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Equivalent expressions for norms in classical Lorentz spaces

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Abstract

We characterize the weights $w$ such that

$$\int_0^\infty f^*(s)^p w(s) \, ds \simeq \int_0^\infty (f^{**}(s) - f^*(s))^p w(s) \, ds.$$ 

Our result generalizes a result due to Bennett–De Vore–Sharpley, where the usual Lorentz $L^{p,q}$ norm is replaced by an equivalent expression involving the functional $f^{**} - f^*$. Sufficient conditions for the boundedness of maximal Calderón–Zygmund singular integral operators between classical Lorentz spaces are also given.

1 Introduction

Let $(\Omega, \Sigma(\Omega), \mu)$ be a nonfinite totally σ-finite resonant measure space, and let $w$ be a strictly nonnegative Lebesgue measurable function on $\mathbb{R}^+ = (0, \infty)$ (briefly a weight). For $1 \leq p < \infty$ the classical Lorentz space $\Lambda^p_\mu(w)$ (see [10] and [6]) is defined by those measurable functions in $\Omega$ such that

$$\|f\|_{\Lambda^p_\mu(w)} := \left( \int_0^\infty f^*_\mu(s) w(s) \, ds \right)^{1/p} < \infty,$$

where $f^*_\mu(t) = \inf \left\{ s : \lambda^\mu_f(s) \leq t \right\}$ is the decreasing rearrangement of $f$, and $\lambda^\mu_f(y) = \mu\left\{ x \in \Omega : |f(x)| > y \right\}$ is the distribution function of $f$ with respect to the measure $\mu$ (we refer the reader to [4] for further information about distribution functions and decreasing rearrangements).

Similarly, the weak Lorentz space $\Lambda^{p,\infty}_\mu(w)$ (see [6]) is defined by the condition

$$\|f\|_{\Lambda^{p,\infty}_\mu(w)} := \sup_{t>0} f^*_\mu(t) W^{-1/p}(t) < \infty,$$

where $W(t) = \int_0^t w(s) \, ds$.

Obviously, the above spaces are invariant under rearrangement and generalize the Lorentz spaces $L^{p,q}_\mu$ since if $w(t) = t^{q/p-1}$, $(1 \leq q, p < \infty)$ then $\Lambda^q_\mu(w) = L^{p,q}_\mu$ and $\Lambda^{q,\infty}_\mu(w)$ coincides with $L^{p,\infty}_\mu$; in particular the Lebesgue space $L^p_\mu$ is the space $\Lambda^p_\mu(w)$ when $w = 1$.

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Let us denote by \( f^{**}_\mu \) the maximal function of \( f^*_\mu \) defined by

\[
f^{**}_\mu(t) := \frac{1}{t} \int_0^t f^*_\mu(s) \, ds.
\]

It is proved in [3] (see also [4], Proposition 7.12) that in the case \( p > 1 \) the usual Lorentz \( L^{p,q}_\mu \) norm can be replaced by an equivalent expression in terms of the functional \( f^{**}_\mu - f^*_\mu \); i.e., if \( f \in L^{p,q}_\mu \), \( 1 < p < \infty \), \( 1 \le q \le \infty \) and \( \lim_{t \to \infty} f^{**}_\mu(t) = 0 \), then

\[
\left( \int_0^\infty \left( \frac{t^{1/p} (f^{**}_\mu(t) - f^*_\mu(t))}{t} \right)^q \frac{dt}{t} \right)^{1/q} \simeq \| f \|_{L^{p,q}_\mu}, \tag{1}
\]

where as usual, by \( A \simeq B \) we mean that \( c^{-1}A \le B \le cA \), for some constant \( c > 0 \) independent of appropriate quantities.

The main purpose of this paper is to extend (1) in the context of the classical Lorentz spaces and describe the weights \( w \) for which

\[
\| f \|_{\Lambda^p_\mu(w)} \simeq \left( \int_0^\infty (f^{**}_\mu(s) - f^*_\mu(s))^p w(s) \, ds \right)^{1/p}. \tag{2}
\]

The work is organized as follows: in Section 2 we provide a brief review of the parts of the theory of \( B_p \) and \( B^*_\infty \) weights that we shall use in this paper and prove some properties of the weights \( w \) that belong to \( B_p \cap B^*_\infty \). In Section 3 we characterize the weights \( w \) for which (2) holds, and as application, we obtain sufficient conditions for the boundedness of maximal Calderón–Zygmund singular integral operators between Lorentz spaces \( \Lambda^p_\mu(w) \), if \( \mu \) is an absolutely continuous measure on \( \mathbb{R}^n \) defined by \( \mu(A) = \int_A u(x) \, dx \) (\( A \in \Sigma(\mathbb{R}^n) \)) where \( u \) belongs to the class of weights \( A^{p_0} \), for some \( p_0 \ge 1 \) (see [8] as a general reference of this class of weights).

