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Weighted Norm Inequalities and Indices

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Abstract. We extend and simplify several classical results on weighted norm inequalities for classical operators acting on rearrangement invariant spaces using the theory of indices. As an application we obtain necessary and sufficient conditions for generalized Hardy type operators to be bounded on $\Lambda_p(w)$, $\Lambda_{p,\infty}(w)$, $\Gamma_p(w)$ and $\Gamma_{p,\infty}(w)$.

1. Introduction

Ariño and Muckenhoupt [**2**] characterized the class of weights B_p for which the Hardy-Littlewood maximal operator is bounded on classical Lorentz spaces $\Lambda_p(w)$ defined by $f \in \Lambda_p(w) \Leftrightarrow \|f\|_{\Lambda_p(w)} < \infty$, where the quasi norm $\|f\|_{\Lambda_p(w)}$ is given by

$$\|f\|_{\Lambda_p(w)} = \left\| f^* w^{1/p} \right\|_{L^p(0,\infty)} = \|f^*\|_{L^p(w)} \tag{1.1}$$

(where f^* is the non-increasing rearrangement of f (cf. section 2 bellow)). In [**2**] it is shown that $w \in B_p$ iff the Hardy operator $Pf(t) = \frac{1}{t}\int_0^t f(x)dx$, restricted to $L^p(w)^d$ = decreasing functions in $L^p(w)$, $(1 \le p < \infty)$ is bounded:

$$w \in B_p \Leftrightarrow P : L^p(w)^d \to L^p(w)^d. \tag{1.2}$$

Moreover, the following explicit characterization of B_p is given in [**2**]:

$$B_p = \left\{ w \ge 0 \ : \exists c > 0 \ \text{ s.t. } \int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le c \int_0^r w(x)dx \right\}. \tag{1.3}$$

Neugebauer [**31**] derived a similar characterization of B_p^*, the class of weights for which the conjugate Hardy operator $Qf(t) = \int_t^\infty f(x)\frac{dx}{x}$ is bounded when restricted to $L^p(w)^d$:

$$w \in B_p^* \Leftrightarrow Q : L^p(w)^d \to L^p(w)^d, \tag{1.4}$$

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namely

$$B_p^* = \left\{ w \geq 0 : \exists c > 0 \text{ s.t. } \int_0^r \left(\log \left(\frac{r}{x} \right) \right)^p w(x)\, dx \;\leq\; c \int_0^r w(x)\, dx \right\}. \tag{1.5}$$

The classes B_p and B_p^* are of interest in analysis since some of the basic rearrangement estimates for classical operators can be formulated in terms of inequalities involving Hardy type operators acting on decreasing functions (cf. section **2** below). A body of literature has been devoted to study the structural properties of these classes of weights (cf. [**3**], [**16**], [**31**], [**32**], [**36**], [**38**], [**40**], [**42**], and the references therein).

A characterization of B_p^d = decreasing weights in B_p, had been given much earlier by Lorentz [**27**]. An alternative very general approach was developed in the late sixties by Boyd [**9**]. Among other things Boyd gave a very simple characterization of the class of rearrangement invariant **Banach** spaces on which the maximal operator of Hardy-Littlewood M acts continuously:

$$M : X \to X \Leftrightarrow P : X \to X \Leftrightarrow \bar{\beta}_X < 1,$$

where $\bar{\beta}_X$ is the upper Boyd index of X (see section **2** for definitions). The difficulty trying to apply this result in our context lies in the fact that Boyd's theory was developed for Banach spaces. In this context a curious situation arises for Lorentz spaces since, while the restriction that w is decreasing is known to be a necessary and sufficient condition for (1.1) to define a norm, a posteriori, it follows from the results in [**36**] that if $w \in B_p$ then $\Lambda_p(w)$ can be realized as a Banach space under the equivalent norm given by $\|f\|_{\Lambda_p(w)} = \|(Pf^*)w^{1/p}\|_{L^p(0,\infty)}$. However, by now it is well understood that Boyd's methods can be extended in a straightforward manner to **quasi-Banach** rearrangement invariant spaces (cf. [**36**], [**25**], [**30**]), thus providing an alternative route to a complete characterization of B_p using Boyd's theory. Given a weight w, let $W(r) = \int_0^r w(x)dx$, then the characterization of B_p that follows from Boyd's theory can be explicitly expressed as follows (cf. [**36**])

$$B_p = \left\{ w : \exists \gamma \in [0,1),\ c > 0 \text{ s.t. } \forall s \in (0,1],\ \sup_{r>0} \frac{W(r)}{W(rs)} \leq c s^{-\gamma p} \right\}. \tag{1.6}$$

If we let $M_W(s) = \sup_{r>0} \frac{W(r)}{W(rs)}$, then (1.6) simply states that $\bar{\beta}_{M_W}$, the upper Matuszewska-Orlicz index of M_W, satisfies

$$\bar{\beta}_{M_W} < p. \tag{1.7}$$

We note in passing that this characterization in turn immediately implies the crucial $B_p \Rightarrow B_{p-\varepsilon}$ property discovered in [**2**] (cf. also [**3**] and the papers quoted therein). Indeed, since $w \in B_p \Leftrightarrow \bar{\beta}_{M_W} < p$, it follows that for each $w \in B_p$ there exists $\varepsilon = \varepsilon(w) > 0$ such that $\bar{\beta}_{M_W} < p - \varepsilon \Rightarrow w \in B_{p-\varepsilon}$.[1]

On a different direction, since $P\chi_{(0,r)}(t) = \min(1, \frac{r}{t})$, we see that (1.3) can be reformulated as

$$w \in B_p \Leftrightarrow \sup_{r>0} \frac{\left\| P\chi_{(0,r)} \right\|_{L^p(w)}}{\left\| \chi_{(0,r)} \right\|_{L^p(w)}} < \infty. \tag{1.8}$$

[1] As it will follow from our development this proof that $B_p \Rightarrow B_{p_{-\varepsilon}}$ also works when $0 < p < \infty$. On a similar vein several self improving results for closely related classes of weights in [**4**] can be derived in the same fashion.

Similarly, since $Q\chi_{(0,r)}(t) = \log\left(\frac{r}{t}\right)\chi_{(0,r)}(t)$, we have (cf. (1.5))

$$w \in B_p^* \Leftrightarrow \sup_{r>0} \frac{\left\|Q\chi_{(0,r)}\right\|_{L^p(w)}}{\left\|\chi_{(0,r)}\right\|_{L^p(w)}} < \infty. \tag{1.9}$$

Thus, in the context of $L^p(w)$ spaces, to check the continuity of P on the class of decreasing functions it is enough to test on the subclass $\chi = \{\chi_r = \chi_{(0,r)} : r > 0\}$. The previous discussion can be summarized as follows:

$$P : L^p(w)^d \to L^p(w)^d \Leftrightarrow \sup_{r>0} \frac{\left\|P\chi_{(0,r)}\right\|_{L^p(w)}}{\left\|\chi_{(0,r)}\right\|_{L^p(w)}} < \infty \Leftrightarrow \bar{\beta}_{\Lambda_p(w)} = \bar{z}_{\Lambda_p(w)} < 1, \tag{1.10}$$

where $\bar{z}_{\Lambda_p(w)}$ is the upper Zippin index of $\Lambda_p(w)$. The Zippin indices are defined exactly as the Boyd indices but considering only the class χ (cf. section **2** below for the details).

In this paper we consider the following extended form of the Ariño-Muckenhoupt problem: given a function space X on $(0,\infty)$ characterize the class of weights W_X for which there exists $c > 0$, such that for all $f \in X^d$ = decreasing functions in X, we have

$$\|(Pf)\, w\|_X \leq c\, \|fw\|_X\, . \tag{1.11}$$

In particular we study in detail the connection between index conditions and properties of the generalized classes W_X. We aim to show that the added generality is useful in as much as it allows to clarify and simplify results in the literature as well as to provide new results for classical function spaces.

A first obstacle towards a general theory is that, in general, the Boyd and the Zippin indices are different and thus a characterization of the type (1.3):

$$w \in W_X \Leftrightarrow \|(P\chi_{(0,r)})w\|_X \leq c\|\chi_{(0,r)}w\|_X,$$

fails, in general. However, it turns out that while the Zippin indices do not control, in general, the action of P on all the decreasing functions, things improve after we consider iterations of P (cf. Theorem 1 below). This phenomenon is thus closely related to the self improving properties of these classes of weights. Our approach clarify the role of Zippin indices of a rearrangement invariant space, roughly speaking, we shall show that in fact, they control the boundeddness of $P \circ P\chi_{(0,r)}$. Another important tool in our development are certain 'reverse' estimates for generalized Hardy operators acting on decreasing functions (cf. [**6**], [**35**]). In the last section of the paper we combine these reverse estimates and the theory of indices to give an extension of a variant of the classical Gehring Lemma[2] in the context of rearrangement invariant spaces.

The paper is organized as follows. In section **2** we shall give some background information and set up our notation. In section **3** we establish the relation between the Zippin indices of a r.i. space and the boundedness of generalized Hardy type operators restricted to the class χ. In section **4** we study the boundednees of Hardy's operators (and its variants) on all sorts of weigthed Lorentz spaces. This problem has been studied by several authors (cf. for example [**1**], [**3**], [**14**], [**15**], [**20**], [**21**], [**34**], [**40**], [**42**], [**43**] and the references quoted therein[3]) however these

[2]Gehring's Lemma lies behind the self improving property of A_p weights: $A_p \Rightarrow A_{p-\varepsilon}$.

