Galois actions on Q-curves and Winding Quotients

Francesc Bars* and Luis Dieulefait †

December 2, 2003

Abstract

We prove two "large images" results for the Galois representations attached to a degree d Q-curve E over a quadratic field K: if K is arbitrary, we prove maximality of the image for every prime $p > 13$ not dividing d, provided that d is divisible by q (but $d \neq q$) with $q = 2$ or 3 or 5 or 7 or 13. If K is real we prove maximality of the image for every odd prime p not dividing dD, where $D = \mathrm{disc}(K)$, provided that E is a semistable Q-curve. In both cases we make the (standard) assumptions that E does not have potentially good reduction at all primes $p \nmid 6$ and that d is square-free.

1 Semistable Q-curves over real quadratic fields

Let K be a quadratic field, and let E be a degree d Q-curve defined over K. Let $D = \mathrm{disc}(K)$. Assume that E is semistable, i.e., that E has good or semistable reduction at every finite place β of K. Recall that we can attach to E a compatible family of Galois representations $\{\sigma_\lambda\}$ of the absolute Galois group of $\mathbb{Q}$: these representations can be seen as those attached to the Weil restriction A of E to $\mathbb{Q}$, which is an abelian surface with real multiplication by $F := \mathbb{Q}(\sqrt{\pm d})$ (cf. [E]). Let us call U the set of primes dividing D. For primes not in U, it is clear that A is also semistable, so in particular for

*supported by BFM2003-06092

†supported by a MECD postdoctoral grant at the Centre de Recerca Matemática from Ministerio de Educación y Cultura

every prime λ of F dividing a prime ℓ not in U the residual representation $\bar{\sigma}_\lambda$ will be a representation "semistable outside U", i.e., it will be semistable (in the sense of [Ri 97]) at ℓ and locally at every prime $q \neq \ell$, $q \notin U$. This is equivalent to say that its Serre's weight will be either 2 or $\ell+1$ and that the restriction to the inertia groups I_q will be unipotent, for every $q \neq \ell$, $q \notin U$ (cf. [Ri97]).
Imitating the argument of [Ri97], we want to show that in this situation, if the image of $\bar{\sigma}_\ell$ is (irreducible and) contained in the normalizer of a Cartan subgroup, then this Cartan subgroup must correspond to the image of the Galois group of K, i.e., the restriction to K of $\bar{\sigma}_\ell$ must be reducible. More precisely:

Theorem 1.1. *Let E be a semistable Q-curve over a quadratic field K as above. If $\ell \nmid 2dD$, $\lambda \mid \ell$, and $\bar{\sigma}_\lambda$ is irreducible with image contained in the normalizer of a Cartan subgroup of* $\mathrm{GL}(2, \bar{\mathbb{F}}_\ell)$*, then the restriction of this residual representation to the Galois group of K is reducible.*

Proof. For any number field X, let us denote by G_X its absolute Galois group. We know that if we take $\ell \notin U$ the residual representation $\bar{\sigma}_\lambda$ is semistable outside U. If this representation is irreducible and its image is contained in the normalizer N of a Cartan subgroup, then there is a quadratic field L such that the restriction of $\bar{\sigma}_\lambda$ to G_L is reducible and the quadratic character ψ corresponding to L is a quotient of $\bar{\sigma}_\lambda$ (cf. [Ri 97]).
Using the description of the restriction of $\bar{\sigma}_\lambda$ to the inertia group I_ℓ in terms of fundamental characters, and the fact that the restriction of $\bar{\sigma}_\lambda$ to the inertia groups I_q, for every $q \neq \ell$, $q \notin U$, is unipotent, we conclude as in [Ri 97] that the quadratic character ψ can only ramify at primes in U, and therefore that the quadratic field L is unramified outside U, the ramification set of K. On the other hand, we know (by Cebotarev) that the restriction to G_K of $\bar{\sigma}_\lambda$ is isomorphic to $\bar{\sigma}_{E,\ell}$. Let us assume that $\bar{\sigma}_{E,\ell}$ is irreducible (*). Its image is contained in N, and since the restriction of $\bar{\sigma}_\lambda$ to G_L is reducible, it follows that the restriction of $\bar{\sigma}_{E,\ell}$ to $G_{L \cdot K}$ is reducible. We are again in the case of "image contained in the normalizer of a Cartan subgroup" but now for a representation of G_K. Once again, the quadratic character ψ' corresponding to the extension $L \cdot K/K$ is a quotient of the residual representation $\bar{\sigma}_{E,\ell}$. Using the fact that the curve E is semistable we know that the restriction of this residual representation to all inertia subgroups at places relatively primes to ℓ give unipotent groups, and this implies as in [Ri97] that ψ' is unramified outside (places above) ℓ. But ψ' corresponds to the extension $L \cdot K/K$, and