For other applications of the functional \( f^{**}_\mu - f^*_\mu \) in rearrangement function inequalities and interpolation theory we refer to [4], [3], [9], [13] and the references quoted therein.

## 2 Preliminaries

If \( h \) is a Lebesgue measurable function defined on \( \mathbb{R}^+ \) the Hardy operator \( P \) and its adjoint \( Q \) are defined by

\[
Ph(t) := \frac{1}{t} \int_0^t h(s) \, ds, \quad Qh(t) := \int_t^\infty h(s) \, \frac{ds}{s}.
\]

Results by M. Ariño and B. Muckenhoupt (see [1]) and C. J. Neugebauer (see [11]) which extend Hardy’s inequalities, ensure that:

- \( Pf^*_\mu \in \Lambda^p_\mu(w) \) for all \( f \in \Lambda^p_\mu(w) \) \((1 \le p < \infty)\) if and only if \( w \in B_p \), i.e. there is a constant \( c > 0 \) such that

\[
\int_r^\infty \left( \frac{t}{s} \right)^p w(s) \, ds \le cW(r) \quad (r > 0).
\]
Lemma 2.1 Let $Qf^*_\mu \in \Lambda^p_\mu (w)$ for all $f \in \Lambda^p_\mu (w)$ $(1 \leq p < \infty)$ if and only if $w \in B^*_\infty$, i.e. there is a constant $c > 0$ such that
\[
\int_0^r \left( \frac{1}{s} \int_0^s w(x)dx \right) ds \leq c W(r) \quad (r > 0).
\]

The boundedness of $P$ on $\Lambda^{p,\infty}_\mu (w)$ was also considered by J. Soria (see [14] Theorem 3.1). Soria’s result ensures that:

- $Pf^*_\mu \in \Lambda^{p,\infty}_\mu (w)$ for all $f \in \Lambda^{p,\infty}_\mu (w)$ $(1 \leq p < \infty)$ if and only if $w \in B_p$. Moreover, another characterization of the $B_p$ class is provided by
\[
w \in B_p \iff \frac{1}{r} \int_0^r \frac{ds}{W^{1/p}(s)} \simeq \frac{1}{W^{1/p}(r)}.
\]

**Lemma 2.1** Let $1 \leq p < \infty$ and $w$ be a weight on $\mathbb{R}^+$. Then, the following are equivalent,

i) $w \in B_p \cap B^*_\infty$.

ii) $\frac{1}{r^p} \int_0^r w(s)ds \simeq \int_r^\infty w(s)\frac{ds}{sp^{1/p}}$ $(r > 0)$.

iii) $\frac{1}{r} \int_0^r \frac{ds}{W^{1/p}(s)} \simeq \int_r^\infty ds\frac{ds}{sW^{1/p}(s)} \simeq \frac{1}{W(r)^{1/p}}$ $(r > 0)$.

**Proof.** i) $\Rightarrow$ ii). Obviously, $\int_r^\infty w(s)\frac{ds}{sp^{1/p}} \leq \frac{1}{r} \int_0^r w(s)ds$ since $w \in B_p$. Conversely, let us write
\[
Q_pw(r) := r^{p-1} \int_r^\infty w(s)\frac{ds}{sp^p}.
\]

Since $w \in B^*_\infty$ and $P \circ Q_p = p^{-1}(P + Q_p)$ it follows that
\[
P(P \circ Q_p w)(t) \leq cP(Q_pw)(t).
\]

For any $a > 1$ we have that
\[
P(Q_pw)(r) = \frac{1}{r \log a} \int_r^{ar} \frac{1}{t} \int_0^r Q_pw(s) ds dt \leq \frac{1}{r \log a} \int_0^{ar} \frac{1}{t} \int_0^t Q_pw(s) ds dt
\]
\[
\leq \frac{c}{r \log a} \int_0^{ar} Q_pw(s) ds,
\]
where the last inequality follows from (4).

