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Integrability properties of maximal convolution operators*

Joaquim Martín and Fernando Soria

Abstract

In this paper we study integrability properties of maximal convolution operators satisfying restricted weak type (p,p) estimates.

1 Introduction

Let $S_n f$ by the *nth* partial sum of the Fourier series of the function f, and let S^*f be the Carleson maximal operator, i.e.

$$S^*f(x) = \sup_n |S_n f(x)|.$$

R. Hunt proved in [5], the following extension of the celebrated Theorem of Carleson [3],

$$t|\{x \in \mathbb{T} : S^*\chi_A(x) > t\}|^{1/p} \leq c\frac{p^2}{p-1}|A|^{1/p}, \quad p > 1 \tag{1}$$

(where $\mathbb{T}$ denotes the one-dimensional Torus which we identify with the interval $[0,1]$, χ_A is the characteristic function of the set A and $|A|$ its Lebesgue measure), and then, via the so called "Yano's extrapolation theorem" (see [10]) he obtained

$$\int_{\mathbb{T}} S^*f \leq c\Big\{1 + \int_{\mathbb{T}} |f|(1 + \log^+ |f|)^2\Big\}. \tag{2}$$

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This last estimate gives as corollary that $S_n f(x)$ converges a.e. for every function of the class $L(\log L(\mathbb{T}))^2$. Carleson and Sjölin in [6], showed that is possible to extrapolate directly from estimate (1) to obtain

$$t|\{x \in \mathbb{T} : S^* f(x) > t\}| \leq c\Big\{1 + \int_{\mathbb{T}} |f|(1 + \log^+ |f|)(1 + \log^+ \log^+ |f|)\Big\}$$

which implies a.e. convergence in $L \log L \log\log L(\mathbb{T})$.

In [7], P. Sjölin showed that S^* has the following integrability property: For $0 < \delta \leq 1$, there exists a constant C_δ such that

$$\int_{\mathbb{T}} \frac{S^* f}{(1 + \log^+ S^* f)^{1-\delta}} \leq C_\delta \Big\{1 + \int_{\mathbb{T}} |f|(1 + \log^+ |f|)^{1+\delta}\Big\}.$$

The case in which $\delta = 0$ was considered by F. Soria in [9], where he obtain the following extension of the previous result

$$\int_{\mathbb{T}} \frac{S^* f}{1 + \log^+ S^* f} \leq c\Big\{1 + \int_{\mathbb{T}} |f|(1 + \log^+ |f|)(1 + \log^+ \log^+ |f|)\Big\}. \tag{3}$$

More recently, N.Y. Antonov in [1] has proved that

$$t|\{x \in \mathbb{T} : S^* f(x) > t\}| \leq c\Big\{1 + \int_{\mathbb{T}} |f|(1 + \log^+ |f|)(1 + \log^+ \log^+ \log^+ |f|)\Big\}$$

and from here one can in fact obtain a.e. convergence of the Fourier series in the biggest class $L \log L \log\log\log L(\mathbb{T})$.

In this paper we consider an extension of estimate (3) for maximal convolution operators of restricted weak type (p, p) (see Theorem 3.1 below), that in the particular case of the Carleson maximal operator S^*, states that

$$\begin{aligned} \int_{\mathbb{T}} \frac{S^* f}{(1 + \log^+ S^* f)(1 + \log^+ \log^+ S^* f)} & \\ \leq c\Big\{1 + \int_{\mathbb{T}} |f|(1 + \log^+ |f|)(1 + \log^+ \log^+ \log^+ |f|)\Big\}. \end{aligned} \tag{4}$$

As usual, the symbol $f \simeq g$ will indicate the existence of a universal positive constant c (independent of all parameters involved) so that $(1/c)f \leq g \leq c\, f$, while the symbol $f \preceq g$ means that $f \leq c\, g$. We write $\|g\|_p$ to denote $\|g\|_{L^p}$, $\lambda_g(y) = |\{x \in \mathbb{R}^n : \; |g(x)| > y\}|$ is the distribution function of g and $g^*(t) = \inf\{s : \lambda_g(s) \leq t\}$ is the decreasing rearrangement (we refer the reader to [2] for further information about distribution functions and decreasing rearrangements).