[3]We apologize in advance to the authors of papers not mentioned here.

authors generally do not use the theory of indices for function spaces. We shall present here an unified approach of this problem based on indices techniques (cf. Theorem 4) that extend and simplify results available in the literature and which allows us to give necessary and sufficient conditions for generalized Hardy type operators to be bounded on $\Lambda_p(w)$, $\Lambda_{p,\infty}(w)$, $\Gamma_p(w)$ and $\Gamma_{p,\infty}(w)$ (cf. section **4** below the definitions of these spaces). Finally, in section **5**, using still another variant of the Boyd indices (cf. [**23**]), we show another application of our generalized reverse Hardy inequalities and prove a version of Gehring's lemma (cf. [**19**]) in the context of rearrangement invariant spaces.

2. Background

We consider the **generalized Hardy operators** defined by

Definition 1. *Let $0 < q < \infty,\ 0 \leq \lambda < 1$, we let $P_{1-\lambda}^{(q)}$ and $Q_\lambda^{(q)}$, be defined by*

$$P_{1-\lambda}^{(q)} f(t) = \left(\frac{1}{t^{1-\lambda}} \int_0^t |f(x)|^q \, \frac{dx}{x^\lambda} \right)^{1/q}, \ Q_\lambda^{(q)} f(t) = \left(\frac{1}{t^\lambda} \int_t^\infty |f(x)|^q \, \frac{dx}{x^{1-\lambda}} \right)^{1/q}, \ t > 0.$$

Note that $Q_\lambda^{(1)}$ is the adjoint of $P_{1-\lambda}^{(1)}$, and if $\lambda = 0$, $q = 1$ then $P_1^{(1)} = P$, $Q_0^{(1)} = Q$.

It is well know that these operators, restricted to decreasing functions, control the basic rearrangement inequalities associated with some of the classical operators of analysis. For example, for the Hardy-Littlewood maximal operator M, defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(s)| \, ds = \sup_{x \in Q} \frac{\|f\chi_Q\|_{L_1}}{\|\chi_Q\|_{L_1}},$$

we have (see [**22**], and also [**7**])

$$(Mf)^*(t) \simeq Pf^*(t) := f^{**}(t),\ t > 0, \tag{2.1}$$

where $f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n :\ |f(x)| > \lambda\}| \leq t\}$ is the non-increasing rearrangement of f with respect to Lebesgue measure on $\mathbb{R}^n$, and as usual $F \simeq G$ indicates the existence of some universal constant $c > 0$ such that $c^{-1}F \leq G \leq cF$.

A commonly used version of the maximal operator, $M_q f = (M(|f|^q))^{1/q}$, is obtained by means of replacing L^1-averages with L^q-averages, from (2.1) we obtain

$$(M_q f)^*(t) \simeq P_1^{(q)} f^*(t).$$

More generally Stein ([**37**]) introduced maximal operators associated with Lorentz space $L^{p,q}-$averages as follows

$$M_{p,q} f(x) = \sup_{x \in Q} \frac{\|f\chi_Q\|_{L_{p,q}}}{\|\chi_Q\|_{p,q}} = \sup_{x \in Q} \frac{\|f\chi_Q\|_{L_{p,q}}}{|Q|^{1/p}}, \quad (1 \leq q \leq p).$$

We have (cf. [**5**])

$$(M_{p,q} f)^*(t) \leq c \left(\frac{1}{t^{q/p}} \int_0^t f^*(x)^q \frac{dx}{x^{1-q/p}} \right)^{1/q} = c P_{q/p}^{(q)} f^*(t).$$

In the same way, if H is the Hilbert transform, then a classical result of O'Neil-Weiss and Calderón (cf. [**33**], [**12**], and also [**9**]) gives

$$(Hf)^*(t) \leq c\,(Pf^*(t) + Qf^*(t))\,,\ t > 0. \tag{2.2}$$

Generalized Hardy operators also play an important role in the weak interpolation theory of Calderón [**12**] and more generally in abstract real interpolation (cf. [**8**], [**11**], [**4**], as well as the references therein.)

Throughout the paper we let L_0 denote the vector space of all (equivalence classes) of Lebesgue measurable real functions on $\mathbb{R}^+ = (0, \infty)$.

We shall say that X is a **quasi-Banach function space** if is a quasi-Banach linear subspace of L_0 with the following properties:

(1) *(Lattice property)* Given $g \in X$ and $f \in L_0$ such that $|f| \leq |g|$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.
(2) *(Fatou property)* $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \|f_n\|_X \leq \|f\|_X$.

A quasi-Banach function space X will be called a **rearrangement invariant space** if $f \in X \Leftrightarrow f^* \in X$ and $\|f\|_X = \|f^*\|_X$.

Remark 1. *If T is one of the above maximal operators and X is a quasi-Banach rearrangement invariant space, then the boundedness of T on X is described by the behavior of these generalized Hardy operators.*

A **decreasing function** means a non-increasing and non-negative function on $\mathbb{R}^+$, $X^d = X \cap L_0^d$, will denote the cone of all decreasing functions of X, and χ will denote the set of all characteristic decreasing functions, that is

$$\chi = \left\{ \chi_r := \chi_{(0,r)},\ r > 0 \right\}.$$

Given a **weight** w on $\mathbb{R}^+$, that is a positive and measurable function, we let $W(r) = \int_0^r w(x)dx$, $r > 0$.

We write $T : X \longrightarrow Y$ to indicate that T is a bounded operator between X and Y.

We shall now briefly review the basic theory of indices that we need for our purposes here, for further information we refer to [**7**], [**26**] and the literature quoted therein.

The **upper** and the **lower Boyd indices** associated with a quasi-Banach function space X are defined as follows. We associate to X a submultiplicative increasing function: $h_X(\alpha)$, which is the norm of the dilation operator $D_{\frac{1}{\alpha}} f(t) = f\left(\frac{t}{\alpha}\right)$ on decreasing functions:

$$h_X(\alpha) = \sup_{f \in X^d} \frac{\left\| D_{\frac{1}{\alpha}} f \right\|_X}{\|f\|_X},\ \alpha > 0.$$

Note that $h_X(\alpha) < \infty$ for all $\alpha > 1$, or $h_X(\alpha) = \infty$ for all $\alpha > 1$. Then, we let

$$\overline{\beta}_X = \inf_{\alpha > 1} \frac{\ln h_X(\alpha)}{\ln \alpha} = \lim_{\alpha \to \infty} \frac{\ln h_X(\alpha)}{\ln \alpha},\ \underline{\beta}_X = \sup_{\alpha < 1} \frac{\ln h_X(\alpha)}{\ln \alpha} = \lim_{\alpha \to 0^+} \frac{\ln h_X(\alpha)}{\ln \alpha}.$$

Similarly, the **upper** and the **lower Zippin indices** (cf. [**44**]) are defined as the Matuszewska-Orlicz indices associated with the submultiplicative increasing function $M_X(\alpha)$, which is the norm of the dilation operator $D_\alpha f(t) = f\left(\frac{t}{\alpha}\right)$ restricted to the set χ, i.e.

$$M_X(\alpha)=\sup_{f\in\chi}\frac{\left\|D_{\frac{1}{\alpha}}f\right\|_X}{\|f\|_X}=\sup_{r>0}\frac{\varphi_X(r\alpha)}{\varphi_X(r)},\ \alpha>0,$$

where $\varphi_X(r)=\|\chi_r\|_X$, is the **fundamental function** of X. The Zippin indices are likewise given by

$$\overline{z}_X=\inf_{\alpha>1}\frac{\ln M_X(\alpha)}{\ln\alpha}=\lim_{\alpha\to\infty}\frac{\ln M_X(\alpha)}{\ln\alpha},\ \underline{z}_X=\sup_{\alpha<1}\frac{\ln M_X(\alpha)}{\ln\alpha}=\lim_{\alpha\to 0^+}\frac{\ln M_X(\alpha)}{\ln\alpha}.$$

It follows readily from the obvious estimate $M_X(\alpha)\leq h_X(\alpha)$ that,

$$0\leq\underline{\beta}_X\leq\underline{z}_X\leq\overline{z}_X\leq\overline{\beta}_X\leq\infty.$$

3. $\chi-$Bounded Hardy type operators

In this section we establish a connection between the theory of Zippin indices and the continuity of iterates of $P_{1-\lambda}^{(q)}$ and $Q_\lambda^{(q)}$ acting on χ.

We first single out the type of continuity that is naturally associated with the Zippin indices (cf. [**41**] for a related type of continuity).

Definition 2. *Let X be a quasi-Banach function space and let T a quasilinear operator. We shall say that T is $\chi-X$ bounded if there exits a constant $C=C(T,X)$ such that for all $f\in\chi$*

$$\|Tf\|_X\leq C\,\|f\|_X\,.$$

Lemma 1. *Let X be a quasi-Banach function space. Let $0<q<\infty$, then*

(1) *If $0\leq\lambda<1$, $P_{1-\lambda}^{(q)}$ is $\chi-X$ bounded if and only if*

$$\left\|\left(\frac{r}{t}\right)^{\frac{1-\lambda}{q}}\chi_{(r,\infty)}(t)\right\|_X\leq C\|\chi_{(0,r)}(t)\|_X \text{ for all } r>0.$$

(2) *If $\lambda\neq 0$, $Q_\lambda^{(q)}$ is $\chi-X$ bounded if and only if*

$$\left\|\left(\frac{r}{t}\right)^{\frac{\lambda}{q}}\chi_{(0,r)}(t)\right\|_X\leq C\|\chi_{(0,r)}(t)\|_X,\ \text{for all } r>0.$$

(3) *$Q_0^{(q)}$ is $\chi-X$ bounded if and only if*

$$\left\|\left(\ln\left(\frac{r}{t}\right)\right)^{\frac{1}{q}}\chi_{(0,r)}(t)\right\|_X\leq C\|\chi_{(0,r)}(t)\|_X \text{ for all } r>0.$$

Proof. Note that by the lattice property if $f,g\geq 0$ then

$$\|f\|_X+\|g\|_X\leq 2\,\|f+g\|_X\leq 2c\left(\|f\|_X+\|g\|_X\right).$$

Using this property and

$$P_{1-\lambda}^{(q)}\chi_{(0,r)}(t)=(1-\lambda)^{-1/q}\min\left(1,\left(\frac{r}{t}\right)^{\frac{1-\lambda}{q}}\right),$$

statement (1) follows immediately.