L is unramified outside U, thus ψ' is also unramified outside (places above primes in) U. This two facts entrain that $\ell \in U$, which is contrary to our hypothesis.
This proves that the assumption (*) contradicts the hypothesis of the theorem, i.e., that the restriction to G_K of $\bar{\sigma}_\lambda$ is reducible, as we wanted. $\square$

Keep the hypothesis of the theorem above, and assume furthermore that the field K is real. Then, the conclusion of the theorem together with a standard trick show that the image of $\bar{\sigma}_\lambda$ can not be (irreducible and) contained in the normalizer of a non-split Cartan subgroup: the reason is simply that the representation σ_λ is odd, thus the image of c, the complex conjugation, has eigenvalues 1 and -1. In odd residual characteristic, this gives an elements which is not contained in a non-split Cartan, but if we assume that K is real, we have c contained in the group G_K, and we obtain a contradiction because as a consequence of theorem 1.1 the restriction of $\bar{\sigma}_\lambda$ to G_K must be contained in the Cartan subgroup. This, combined with Ellenberg's generalizations of the results of Mazur and Momose (cf. [E]), shows that the image has to be large except for very particular primes. In fact, we have the following:

Corollary 1.2. *Let E be a semistable Q-curve over a real quadratic field K of square-free degree d. Assume that E does not have potentially good reduction at all primes not dividing* 6*. Then, if D is the discriminant of K, for every $\ell \nmid dD$, $\ell > 13$ and $\lambda \mid \ell$, the image of the projective representation $P(\bar{\sigma}_\lambda)$ is the full* $\mathrm{PGL}(2, \mathbb{F}_\ell)$.

2 Q-curves of composite degree over quadratic fields

Let E be a Q-curve over a quadratic field K of square-free degree d. Let λ be a prime of K and let us consider the projective representation $P(\overline{\sigma}_\lambda)$ coming from E. We can characterize the image in a subgroup of $PGL_2(\mathbb{F}_l)$ with $\lambda | l$ of the projective representation $P(\overline{\sigma}_\lambda)$ by points of modular curves as follows (proposition 2.2 [E]):

1. $P(\overline{\sigma}_\lambda)$ lies in a Borel subgroup, then E is a point of $X_0(dl)^K(\mathbb{Q})$,

2. $P(\overline{\sigma}_\lambda)$ lies in the normalizer of a split Cartan subgroup then E is a point of $X_0^s(d;l)^K(\mathbb{Q})$,

3. $P(\overline{\sigma}_\lambda)$ lies in the normalizer of a non-split Cartan subgroup, then E is a point of $X_0^{ns}(d;l)^K(\mathbb{Q})$;

where $X^K(\mathbb{Q})$ is the subset of $P \in X(K)$ such that $P^\sigma = w_d P$ for σ a generator of $Gal(K/\mathbb{Q})$ where w_d is the Fricke or Atkin-Lehner involution.