2 Preliminaries

The key in the proof of Antonov's result (see [1]) about a.e. convergence of Fourier series in $L \log L \log\log\log L(\mathbb{T})$ follows from the following approximation argument which uses the fact that the convolution kernels defining S^*, the Dirichlet kernels, are smooth.

Lemma 2.1 *(**Antonov**) Let $S^N f(x) = \sup_{0\le n\le N} |S_n f(x)|$. Then, for every $\varepsilon > 0$, every $N \in \mathbb{N}$ and every $0 \le f(x) \le 1$, there exists a measurable set F such that $|F| = \|f\|_1$ and $\|S^N(f - \chi_F)\|_\infty \le \varepsilon$.*

Antonov's lemma has been recently extended by P. Sjölin and F. Soria (see [8] Lemma 5) in the sense that one can approximate a given function in the spirit of Antonov's lemma but with no smooth assumption at all on the kernels. This extension is contained in the following general approximation principle

Lemma 2.2 *Let $\{K_j\}$ be a sequence of integrable functions in $\mathbb{R}^n$ (or $\mathbb{T}^n$) and define the maximal operator K^* by*

$$K^* f(x) = \sup_j |K_j * f(x)|, \qquad (x \in \mathbb{R}^n).$$

Let $K^N f(x) = \sup_{0\le j\le N} |K_j * f(x)|$. Then, given $\varepsilon > 0$, $a > 0$, $N \in \mathbb{N}$ and a measurable function $f \in L^1(\mathbb{R}^n)$ with $0 \le f(x) \le a$, there exists a set $F \subset$supp f such that

i) $\|K^N(f - a\chi_F)\|_\infty \le \varepsilon$

ii) $\int_{\mathbb{R}^n} f = a|F|$.

Proof: In [8] Lemma 5, condition i) reads $\|K^N(f - a\chi_F)\|_1 \le \varepsilon$, however, it is easy to see that if the kernels $\{K_j\}$ are uniform continuous then the same proof works to obtain $\|K^N(f - a\chi_F)\|_\infty \le \varepsilon$. In the general case, for each $j = 1, 2, \cdots, N$, let g_j be a continuous function of compact support such that

$$\|K_j - g_j\|_1 \le \frac{\varepsilon}{a}.$$

With $G^N f(x) = \sup_{0\le j\le N} |g_j * f(x)|$, we therefore obtain

$$K^N f(x) \le G^N f(x) + \max_{1\le j\le N} \|K_j - g_j\|_1 \|f\|_\infty \le G^N f(x) + \varepsilon.$$

Similarly, G^N can be estimate by K^N, and so

$$|K^N f(x) - G^N f(x)| \le \varepsilon. \qquad \square$$

We shall denote by $L(\Psi_m)$ the set of measurable functions such that

$$\int_{\mathbb{R}^n} \Psi_m(|f(x)|)\, dx < \infty,$$

where $m > 0$ and $\Psi_m(t) = t(1+\log^+ t)^m(1+\log^+\log^+\log^+ t)$ $(t > 0)$.

Proposition 2.1 *$L(\Psi_m)$ is a Banach space whose norm is given by*

$$\|f\|_{L(\Psi_m)} = \int_0^\infty f^*(s)\Big(1+\log^+\frac{1}{s}\Big)^m\Big(1+\log^+\log^+\log^+\frac{1}{s}\Big)\, ds.$$