Since
\[
\int_r^{ar} Q_pw(s)ds = \int_r^{ar} t^{p-1} \int_t^\infty w(s)\frac{ds}{sp^p} dt \leq \left( \int_r^{ar} t^{p-1} dt \right) \left( \int_r^\infty w(s)\frac{ds}{sp^p} \right)
\]
\[
= \frac{a^p - 1}{p} Q_pw(r)
\]
we have that
\[
P(Q_pw)(r) \leq \frac{c}{r \log a} \left( \int_0^r Q_pw(s) ds + \int_r^{ar} Q_pw(s) ds \right)
\]
\[
\leq \frac{c}{\log a} \left( \frac{1}{r} \int_0^r Q_pw(s) ds + \frac{a^p - 1}{p} Q_pw(r) \right).
\]
Hence
\[
\left(1 - \frac{c}{\log a}\right) P(Q_p w)(r) \leq \frac{c(a^p - 1)}{p \log a} Q_p w(r).
\]
Now if we take \(a = e^{2c}\) we obtain a constant \(C\) (depending only on \(p\)) such that
\[
P(Q_p w)(r) \leq C Q_p w(r).
\]
Finally, since \(P w(r) \leq p P(Q_p w)(r)\) it follows that
\[
\frac{1}{r^p} \int_0^r w(s) \, ds \leq c \int_r^\infty w(s) \frac{ds}{s^p}.
\]
\(\Rightarrow\) ii). It is enough to check that \(w \in B^*_{\infty}\), that is, there is a \(c > 0\) such that \(P \circ P w \leq c P w\).

We observe that condition ii) is
\[
P w(r) \simeq Q_p w(r)
\]
hence by Fubini’s theorem
\[
P(P w)(r) \simeq P(Q_p w)(r) = \frac{1}{p} (P w(r) + Q_p w(r)) \leq c P w(r).
\]
\(\Rightarrow\) iii). If \(w \in B_p\) by (3) \(\frac{1}{r} \int_0^r \frac{ds}{W^{1/p}(s)} \simeq \frac{1}{W^{1/p}(r)}\), hence only we need to see that
\[
\int_r^\infty \frac{ds}{s W(s)^{1/p}} \simeq \frac{1}{W^{1/p}(r)}
\]
which by [7] (Theorem 3.2) is equivalent to
\[
\int_r^\infty \frac{ds}{s W(s)} \simeq \frac{1}{W(r)}
\]
and by Sagher’s Lemma (see [12]), this happens if and only if
\[
\int_0^r \left(\frac{1}{s} \int_0^s w(x) \, dx\right) \, ds \simeq W(r)
\]
which follows from the fact that \(w \in B_p \cap B^*_{\infty}\) since
\[
P w \leq P w + Q_p w \simeq P(Q_p w) \leq c P(P w) \leq C P w.
\]
\(\Rightarrow\) i). If
\[
\frac{1}{r} \int_0^r \frac{ds}{W(s)^{1/p}} \simeq \frac{1}{W^{1/p}(r)}
\]
by (3) we have that \(w \in B_p\).

On the other hand, as we have seen before, condition
\[
\int_r^\infty \frac{ds}{s W(s)^{1/p}} \simeq \frac{1}{W^{1/p}(r)}
\]
is equivalent to
\[
\int_0^r \left(\frac{1}{s} \int_0^s w(x) \, dx\right) \, ds \simeq W(r)
\]
i.e. \(w \in B^*_{\infty}\). \(\square\)
3 The main result

Theorem 3.1 Let $1 \leq p < \infty$ and $w$ be a weight in $\mathbb{R}^+$. Then, the following are equivalent,

i) $w \in B_p \cap B^*_\infty$,

ii) $\|f\|_{\Lambda_p(w)} \simeq \left( \int_0^\infty (f^{**}_\mu(s) - f^*_\mu(s))^p w(s) ds \right)^{1/p}$,

iii) $\|f\|_{\Lambda_p,\infty(w)} \simeq \sup_{s>0}(f^{**}_\mu(s) - f^*_\mu(s))W^{1/p}(s)$,

where the equivalence constants do not depend on $\mu$.