We have the following results ([E], propositions 3.2, 3.4):

Proposition 2.1. *Let E be a Q-curve of square-free degree d over K a quadratic field. We have:*

1. *Suppose $P(\overline{\sigma}_\lambda)$ is reducible for some $l = 11$ or $l > 13$ with $(p,d) = 1$ where $\lambda | l$. Then E has potentially good reduction at all primes of K of characteristic greater than 3.*
2. *Suppose $P(\overline{\sigma}_\lambda)$ lies in the normalizer of a split Cartan subgroup of $PGL_2(\mathbb{F}_l)$ where $\lambda | l$ for $l = 11$ or $l > 13$ with $(l,d) = 1$. Then E has good reduction at all primes of K not dividing 6.*

After this result we need to study what happens when the image lies in the non-split Cartan situation. For this case, Ellenberg obtains for the situation of K an imaginary quadratic field, that there is a constant depending of the degree d and the quadratic imaginary field K such that if the image of $P(\overline{\sigma}_\lambda)$ lies in a non-split Cartan and $l > M_{d,K}$ then E has potentially good reduction at all primes of K, see proposition 3.6 [E]. He centers in the twisted version for X^K to obtain this result. We obtain a similar result in a non-twisted situation for X^K, and with K non necessarily imaginary.

We impose once for all that d, the degree, is even. We denote $d = 2\tilde{d}$. First, let us construct an abelian variety quotient of the Jacobian of $X_0^{ns}(2\tilde{d};l)$ on which $w_{2\tilde{d}}$ acts as 1 and having $\mathbb{Q}$-rang zero. Then using "standard" arguments, that we will reproduce here for reader's convenience, we obtain our result on the non-split Cartan situation.

By the Chen-Edixhoven theorem, we have an isogeny between $J_0^{ns}(2;l)$ and $J_0(2l^2)/w_{l^2}$. Darmon and Merel [DM, prop.7.1] construct an optimal quotient A_f with $\mathbb{Q}$-rang zero. They construct A_f as the associated abelian variety to a form $f \in S_2(\Gamma_0(2l^2))$ with $w_{l^2} f = f$ and $w_2 f = -f$.

Then, in this situation, we construct now a quotient morphism

$$\pi_f : J_0(2\tilde{d}l^2) \to A'_f$$

such that the actions of $w_{2\tilde{d}}$ and w_{l^2} on $J_0(2\tilde{d}l^2)$ give both the identity on A'_f if $\tilde{d} \neq 1$. Moreover, we can see that A'_f is preserved by the whole group W of Atkin-Lehner involutions. We construct A'_f from $f \in S_2(\Gamma_0(2l^2))$ and we go to increase the level.

We denote by B_n the operator on modular forms of weight 2 that acts as: $f|_{B_n}(\tau) = f(n\tau) = n^{-1} f|_{A_n}$, where $A_n = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$ from level M to level Mk with $n|k$. We denote by

$$B_n : J_0(M) \to J_0(Mk)$$

the induced map on jacobians.

Lemma 2.2. *With the above notation and supposing that $(\tilde{n}, k) = 1$ and g is a modular form which is an eigenform for the Atkin-Lehner involution $w_{\tilde{n}}$ in $J_0(M)$, then $g|_{B_n}$ is also an eigenform for the Atkin-Lehner involution $w_{\tilde{n}}$ in $J_0(Mk)$ with the same eigenvalue.*

Proof. We only need to show that there exist $w_{\tilde{n},M}$ and $w_{\tilde{n},Mk}$, the Atkin-Lehner involution of $\tilde{n}$ at level M and Mk respectively, such that:

$$A_n w_{\tilde{n},Mk} = w_{\tilde{n},M} A_n$$

which is easy to check.