Proof. We shall see that the functional $\|\cdot\|_{L(\Psi_m)}$ is finite precisely in the set $L(\Psi_m)$. Effectively, given a measurable function f, consider the set $A = \{t \in [0,1] : f^*(t)^2 > 1/t\}$, then

$$\begin{aligned}
\|f\|_{L(\Psi_m)} &\preceq \|f\|_1 + \int_0^1 f^*(s)\Big(\log^+\frac{1}{s}\Big)^m\Big(\log^+\log^+\log^+\frac{1}{s}\Big)\, ds \\
&\preceq \|f\|_1 + \int_A f^*(s)(\log^+ f^*(s))^m(\log^+\log^+\log^+ f^*(s)) \\
&\quad + \int_0^1 \frac{1}{\sqrt{s}}\Big(\log^+\frac{1}{s}\Big)^m\Big(\log^+\log^+\log^+\frac{1}{s}\Big)\, ds \\
&\preceq 1 + \int_{\mathbb{R}^n} \Psi_m(|f(x)|)\, dx.
\end{aligned}$$

On the other hand, if $f \in L(\Psi_m)$ then $\|f\|_1 < \infty$, and since $tf^*(t) \le \|f\|_1$, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \Psi_m(|f(x)|)\, dx &= \int_0^\infty \Psi_m(f^*(t))dt \\
&\le \int_0^\infty f^*(s)\Big(1+\log^+\frac{\|f\|_1}{s}\Big)^m\Big(1+\log^+\log^+\log^+\frac{\|f\|_1}{s}\Big)\, ds \\
&\le \min(1, \|f\|_1)\|f\|_{L(\Psi_m)}. \qquad \square
\end{aligned}$$

Finally, we also consider the function $\Phi : [0,\infty) \to [0,\infty)$ defined by

$$\Phi(t) = \frac{t}{(1+\log^+ t)(1+\log^+\log^+ et)}.$$

Notice that since $\Phi(t)$ is increasing and $\Phi(t)/t$ is decreasing we can find a concave function $\hat{\Phi}$ (see [2] Proposition 5.10 pag. 71) such that

$$\Phi(t) \simeq \hat{\Phi}(t). \tag{5}$$

3 The main result

Theorem 3.1 *Let $\{K_j\}$ be a sequence of kernels in $L^1(\mathbb{R}^n)$. Assume that the maximal operator K^* is of restricted weak type (p,p) with a constant which grows like $(p-1)^{-m}$ as $p \to 1^+$, that is*

$$\lambda |\{x \in \mathbb{R}^n : K^* \chi_A(x) > \lambda\}|^{1/p} \leq \Big(\frac{C}{p-1}\Big)^m |A|^{1/p} \tag{6}$$

for some $m > 0$ and for all $1 < p \leq 2$, with C independent of p, $\lambda > 0$ and of every measurable set A.

Then, for $R > 2$

$$\int_{|x|\leq R} \Phi(K^* f(x))\, dx \leq c(\log R)^{m+1}\Big(1 + \|f\|_{L(\Psi_m)}\Big).$$

Observe that the dependence on R is logarithmic. This gives in particular the global integrability: $\forall \varepsilon > 0$, $\exists C_\varepsilon$ such that

$$\int_{\mathbb{R}^n} \Phi(K^* f(x)) \frac{dx}{(1+|x|)^\epsilon} \leq C_\varepsilon \{1 + \|f\|_{L(\Psi_m)}\}.$$

Proof: Using the above hypothesis (6) on characteristic functions and Lemma 2.2 we can conclude that

$$\lambda |\{x \in \mathbb{R}^n : K^* f(x) > \lambda\}|^{1/p} \leq \Big(\frac{C}{p-1}\Big)^m \|f\|_1^{1/p}, \quad \|f\|_\infty \leq 1 \tag{7}$$

since given $\varepsilon = \lambda/2$, $N \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^n)$ such that $0 \leq f(x) \leq 1$, by Lemma 2.2 there exists a measurable set F such that $|F| = \|f\|_1$ and $\|K^N(f - \chi_F)\|_\infty \leq \varepsilon$, then

$$K^N f(x) \leq K^N(f - \chi_F)(x) + K^N \chi_F(x) \leq \lambda/2 + K^N \chi_F(x) \leq \lambda/2 + K^* \chi_F(x).$$

Hence $\{x \in \mathbb{R}^n : K^N f(x) > \lambda\} \subset \{x \in \mathbb{R}^n : K^* \chi_F(x) > \lambda/2\}$ which implies that

$$\lambda |\{x \in \mathbb{R}^n : K^N f(x) > \lambda\}|^{1/p} \leq \Big(\frac{C}{p-1}\Big)^m \|f\|_1^{1/p}.$$

Using now that $K^N f \nearrow K^* f$ a.e. we obtain (7).