Proof. i) $\Rightarrow$ ii). Since $w \in B_p \cap B^*_\infty$ by Lemma 2.1, there is $c > 0$ such that

$$\frac{1}{r} \int_0^r w(s) ds \leq c \int_r^\infty w(s) ds.$$

Hence

$$\frac{1}{r} \int_0^r w(s) ds \leq \frac{c}{c+1} \left( \frac{1}{r} \int_0^r w(s) ds + r^{p-1} \int_r^\infty w(s) ds \right).$$

Thus by Hardy’s Lemma (see [4] Proposition 3.6, pag. 56) and Fubini

$$\int_0^\infty f^*_\mu(s)^p w(s) ds \leq \frac{cp}{c+1} \int_0^\infty f^*_\mu(s)^p s^{p-1} \int_s^\infty w(x) \frac{dx}{xp} ds,$$

$$\leq \frac{cp}{c+1} \int_0^\infty \left( \int_0^s f^*_\mu(x)^p x^{p-1} dx \right) w(s) ds.$$

Since $\left( \int_0^s f^*_\mu(x)^p x^{p-1} dx \right)^{1/p} \leq 1/p \int_0^s f^*_\mu(x) dx$ (see [15], Theorem 3.11, pag 192) we have that

$$\int_0^\infty f^*_\mu(s)^p w(s) ds \leq \frac{c}{c+1} \int_0^\infty \left( \frac{1}{s} \int_0^s f^*_\mu(x) dx \right)^p w(s) ds.$$

Hence

$$\left( \int_0^\infty f^*_\mu(s)^p w(s) ds \right)^{1/p} \leq \left( \frac{c}{c+1} \int_0^\infty (f^{**}(s) - f^*_\mu(s))^p w(s) ds \right)^{1/p} + \left( \frac{c}{c+1} \int_0^\infty f^*_\mu(s)^p w(s) ds \right)^{1/p}.$$

Collecting terms, we get

$$\left( \int_0^\infty f^*_\mu(s)^p w(s) ds \right)^{1/p} \leq \frac{c^{1/p}}{(c+1)^{1/p} - c^{1/p}} \left( \int_0^\infty (f^{**}_\mu(s) - f^*_\mu(s))^p w(s) ds \right)^{1/p}.$$

The reverse inequality follows by the triangular inequality and condition $B_p$.

ii) $\Rightarrow$ i). This is a direct consequence of Lemma 2.1 since if we apply condition ii) to the characteristic function $\chi_A$ with $\mu(A) = r$, we obtain

$$\frac{1}{r^p} \int_0^r w(s) ds \simeq \int_r^\infty w(s) ds.$$

$$\frac{s}{sp}.$$
i) ⇒ iii). First observe that if \( w \in B_p \cap B^*_\infty \) then \( \lim_{x \to \infty} W(x) = \infty \) (otherwise the constant function 1 ∈ \( \Lambda_p^\infty(w) \) and then \( w \notin B^*_\infty \)) and hence if \( f \in \Lambda_p^\infty(w) \) we get \( \lim_{t \to \infty} f^{**}(t) = 0 \).

Now using the elementary identity (see [3])
\[
f^{**}(t) - f^{**}(s) = \int_t^s (f^{**}(x) - f^*(x)) \frac{dx}{x} \quad (0 < t \leq s < \infty)
\]
and letting \( s \to \infty \) we find that if \( f \in \Lambda_p^\infty(w) \)
\[
f^{**}(t) = \int_t^\infty (f^{**}(x) - f^*(x)) \frac{dx}{x}.
\]
Hence
\[
\|f\|_{\Lambda_p^\infty(w)} \leq \sup_{t>0} \left( \int_t^\infty (f^{**}(x) - f^*(x)) \frac{dx}{x} \right) W^{1/p}(t)
\]
\[
\leq \sup_{x>0} \left( (f^{**}(x) - f^*(x)) W^{1/p}(x) \right) \sup_{t>0} \left( W^{1/p}(t) \int_t^\infty \frac{dx}{W^{1/p}(x)x} \right)
\]
with \( \sup_{t>0} \left( W^{1/p}(t) \int_t^\infty \frac{dx}{W^{1/p}(x)x} \right) \leq C \) by Lemma 2.1.

