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Equivalent expressions for norms in classical Lorentz spaces

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Abstract

We characterize the weights w such that

$$\int_0^\infty f^*(s)^p w(s)\,ds \simeq \int_0^\infty \big(f^{**}(s) - f^*(s)\big)^p w(s)\,ds.$$

Our result generalizes a result due to Bennett–De Vore–Sharpley, where the usual Lorentz $L^{p,q}$ norm is replaced by an equivalent expression involving the functional $f^{**} - f^*$. Sufficient conditions for the boundedness of maximal Calderón–Zygmund singular integral operators between classical Lorentz spaces are also given.

1 Introduction

Let $(\Omega, \Sigma(\Omega), \mu)$ be a nonfinite totally $\sigma-$finite resonant measure space, and let w be a strictly nonnegative Lebesgue measurable function on $\mathbb{R}^+ = (0,\infty)$ (briefly a weight). For $1 \le p < \infty$ the classical Lorentz space $\Lambda^p_\mu(w)$ (see [10] and [6]) is defined by those measurable functions in Ω such that

$$\|f\|_{\Lambda^p_\mu(w)} := \left(\int_0^\infty f^*_\mu(s)^p w(s) ds\right)^{1/p} < \infty,$$

where $f^*_\mu(t) = \inf\big\{s : \lambda^\mu_f(s) \le t\big\}$ is the decreasing rearrangement of f, and $\lambda^\mu_f(y) = \mu\big(\{x \in \Omega : \ |f(x)| > y\}\big)$ is the distribution function of f with respect to the measure μ (we refer the reader to [4] for further information about distribution functions and decreasing rearrangements).

Similarly, the weak Lorentz space $\Lambda^{p,\infty}_\mu(w)$ (see [6]) is defined by the condition

$$\|f\|_{\Lambda^{p,\infty}_\mu(w)} := \sup_{t>0} f^*_\mu(t) W^{1/p}(t) < \infty,$$

where $W(t) = \int_0^t w(s) ds$.

Obviously, the above spaces are invariant under rearrangement and generalize the Lorentz spaces $L^{p,q}_\mu$ since if $w(t) = t^{q/p-1}$, $(1 \le q, p < \infty)$ then $\Lambda^q_\mu(w) = L^{p,q}_\mu$ and $\Lambda^{q,\infty}_\mu(w)$ coincides with $L^{p,\infty}_\mu$; in particular the Lebesgue space L^p_μ is the space $\Lambda^p_\mu(w)$ when $w = 1$.

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Let us denote by f_μ^{**} the maximal function of f_μ^* defined by

$$f_\mu^{**}(t) := \frac{1}{t}\int_0^t f_\mu^*(s)ds.$$

It is proved in [3] (see also [4], Proposition 7.12) that in the case $p > 1$ the usual Lorentz $L_\mu^{p,q}$ norm can be replaced by an equivalent expression in terms of the functional $f_\mu^{**} - f_\mu^*$ i.e. if $f \in L_\mu^{p,q}$, $1 < p < \infty$, $1 \le q \le \infty$ and $\lim_{t\to\infty} f_\mu^{**}(t) = 0$, then

$$\Big(\int_0^\infty (t^{1/p}(f_\mu^{**}(t) - f_\mu^*(t)))^q \frac{dt}{t}\Big)^{1/q} \simeq \|f\|_{L_\mu^{p,q}}, \tag{1}$$

where as usual, by $A \simeq B$ we mean that $c^{-1}A \le B \le cA$, for some constant $c > 0$ independent of appropriate quantities.