Let $R > 2$, by Hölder's inequality we get

$$\lambda |\{x \in \mathbb{R}^n : |x| < R, K^* f(x) > \lambda\}| \leq C R^{n(p-1)/p)} |\{x \in \mathbb{R}^n : K^* f(x) > \lambda\}|^{1/p},$$

and now, using (7) with $p = 1 + \big(\log R(1 + \log \frac{1}{\|f\|_1})\big)^{-1}$ we obtain

$$\lambda|\{x\in\mathbb{R}^n : |x|<R, K^*f(x)>\lambda\}| \preceq (\log R)^m\varphi_m(\|f\|_1), \quad \|f\|_\infty \le 1$$

where $\varphi_m(t)=t(1+\log^+\frac{1}{t})^m \ \ (t>0)$.

Equivalently, for every $f\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$,

$$\lambda|\{x\in\mathbb{R}^n : |x|<R, K^*f(x)>\lambda\}| \preceq (\log R)^m\|f\|_\infty\varphi_m\Big(\frac{\|f\|_1}{\|f\|_\infty}\Big). \tag{8}$$

Let us assume that $\|f\|_{L(\Psi_m)}\le 1$ and let us write

$$f=f\chi_{\{|f|\le 2\}}+\sum_{i=0}^\infty f\chi_{\{2^{2^i}<|f|\le 2^{2^{i+1}}\}} := \bar f+\sum_{i=0}^\infty f_i.$$

As we shall see later, we have that

$$\sum_{i=0}^\infty \|f_i\|_\infty\varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \preceq \|f\|_{L(\Psi_m)} \le 1, \tag{9}$$

hence, we may assume without lost of generality that

$$\|f_i\|_\infty\varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \le 1, \quad i\ge 0.$$

Since $\Phi(t)\simeq\hat\Phi(t)$ and $\hat\Phi$ is subadditive (since it is concave (see (5)) we get

$$\int_{|x|\le R}\Phi(K^*f(x))\,dx \preceq \int_{|x|\le R}\hat\Phi(K^*\bar f(x))dx+\sum_{i=0}^\infty\int_{|x|\le R}\hat\Phi(K^*f_i(x))\,dx.$$

Now, by the concavity of $\hat\Phi$ we get $\hat\Phi'(t)\le\hat\Phi(t)/t\simeq\Phi(t)/t$, and so

$$\begin{aligned}
&\int_{|x|\le R}\hat\Phi(K^*f_i(x))\,dx\\
&\quad=\int_0^\infty |\{x\in\mathbb{R}^n,\ |x|\le R : K^*f_i(x)>\hat\Phi^{-1}(t)\}|\,dt\\
&\quad=\int_0^\infty |\{x\in\mathbb{R}^n,\ |x|\le R : K^*f_i(x)>t\}|\hat\Phi'(t)\,dt\\
&\quad\preceq\int_0^\infty |\{x\in\mathbb{R}^n,\ |x|\le R : K^*f_i(x)>t\}|\frac{\Phi(t)dt}{t}\\
&\quad=\Big(\int_{A_1}+\int_{A_2}+\int_{A_3}+\int_{A_4}\Big)|\{x\in\mathbb{R}^n,\ |x|\le R : K^*f_i(x)>t\}|\frac{\Phi(t)dt}{t}\\
&\quad=I+II+III+IV,
\end{aligned}$$

where, $A_1 = (0, \|f_i\|_\infty \varphi_m(\frac{\|f_i\|_1}{\|f_i\|_\infty}))$, $A_2 = (\|f_i\|_\infty \varphi_m(\frac{\|f_i\|_1}{\|f_i\|_\infty}), 1)$, $A_3 = (1, 2^{2^{i+1}})$ and $A_4 = (2^{2^{i+1}}, \infty)$.