On the other hand, since \( w \in B_p \) the boundedness of the Hardy operator in \( \Lambda_p^\infty(w) \) implies that
\[
\sup_{s>0} (f^{**}(s) - f^*(s)) W^{1/p}(s) \leq c \|f\|_{\Lambda_p^\infty(w)}.
\]

iii) ⇒ i). Since \( 1/W^{1/p} \) is decreasing and \( \lim_{x \to \infty} 1/W^{1/p}(x) = 0 \); as a consequence of Ryff’s Theorem (see [4] Corollary 7.6. pag. 83) there is a \( \mu \)-measurable function \( f \) on \( \Omega \) such that \( f^*_\mu = 1/W^{1/p} \), then by hypothesis
\[
(f^{**}(x) - f^*(x)) W^{1/p}(x) \leq c \|f\|_{\Lambda_p^\infty(w)} = c.
\]
Thus
\[
\frac{1}{W^{1/p}(r)} \leq \frac{1}{r} \int_0^r \frac{ds}{W^{1/p}(s)} \leq \frac{c + 1}{W^{1/p}(r)} \quad (5)
\]
and by (3) \( w \in B_p \).

Given \( a > 0 \) and \( s > 1 \), define
\[
h(t) = \begin{cases} 
1 & \text{if } 0 < t < a, \\
\left( \frac{W(a)}{W(t)} \right)^{1/p} & \text{if } a < t < sa, \\
0 & \text{if } t > sa,
\end{cases}
\]
and let \( g(t) = Qh(t) \). Since \( g \) is decreasing and \( \lim_{x \to \infty} g(t) = 0 \), again by Corollary 7.6. pag. 83 of [4], we can find \( f \) such that \( f^*_\mu = g \). Then
\[
f^{**} - f^*_\mu = P(Qh) - Qh = Ph + Qh - Qh = Ph
\]
thus, by iii), since \( h \) is decreasing and \( w \in B_p \),
\[
\|f\|_{\Lambda_p^\infty(w)} = \sup_{t>0} Qh(t) W(t)^{1/p} \simeq \sup_{t>0} Ph(t) W(t)^{1/p} \leq c \sup_{t>0} h(t) W(t)^{1/p} \leq cW^{1/p}(a)
\]

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we get that in particular
\[ Qh(a)W(a)^{1/p} = \int_a^{\infty} \frac{W^2(a)}{W^{1/p}(t)} dt \leq cW^{1/p}(a) \]
which implies that
\[ \int_a^{\infty} \frac{dt}{W^{1/p}(t)t} \leq \frac{c}{W^{1/p}(a)}. \]

Now by (5)
\[ \int_a^{\infty} \frac{dt}{W^{1/p}(t)t} \approx \int_a^{\infty} \frac{1}{r^2} \int_0^r \frac{ds}{W^{1/p}(s)} dr \geq \left( \int_a^{\infty} \frac{1}{r^2} dr \right) \left( \int_0^a \frac{ds}{W^{1/p}(s)} \right) \]
\[ = \frac{1}{a} \int_0^a \frac{ds}{W^{1/p}(s)} \approx \frac{1}{W^{1/p}(a)}. \]

Summarizing we have proved that
\[ \frac{1}{a} \int_0^a \frac{ds}{W^{1/p}(s)} \approx \int_a^{\infty} \frac{dt}{W^{1/p}(t)t} \approx \frac{1}{W^{1/p}(a)} \]
which by Lemma 2.1 implies that \( w \in B_p \cap B_\infty^*. \)

Observe that in the above theorem we have proved the norm equivalence between \( f \) (in the classical Lorentz space \( \Lambda_\mu^p(w) \)) and \( f_{\mu}^{**} - f_{\mu}^{*} \) in the weighted \( L^p(w) \) space. In fact we have the following

**Proposition 3.1** The following statements are equivalent,

i) \( w \in B_p \cap B_\infty^* \),

\[ \|f\|_{\Lambda_\mu^p(w)} \approx \left( \int_0^{\infty} ((f_{\mu}^{**} - f_{\mu}^{*})(s))^p w(s) ds \right)^{1/p}, \]

\[ \|f\|_{\Lambda_\mu^{p,\infty}(w)} \approx \sup_{s>0} (f_{\mu}^{**} - f_{\mu}^{*})^{*}(s)W^{1/p}(s), \]

where the rearrangement \( (f_{\mu}^{**} - f_{\mu}^{*})^{*} \) is taken with respect the Lebesgue measure in \( \mathbb{R}^+ \).