□

With the above lemma we can rewrite lemma 26 in [AL] as follows

Lemma 2.3 (Atkin-Lehner). *Let g a form in $\Gamma_0(M)$, eigenform for all the Atkin-Lehner involutions w_l at this level. Let q be a prime. Then the form*

$$g|_{B_{q^\alpha}} \pm q^{(\delta - 2\alpha)} g|_{B_1 = Id}$$

is a form in $\Gamma_0(Mq^\alpha)$ which is an eigenform for all Atkin-Lehner involutions at level Mq^α where δ is defined by $q^{\gamma-\delta}||M$ and $q^\gamma||Mq^\alpha$. Moreover, let us impose that $\delta \neq 2\alpha$. Then the eigenvalue of this form for $w_{q^{v_q(Mq^\alpha)}}$ is $\pm$ the eigenvalue of $w_{q^{v_q(M)}}$ on g.

Remark 2.4 (AL). *In the case $\delta = 2\alpha$ let us take the form $g|_{B_q^\alpha}$. Then it satisfies the following: it is an eigenform for the Atkin-Lehner involutions at level Mq^α with eigenvalue for the Atkin-Lehner involution at q equal to that of the Atkin-Lehner involution at q on g (g of level M).*

Let us remark that if the condition $\delta \neq 2\alpha$ is satisfied we can choose a form in level Mq^α with eigenvalue of the Atkin-Lehner involution at q as one wishes: 1 or -1. This condition is always satisfied if $(M,q)=1$, situation that we will use in this article. With this remarks the following lemma is clear by induction:

Lemma 2.5. *Let g be a modular form of level M which is an eigenvector for all the Atkin-Lehner involutions at level M. Then we can construct by the above lemma a modular form f of level Mk ($k \in \mathbb{N}$) which is an eigenvector for all the Atkin-Lehner involutions at level Mk, and moreover the eigenvalue at the primes $q|M$ with $(q,k)=1$ is the same that the one for the Atkin-involution of this prime at g at level M, and we can choose (1 or -1) the eigenvalue for the Atkin-Lehner involution at the primes q with $(q,k)\neq 1$ if this prime satisfies the condition $\delta \neq 2\alpha$ of the above lemma.*

Let us write a result in the form that will be usefull for our exposition, noting here that the even level condition can be removed.

Corollary 2.6. *Let us write $\tilde{d} = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with $(\tilde{d}, 2p^2) = 1$. We have a map*

$$I_{\chi_{p_1},\dots,\chi_{p_r}} : J_0(2p^2) \to J_0(2\tilde{d}p^2)$$

whose image is stable under the action of W, and we can choose the action of $w_{2\tilde{d}}$ on the quotient as $\pm$ the action of w_2 for an initial form $g \in S_2(\Gamma_0(2p^2))$ eigenform for the Atkin-Lehner involutions at level $2p^2$.

Proof. From lemma 27 in [AL], we have a base for the modular forms which are eigenforms for the Atkin-Lehner involutions. Applying the lemma of Atkin-Lehner above we have the result for $\tilde{d} = p_1^{\alpha_1}$, we have to consider $I_{\chi_{p_1}} = |_{B_{p_1^{\alpha_1}}} + \chi(p_1) p_1^{-\alpha_1} |_{B_1 = Id}$, where we can choose $\chi(p_1)$ as 1 or -1 depending on how we want the Atkin-Lehner involution at the prime p_1 to act on the quotient. Inductively we obtain the result. □

Applying the above corollary with $\tilde{d}$ square-free ($\alpha_i = 1$) in our situation ($\tilde{d} \neq 1$) and choosing $w_{2\tilde{d}} = 1$, we take

$$A'_f := I_{\chi_{p_1},\dots,\chi_{p_r}}(A_f),$$

which is by construction a subvariety of $J_0(2\tilde{d}l^2)$ isogenous to A_f which is stable under W (at level $2\tilde{d}l^2$) on which $w_{2\tilde{d}}$ and w_{l^2} acts as identity. In particular the $\mathbb{Q}$-rank of A'_f is zero (recall that we started with an A_f of $\mathbb{Q}$-rank zero).