A computation shows that

$$Q_\lambda^{(q)}\chi_{(0,r)}(t)=\lambda^{-1/q}\left[\left(\frac{r}{t}\right)^{\lambda}-1\right]^{1/q}\chi_{(0,r)}(t),$$

consequently $Q_{\lambda}^{(q)}$ is $\chi - X$ bounded if and only if

$$\left\| \left[\left(\frac{r}{t} \right)^{\lambda} - 1 \right]^{1/q} \chi_{(0,r)}(t) \right\|_X \leq C \|\chi_{(0,r)}\|_X.$$

Since $(a+b)^{1/q} \simeq a^{1/q} + b^{1/q}$ $(a \geq 0, b \geq 0)$ we get that

$$\left(\frac{r}{t} \right)^{\lambda/q} \simeq \left[\left(\frac{r}{t} \right)^{\lambda} - 1 \right]^{1/q} + 1$$

and (2) follows readily.

Finally, using

$$Q_0^{(q)} \chi_{(0,r)}(t) = \left(\ln \left(\frac{r}{t} \right) \right)^{1/q} \chi_{(0,r)}(t),$$

part (3) follows immediately. □

In this section we will quantify the extent to which the Zippin indices can be used to characterize the χ−boundedness of Hardy type operators. Observe that if X is a rearrangement invariant quasi-Banach space, then (cf. Theorem 1 below),

$$\overline{z}_X < \frac{1-\lambda}{q} \Rightarrow P_{1-\lambda}^{(q)} \text{ is } \chi - X \text{ bounded,} \tag{3.1}$$

$$\underline{z}_X > \frac{\lambda}{q} \Rightarrow Q_{\lambda}^{(q)} \text{ is } \chi - X \text{ bounded.} \tag{3.2}$$

However (cf. Remark 3 below), if $P_{1-\lambda}^{(q)}$ (resp. $Q_{\lambda}^{(q)}$) is $\chi - X$ bounded then, in general, we can only deduce that $\overline{z}_X \leq \frac{1-\lambda}{q}$ (resp. $\underline{z}_X \geq \frac{\lambda}{q}$). For example, if $X = L^{2,\infty}$ is the space weak-L^2, i.e.

$$L^{2,\infty} = \{ f \in L^0 : \|f\|_{L^{2,\infty}} = \sup_{0<t<\infty} f^*(t) t^{1/2} < \infty \}$$

then $\|\chi_r\|_{L^{2,\infty}} \simeq r^{1/2}$, $\|P_{1/2}^{(1)} \chi_r\|_{L^{2,\infty})} \simeq r^{1/2}$, and therefore $P_{1/2}^{(1)}$ is $\chi - L^{2,\infty}$ bounded, however $\overline{z}_{L^{2,\infty}} = 1/2$.

Our main result in this section shows however that the $\chi - X$ boundedness of *iterates* of $P_{1-\lambda}^{(q)}$ or $Q_{\lambda}^{(q)}$ is controlled by the Zippin indices of the quasi-Banach function space X. The proof relies on a suitable modification of the classical iteration methods of Boyd and Montgomery-Smith (cf. [**9**] and [**30**, Theorem 2]).

We shall need the following version of the Aoki-Rolewicz Theorem :

Remark 2. *(Cf. [**30**, Lemma 6]) Let X be a quasi-Banach function space, then for all $q > 0$, there is a number $0 < s \leq q$ such that*

$$\left\| \left(\sum_{n=0}^{\infty} |f_n|^q \right)^{1/q} \right\|_X \leq c \left(\sum_{n=0}^{\infty} \|f_n\|_X^s \right)^{1/s}.$$

The next two Lemmas collect some results that will be useful in what follows.

Lemma 2. *Let $0 < q < \infty,\ 0 \leq \lambda < 1$, $0 < \varepsilon$, then if $\lambda + \varepsilon < 1$*

(1) $P_{1-\lambda-\varepsilon}^{(q)} \chi_r(t) = \left(\sum\limits_{n=0}^{\infty} \varepsilon^n \left[\left(P_{1-\lambda}^{(q)} \right)^{n+1} \chi_r(t) \right]^q \right)^{1/q}.$

(2) $Q_{\lambda+\varepsilon}^{(q)} \chi_r(t) = \left(\sum_{n=0}^{\infty} \varepsilon^n \left[\left(Q_{\lambda}^{(q)} \right)^{n+1} \chi_r(t) \right]^q \right)^{1/q}.$

Proof. By induction and a straightforward application of Fubini theorem we obtain

(3.3)
$$\left[\left(P_{1-\lambda}^{(q)}\right)^{n+1}\chi_r(t)\right]^q = \left(\overbrace{P_{1-\lambda}^{(q)}\circ\cdots\circ P_{1-\lambda}^{(q)}}^{n+1}\chi_r(t)\right)^q = \frac{1}{t^{1-\lambda}}\int_0^t \chi_r(x)\,\frac{\ln(\frac{t}{x})^n}{n!}\frac{dx}{x^\lambda}.$$

Therefore by the monotone converge theorem and a simple summation of series we see that for $0<\lambda+\varepsilon<1$ we get the desired result.

Statement (2) follows from

$$\left[\left(Q_\lambda^{(q)}\right)^{n+1}\chi_r(t)\right]^q = \frac{1}{t^\lambda}\int_t^\infty \chi_r(x)\,\frac{\ln(\frac{x}{t})^n}{n!}\frac{dx}{x^{1-\lambda}}. \tag{3.4}$$

□

Lemma 3. *Let $\alpha>1$, then*

(1) $P_{1-\lambda}^{(q)}\circ P_{1-\lambda}^{(q)}\chi_r(t) \geq \left(\frac{\ln\alpha}{(1-\lambda)\alpha^{1-\lambda}}\right)^{1/q}\chi_{\alpha r}(t)$, $(0\leq\lambda<1)$.

(2) $Q_\lambda^{(q)}\circ Q_\lambda^{(q)}\chi_r(t) \geq \left(\frac{\alpha(\ln\alpha-1)+1}{\alpha^{1-\lambda}}\right)^{1/q}\chi_{\frac{r}{\alpha}}(t)$, $(\lambda\neq 0)$.

(3) $Q_0^{(q)}\chi_r(t)\geq(\ln\alpha)^{1/q}\chi_{\frac{r}{\alpha}}(t)$.

Proof. Using (3.3) and the fact that $\chi_r(x)\ln(\frac{t}{x})$ is a decreasing function of x, we get

$$P_{1-\lambda}^{(q)}\circ P_{1-\lambda}^{(q)}\chi_r(t) \geq \left(\frac{1}{t^{1-\lambda}}\int_0^{t/\alpha}\chi_r(x)\ln\left(\frac{t}{x}\right)\frac{dx}{x^\lambda}\right)^{1/q} \geq \left(\frac{\chi_r(t/\alpha)\ln\alpha}{(1-\lambda)\alpha^{1-\lambda}}\right)^{1/q}.$$

Statement (2) follows from (3.4), since

$$\begin{aligned} Q_\lambda^{(q)}\circ Q_\lambda^{(q)}\chi_r(t) &\geq \left(\frac{1}{t^\lambda}\int_t^{\alpha t}\chi_r(x)\;\ln\left(\frac{x}{t}\right)\frac{dx}{x^{1-\lambda}}\right)^{1/q} \\ &\geq \left(\frac{\chi_r(\alpha t)}{\alpha^{1-\lambda}}\frac{1}{t}\int_t^{\alpha t}\ln\left(\frac{x}{t}\right)dx\right)^{1/q} \\ &= \left(\chi_{\frac{r}{\alpha}}(t)\,\frac{\alpha(\ln\alpha-1)+1}{\alpha^{1-\lambda}}\right)^{1/q}. \end{aligned}$$

Finally (3) follows from

$$\left(\int_t^\infty \chi_r(x)\frac{dx}{x}\right)^{1/q} \geq \left(\int_t^{\alpha t}\chi_r(x)\frac{dx}{x}\right)^{1/q} \geq (\ln\alpha)^{1/q}\chi_{\frac{r}{\alpha}}(t).$$

□

Theorem 1. *Let X be a quasi-Banach function space, $0<q<\infty$, then:*

(a) $P_{1-\lambda}^{(q)}\circ P_{1-\lambda}^{(q)}$ *is $\chi-X$ bounded if and only if* $\overline{z}_X<\frac{1-\lambda}{q}$, $(0\leq\lambda<1)$.

(b) $Q_\lambda^{(q)}\circ Q_\lambda^{(q)}$ *is $\chi-X$ bounded if and only if* $\underline{z}_X>\frac{\lambda}{q}$, $(\lambda\neq 0)$.

(c) $Q_0^{(q)}$ *is $\chi-X$ bounded if and only if* $\underline{z}_X>0$.