Let us see first that for $R > 2$

$$I \preceq (\log R)^{m+1} \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{|f_i\|_\infty}).$$

Effectively,

$$I \leq \int_{A_1} |\{x \in \mathbb{R}^n,\, |x| \leq R : K^* f_i(x) > t\}| \, dt$$

and from (8) we get

$$\begin{aligned} I \quad &\preceq \quad \int_{A_1} \min\Big(R^n, \frac{(\log R)^m \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big)}{t}\Big)\, dt \\ &\preceq \quad (\log R)^m \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \int_0^1 \min\,(R^n, 1/t)\, dt \\ &\preceq \quad (\log R)^{m+1}\, \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big). \end{aligned}$$

On the other hand estimate (8) gives again

$$\begin{aligned} II \quad &= \quad \int_{\|f_i\|_\infty \varphi_m\left(\frac{\|f_i\|_1}{\|f_i\|_\infty}\right)}^{1} |\{x \in \mathbb{R}^n,\, |x| \leq R : K^* f_i(x) > t\}| \, dt \\ &\preceq \quad \int_{\|f_i\|_\infty \varphi_m\left(\frac{\|f_i\|_1}{\|f_1\|_\infty}\right)}^{1} (\log R)^m \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \frac{dt}{t} \\ &\preceq \quad (\log R)^m \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \log \frac{1}{\|f_i\|_\infty \varphi_m\left(\frac{\|f_i\|_1}{\|f_i\|_\infty}\right)} \end{aligned}$$

and

$$\begin{aligned} III \quad &= \quad \int_1^{2^{2^{i+1}}} |\{x \in \mathbb{R}^n,\, |x| \leq R : K^* f_i(x) > t\}| \frac{dt}{(1+\log^+ t)(1+\log^+ \log^+ et)} \\ &\preceq \quad \int_1^{2^{2^{i+1}}} (\log R)^m \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \frac{dt}{t(1+\log^+ t)(1+\log^+ \log^+ et)} \\ &= \quad (\log R)^m \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \log(1 + \log(1 + 2^{i+1} \log 2)) \\ &\preceq \quad (\log R)^m \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \log(i+2). \end{aligned}$$

Finally, estimate (7) for $p=2$ implies

$$\begin{aligned}
IV &\le \int_{2^{2^{i+1}}}^{\infty} |\{x\in\mathbb{R}^n : K^*f_i(x)>t\}| \frac{dt}{(1+\log^+ t)(1+\log^+\log^+ et)} \\
&\preceq \|f_i\|_\infty \|f_i\|_1 \int_{2^{2^{i+1}}}^{\infty} \frac{dt}{t^2(1+\log^+ t)(1+\log^+\log^+ et)} \\
&\preceq \|f_i\|_1 \le \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big).
\end{aligned}$$

Therefore, for $R>2$

$$\begin{aligned}
&\sum_{i=0}^{\infty} \frac{1}{(\log R)^{m+1}} \int_{|x|\le R} \Phi(K^*f_i(x))\,dx \\
&\quad \preceq \sum_{i=0}^{\infty} \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \left(1+\log \frac{1}{\|f_i\|_\infty \varphi_m\big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\big)}\right) \\
&\qquad + \sum_{i=0}^{\infty} \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big)(1+\log(i+2)) \\
&\quad = \sum_{i=0}^{\infty} \varphi_1\Big(\|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big)\Big) \\
&\qquad + \sum_{i=0}^{\infty} \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big)(1+\log(i+2)).
\end{aligned}$$