The main purpose of this paper is to extend (1) in the context of the classical Lorentz spaces and describe the weights w for which

$$\|f\|_{\Lambda_\mu^p(w)} \simeq \Big(\int_0^\infty \big(f_\mu^{**}(s) - f_\mu^*(s)\big)^p w(s)ds\Big)^{1/p}. \tag{2}$$

The work is organized as follows: in Section 2 we provide a brief review of the parts of the theory of B_p and B_∞^* weights that we shall use in this paper and prove some properties of the weights w that belong to $B_p \cap B_\infty^*$. In Section 3 we characterize the weights w for which (2) holds, and as application, we obtain sufficient conditions for the boundedness of maximal Calderón–Zygmund singular integral operators between Lorentz spaces $\Lambda_\mu^p(w)$, if μ is an absolutely continuous measure on $\mathbb{R}^n$ defined by $\mu(A) = \int_A u(x)\,dx$, $(A \in \Sigma(\mathbb{R}^n))$ where u belongs to the class of weights $\mathcal{A}^{p_0}$, for some $p_0 \ge 1$ (see [8] as a general reference of this class of weights).

For other applications of the functional $f_\mu^{**} - f_\mu^*$ in rearrangement function inequalities and interpolation theory we refer to [4], [3], [9], [13] and the references quoted therein.

2 Preliminaries

If h is a Lebesgue measurable function defined on $\mathbb{R}^+$ the Hardy operator P and its adjoint Q are defined by

$$Ph(t) := \frac{1}{t}\int_0^t h(s)ds, \quad Qh(t) := \int_t^\infty h(s)\frac{ds}{s}.$$

Results by M. Ariño and B. Muckenhoupt (see [1]) and C. J. Neugebauer (see [11]) which extend Hardy's inequalities, ensure that:

- $Pf_\mu^* \in \Lambda_\mu^p(w)$ for all $f \in \Lambda_\mu^p(w)$ $(1 \le p < \infty)$ if and only if $w \in B_p$, i.e. there is a constant $c > 0$ such that

$$\int_r^\infty \Big(\frac{r}{s}\Big)^p w(s)ds \le cW(r) \quad (r > 0).$$

- $Qf^*_\mu \in \Lambda^p_\mu(w)$ for all $f \in \Lambda^p_\mu(w)$ $(1 \le p < \infty)$ if and only if $w \in B^*_\infty$, i.e. there is a constant $c > 0$ such that
$$\int_0^r \Big(\frac{1}{s}\int_0^s w(x)dx\Big)\, ds \le cW(r) \quad (r > 0).$$

The boundedness of P on $\Lambda^{p,\infty}_\mu(w)$ was also considered by J. Soria (see [14] Theorem 3.1). Soria's result ensures that:

- $Pf^*_\mu \in \Lambda^{p,\infty}_\mu(w)$ for all $f \in \Lambda^{p,\infty}_\mu(w)$ $(1 \le p < \infty)$ if and only if $w \in B_p$. Moreover, another characterization of the B_p class is provided by
$$w \in B_p \Leftrightarrow \frac{1}{r}\int_0^r \frac{ds}{W^{1/p}(s)} \simeq \frac{1}{W^{1/p}(r)}. \tag{3}$$

Lemma 2.1 *Let $1 \le p < \infty$ and w be a weight on $\mathbb{R}^+$. Then, the following are equivalent,*

i) $w \in B_p \cap B^*_\infty$.

ii) $\displaystyle \frac{1}{r^p}\int_0^r w(s)ds \simeq \int_r^\infty w(s)\frac{ds}{s^p} \quad (r > 0).$

iii) $\displaystyle \frac{1}{r}\int_0^r \frac{ds}{W(s)^{1/p}} \simeq \int_r^\infty \frac{ds}{sW(s)^{1/p}} \simeq \frac{1}{W(r)^{1/p}} \quad (r > 0).$

Proof. *i)* $\Rightarrow$ *ii)*. Obviously, $\int_r^\infty w(s)\frac{ds}{s^p} \le \frac{c}{r^p}\int_0^r w(s)ds$ since $w \in B_p$. Conversely, let us write
$$Q_p w(r) := r^{p-1}\int_r^\infty w(s)\frac{ds}{s^p}.$$
Since $w \in B^*_\infty$ and $P \circ Q_p = p^{-1}(P + Q_p)$ it follows that
$$P\big(P \circ Q_p w\big)(t) \le cP\big(Q_p w\big)(t). \tag{4}$$