PROOF. To see (a) suppose that $P_{1-\lambda}^{(q)}\circ P_{1-\lambda}^{(q)}$ is $\chi-X$ bounded, and let $\alpha>1$. By Lemma 3-(1) we get

$$\begin{aligned}C &\geq \sup_{r>0}\frac{\left\|P_{1-\lambda}^{(q)}\circ P_{1-\lambda}^{(q)}\chi_r\right\|_X}{\|\chi_r\|_X}\geq\left(\frac{\ln\alpha}{(1-\lambda)\alpha^{1-\lambda}}\right)^{1/q}\sup_{r>0}\frac{\|\chi_{\alpha r}\|_X}{\|\chi_r\|_X}\\ &=\left(\frac{\ln\alpha}{(1-\lambda)\alpha^{1-\lambda}}\right)^{1/q}M_X(\alpha)\geq\left(\frac{\ln\alpha}{(1-\lambda)\alpha^{1-\lambda}}\right)^{1/q}\alpha^{\overline{z}_X}.\end{aligned}$$

Thus $\overline{z}_X<\frac{1-\lambda}{q}$. To prove the converse we first prove that $P_{1-\lambda}^{(q)}$ is $\chi-X$ bounded. To this end note that

$$P_{1-\lambda}^{(q)}\chi_r(t)=\left(\int_0^1\chi_r(\xi t)\frac{d\xi}{\xi^\lambda}\right)^{1/q}\leq C\left(\sum_{n=0}^\infty 2^{-n(1-\lambda)}\chi_{2^n r}(t)\right)^{1/q}.$$

Therefore, if we pick s, $0<s\leq q$, as in Remark 2, we have

$$\begin{aligned}\left\|P_{1-\lambda}^{(q)}\chi_r\right\|_X &\leq c\left(\sum_{n=0}^\infty 2^{-\frac{ns(1-\lambda)}{q}}\varphi_X(r2^n)^s\right)^{1/s}\\ &\leq c\varphi_X(r)\left(\sum_{n=0}^\infty 2^{-\frac{ns(1-\lambda)}{q}}M_X(2^n)^s\right)^{1/s}.\end{aligned}$$

Let $\varepsilon>0$, be such that $\overline{z}_X+\varepsilon<\dfrac{1-\lambda}{q}$, then by definition it follows that there exists $C>0$ such that $M_X(2^n)\leq C2^{n(\overline{z}_X+\varepsilon)}$, whence $\left(\sum_{n=0}^\infty 2^{-\frac{ns(1-\lambda)}{q}}M_X(2^n)^s\right)^{1/s}<\infty$. Consequently $P_{1-\lambda}^{(q)}$ is $\chi-X$ bounded for any $0\leq\lambda<1$ such that $\overline{z}_X<\frac{1-\lambda}{q}$. The fact that $P_{1-\lambda}^{(q)}$ is $\chi-X$ bounded does not, a priori, imply that its iterates are $\chi-X$ bounded. In fact note that $P_{1-\lambda}^{(q)}\chi_r$ does not belong to χ. However, since we have proved that a family of operators $P_{1-\lambda}^{(q)}$ is $\chi-$bounded, the boundedness of iterates can be extrapolated as follows. By our previous discussion if we select ε such that $\overline{z}_X<\dfrac{1-\lambda-\varepsilon}{q}<\dfrac{1-\lambda}{q}$, it follows that $P_{1-\lambda-\varepsilon}^{(q)}$ is $\chi-X$ bounded. By Lemma 2-(1) it follows that

$$\varepsilon^{1/q}P_{1-\lambda}^{(q)}\circ P_{1-\lambda}^{(q)}\chi_r(t)\leq P_{1-\lambda-\varepsilon}^{(q)}\chi_r(t),$$

therefore $P_{1-\lambda}^{(q)}\circ P_{1-\lambda}^{(q)}$ is $\chi-X$ bounded. The proof of parts (b) and (c) are almost identical to the case (a) if we use Lemma 2-(2) and Lemma 3 parts (2) and (3). $\square$

REMARK 3.

$$\text{If } P_{1-\lambda}^{(q)} \text{ is } \chi-X \text{ bounded} \Rightarrow \underline{z}_X\leq\frac{1-\lambda}{q}. \tag{3.5}$$

$$\text{If } Q_{\lambda}^{(q)} \text{ is } \chi-X \text{ bounded} \Rightarrow \underline{z}_X\geq\frac{\lambda}{q},\ (\lambda\neq 0). \tag{3.6}$$

Note that if X is a quasi-Banach function space then $X^d\subset L^\infty(\varphi_X)^d$. Indeed, if $f\in X^d$, $r>0$, then we have

$$f(r)\left\|\chi_r\right\|_X=\left\|f(r)\chi_r\right\|_X\leq\left\|f\chi_r\right\|_X\leq\|f\|_X.$$

Moreover X and $L^{\infty}(\varphi_X)$ have the same fundamental function. Obviously if $P^{(q)}_{1-\lambda}$ is $\chi - X$ bounded then $P^{(q)}_{1-\lambda}$ is $\chi - L^{\infty}(\varphi_X)$ bounded. Thus for all $t > 0$ we have,

$$\left(\chi_{(0,r)}(t) + \left(\frac{r}{t}\right)^{\frac{1-\lambda}{q}} \chi_{(r,\infty)}(t)\right) \varphi_X(t) \leq C\varphi_X(r).$$

If we now let $t = \alpha r$, $\alpha > 1$ then we see that

$$\frac{\varphi_X(\alpha r)}{\varphi_X(r)} \leq C\alpha^{\frac{1-\lambda}{q}},$$

and (3.5) follows. To prove (3.6) we proceed as in Theorem 1 using the inequality

$$Q^{(q)}_{\lambda}\chi_r(t) \geq \left(\frac{1}{t^{\lambda}} \int_t^{\alpha t} \chi_r(x) \, \frac{dx}{x^{1-\lambda}}\right)^{1/q} \geq \left(\frac{\chi_r(\alpha t)\left(\alpha^{\lambda} - 1\right)}{\lambda}\right)^{1/q}.$$

The following corollary can be proved using the same methods.

Corollary 1. *Let X be a quasi-Banach function space, $0 < q < \infty$, then the following statements are equivalent:*

(a) *$P^{(q)}_{1-\lambda} \circ P^{(q)}_{1-\lambda}$ is $\chi - X$ bounded $\Leftrightarrow$ $\exists \varepsilon > 0$ such that $P^{(q)}_{1-\lambda-\varepsilon}$ is $\chi - X$ bounded $\Leftrightarrow$ $P^{(q)}_{1-\lambda} \circ ..^{(n}.. \circ P^{(q)}_{1-\lambda}$ is $\chi - X$ bounded for all $n \geq 1, (0 \leq \lambda < 1)$.*

(b) *$Q^{(q)}_{\lambda} \circ Q^{(q)}_{\lambda}$ is $\chi - X$ bounded $\Leftrightarrow$ $\exists \varepsilon > 0$ such that $Q^{(q)}_{\lambda+\varepsilon}$ is $\chi - X$ bounded $\Leftrightarrow$ $Q^{(q)}_{\lambda} \circ ..^{(n}.. \circ Q^{(q)}_{\lambda}$ is $\chi - X$ bounded for all $n \geq 1, (\lambda \neq 0)$.*

(c) *$Q^{(q)}_{0}$ is $\chi - X$ bounded $\Leftrightarrow$ $\exists \varepsilon > 0$ such that $Q^{(q)}_{\varepsilon}$ is $\chi - X$ bounded $\Leftrightarrow$ $Q^{(q)}_{0} \circ ..^{(n}.. \circ Q^{(q)}_{0}$ is $\chi - X$ bounded for all $n \geq 1$.*

Remark 4. *Note that, in general, if $P^{(q)}_{1-\lambda} \circ P^{(q)}_{1-\lambda}$ (resp. $Q^{(q)}_{\lambda} \circ Q^{(q)}_{\lambda}$) is $\chi-$bounded we cannot deduce that $P^{(q)}_{1-\lambda}$ (resp. $Q^{(q)}_{1-\lambda}$) is bounded on decreasing functions. For example, in* [**39**] *the author exhibits a rearrangement invariant Banach space X such that $\underline{z}_X = \overline{z}_X = \frac{1}{2}$, but $\underline{\beta}_X = 0$ and $\overline{\beta}_X = 1$, therefore while Theorem 1 implies that P and Q are $\chi - X$ bounded, by Boyd's theory P and Q are not bounded on X, (cf.* [**9**] *or* [**7**, Chapter 3, Theorem 5.15]*).*

If the Boyd and Zippin indices coincide the $\chi-$ boundedness of $P^{(q)}_{1-\lambda} \circ P^{(q)}_{1-\lambda}$ (resp. $Q^{(q)}_{\lambda} \circ Q^{(q)}_{\lambda}$, or $Q^{(q)}_{0}$) is equivalent to the boundedness of $P^{(q)}_{1-\lambda}$ (resp. $Q^{(q)}_{\lambda}$) on decreasing functions. The next Theorem thus follows immediately from Theorem 1 and [**30**, Theorem 2].