Using $\|f_i\|_\infty \le 2^{2^{i+1}}$, and that $t\varphi_m(1/t)$ and $\varphi_1(t\varphi_m(1/t))$ are both increasing, we get

$$\begin{aligned}
\sum_{i=0}^{\infty} \frac{1}{(\log R)^{m+1}} \int_{|x|\le R} \Phi(K^*f_i(x))\,dx &\preceq \sum_{i=0}^{\infty} \varphi_1\left(2^{2^{i+1}} \varphi_m\Big(\frac{\|f_i\|_1}{2^{2^{i+1}}}\Big)\right) \\
&\quad + \sum_{i=0}^{\infty} 2^{2^{i+1}} \varphi_m\Big(\frac{\|f_i\|_1}{2^{2^{i+1}}}\Big)(1+\log(i+2)) \\
&= S_0 + S_1.
\end{aligned}$$

Define

$$B = \Big\{ i \ge 0 : 2^{2^{i+1}} \varphi_m\Big(\frac{\|f_i\|_1}{2^{2^{i+1}}}\Big) \le \frac{1}{1+i^2} \Big\},$$

and split the sum S_0 as

$$S_0 = \sum_{i\in B} + \sum_{i\notin B}.$$

Using again that $\varphi_1(s)$ is increasing, we get

$$\sum_{i\in B} \leq \sum_{i=0}^{\infty} \frac{1}{1+i^2}(1+\log(1+i^2)) < \infty.$$

On the other hand,

$$\sum_{i\notin B} \leq \sum_{i=0}^{\infty} 2^{2^{i+1}} \varphi_m\Big(\frac{\|f_i\|_1}{2^{2^{i+1}}}\Big)(1+\log(1+i^2)) = S_1.$$

Summarizing, we have that

$$\sum_{i=0}^{\infty} \frac{1}{(\log R)^{m+1}} \int_{|x|\leq R} \Phi(K^* f_i(x))dx \preceq 1 + S_1.$$

To estimate S_1 we follow the steps of the corresponding Theorem 2.2 in [4]. Since $\|f_i\|_1 \leq 2\sum_{j=2^i}^{2^{i+1}-1} 2^j \lambda_f(2^j)$ and φ_m is concave

$$S_1 \preceq \sum_{i=0}^{\infty} 2^{2^{i+1}} \sum_{j=2^i}^{2^{i+1}-1} \varphi_m\Big(\frac{2^j}{2^{2^{i+1}}}\lambda_f(2^j)\Big)(1+\log(i+2)).$$

Now, using $\varphi_m(s)/s$ decreases, and that $2^i \leq j < 2^{i+1}$, we obtain that

$$2^{2^{i+1}} \varphi_m\Big(\frac{2^j}{2^{2^{i+1}}}\lambda_f(2^j)\Big) \leq (2^j)^2 \varphi_m\Big(\frac{\lambda_f(2^j)}{2^j}\Big).$$

Hence

$$\begin{aligned} S_1 &\preceq \sum_{i=0}^{\infty} \sum_{j=2^i}^{2^{i+1}-1} (2^j)^2 \varphi_m\Big(\frac{\lambda_f(2^j)}{2^j}\Big)(1+\log^+\log^+\log^+ 2^j) \\ &\preceq \int_1^{\infty} s\varphi_m\Big(\frac{\lambda_f(s)}{s}\Big)(1+\log^+\log^+\log + s)\, ds. \end{aligned}$$

Since $s\lambda_f(s) \leq \|f\|_1 \leq \|f\|_{L(\Psi_m)} \leq 1$, we get

$$\frac{s}{\lambda_f(s)} \leq \Big(\frac{1}{\lambda_f(s)}\Big)^2$$

and, since $s\varphi_m(1/s)$ increases, and $\varphi_m(s^2) \leq 2^m s\varphi_m(s)$,

$$s\varphi_m\Big(\frac{\lambda_f(s)}{s}\Big) \leq \frac{1}{\lambda_f(s)}\varphi_m((\lambda_f(s))^2) \preceq \varphi_m(\lambda_f(s)).$$

Thus,

$$\begin{aligned} S_1 &\preceq \int_1^\infty \varphi_m(\lambda_f(s))(1+\log^+\log^+\log^+ s)\,ds \\ &\leq \int_0^\infty \varphi_m(\lambda_f(s))(1+\log^+\log^+\log^+ s)\,ds \\ &= \int_0^\infty \left(\int_0^{f^*(s)} (1+\log^+\log^+\log^+ t)\,dt\right) d\varphi_m(s). \end{aligned}$$