For any $a > 1$ we have that
$$\begin{aligned} P\big(Q_p w\big)(r) &= \frac{1}{r\log a}\int_r^{ar}\frac{1}{t}\int_0^r Q_p w(s)\,ds\,dt \le \frac{1}{r\log a}\int_0^{ar}\frac{1}{t}\int_0^t Q_p w(s)\,ds\,dt \\ &\le \frac{c}{r\log a}\int_0^{ar} Q_p w(s)\,ds, \end{aligned}$$
where the last inequality follows from (4).

Since
$$\begin{aligned} \int_r^{ar} Q_p w(s)ds &= \int_r^{ar} t^{p-1}\int_t^\infty w(s)\frac{ds}{s^p}dt \le \Big(\int_r^{ar} t^{p-1}dt\Big)\Big(\int_r^\infty w(s)\frac{ds}{s^p}\Big) \\ &= r\frac{a^p-1}{p}Q_p w(r) \end{aligned}$$
we have that
$$\begin{aligned} P\big(Q_p w\big)(r) &\le \frac{c}{r\log a}\Big(\int_0^r Q_p w(s)\,ds + \int_r^{ar} Q_p w(s)\,ds\Big) \\ &\le \frac{c}{\log a}\Big(\frac{1}{r}\int_0^r Q_p w(s)\,ds + \frac{a^p-1}{p}Q_p w(r)\Big). \end{aligned}$$

Hence

$$\left(1-\frac{c}{\log a}\right)P\big(Q_pw\big)(r)\le\frac{c(a^p-1)}{p\log a}Q_pw(r).$$

Now if we take $a=e^{2c}$ we obtain a constant C (depending only on p) such that

$$P\big(Q_pw\big)(r)\le CQ_pw(r).$$

Finally, since $Pw(r)\le pP\big(Q_pw\big)(r)$ it follows that

$$\frac{1}{r^p}\int_0^r w(s)\,ds\le c\int_r^\infty w(s)\frac{ds}{s^p}.$$

ii) $\Rightarrow$ *i)*. It is enough to check that $w\in B_\infty^*$, that is, there is a $c>0$ such that $P\circ Pw\le cPw$.

We observe that condition *ii)* is

$$Pw(r)\simeq Q_pw(r)$$

hence by Fubini's theorem

$$P\big(Pw\big)(r)\simeq P\big(Q_pw\big)(r)=\frac{1}{p}(Pw(r)+Q_pw(r))\le cPw(r).$$

i) $\Rightarrow$ *iii)*. If $w\in B_p$ by (3) $\frac{1}{r}\int_0^r\frac{ds}{W^{1/p}(s)}\simeq\frac{1}{W^{1/p}(r)}$, hence only we need to see that

$$\int_r^\infty\frac{ds}{sW(s)^{1/p}}\simeq\frac{1}{W^{1/p}(r)}$$

which by [7] (Theorem 3.2) is equivalent to

$$\int_r^\infty\frac{ds}{sW(s)}\simeq\frac{1}{W(r)}$$

and by Sagher's Lemma (see [12]), this happens if and only if

$$\int_0^r\left(\frac{1}{s}\int_0^s w(x)\,dx\right)ds\simeq W(r)$$

which follows from the fact that $w\in B_p\cap B_\infty^*$ since

$$Pw\le Pw+Q_pw\simeq P(Q_pw)\le cP(Pw)\le CPw.$$

iii) $\Rightarrow$ *i)*. If

$$\frac{1}{r}\int_0^r\frac{ds}{W(s)^{1/p}}\simeq\frac{1}{W(r)^{1/p}}$$

by (3) we have that $w\in B_p$.