Theorem 2. *Let X be a quasi-Banach function space, $0 < q < \infty$. Then,*

(a) *If $\overline{\beta}_X = \overline{z}_X$, $P^{(q)}_{1-\lambda}$ is bounded when restricted to decreasing functions if and only if $P^{(q)}_{1-\lambda} \circ P^{(q)}_{1-\lambda}$ is $\chi - X$ bounded, $(0 \leq \lambda < 1)$.*

(b) *If $\underline{\beta}_X = \underline{z}_X$, $Q^{(q)}_{\lambda}$ is bounded when restricted to decreasing functions if and only if $Q^{(q)}_{\lambda} \circ Q^{(q)}_{\lambda}$ is $\chi - X$ bounded, $(\lambda \neq 0)$.*

(c) *If $\underline{\beta}_X = \underline{z}_X$, $Q^{(q)}_{0}$ is bounded when restricted to decreasing functions if and only if $Q^{(q)}_{0}$ is $\chi - X$ bounded.*

4. Applications

Definition 3. *We say that a quasi-Banach function lattice X belongs to the class $W_{\lambda,q}$ $(0 \leq \lambda < 1)$ iff*

$$P^{(q)}_{1-\lambda} \text{ is } \chi - X \text{ bounded} \Rightarrow P^{(q)}_{1-\lambda} \circ P^{(q)}_{1-\lambda} \text{ is } \chi - X \text{ bounded.}$$

In the same way, we say that X belongs to the class $W^*_{\lambda,q}$ $(0 \leq \lambda < 1)$ iff

$$Q^{(q)}_{\lambda} \text{ is } \chi - X \text{ bounded} \Rightarrow Q^{(q)}_{\lambda} \circ Q^{(q)}_{\lambda} \text{ is } \chi - X \text{ bounded.}$$

In terms of the Zippin indices the above definition is equivalent to (cf. Theorem 1)

$$X \in W_{\lambda,q} \Leftrightarrow \overline{z}_X < \frac{1-\lambda}{q}, \text{ and } X \in W^*_{\lambda,q} \Leftrightarrow \underline{z}_X > \frac{\lambda}{q}.$$

Remark that by Theorem 1-(c) every quasi-Banach function lattice X belongs to the class $W^*_{0,q}$. However as we have seen in Remark 3 the $P^{(q)}_{1-\lambda}$ $\chi - X$ boundedness (resp. $Q^{(q)}_{\lambda}$ $\chi - X$ boundedness) does not imply that $X \in W_{\lambda,q}$ (resp. $X \in W^*_{\lambda,q}$). In this general context the property $X \in W_{\lambda,q}$ (resp. $X \in W^*_{\lambda,q}$) is related with some self improving properties of classes of weights that we shall now discuss.

Definition 4. *Let $0 < p < \infty$, $0 \leq \lambda < 1$, we consider the following classes of weights*

$$B_p = \left\{ w \geq 0 \ : \exists c > 0 \ s.t. \ \int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \leq c \int_0^r w(x)dx \right\}. \tag{4.1}$$

(4.2)

$$B^*_{\lambda,p} = \left\{ w \geq 0 : \exists c > 0 \ s.t. \ \begin{cases} \displaystyle\int_0^r \left(\frac{r}{x}\right)^{\lambda p} w(x)\,dx \ \leq \ c\int_0^r w(x)\,dx\,, & \lambda \neq 0, \\ \displaystyle\int_0^r \left(\log\left(\frac{r}{x}\right)\right)^p w(x)\,dx \ \leq \ c\int_0^r w(x)dx, & \lambda = 0. \end{cases} \right\}$$

Remark 5. *If $0 < p < \infty$, $0 \leq \lambda < 1$, from Lemma 1 we have*

(1) $P^{(q)}_{1-\lambda}$ *is* $\chi - L^p(w)$ *bounded* $\Leftrightarrow w \in B_{\frac{(1-\lambda)p}{q}}$.

(2) $Q^{(q)}_{\lambda}$ *is* $\chi - L^p(w)$ *bounded* $\Leftrightarrow w \in B^*_{\lambda,\frac{p}{q}}$.

These classes of weights have some self improving properties that have been proved by several authors (cf. [**2**], [**3**], [**38**] and the papers quoted therein for the B_p class, and [**31**, Theorem 3.2] for the $B^*_{\lambda,p}$ weights). In the next Lemma we collect these properties and give an alternative proof which gives the optimal range of $\varepsilon' s$ for which the implication "$B^*_{\lambda,p} \Rightarrow B^*_{\lambda+\varepsilon,p}$" holds.

Lemma 4. *Let $0 < p < \infty$, $0 < \lambda < 1$. The following statements hold*

(1) *(cf. [**38**]) If $w \in B_p$ then there exists $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$. Moreover, if c is the best constant such that (4.1) holds, then $\varepsilon < \frac{p}{c+1}$.*

(2) *If $w \in B^*_{\lambda,p}$ then there exists $\varepsilon > 0$ such that $w \in B^*_{\lambda+\varepsilon,p}$. Moreover, if c is the best constant such that (4.2) holds, then $\varepsilon < \frac{\lambda}{c-1}$.*

In both cases the upper bound on ε is the best possible.

Proof. For (1) see [**38**, Lemma 3].

To see (2) recall that $w\in B^{*}_{\lambda,p}$ is equivalent to

$$\int_0^x \frac{w(t)}{t^{p\lambda}}dt \le \frac{c}{x^{p\lambda}}\int_0^x w(t)dt,\ x>0. \tag{4.3}$$

Choose $0<\varepsilon<\frac{\lambda}{c-1}$, multiply (4.3) by $x^{-\varepsilon p-1}$ and integrate from 0 to r. Changing the order of integration we arrive to

$$\frac{1}{\varepsilon p}\int_0^r \frac{w(t)}{t^{p\lambda}}\left(t^{-\varepsilon p}-r^{-\varepsilon p}\right)dt \le \frac{c}{p(\lambda+\varepsilon)}\left(\int_0^r w(t)\left(t^{-p(\lambda+\varepsilon)}-r^{-p(\lambda+\varepsilon)}\right)dt\right).$$

Therefore using (4.3) once again we get

$$\left(\frac{1}{\varepsilon}-\frac{c}{(\lambda+\varepsilon)}\right)\int_0^r \frac{w(t)}{t^{p(\lambda+\varepsilon)}}dt \le \left(\frac{1}{\varepsilon}-\frac{1}{(\lambda+\varepsilon)}\right)\frac{c}{r^{p(\lambda+\varepsilon)}}\int_0^r w(t)dt.$$

To show that $\frac{\lambda}{c-1}$ is best possible consider $w(x)=x^{\alpha}$, $\alpha>\lambda p-1$. $\square$

Theorem 3. *Let $0<p,q<\infty,\ 0\le\lambda<1$, then*

(a) $L^p(w)\in W_{\lambda,q}\Leftrightarrow L^{p,\infty}(w)\in W_{\lambda,q}\Leftrightarrow w\in B_{\frac{p(1-\lambda)}{q}}$.

(b) $L^p(w)\in W^{*}_{\lambda,q}\Leftrightarrow L^{p,\infty}(w)\in W^{*}_{\lambda,q}\Leftrightarrow w\in B^{*}_{\lambda,\frac{p}{q}}$.

Proof. (a) Since $L^p(w)\in W_{\lambda,q}$ is equivalent to $\overline{z}_{L^p(w)}<\frac{1-\lambda}{q}$, to prove the first equivalence it is enough to see that

$$\overline{z}_{L^p(w)}=\overline{z}_{L^{p,\infty}(w)}$$

which follows readily from

$$M_{L^{p,\infty}(w)}(\alpha)=\sup_{r>0}\left(\frac{W(\alpha r)}{W(r)}\right)^{\frac{1}{p}}=M_{L^p(w)}(\alpha).$$

Assume now that $w\in B_{\frac{p(1-\lambda)}{q}}$, by Lemma 4-(1) there exists $\delta>0$ such that $w\in B_{\frac{p(1-\lambda)}{q}-\frac{\delta q}{p}}$, therefore letting $\varepsilon=\frac{\delta q}{p}$ we have $w\in B_{\frac{p(1-\lambda-\varepsilon)}{q}}$. It follows that $P^{(q)}_{1-\lambda-\varepsilon}$ is $\chi-L^p(w)$ bounded and by Corollary 1-(a) we get that $P^{(q)}_{1-\lambda}\circ P^{(q)}_{1-\lambda}$ is $\chi-L^p(w)$ bounded which is equivalent to $L^p(w)\in W_{\lambda,q}$. The converse is obvious.

Let us now to see (b). Since

$$\underline{z}_{L^p(w)}=\underline{z}_{L^{p,\infty}(w)}$$

the first equivalence follows as in the previous case. If $\lambda=0$ the second equivalence follows from Theorem 1-(c). Suppose that $w\in B^{*}_{\lambda,\frac{p}{q}}$, $\lambda\neq 0$. By Lemma 4-(2), there exists $\varepsilon>0$ such that $w\in B^{*}_{\lambda+\varepsilon,\frac{p}{q}}$ thus $Q^{(q)}_{\lambda+\varepsilon}$ is $\chi-L^p(w)$ bounded and by Corollary 1-(b), $Q^{(q)}_{\lambda}\circ Q^{(q)}_{\lambda}$ is $\chi-L^p(w)$ bounded, as required. $\square$

The main result of this section gives a characterization of the continuity of $P^{(q)}_{1-\lambda}$ and $Q^{(q)}_{\lambda}$on $L^p(w)^d$, $L^{p,\infty}(w)^d$, $\Gamma_p(w)$ and $\Gamma_{p,\infty}(w)$ spaces ($0<p<\infty$). We start by recalling the definitions of the spaces $\Gamma_p(w)$ and $\Gamma_{p,\infty}(w)$.

$$\Gamma_p(w)=\left\{f\in L_0:\|f\|_{\Gamma_p(w)}=\left(\int_0^{\infty} f^{**}(x)^p w(x)\ dx\right)^{1/p}<\infty\right\},$$

and

$$\Gamma_{p,\infty}(w) = \left\{ f \in L_0 : \|f\|_{\Gamma_{p,\infty}(w)} = \sup_{0<t<\infty} f^{**}(t) \left(\int_0^t w(x)\, dx \right)^{1/p} < \infty \right\},$$

where as usual we let $f^{**} = Pf^*$.