Finally, since $(1+\log^+\log^+\log^+ t)$ is increasing and $sf^*(s)\leq 1$

$$\begin{aligned} S_1 &\preceq \int_0^\infty f^*(s)(1+\log^+\log^+\log^+ f^*(s))\,d\varphi_m(s) \\ &\simeq \int_0^\infty f^*(s)\Big(1+\log^+\frac{1}{s}\Big)^m\Big(1+\log^+\log^+\log^+\frac{1}{s}\Big)\,ds. \end{aligned}$$

Notice that claim (9) follows from the fact that

$$\sum_{i=0}^\infty \|f_i\|_\infty \varphi_m\Big(\frac{\|f_i\|_1}{\|f_i\|_\infty}\Big) \leq S_1 \preceq \|f\|_{L(\Psi_m)} \leq 1.$$

To estimate $\int_{|x|\leq R} \hat{\Phi}(K^*\bar{f}(x))\,dx$ we use the same argument that above, considering the sets $A_1 = \Big(0, \|\overline{f}\|_\infty \varphi_m\Big(\frac{\|\overline{f}\|_1}{\|\overline{f}\|_\infty}\Big)\Big)$, $A_2 = \Big(\|\overline{f}\|_\infty \varphi_m\Big(\frac{\|\overline{f}\|_1}{\|\overline{f}\|_\infty}\Big), 1\Big)$, $A_3 = (1,2)$, $A_4 = (2,\infty)$ (or $A_1 = (0,1)$, $A_3 = (1,2)$, $A_4 = (2,\infty)$ if $\|\overline{f}\|_\infty \varphi_m\Big(\frac{\|\overline{f}\|_1}{\|\overline{f}\|_\infty}\Big) > 1$) and the fact that $\|\overline{f}\|_\infty \leq 2$ and $\|\overline{f}\|_1 \leq 1$. Then, one can easily check that for $R>2$,

$$\int_{|x|\leq R} \hat{\Phi}(K^*\bar{f}(x))\,dx \preceq (\log R)^{m+1}.$$

Summarizing, if $\|f\|_{L(\Psi_m)} \leq 1$ we have proved that

$$\|f\|_{L(\Psi_m)} \int_{|x|\leq R} \Phi(K^* f(x))\,dx \leq c(\log R)^{m+1}\{1+\|f\|_{L(\Psi_m)}\}.$$

Finally, the case $\|f\|_{L(\Psi_m)} > 1$ follows from the previous one since

$$\frac{1}{\|f\|_{L(\Psi_m)}} \int_{|x|\leq R} \Phi(K^* f(x))\,dx \leq \int_{|x|\leq R} \Phi\Big(K^* \frac{f(x)}{\|f\|_{L(\Psi_m)}}\Big)\,dx. \qquad \square$$

Remark 3.1 *Theorem 3.1 is also true for kernels $\{K_j\}$ in $L^1(\mathbb{T}^n)$, hence if $K^* = S^*$ we obtain inequality (4).*

Remark 3.2 *Under the same hypothesis of Theorem 3.1, the following weak type inequalities were obtained in [8] and [4]*

$$|\{x \in \mathbb{R}^n,\, |x| \leq R : K^* f_i(x) > t\}| \preceq 1 + \|f\|_{L(\Psi_m)}$$

and

$$t \frac{|\{x \in \mathbb{R}^n : K^* f(x) > t\}|}{(1 + \log^+ t)^m} \preceq 1 + \|f\|_{L(\Psi_m)}.$$

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Joaquim Martín. Departament de Matemàtiques
Universitat Autònoma de Barcelona, E–08139 Bellaterra (Barcelona) SPAIN.
E–mail: jmartin@mat.uab.es

Fernando Soria. Departamento de Matemáticas.
Universidad Autónoma de Madrid, E–28049 Madrid, SPAIN.
E–mail: fernando.soria@uam.es