the convergence of the series. Moreover,

$$P\widetilde{w}(s) = Pw(s) + \sum_{n=1}^{\infty} \epsilon^{n-1} P^{(n+1)} w(s) \leq C \sum_{n=1}^{\infty} \epsilon^{n} P^{(n)} w(s) \leq C\widetilde{w}(s),$$

which means that $\widetilde{w} \in M^1 \subset M^p$.

Finally, since $w \leq \widetilde{w}$ and

$$\begin{aligned} P\widetilde{w}(s) &= Pw(s) + \sum_{n=1}^{\infty} \epsilon^{n-1} P^{(n+1)} w(s) \leq Pw(s) + \sum_{n=1}^{\infty} \epsilon^{n-1} C^n Pw(s) \\ &= \frac{1 + C(1-\epsilon)}{1 - C\epsilon} Pw(s), \end{aligned} \tag{6}$$

we have that $\int_0^s w(x)dx \approx \int_0^s \tilde{w}(x)dx$ and applying Lemma 1.1 we are done.

PART 3.

For $w \in B_p \cap B^*_\infty$, let us consider

$$\widetilde{w}(x) = Q_{p-\epsilon}w(x) + P_\epsilon w(x) \approx (Q_{p-\epsilon} \circ P_\epsilon)w(x) = (P_\epsilon \circ Q_{p-\epsilon})w(x),$$

where $\epsilon > 0$ is small enough in order to ensure that $\omega \in B_{p-\epsilon}$ and also, as in part 2, that $\epsilon C < 1$ for C as in (3).

To prove that $\widetilde{w} \in M_p$, we proceed as follows,

$$Q_p\widetilde{w} \approx Q_p(Q_{p-\epsilon} \circ P_\epsilon)(w) = P_\epsilon(Q_p \circ Q_{p-\epsilon})w.$$

As we have seen in (4), we obtain that

$$\begin{aligned} (Q_p \circ Q_{p-\epsilon})w(t) &= t^{p-1} \int_t^\infty \frac{Q_{p-\epsilon}w(x)}{x^p}\, dx \\ &\leq Ct^{p-1-\epsilon} \int_t^\infty \frac{w(x)}{x^{p-\epsilon}}\, dx = CQ_{p-\epsilon}w(t). \end{aligned}$$

So then,

$$Q_p\widetilde{w} \leq C(P_\epsilon \circ Q_{p-\epsilon})w = C\widetilde{w}. \tag{7}$$

In the case $p = 1$ we are done, because this inequality implies that $\widetilde{w} \in M_1$. We observe that condition (7) can be explicitely written as

$$\int_t^\infty \frac{\widetilde{w}(x)}{x^p}\, dx \leq Ct^{-\epsilon} \int_t^\infty \frac{P_\epsilon w(x)}{x^{p-\epsilon}}\, dx.$$

Following the same reasoning as in (5), we also have that, for $p > 1$,

$$\int_0^t \widetilde{w}(x)^{1-p'} dx = \int_0^t (Q_{p-\epsilon} \circ P_\epsilon w(x))^{1-p'} dx \leq Ct^{\epsilon p'/p} \left(\int_t^\infty \frac{P_\epsilon w(s)}{s^{p-\epsilon}} ds \right)^{-p'/p}.$$

Combining these two last inequalities we obtain that $\widetilde{w} \in M_p$.

Next, we will check the condition M^1 for $\widetilde{w}$, that is $P\widetilde{w} \leq C\widetilde{w}$. First, we observe that

$$P\widetilde{w} \approx P(Q_{p-\epsilon} \circ P_\epsilon w) \approx Q_{p-\epsilon}(P \circ P_\epsilon w),$$

and, in the same way that (6), it can be easily deduced,

$$(P \circ P_\epsilon)w \leq CP_\epsilon w.$$

Hence, we can conclude that

$$P\widetilde{w} \leq C(Q_{p-\epsilon} \circ P_\epsilon)w = C\widetilde{w},$$

that means $\widetilde{w} \in M^1 \subset M^p$.

Finally, from the fact that

$$P\widetilde{w} = P(Q_{p-\epsilon} \circ P_\epsilon)w \approx Pw + Q_{p-\epsilon}w + (P \circ P_\epsilon)w, \tag{8}$$

it follows that $Pw \leq CP\widetilde{w}$.

On the other hand, since $w \in B_{p-\epsilon}$,

$$Q_{p-\epsilon} w \leq CPw.$$

Also, as we have seen before

$$P \circ P_{\epsilon} w \leq CPw.$$

These two facts and (8) give us $P\widetilde{w} \leq CPw$. So then, $Pw \approx P\widetilde{w}$ and again an application of Lemma 1.1 ends the proof. □

Remark 2.1 *From part 1 of Theorem 1.1 we obtain a new proof of the Ariño-Muckenhoupt theorem about the boundedness of* P *on* $L^p(w)^{dec}$*, since we have seen that* $L^p(w)^{dec} = L^p(\tilde{w})^{dec}$ *with* $\tilde{w} \in M_p$*, and then Muckenhoupt's theory applies. Similarly, Neugebauer's result about the boundedness of the adjoint operator* Q *on* $L^p(w)^{dec}$ *follows from part 2.*

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SANTIAGO BOZA. Departament de Matemàtica Aplicada IV, E.U.P.V.G.,
Avda.Victor Balaguer s/n, E-08800 Vilanova i Geltrú, Barcelona (Spain).
E–mail: boza@mat.upc.es
JOAQUIM MARTIN. Departament de Matemàtiques.
Universitat Autònoma de Barcelona, Edifici C 08193 Bellaterra, Barcelona, (Spain).
E–mail: jmartin@mat.uab.es