By the Chen-Edixhoven isomorphism, we obtain a quotient map

$$\pi'_f : J_0^{ns}(2\tilde{d}; l) \to A'_f.$$

π'_f is compatible with the Hecke operators T_n with $(n, 2\tilde{d}l) = 1$ (see for example lemma 17 [AL]) and moreover $\pi'_f \circ w_{2\tilde{d}} = \pi'_f$. Let us recall that we are interested in points on $X_0^{ns}(2\tilde{d}; l)^K(\mathbb{Q})$ (we want to study the non-split Cartan situation). We have the following commutative diagram:

$$\begin{array}{ccc} J_0^{ns}(2\tilde{d}; l) & \to & A'_f \\ \downarrow i & & \downarrow id \\ J := J_0^{ns}(2\tilde{d}; l)^K & \to & A'_f \end{array}$$

where i is an isomorphism such that $i^\sigma = w_{2\tilde{d}} \circ i$ with σ the non-trivial element of $Gal(K/\mathbb{Q})$. Observe that $\psi_f := \pi'_f \circ i^{-1} : J \to A'_f$ is defined over $\mathbb{Q}$ because,

$$\psi_f^\sigma = (\pi'_f)^\sigma \circ (i^{-1})^\sigma = \pi'_f \circ w_{2\tilde{d}} \circ i^{-1} = \pi'_f \circ i^{-1} = \psi_f.$$

Let R_0 be the ring of integers of $K(\zeta_l + \zeta_i^{-1})$ and $R = R_0[1/2\tilde{d}l]$, then $X_0^{ns}(2\tilde{d}; l)$ has a smooth model over R and the cusp ∞ of $X_0^{ns}(2\tilde{d}; l)$ is defined over R [DM]. We define

$$h : X_0^{ns}(2\tilde{d}; l)/R \to J_0^{ns}(2\tilde{d}; l)/R$$

by $h(P) = [P] - [\infty]$. Then it follows by lemma 3.8 [E]

Lemma 2.7. *Let β be a prime of R. Then the map,*

$$\pi'_f \circ h : X_0^{ns}(2\tilde{d}; l)/R \to A'_f/R$$

is a formal immersion at the point $\overline{\infty}$ of $X_0^{ns}(2\tilde{d}; l)(\mathbb{F}_\beta)$.

We can prove a result for the non-split Cartan situation with a constant independent of the quadratic field.

Proposition 2.8. *Let K be a quadratic field, and E/K be a Q-curve of square-free degree $d = 2\tilde{d}$, with $\tilde{d} > 1$. Suppose that the image of $P(\overline{\sigma}_\lambda)$ lies in the normalizer of a non-split Cartan subgroup of $PGL_2(\mathbb{F}_l)$ with $\lambda | l$ for $l > 13$ with $(2\tilde{d}, l) = 1$. Then E has potentially good reduction at all primes of K.*

Proof. We can follow closely the proof of prop.3.6 in [E], let us reproduce it here for reader's convenience. Take β a prime of K where E has potentially multiplicative reduction, if $\beta | l$ then the image of the decomposition group G_β under $P(\overline{\sigma}_\lambda)$ lies in a Borel subgroup. By hypothesis this image lies in the normalizer of a non-split Cartan subgroup. We conclude that the size of this image has order at most 2, which means that K_β contains $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$, which is impossible once $l \geq 7$.