On the other hand, as we have seen before, condition

$$\int_r^\infty\frac{ds}{sW(s)^{1/p}}\simeq\frac{1}{W^{1/p}(r)}$$

is equivalent to

$$\int_0^r\left(\frac{1}{s}\int_0^s w(x)\,dx\right)ds\simeq W(r)$$

i.e. $w\in B_\infty^*$. □

3 The main result

Theorem 3.1 *Let $1 \le p < \infty$ and w be a weight in $\mathbb{R}^+$. Then, the following are equivalent,*

i) $w \in B_p \cap B^*_\infty$,

ii) $\|f\|_{\Lambda^p_\mu(w)} \simeq \left(\int_0^\infty (f^{**}_\mu(s) - f^*_\mu(s))^p w(s) ds\right)^{1/p}$,

iii) $\|f\|_{\Lambda^{p,\infty}_\mu(w)} \simeq \sup_{s>0}(f^{**}_\mu(s) - f^*_\mu(s))W^{1/p}(s)$,

where the equivalence constants do not depend on μ.

Proof. *i)* $\Rightarrow$ *ii)*. Since $w \in B_p \cap B^*_\infty$ by Lemma 2.1, there is $c > 0$ such that $\frac{1}{r^p}\int_0^r w(s)\,ds \le c\int_r^\infty w(s)\frac{ds}{s^p}$. Hence

$$\begin{aligned}\frac{1}{r}\int_0^r w(s)\,ds &\le \frac{c}{c+1}\left(\frac{1}{r}\int_0^r w(s)\,ds + r^{p-1}\int_r^\infty w(s)\frac{ds}{s^p}\right)\\ &= \frac{cp}{c+1}\frac{1}{r}\int_0^r s^{p-1}\int_s^\infty w(x)\frac{dx}{x^p}\,ds.\end{aligned}$$

Thus by Hardy's Lemma (see [4] Proposition 3.6, pag. 56) and Fubini

$$\begin{aligned}\int_0^\infty f^*_\mu(s)^p w(s)\,ds &\le \frac{cp}{c+1}\int_0^\infty f^*_\mu(s)^p s^{p-1}\int_s^\infty w(x)\frac{dx}{x^p}\,ds\\ &= \frac{cp}{c+1}\int_0^\infty \left(\int_0^s f^*_u(x)^p x^{p-1}\,dx\right)\frac{w(s)}{s^p}\,ds.\end{aligned}$$

Since $\left(\int_0^s f^*_\mu(x)^p x^{p-1}\,dx\right)^{1/p} \le 1/p\int_0^s f^*_\mu(x)\,dx$ (see [15], Theorem 3.11. pag 192) we have that

$$\int_0^\infty f^*_\mu(s)^p w(s)\,ds \le \frac{c}{c+1}\int_0^\infty \left(\frac{1}{s}\int_0^s f^*_\mu(x)\,dx\right)^p w(s)ds.$$

Hence

$$\begin{aligned}\left(\int_0^\infty f^*_u(s)^p w(s)\,ds\right)^{1/p} &\le \left(\frac{c}{c+1}\int_0^\infty \left(f^{**}_\mu(s) - f^*_\mu(s)\right)^p w(s)\,ds\right)^{1/p}\\ &\quad + \left(\frac{c}{c+1}\int_0^\infty f^*_\mu(s)^p w(s)\,ds\right)^{1/p}.\end{aligned}$$

Collecting terms, we get

$$\left(\int_0^\infty f^*_\mu(s)^p w(s)ds\right)^{1/p} \le \frac{c^{1/p}}{(c+1)^{1/p} - c^{1/p}}\left(\int_0^\infty \left(f^{**}_\mu(s) - f^*_\mu(s)\right)^p w(s)ds\right)^{1/p}.$$

The reverse inequality follows by the triangular inequality and condition B_p.