We also let

$$\Lambda_p(w) = \left\{ f \in L_0 : \|f\|_{\Lambda_p(w)} = \left(\int_0^\infty f^*(x)^p w(x)\, dx \right)^{1/p} < \infty \right\},$$

(note that $\Lambda_p(w)^d = L^p(w)^d$), and

$$\Lambda_{p,\infty}(w) = \left\{ f \in L_0 : \|f\|_{\Lambda_{p,\infty}(w)} = \sup_{0<t<\infty} f^*(t) \left(\int_0^t w(x)\, dx \right)^{1/p} < \infty \right\}.$$

Observe that if f is decreasing then

$$\|f\|_{L^{p,\infty}(w)} = \sup_{t>0} \left(t^{1/p} \int_{\{s: f(s)>t\}} w(s)\, ds \right) = \sup_{t>0} f(t) \left(\int_0^t w(s) ds \right)^{1/p}, \tag{4.4}$$

(therefore we have $\Lambda_{p,\infty}(w)^d = L^{p,\infty}(w)^d$.)

REMARK 6. *The spaces $\Lambda_p(w)$ and $\Gamma_p(w)$ are called classical Lorentz spaces. The spaces $\Lambda_{p,\infty}(w)$ and $\Gamma_{p,\infty}(w)$ are called weak Lorentz spaces. The spaces $\Lambda_p(w)$ were introduced by Lorentz in 1951 in* [**27**]. *The spaces $\Gamma_p(w)$ were first used by Sawyer in* [**41**]. *The weak Lorentz spaces were introduced in* [**15**].

We assume in what follows that if $X = \Gamma_p(w)$ and $0 < p < 1$, then w satisfies the non-degeneracy conditions (cf. [**20**])

$$\int_0^\infty \frac{w(s)}{(s+1)^p} ds < \infty, \quad \int_0^1 \frac{w(s)}{s^p} ds = \int_1^\infty w(s) ds = \infty. \tag{4.5}$$

It will be convenient to state and prove the following

LEMMA 5. *If $X = L^p(w)$, $X = L^{p,\infty}(w)$, $X = \Gamma_{p,\infty}(w)$ or $X = \Gamma_p(w)$ then $\overline{\beta}_X = \overline{z}_X$ and $\underline{\beta}_X = \underline{z}_X$.*

PROOF. Note that the result will follow if can show that, there is a constant $c > 0$ such that $h_X(\alpha) \le c M_X(\alpha)$. We shall prove this claim for each of the spaces under consideration. For $X = L^p(w)$, the result is known for w *decreasing* (cf. [**26**, page 98]), however the same proof works for general w. We now consider the case $X = L^{p,\infty}(w)$. Let f be a given decreasing function, then by (4.4) we have

$$\begin{aligned} \left\| D_{\frac{1}{\alpha}} f \right\|_{L^{p,\infty}(w)} &= \sup_{t>0} f\left(\frac{t}{\alpha}\right) W(t)^{1/p} = \sup_{z>0} f(z) W(\alpha z)^{1/p} \\ &\le M_{L^{p,\infty}(w)}(\alpha) \sup_{z>0} f(z) W(z)^{1/p}. \end{aligned}$$

Thus,

$$h_{L^{p,\infty}(w)}(\alpha) \le M_{L^{p,\infty}(w)}(\alpha).$$

In the case that $X = \Gamma_{p,\infty}(w)$, since

$$\varphi_{\Gamma_{p,\infty}(w)}(t) = t \sup_{s \ge t} \frac{W(s)^{1/p}}{s}, \quad t > 0,$$

we get

$$M_{\Gamma_{p,\infty}(w)}(\alpha)=\sup_{t>0}\frac{\varphi_{\Gamma_{p,\infty}(w)}(\alpha t)}{\varphi_{\Gamma_{p,\infty}(w)}(t)}=\sup_{t>0}\left(\frac{t\alpha\sup\limits_{s\geq t\alpha}\frac{W(s)^{1/p}}{s}}{\sup\limits_{s\geq t}\frac{W(s)^{1/p}}{s}}\right)=\sup_{t>0}\left(\frac{\sup\limits_{s\geq t}\frac{W(\alpha s)^{1/p}}{s}}{\sup\limits_{s\geq t}\frac{W(s)^{1/p}}{s}}\right).$$

Then

$$\begin{aligned}\|D_{\frac{1}{\alpha}}f\|_{\Gamma_{p,\infty}(w)}&=\sup_{t>0}f^{**}(\frac{t}{\alpha})W(t)^{1/p}=\sup_{t>0}f^{**}(t)W(\alpha t)^{1/p}\\&\leq\sup_{t>0}\int_0^t f^*(s)\,ds\ \sup_{s\geq t}\frac{W(\alpha s)^{1/p}}{s}\\&\leq\sup_{t>0}\left(\frac{\sup\limits_{s\geq t}\frac{W(\alpha s)^{1/p}}{s}}{\sup\limits_{s\geq t}\frac{W(s)^{1/p}}{s}}\right)\sup_{t>0}\int_0^t f^*(s)\,ds\ \sup_{s\geq t}\frac{W(s)^{1/p}}{s}\\&=M_{\Gamma_{p,\infty}(w)}(\alpha)\sup_{t>0}\int_0^t f^*(s)\,ds\ \sup_{s\geq t}\frac{W(s)^{1/p}}{s}.\end{aligned}$$

Since (see [**13**, Remark 2.11 (iv)])

$$\|f\|_{\Gamma_{p,\infty}(w)}=\sup_{t>0}\int_0^t f^*(s)\,ds\ \sup_{s\geq t}\frac{W(s)^{1/p}}{s}$$

we obtain,

$$h_{\Gamma_{p,\infty}(w)}(\alpha)\leq M_{\Gamma_{p,\infty}(w)}(\alpha).$$

Finally, if $X=\Gamma_p(w)$, then we compute

$$M_{\Gamma_p(w)}(\alpha)=\sup_{r>0}\frac{\left\|D_{\frac{1}{\alpha}}\chi_r\right\|_{\Gamma_p(w)}}{\|\chi_r\|_{\Gamma_p(w)}}=\sup_{r>0}\left(\frac{\int_0^{\alpha r}w(x)dx+\alpha^p r^p\int_{\alpha r}^\infty w(x)\frac{dx}{x^p}}{\int_0^r w(x)dx+r^p\int_r^\infty w(x)\frac{dx}{x^p}}\right)^{1/p}.$$

Now, if $1\leq p<\infty$, by [**21**, Theorem 3.7] we get that

$$\begin{aligned}\left(\int_0^\infty D_{\frac{1}{\alpha}}f^{**}(x)^p w(x)dx\right)^{1/p}&=\left(\int_0^\infty f^{**}(x)^p\alpha w(\alpha x)dx\right)^{1/p}\\&\leq M_{\Gamma_p(w)}(\alpha)\left(\int_0^\infty f^{**}(x)^p w(x)dx\right)^{1/p}.\end{aligned}$$

If $0<p<1$, since w satisfies the non-degeneracy conditions (4.5), by [**20**, Theorem 3.2] there is a constant $c=c(p)$ (which only depends of p)[4] such that

$$\left(\int_0^\infty f^{**}(x)^p\alpha w(\alpha x)dx\right)^{1/p}\leq cM_{\Gamma_p(w)}(\alpha)\left(\int_0^\infty f^{**}(x)^p w(x)dx\right)^{1/p},$$

therefore in both cases,

$$h_{\Gamma_p(w)}(\alpha)\leq cM_{\Gamma_p(w)}(\alpha).$$

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[4]A perusal of the constants in the proof of [**20**, Theorem 3.2] shows that if we fix $a>4$, then

$$c(p)=\frac{2^{1+1/p}a}{(a^p-1)^{1/p}}\left(\frac{1}{2^{-p}-(\frac{2}{a})^p}\right)^{1/p}.$$

□

THEOREM 4. *Let* $0 < p, q < \infty$, $0 \le \lambda < 1$. *Then,*

(a) $P^{(q)}_{1-\lambda} : \Lambda_p(w) \to \Lambda_p(w) \Leftrightarrow w \in B_{\frac{p(1-\lambda)}{q}} \Leftrightarrow P^{(q)}_{1-\lambda} : \Lambda_{p,\infty}(w) \to \Lambda_{p,\infty}(w)$.

(b) $Q^{(q)}_{\lambda} : L^p(w)^d \to L^p(w) \Leftrightarrow w \in B^*_{\lambda,\frac{p}{q}} \Leftrightarrow Q^{(q)}_{\lambda} : L^{p,\infty}(w)^d \to L^{p,\infty}(w)$. *(Moreover* $w \in B^*_{0,\frac{p}{q}} \Leftrightarrow P \circ Pw(r) \le CPw(r)$, *i.e.* $B^*_{0,\frac{p}{q}} = B^*_{0,1}$*).*

(c) $P^{(q)}_{1-\lambda} : \Gamma_p(w) \to \Gamma_p(w) \Leftrightarrow \left\| P^{(q)}_{1-\lambda} \circ P^{(q)}_{1-\lambda}\chi_r \right\|_{\Gamma_p(w)} \le C \left\|\chi_r\right\|_{\Gamma_p(w)}$.

(d) $Q^{(q)}_{\lambda} : \Gamma_p(w)^d \mapsto \Gamma_p(w) \Leftrightarrow \left\| Q^{(q)}_{\lambda} \circ Q^{(q)}_{\lambda}\chi_r \right\|_{\Gamma_p(w)} \le C \left\|\chi_r\right\|_{\Gamma_p(w)}$, $(\lambda \ne 0)$.