Now let us suppose that E has potentially multiplicative reduction over β with $\beta \nmid l$, denote by l' the prime of $\mathbb{Q}$ such that $\beta | l'$. It corresponds to a cusp on $X_0^{ns}(2\tilde{d}; l)$ where we will take reduction modulo β. The cusps of $X_0^{ns}(2\tilde{d}; l)$ have minimal field of definition $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$ [DM,§5], and K is linearly disjoint from $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$; it follows that the cusps of $X_0^{ns}(2\tilde{d}; l)$ which lie over $\infty \in X_0(2\tilde{d})$ form a single orbit under G_K. If $\tilde{\beta}$ is a prime of $L = K(\zeta_l + \zeta_l^{-1})$ over β, then the point $P \in X_0^{ns}(2\tilde{d}; l)(K)$ parametrizing E reduces mod $\tilde{\beta}$ to some cusp c. By applying Atkin-Lehner involutions at the primes dividing $2\tilde{d}$, we can ensure that P reduces to a cusp which lies over ∞ in $X_0(2\tilde{d})$. By the transitivity of the Galois action, we can choose $\tilde{\beta}$ so that P actually reduces to the cusp ∞ mod $\tilde{\beta}$. Using the condition that a K-point of $X_0^{ns}(2\tilde{d}; l)$ reduces to ∞, we have then that the residue field $\mathcal{O}_K/\beta$ contains $\zeta_l + \zeta_l^{-1}$, and this implies that $(l')^4 \equiv 1 \bmod l$, in particular $l' \neq 2, 3$ when $l \geq 7$.
We have constructed a form f and an abelian variety A'_f isogenous to the one associated to f with $\mathbb{Q}$-rank zero and $w_{2\tilde{d}}$ acting as 1 on it, and we have a formal immersion $\phi = \pi'_f \circ h$ at $\overline{\infty}$

$$X_0^{ns}(d; l)^K / R \to A'_f / R.$$

Let us consider $y = P$ our point from the Q-curve and $x = \infty$ at the curve $X = X_0^{ns}(2\tilde{d}; l)/R_\beta$, we know that they reduce at β to the same cusp if P has multiplicative reduction. Let us consider then $\phi(P)$ the point in $A'_f(L)$ with $L = K(\zeta_l + \zeta_l^{-1})$. Let n be an integer which kills the subgroup of $J_0^{ns}(2\tilde{d}; l)$ generated by cusps, it exists by Drinfeld-Manin, then $nh(P) \in J_0^{ns}(2\tilde{d}; l)$ and let $\tau \in Gal(L/\mathbb{Q})$ and not fixing K, then $P^\tau = w_{2\tilde{d}} P$ and we obtain that

$n\phi(P)^\tau = n\phi(P)$ then lies in $A'_f(\mathbb{Q})$ which is a finite group and then torsion, concluding that $\phi(P)$ is torsion (this is getting a standard argument [DM, lemma 8.3]).

Since $l' > 3$ the absolute ramification index of R_β at l' is at most 2. Then it follows from known properties of integer models (see for example [E, prop.3.1]) that x and y reduce to distinct point of X at β, in contradiction with our hypothesis on E.

□

Putting together propositions 2.1 and 2.8, we obtain:

Corollary 2.9. *Let E be a Q-curve over a quadratic field K of square-free composite degree $d = 2\tilde{d}$, with $\tilde{d} > 1$. Assume that E does not have potentially good reduction at all primes not dividing 6. Then, for every $\ell \nmid 2\tilde{d}$, $\ell > 13$ and $\lambda \mid \ell$, the image of the projective representation $P(\bar{\sigma}_\lambda)$ is the full $\mathrm{PGL}(2, \mathbb{F}_\ell)$.*

To conclude, observe that if we take a Q-curve over a quadratic field whose degree d is odd and composite (and square-free), there are more cases where the above result still holds: for example if $3 \mid d$ the result holds because all the required results from [DM] (in particular, the existence of a non-trivial Winding Quotient in $S_2(3p^2)$) are also proved in this case. Moreover, since the only property of the small primes $q = 2$ or 3 required for all the results we need from [DM] to hold is the fact that the modular curve $X_0(q)$ has genus 0, we can apply them to any of $q = 2, 3, 5, 7, 13$, and so we conclude that the above result applies whenever d is composite (and square-free) and divisible by one such prime q.

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Francesc Bars Cortina, Depart. Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra. E-mail: francesc@mat.uab.es

Luis Dieulefait, Depart. d'Algebra i Geometria, Facultat Matemàtiques, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona. E-mail: luisd@mat.ub.es