ii) $\Rightarrow$ *i)*. This is a direct consequence of Lemma 2.1 since if we apply condition *ii)* to the characteristic function χ_A with $\mu(A) = r$, we obtain

$$\frac{1}{r^p}\int_0^r w(s)\,ds \simeq \int_r^\infty w(s)\frac{ds}{s^p}.$$

$i) \Rightarrow iii)$. First observe that if $w \in B_p \cap B_\infty^*$ then $\lim_{x\to\infty} W(x) = \infty$ (otherwise the constant function $1 \in \Lambda_\mu^p(w)$ and then $w \notin B_\infty^*$) and hence if $f \in \Lambda_\mu^{p,\infty}(w)$ we get $\lim_{t\to\infty} f_\mu^{**}(t) = 0$.

Now using the elementary identity (see [3])

$$f_\mu^{**}(t) - f_\mu^{**}(s) = \int_t^s \left(f_\mu^{**}(x) - f_\mu^*(x)\right)\frac{dx}{x} \qquad (0 < t \le s < \infty)$$

and letting $s \to \infty$ we find that if $f \in \Lambda_\mu^{p,\infty}(w)$

$$f_\mu^{**}(t) = \int_t^\infty \left(f_\mu^{**}(x) - f_\mu^*(x)\right)\frac{dx}{x}.$$

Hence

$$\begin{aligned}\|f\|_{\Lambda_\mu^{p,\infty}(w)} &\le \sup_{t>0}\left(\int_t^\infty \left(f_\mu^{**}(x) - f_\mu^*(x)\right)\frac{dx}{x}\right) W^{1/p}(t)\\ &\le \sup_{x>0}\left((f_\mu^{**}(x) - f_\mu^*(x))W^{1/p}(x)\right)\sup_{t>0}\left(W^{1/p}(t)\int_t^\infty \frac{dx}{W^{1/p}(x)x}\right)\end{aligned}$$

with $\sup_{t>0}\left(W^{1/p}(t)\int_t^\infty \frac{dx}{W^{1/p}(x)x}\right) \le C$ by Lemma 2.1.

On the other hand, since $w \in B_p$ the boundedness of the Hardy operator in $\Lambda_\mu^{p,\infty}(w)$ implies that

$$\sup_{s>0}\left(f_\mu^{**}(s) - f_\mu^*(s)\right)W^{1/p}(s) \le \sup_{s>0} f_\mu^{**}(s)W^{1/p}(s) \le c\|f\|_{\Lambda_\mu^{p,\infty}(w)}.$$

$iii) \Rightarrow i)$. Since $1/W^{1/p}$ is decreasing and $\lim_{x\to\infty} 1/W^{1/p}(x) = 0$; as a consequence of Ryff's Theorem (see [4] Corollary 7.6. pag. 83) there is a μ–measurable function f on Ω such that $f_\mu^* = 1/W^{1/p}$, then by hypothesis

$$\left(f_\mu^{**}(x) - f_\mu^*(x)\right)W^{1/p}(x) \le c\,\|f\|_{\Lambda_\mu^{p,\infty}(w)} = c.$$

Thus

$$\frac{1}{W^{1/p}(r)} \le \frac{1}{r}\int_0^r \frac{ds}{W^{1/p}(s)} \le \frac{c+1}{W^{1/p}(r)} \qquad (5)$$

and by (3) $w \in B_p$.

Given $a > 0$ and $s > 1$, define

$$h(t) = \begin{cases} 1 & \text{if } 0 < t < a,\\ \left(\frac{W(a)}{W(t)}\right)^{1/p} & \text{if } a < t < sa,\\ 0 & \text{if } t > sa,\end{cases}$$

and let $g(t) = Qh(t)$. Since g is decreasing and $\lim_{x\to\infty} g(t) = 0$, again by Corollary 7.6. pag. 83 of [4], we can find f such that $f_\mu^* = g$. Then