(e) $Q^{(q)}_{0} : \Gamma_p(w)^d \to \Gamma_p(w) \Leftrightarrow \left\| Q^{(q)}_{0}\chi_r \right\|_{\Gamma_p(w)} \le C \left\|\chi_r\right\|_{\Gamma_p(w)}$.

(f) *If we replace* $\Gamma_p(w)$ *by* $\Gamma_{p,\infty}(w)$ *in (c), and* $\Gamma_p(w)^d$ *by* $\Gamma_{p,\infty}(w)^d$ *in (d) and (e) the corresponding statements remain true.*

PROOF. (a) Since $\left(P^{(q)}_{1-\lambda} f\right)^* \le P^{(q)}_{1-\lambda} f^*$ we only need to prove the boundedness of $P^{(q)}_{1-\lambda}$ on decreasing functions. By Theorem 3 we know that

$$w \in B_{\frac{p(1-\lambda)}{q}} \Leftrightarrow \overline{z}_{\Lambda^p(w)} < \frac{1-\lambda}{q}.$$

By Lemma 5, $\overline{z}_{\Lambda^p(w)} = \overline{\beta}_{\Lambda^p(w)}$, thus Theorem 2 applies.

To see the claim in (b) stating that

$$w \in B^*_{0,\frac{p}{q}} \Leftrightarrow P \circ Pw(r) \le CPw(r),$$

let us observe that, by Theorem 1-(c), $w \in B^*_{0,\frac{p}{q}}$ is equivalent to $\underline{z}_{L^p(w)} > 0$. Moreover, using the readily seen fact that $\underline{z}_{L^p(w)} = \frac{1}{p}\underline{z}_{L^1(w)}$, we have that $\underline{z}_{L^p(w)} > 0$ if and only if $\underline{z}_{L^1(w)} > 0$ (i.e. $Q : L^1(w)^d \longrightarrow L^1(w)$). Therefore,

$$Q^{(q)}_0 : L^p(w)^d \longrightarrow L^p(w) \Leftrightarrow Q : L^1(w)^d \longrightarrow L^1(w),$$

in other words

$$w \in B^*_{0,\frac{p}{q}} \Leftrightarrow w \in B^*_{0,1},$$

as we wished to show.[5]

To see (c) by Theorem 1

$$\left\| P^{(q)}_{1-\lambda} \circ P^{(q)}_{1-\lambda}\chi_r \right\|_{\Gamma_p(w)} \le C \left\|\chi_r\right\|_{\Gamma_p(w)} \Leftrightarrow \overline{z}_{\Gamma_p(w)} < \frac{1-\lambda}{q}$$

and since (cf. Lemma 5) $\overline{z}_{\Gamma_p(w)} = \overline{\beta}_{\Gamma_p(w)}$, Theorem 2 applies.

Finally, parts (d), (e) and (f) can be proved in the same way. □

COROLLARY 2. *If* w *is decreasing then we have (cf.* [**16**]*)*

$$Q^{(q)}_0 : L^p(w)^d \longrightarrow L^p(w) \Leftrightarrow P \circ Pw(r) \le CPw(r) \Leftrightarrow Pw(r) \le Cw(r).$$

[5] It follows that the boundedness of $Q^{(q)}_0$ on $L^p(w)$ on decreasing functions does not depend on p. (cf. [**31**, Theorem 3.3] for $L^p(w)$, $1 \le p < \infty$ and $q = 1$ and [**17**] if $0 < p < 1$ and $q = 1$).

Remark 7. *Theorem 4 extends and simplifies several results that appear in the literature. Note that if* $q = 1$, $\lambda = 0$ *and* $1 \leq p < \infty$, *the first equivalence in (a) is the result of Ariño-Muckenhoupt* [**2**] *(cf.* [**31**, Theorem 2.3] *for the case* $q = 1$, $0 < \lambda < 1$, $1 \leq p < \infty$ *and* [**38**, Theorem 7] *if* $1 < q < \infty$, $\lambda = 0$ *and* $p = 1$*). If* $q = 1$, $\lambda = 0$ *the second equivalence in (a) was proved by Soria* [**40**, Theorem 3.1]. *If* $q = 1$, $0 < \lambda < 1$ *and* $1 \leq p < \infty$, *the first equivalence in (b) was proved by Neugebauer* [**31**, Theorem 3.1]. *Finally observe that condition* B_p *implies* $B_{p-\varepsilon}$ *is equivalent to the* $\chi - L^p(w)$ *boundedness of* $P \circ P$, *and from this condition it now follows immediately that* P *is bounded on* $\Lambda_p(w)$.

Remark 8. *If* X *is a rearrangement invariant **Banach** space then the inequality (cf.* [**26**, page 126]*)*

$$\frac{1}{x}\int_0^x P\chi_r(t)\ dt \geq \left(\frac{\ln\alpha + 1}{\alpha}\right)\frac{1}{x}\int_0^x D_{\frac{1}{\alpha}}\chi_r(t)dt,\ \ (\alpha > 1,\ x > 0)$$

implies (cf. [**12**]*)*

$$\|P\chi_r\|_X \geq \frac{\ln\alpha}{\alpha}\left\|D_{\frac{1}{\alpha}}\chi_r\right\|_X .$$

Hence if P *is* $\chi - X$ *bounded we have*

$$C \geq \sup_{r>0}\frac{\|P\chi_r\|_X}{\|\chi_r\|_X} \geq \frac{\ln\alpha}{\alpha}\sup_{r>0}\frac{\|\chi_{\alpha r}\|_X}{\|\chi_r\|_X} = \frac{\ln\alpha}{\alpha}M_X(\alpha) \geq \frac{\ln\alpha}{\alpha}\alpha^{\overline{z}_X},$$

which implies that $\overline{z}_X < 1$. *Therefore, every rearrangement invariant **Banach** space* X *belongs to* $W_{0,1}$. *Applying this observation to* $\Gamma_p(w)$, $1 \leq p < \infty$, *we find that*

$$P : \Gamma_p(w) \to \Gamma_p(w) \Leftrightarrow \|P\chi_r\|_{\Gamma_p(w)} \leq C\,\|\chi_r\|_{\Gamma_p(w)} .$$

Moreover the condition to the right is readily seen to be equivalent to

(4.6)

$$\sup_{t>0} t\left(\int_t^\infty x^{-p}\log(x/t)\,w(x)dx\right)^{1/p}\left(\int_0^t w(x)\ dx + t^p\int_t^\infty w(x)\ \frac{dx}{x^p}\right)^{-1/p} < \infty.$$

Consequently we obtain a new proof of Stepanov's result (cf. [**42**, Theorem 5.1]*):*

$$P : \Gamma_p(w) \longrightarrow \Gamma_p(w) \Leftrightarrow \text{(4.6) holds.}$$

It is of interest to point out that the Zippin indices of a rearrangement invariant Banach space X are related to the best B_1-constant associated to its fundamental function φ_X. Let X be a rearrangement invariant Banach space with concave fundamental function (cf. [**7**, Chapter 2, Proposition 5.11]). Suppose that $\varphi_X(0^+) = 0$, then

$$\varphi_X(t) = \int_0^t \phi_X(s)ds, \tag{4.7}$$

where ϕ_X is a decreasing function. Consider the rearrangement invariant Banach space $\Lambda_1(\phi_X)$, it is known (cf. [**10**]) that

$$\overline{\beta}_{\Lambda_1(\phi_X)} = \inf_{\lambda>0}\left\{1 - \frac{1}{\lambda} : P_{1-\frac{1}{\lambda}} \text{ is bounded on } \Lambda_1(\phi_X)\right\},$$

which in turn is equivalent to

$$\overline{\beta}_{\Lambda_1(\phi_X)} = \inf_{\lambda>0}\left\{1 - \frac{1}{\lambda} : \phi_X \in B_{1-\frac{1}{\lambda}}\right\}. \tag{4.8}$$

If $\phi_X \notin B_1$ then the infimum in (4.8) is equal to 1. If $\phi_X \in B_1$ by Lemma 4-1, there exists $\varepsilon > 0$ such that $\phi_X \in B_{1-\varepsilon}$ and we have

$$\sup\{\varepsilon : \phi_X \in B_{1-\varepsilon}\} = \frac{1}{\|\phi_X\|_{B_1} + 1},$$

where

$$\|\phi_X\|_{B_1} = \inf\left\{c > 0; \int_r^\infty \phi_X(s)\frac{ds}{s} \ \leq \ \frac{c}{r}\int_0^r \phi_X(s)\,ds,\ r > 0\right\}.$$

Moreover, since (cf. [**32**, Theorem 2.2])

$$\|\phi_X\|_{B_1} + 1 = \|P\|_{\Lambda_1(\phi_X)}$$

it follows from that

$$\overline{\beta}_{\Lambda_1(\phi_X)} = 1 - \frac{1}{\|P\|_{\Lambda_1(\phi_X)}}.$$

But by Lemma 5, $\overline{z}_{\Lambda_1(\phi_X)} = \overline{\beta}_{\Lambda_1(\phi_X)}$. Moreover, since X and $\Lambda_1(\phi_X)$ have both the same fundamental function (cf. (4.7)),

$$\overline{z}_X = 1 - \frac{1}{\|P\|_{\Lambda_1(\phi_X)}}. \tag{4.9}$$

Boyd [**10**] proves that

$$\frac{1}{1 - \overline{\beta}_X} = \rho_X(P),$$

(where $\rho_X(P)$ is the spectral radius of P on X). Hence using Gelfand's formula for the spectral radius and (4.9) we obtain

$$\|P\|_{\Lambda_1(\phi_X)} = \frac{1}{1 - \overline{z}_X} \leq \frac{1}{1 - \overline{\beta}_X} = \rho_X(P) = \lim_{n\to\infty} \|P^n\|_X^{1/n}.$$

Thus,

$$\overline{\beta}_X = \overline{z}_X \Leftrightarrow \|P\|_{\Lambda_1(\phi_X)} = \inf_{n\geq 1} \|P^n\|_X^{1/n}.$$

Let us also note that since $P : X \to X$ if and only if $Q : X' \to X'$ (where X' is the associate space of X (cf. [**7**]) and $\underline{\beta}_X = 1 - \overline{\beta}_{X'}$ and $\underline{z}_X = 1 - \overline{z}_{X'}$, it is easy to derive a similar result for the lower Zippin indices.