**Proof.** i) \( \Rightarrow \) ii). Since \( w \in B_p \cap B_\infty^* \), it follows from Theorem 3.1 that \( \lim_{t \to \infty} f_{\mu}^{**}(t) = 0 \), for every \( f \in \Lambda_\mu^p(w) \). Hence,
\[ f_{\mu}^{**}(t) = \int_t^{\infty} (f_{\mu}^{**}(x) - f_{\mu}^{*}(x)) \frac{dx}{x}. \]

Then
\[ f_{\mu}^{*}(t) \leq Pf_{\mu}^{**}(t) = \frac{1}{t} \int_0^t \left( \int_s^{\infty} (f_{\mu}^{**}(x) - f_{\mu}^{*}(x)) \frac{dx}{x} \right) ds = S(f_{\mu}^{**} - f_{\mu}^{*})(t). \]

where \( S := P \circ Q \) is the Calderón operator.

Since if \( h \) is a nonnegative function on \( \mathbb{R}^+ \) then \( S(h)(t) \) is decreasing, for each \( t > 0 \), by taking rearrangement (with respect the Lebesgue measure in \( \mathbb{R}^+ \)) we get (see [4] Proposition 5.2, pag. 142)
\[ S(h)(t) = (S(h))^{*}(t) \leq S(h^{*})(t). \]
Thus
\[ f_\mu^*(t) \leq P f_\mu^{**}(t) \leq \frac{1}{t} \int_0^t \left( \int_s^\infty (f_\mu^{**} - f_\mu^*)^*(x) \frac{dx}{x} \right) ds. \]

Now, since \( w \in B_p \cap B_{\infty}^* \), we have that
\[ \|f\|_{\Lambda_p^*(w)} \leq c \left( \int_0^\infty ((f_\mu^{**} - f_\mu^*)^*)(s) w(s) ds \right)^{1/p}. \]

On the other hand, since \((f_\mu^{**} - f_\mu^*)^*(s) \leq f_\mu^{**}(s)\), the reverse inequality follows by condition \( B_p \).

\( ii \Rightarrow i \). If \( f = \chi_A \) with \( \mu(A) = r \), then
\[ (f_\mu^{**} - f_\mu^*)^*(s) = \frac{r}{r + s} \chi_{(0,\infty)}(s) \]

Applying condition \( ii \) we get
\[ W(r) \simeq \int_0^\infty (\frac{r}{r + s})^p w(s) ds \geq \int_r^\infty (\frac{r}{r + s})^p w(s) ds \geq \int_r^\infty (\frac{r}{2})^p \frac{w(s)}{s^p} ds \]

i.e. \( w \in B_p \).

Given \( a > 0 \) and \( s > 1 \), define
\[ h(t) = \begin{cases} 1 & \text{if } 0 < t < a, \\ a/t & \text{if } a < t < sa, \\ 0 & \text{if } t > sa, \end{cases} \]

and let \( g(t) = Qh(t) \). Since \( g \) is decreasing and \( \lim_{x \to \infty} g(t) = 0 \), using again Ryff’s Theorem (see [4] Corollary 7.6, pag. 83) we can find \( f \) such that \( f_\mu^* = g \). Then
\[ f_\mu^{**} - f_\mu^* = P(Qh) - Qh = Ph + Qh - Qh = Ph, \]

thus, by condition \( ii \), since \( h \) is decreasing and since \( w \in B_p \), we get that
\[ \left( \int_0^\infty Qh(x)^p w(x) dx \right)^{1/p} = \|f\|_{\Lambda_p^*(w)} \simeq \left( \int_0^\infty ((f_\mu^{**} - f_\mu^*)^*(x))^p w(x) dx \right)^{1/p} \]
\[ = \left( \int_0^\infty Ph(x)^p w(x) dx \right)^{1/p} \leq c \left( \int_0^\infty h(x)^p w(x) dx \right)^{1/p}. \]

A simple computation shows that
\[ \int_0^a h(x)^p w(x) dx = W(a) + a^p \int_a^{sa} \frac{w(x)}{x^p} dx \]

and
\[ \int_0^a \left( \log \frac{a}{x} \right)^p w(x) dx \leq \int_0^\infty Qh(x)^p w(x) dx. \]

Then, since \( w \in B_p \)
\[ \int_0^a \left( \log \frac{a}{x} \right)^p w(x) dx \leq c \left( W(a) + a^p \int_a^{sa} \frac{w(x)}{x^p} dx \right) \leq c \left( W(a) + a^p \int_a^\infty \frac{w(x)}{x^p} dx \right) \leq cW(a) \]