$$f_\mu^{**} - f_\mu^* = P(Qh) - Qh = Ph + Qh - Qh = Ph$$

thus, by iii), since h is decreasing and $w \in B_p$,

$$\begin{aligned}\|f\|_{\Lambda_\mu^{p,\infty}(w)} &= \sup_{t>0} Qh(t)W(t)^{1/p} \simeq \sup_{t>0} Ph(t)W(t)^{1/p} \le c\sup_{t>0} h(t)W(t)^{1/p}\\ &\le cW^{1/p}(a)\end{aligned}$$

we get that in particular

$$Qh(a)W(a)^{1/p} = \int_a^{sa} \frac{W^{2/p}(a)\ dt}{W^{1/p}(t)t} \leq cW^{1/p}(a)$$

which implies that

$$\int_a^{\infty} \frac{dt}{W^{1/p}(t)t} \leq \frac{c}{W^{1/p}(a)}.$$

Now by (5)

$$\begin{aligned} \int_a^{\infty} \frac{dt}{W^{1/p}(t)t} &\simeq \int_a^{\infty} \frac{1}{r^2} \int_0^r \frac{ds}{W^{1/p}(s)} dr \geq \left(\int_a^{\infty} \frac{1}{r^2} dr \right) \left(\int_0^a \frac{ds}{W^{1/p}(s)} \right) \\ &= \frac{1}{a} \int_0^a \frac{ds}{W^{1/p}(s)} \simeq \frac{1}{W^{1/p}(a)}. \end{aligned}$$

Summarizing we have proved that

$$\frac{1}{a} \int_0^a \frac{ds}{W^{1/p}(s)} \simeq \int_a^{\infty} \frac{dt}{W^{1/p}(t)t} \simeq \frac{1}{W^{1/p}(a)}$$

which by Lemma 2.1 implies that $w \in B_p \cap B_\infty^*$. □

Observe that in the above theorem we have proved the norm equivalence between f (in the classical Lorentz space $\Lambda_\mu^p(w)$) and $f_\mu^{**} - f_\mu^*$ in the weighted $L^p(w)$ space. In fact we have the following

Proposition 3.1 *The following statements are equivalent,*

i) $w \in B_p \cap B_\infty^*$,

ii) $\|f\|_{\Lambda_\mu^p(w)} \simeq \left(\int_0^{\infty} \left((f_\mu^{**} - f_\mu^*)^*(s) \right)^p w(s)\, ds \right)^{1/p}$,

iii) $\|f\|_{\Lambda_\mu^{p,\infty}(w)} \simeq \sup_{s>0} \left(f_\mu^{**} - f_\mu^* \right)^*(s) W^{1/p}(s)$,

where the rearrangement $(f_\mu^{**} - f_\mu^*)^*$ *is taken with respect the Lebesgue measure in* $\mathbb{R}^+$.

Proof. $i) \Rightarrow ii)$. Since $w \in B_p \cap B_\infty^*$, it follows from Theorem 3.1 that $\lim_{t\to\infty} f_\mu^{**}(t) = 0$, for every $f \in \Lambda_\mu^p(w)$. Hence,

$$f_\mu^{**}(t) = \int_t^{\infty} \left(f_\mu^{**}(x) - f_\mu^*(x) \right) \frac{dx}{x}.$$

Then

$$f_\mu^*(t) \leq P f_\mu^{**}(t) = \frac{1}{t} \int_0^t \left(\int_s^{\infty} \left(f_\mu^{**}(x) - f_\mu^*(x) \right) \frac{dx}{x} \right) ds = S\big(f_\mu^{**} - f_\mu^*\big)(t).$$

where $S := P \circ Q$ is the Calderón operator.

Since if h is a nonnegative function on $\mathbb{R}^+$ then $S\big(h\big)(t)$ is decreasing, for each $t > 0$, by taking rearrangement (with respect the Lebesgue measure in $\mathbb{R}^+$) we get (see [4] Proposition 5.2. pag. 142)

$$S\big(h\big)(t) = \big(S\big(h\big)\big)^*(t) \leq S\big(h^*\big)(t).$$

Thus

$$f_\mu^*(t) \le P f_\mu^{**}(t) \le \frac{1}{t}\int_0^t \left(\int_s^\infty \left(f_\mu^{**} - f_\mu^*\right)^*(x) \frac{dx}{x}\right) ds.$$

Now, since $w \in B_p \cap B_\infty^*$, we have that

$$\|f\|_{\Lambda_\mu^p(w)} \le c \left(\int_0^\infty \left((f_\mu^{**} - f_\mu^*)^*(s)\right)^p w(s)\, ds \right)^{1/p}.$$

On the other hand, since $\left(f_\mu^{**} - f_\mu^*\right)^*(s) \le f_\mu^{**}(s)$, the reverse inequality follows by condition B_p.
$ii) \Rightarrow i)$. If $f = \chi_A$ with $\mu(A) = r$, then

$$\left(f_\mu^{**} - f_\mu^*\right)^*(s) = \frac{r}{r+s}\chi_{(0,\infty)}(s)$$

Applying condition $ii)$ we get

$$W(r) \simeq \int_0^\infty \left(\frac{r}{r+s}\right)^p w(s)\, ds \ge \int_r^\infty \left(\frac{r}{r+s}\right)^p w(s)\, ds \ge \int_r^\infty \left(\frac{r}{2}\right)^p \frac{w(s)}{s^p}\, ds$$

i.e. $w \in B_p$.

Given $a > 0$ and $s > 1$, define

$$h(t) = \begin{cases} 1 & \text{if } 0 < t < a, \\ a/t & \text{if } a < t < sa, \\ 0 & \text{if } t > sa, \end{cases}$$

and let $g(t) = Qh(t)$. Since g is decreasing and $\lim_{x\to\infty} g(t) = 0$, using again Ryff's Theorem (see [4] Corollary 7.6, pag. 83) we can find f such that $f_\mu^* = g$. Then

$$f_\mu^{**} - f_\mu^* = P\big(Qh\big) - Qh = Ph + Qh - Qh = Ph,$$

thus, by condition $ii)$, since h is decreasing and since $w \in B_p$, we get that

$$\begin{aligned} \left(\int_0^\infty Qh(x)^p w(x)\, dx \right)^{1/p} &= \|f\|_{\Lambda_\mu^p(w)} \simeq \left(\int_0^\infty \left((f_\mu^{**} - f_\mu^*)^*(x)\right)^p w(x)\, dx \right)^{1/p} \\ &= \left(\int_0^\infty Ph(x)^p w(x)\, dx \right)^{1/p} \le c\left(\int_0^\infty h(x)^p w(x)\, dx \right)^{1/p}. \end{aligned}$$

A simple computation shows that

$$\int_0^\infty h(x)^p w(x)\, dx = W(a) + a^p \int_a^{sa} \frac{w(x)}{x^p}\, dx$$

and

$$\int_0^a \left(\log \frac{a}{x}\right)^p w(x)\, dx \le \int_0^\infty Qh(x)^p w(x)\, dx.$$

Then, since $w \in B_p$

$$\begin{aligned} \int_0^a \left(\log \frac{a}{x}\right)^p w(x)\, dx &\le c\left(W(a) + a^p \int_a^{sa} \frac{w(x)}{x^p}\, dx\right) \\ &\le c\left(W(a) + a^p \int_a^{\infty} \frac{w(x)}{x^p}\, dx\right) \le cW(a) \end{aligned}$$

which is equivalent to $P \circ Pw \le cPw$ (see [11], Theorem 3.3) i.e. $w \in B^*_\infty$.
$iii) \Leftrightarrow i)$. Is proved in the same way. $\square$

Given $u \in \mathcal{A}^p$ $(1 \le p < \infty)$ we shall denote by μ its associate measure on $\mathbb{R}^n$, i.e.