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which is equivalent to $P \circ Pw \leq cPw$ (see [11], Theorem 3.3) i.e. $w \in B_{\infty}^*$. $\Box$

**ii)** $\iff$ $i$). Is proved in the same way.

Given $u \in A^p (1 \leq p < \infty)$ we shall denote by $\mu$ its associate measure on $\mathbb{R}^n$, i.e.

$$\mu(A) = \int_A u(x) \, dx, \quad A \in \Sigma(\mathbb{R}^n).$$

**Theorem 3.2** Let $T$ be a Calderón–Zygmund maximal operator, $u \in A^{p_0} (1 \leq p_0 < \infty)$ and $\mu$ its associate measure. Then $T$ is bounded from $\Lambda_p^p(w)$ to $\Lambda_p^p(w)$ (resp. from $\Lambda_{p,\infty}^p(w)$ to $\Lambda_{p,\infty}^p(w)$) for any $w \in B_{p/p_0} \cap B_{\infty}^* (1 \leq p < \infty)$.

**Proof.** If $T$ is a Calderón–Zygmund maximal operator, $u \in A^{p_0}$ and $\mu$ its associate measure then (see [2])

$$(Tf)_{\mu}^*(s) - (Tf)_{\mu}^*(2s) \leq c(Mf)_{\mu}^*(s/2)$$

(6)

where $M$ is the Hardy–Littlewood maximal function with respect to Lebesgue measure. Hence

$$I := \frac{1}{t} \int_0^t (Tf)_{\mu}^*(s) \, ds - \frac{1}{t} \int_0^t (Tf)_{\mu}^*(2s) \, ds \leq \frac{c}{t} \int_0^t (Mf)_{\mu}^*(s/2) \, ds.$$

But

$$I = \frac{1}{t} \int_0^t (Tf)_{\mu}^*(s) \, ds - \frac{1}{2t} \int_0^{2t} (Tf)_{\mu}^*(s) \, ds$$

$$= \frac{1}{2t} \int_0^t (Tf)_{\mu}^*(s) \, ds - \frac{1}{2t} \int_{t}^{2t} (Tf)_{\mu}^*(s) \, ds$$

$$\geq \frac{1}{2t} \int_0^t (Tf)_{\mu}^*(s) \, ds - \frac{(Tf)_{\mu}^*(t)}{2}$$

Hence

$$\frac{1}{t} \int_0^t (Tf)_{\mu}^*(s) \, ds - (Tf)_{\mu}^*(t) \leq \frac{2c}{t} \int_0^t (Mf)_{\mu}^*(s/2) \, ds.$$

On the other hand, by [11] Lemma 2.2

$$(Mf)_{\mu}^*(s) \leq c \frac{1}{s^{1/p_0}} \int_0^s f_{\mu}^*(x) \frac{dx}{x^{1-1/p_0}} = cA_{p_0}f_{\mu}^*(s)$$

Hence

$$\frac{1}{t} \int_0^t (Tf)_{\mu}^*(s) \, ds - (Tf)_{\mu}^*(t) \leq CP(A_{p_0}f_{\mu}^*)(t/2).$$

Thus, by Theorem 3.1,

$$\|Tf\|_{\Lambda_p^p(w)} \leq c \left( \int_0^\infty \left( \frac{(Tf)_{\mu}^*(s) - (Tf)_{\mu}^*(s)}{w(s)} \right)^p \, ds \right)^{1/p}$$

$$\leq c\|P(A_{p_0}f_{\mu}^*)\|_{\Lambda_p^p(w)} \leq c\|f\|_{\Lambda_p^p(w)} + \epsilon,$$

this last inequality is due to the fact that $w \in B_{p/p_0} \subset B_p$, and so $P$ is bounded in $\Lambda_{p}^p(w)$. We have also used that $A_{p_0}$ is bounded in $\Lambda_{p}^p(w)$ since $w \in B_{p/p_0}$ (see [11] Theorem 2.3). $\Box$
Remark 3.1 If $1 < p_0 < \infty$ using the same proof that above and Theorem 3.3.9 of [5], one can easily check that Theorem 3.2 holds for every $w$ in the biggest class $B_{p/p_0, \infty} \cap B^*_\infty$ where

$$w \in B_{q, \infty} \Leftrightarrow W(r)/r^q \leq cW(s)/s^q \quad 0 < s < r < \infty, \quad (q > 0).$$

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References


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