$$\mu(A) = \int_A u(x)\,dx, \quad A \in \Sigma(\mathbb{R}^n).$$

Theorem 3.2 *Let T be a Calderón–Zygmund maximal operator, $u \in \mathcal{A}^{p_0}$ $(1 \le p_0 < \infty)$ and μ its associate measure. Then T is bounded from $\Lambda^p_\mu(w)$ to $\Lambda^p_\mu(w)$ (resp. from $\Lambda^{p,\infty}_\mu(w)$ to $\Lambda^{p,\infty}_\mu(w)$) for any $w \in B_{p/p_0} \cap B^*_\infty$ $(1 \le p < \infty)$.*

Proof. If T is a Calderón–Zygmund maximal operator, $u \in \mathcal{A}^{p_0}$ and μ its associate measure then (see [2])

$$(Tf)^*_\mu(s) - (Tf)^*_\mu(2s) \le c\big(\mathcal{M}f\big)^*_\mu(s/2) \tag{6}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function with respect to Lebesgue measure.

Hence

$$I := \frac{1}{t}\int_0^t (Tf)^*_\mu(s)ds - \frac{1}{t}\int_0^t (Tf)^*_\mu(2s)\,ds \le \frac{c}{t}\int_0^t \big(\mathcal{M}f\big)^*_\mu(s/2)\,ds.$$

But

$$\begin{aligned} I &= \frac{1}{t}\int_0^t \big(Tf\big)^*_\mu(s)\,ds - \frac{1}{2t}\int_0^{2t} (Tf)^*_\mu(s)\,ds \\ &= \frac{1}{2t}\int_0^t (Tf)^*_\mu(s)\,ds - \frac{1}{2t}\int_t^{2t} (Tf)^*_\mu(s)\,ds \\ &\ge \frac{1}{2t}\int_0^t (Tf)^*_\mu(s)\,ds - \frac{(Tf)^*_\mu(t)}{2} \end{aligned}$$

Hence

$$\frac{1}{t}\int_0^t (Tf)^*_\mu(s)\,ds - (Tf)^*_\mu(t) \le \frac{2c}{t}\int_0^t \big(\mathcal{M}f\big)^*_\mu(s/2)ds.$$

On the other hand, by [11] Lemma 2.2

$$\big(\mathcal{M}f\big)^*_\mu(s) \le c\frac{1}{s^{1/p_0}}\int_0^s f^*_\mu(x)\frac{dx}{x^{1-1/p_0}} = cA_{p_0}f^*_\mu(s)$$

Hence

$$\frac{1}{t}\int_0^t (Tf)^*_\mu(s)ds - (Tf)^*_\mu(t) \le CP(A_{p_0}f^*_\mu)(t/2).$$

Thus, by Theorem 3.1,

$$\begin{aligned} \|Tf\|_{\Lambda^p_\mu(w)} &\le c\left(\int_0^\infty \big((Tf)^{**}_\mu(s) - (Tf)^*_\mu(s)\big)^p w(s)\,ds\right)^{1/p} \\ &\le c\|P(A_{p_0}f^*_\mu)\|_{\Lambda^p_\mu(w)} \le c\|f\|_{\Lambda^p_\mu(w)}, \end{aligned}$$

this last inequality is due to the fact that $w \in B_{p/p_0} \subset B_p$, and so P is bounded in $\Lambda^p_\mu(w)$. We have also used that A_{p_0} is bounded in $\Lambda^p_\mu(w)$ since $w \in B_{p/p_0}$ (see [11] Theorem 2.3). $\square$

Remark 3.1 *If* $1 < p_0 < \infty$ *using the same proof that above and Theorem 3.3.9 of [5], one can easily check that Theorem 3.2 holds for every* w *in the biggest class* $B_{p/p_0,\infty} \cap B_\infty^*$ *where*

$$w \in B_{q,\infty} \Leftrightarrow W(r)/r^q \leq cW(s)/s^q \quad 0 < s < r < \infty, \quad (q > 0).$$

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