The next proposition follows readily from the above definitions

Proposition 1. *Let X be a rearrangement invariant Banach space and let $0 < q < \infty,\ 0 \leq \lambda < 1$. Then there are equivalent*

(1) $X \in W_{\lambda,q}$ *(resp.* $X \in W^*_{\lambda,q}$*).*
(2) $\Lambda_1(\phi_X) \in W_{\lambda,q}$ *(resp.* $\Lambda_1(\phi_X) \in W^*_{\lambda,q}$*).*
(3) $\phi_X \in B_{\frac{(1-\lambda)}{q}}$ *(resp.* $\phi_X \in B^*_{\lambda,q}$*).*

For Orlicz spaces the Boyd and Zippin indices coincide (cf. [**7**, Chapter 4, Lemma 8.17 and Theorem 8.18]) and therefore by combining the above proposition and Theorem 2 we can easily prove the following

Proposition 2. *Let Φ a Young's function. Then $P^{(q)}_{1-\lambda} : L^\Phi \longrightarrow L^\Phi$ is bounded (where L^Φ is a Orlicz space endowed with the Luxemburg norm) if and only if $\frac{1}{\Phi^{-1}(\frac{1}{t})}$ belongs to the Muckenhoupt class $A_{\frac{(1-\lambda)}{q}+1}$.*

PROOF. It is well know (cf. [**7**, Chapter 4, Lemma 8.17]) that

$$\varphi_{L^{\Phi}}(t) = \frac{1}{\Phi^{-1}(\frac{1}{t})}$$

then by Theorem 2, $P_{1-\lambda}^{(q)} : L^{\Phi} \longrightarrow L^{\Phi}$ is bounded if and only if $\overline{z}_{L^{\Phi}} < \frac{1-\lambda}{q}$. By Proposition 1 this is equivalent to $\frac{\partial}{\partial t}\varphi_{L^{\Phi}}(t) \in B_{\frac{(1-\lambda)}{q}}$, and now [**18**, Theorem 3.1] applies. $\square$

5. Gehring's Lemma for rearrangement invariant spaces.

Let X be a rearrangement invariant Banach space. For $1 \leq r < \infty$, we consider the space

$$X^{(r)} = \left\{ f \in L_0;\ \|f\|_{X^{(r)}} = \left\| |f|^r \right\|_X^{1/r} < \infty \right\}.$$

It is easy to see that $(X^{(r)}, \|.\|_{X^{(r)}})$ is also a rearrangement invariant Banach space. In this section it will be technically convenient to work with still a somewhat different class of indices apparently introduced by Herz [**23**] who calls them *"exponents of X."*

Let X be a rearrangement invariant Banach lattice, then following [**23**] we define the **lower** and the **upper exponents** of X as

$$\underline{\gamma}_X = \lim_{\alpha \to 0^+} \frac{1 - h_X(1-\alpha)}{\alpha} \quad \text{and} \quad \overline{\gamma}_X = \lim_{\alpha \to 0^+} \frac{h_X(1+\alpha) - 1}{\alpha}.$$

Since [**23**] is not readily available in the next Lemma we reproduce some results from the manuscript that we needed here.

LEMMA 6. *(cf.* [**23**]*)*

(1) $0 \leq \underline{\gamma}_X \leq \underline{\beta}_X \leq \overline{\beta}_X \leq \overline{\gamma}_X \leq 1.$
(2) $\max(\alpha^{\underline{\beta}_X}, \alpha^{\overline{\beta}_X}) \leq h_X(\alpha) \leq \max(\alpha^{\underline{\gamma}_X}, \alpha^{\overline{\gamma}_X}).$
(3) $\|P\|_X \leq \frac{1}{1-\overline{\gamma}_X}.$
(4) $\underline{\gamma}_{X^{(r)}} = \frac{\underline{\gamma}_X}{r}$ *and* $\overline{\gamma}_{X^{(r)}} = \frac{\overline{\gamma}_X}{r}.$

PROOF. Define $\delta : \mathbb{R} \to \mathbb{R}$ by $\delta(x) = \ln h_X(e^x)$. Since h_X is a quasi-concave increasing, submultiplicative, function, it follows that δ is increasing, $x \to x - \delta(x)$ is decreasing, $\delta(0) = 0$, and δ is subadditive (i.e. $\delta(x+y) \leq \delta(x) + \delta(y)$).[6] For $h > 0$ we have $0 \leq h^{-1}\delta(h) \leq 1$ and $\delta(h) \geq -\delta(-h) \geq 0$. It follows that $\overline{\gamma}_X = \lim \inf_{h \to 0+} h^{-1}\delta(h)$ and $\underline{\gamma}_X = \lim \sup_{h \to 0+} -h^{-1}\delta(-h)$. And similarly $\overline{\beta}_X = \lim \sup_{h \to \infty} h^{-1}\delta(h)$ and $\underline{\beta}_X = \lim \inf_{h \to -\infty} h^{-1}\delta(h)$. Now properties (1) and (2) follow immediately. Part (3) follows from the inequality $\|P\|_X \leq \int_1^{\infty} h_X(t) t^{-2} dt \leq \frac{1}{1-\overline{\gamma}_X}$. For (4) we use the fact that $h_{X^{(r)}}(\alpha) = (h_X(\alpha))^{1/r}$. $\square$

THEOREM 5. *Let $1 < p < \infty$ and suppose that g and h are positive functions such that there exists a constant $c > 0$ such that for all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, we have*

$$\left(\frac{1}{|Q|} \int_Q g(x)^p dx \right)^{1/p} \leq c \frac{1}{|Q|} \int_Q g(x) dx + c \left(\frac{1}{|Q|} \int_Q h(x)^p dx \right)^{1/p}$$

[6] Such a function is necessarily continuous.

Then, there exist $\varepsilon > 0$ such that for all rearrangement invariant Banach spaces X such that $1-\varepsilon \leq \underline{\gamma}_X \leq \overline{\gamma}_X < 1$, if $g,\ h \in X^{(p)}$ there exist a constant $C = C(c, X)$ such that

$$\|g\|_{X^{(p)}} \leq C\,\|h\|_{X^{(p)}}\,.$$

Proof. We adapt the method used in [**29**]. In terms of the maximal operator of Hardy-Littlewood M, we have

$$M_p g \leq c(Mg + M_p h),$$

where $M_p = M(|f|^p)^{1/p}$. Therefore by Herz's inequality (cf. [**7**])

$$P_1^{(p)} g^*(t) \leq c(Pg^*(t/2) + P_1^{(p)} h^*(t/2)).$$

Using (cf. [**26**, page 126])

$$\frac{1}{r}\int_0^r \left(P_1^{(p)} g^*(t)\right)^p\ dt \geq \left(\frac{\ln\alpha + 1}{\alpha}\right)\frac{1}{r}\int_0^r D_{\frac{1}{\alpha}} g^*(t)^p dt, \qquad \alpha > 1$$

and the fact that $X^{(p)}$ is a rearrangement invariant Banach space, we get (cf. Remark 8)

$$\left(\frac{\ln\alpha+1}{\alpha}\right)^{1/p}\left\|D_{\frac{1}{\alpha}} g\right\|_{X^{(p)}} \leq \left\|P_1^{(p)} g\right\|_{X^{(p)}} \leq c(\left\|PD_{\frac{1}{2}} g\right\|_{X^{(p)}} + \left\|P_1^{(p)} D_{\frac{1}{2}} h\right\|_{X^{(p)}}).$$

Since $\overline{\beta}_{X^{(p)}} \leq \overline{\gamma}_{X^{(p)}} = \dfrac{1}{p}\overline{\gamma}_X < \dfrac{1}{p}$, $P_1^{(p)}$ is bounded in $X^{(p)}$, furthermore, by Lemma 6-(4), $\|P\|_{X^{(p)}} \leq \dfrac{1}{1-\frac{\overline{\gamma}_X}{p}}$. Thus

$$\left(\frac{\ln\alpha+1}{\alpha}\right)^{1/p}\|D_{\frac{1}{\alpha}} g\|_{X^{(p)}} \leq 2c\frac{1}{1-\frac{\overline{\gamma}_X}{p}}\|g\|_{X^{(p)}} + C(X,p,c)\,\|h\|_{X^{(p)}}\,.$$

Since $\|g\|_{X^{(p)}} = \|D_\alpha D_{\frac{1}{\alpha}} g\|_{X^{(p)}} \leq \|D_\alpha\|_{X^{(p)}}\|D_{\frac{1}{\alpha}} g\|_{X^{(p)}}$, it follows that

$$\left(\frac{\ln\alpha+1}{\alpha}\right)^{1/p}\frac{\|g\|_{X^{(p)}}}{\|D_\alpha\|_{X^{(p)}}} \leq 2c\frac{1}{1-\frac{\overline{\gamma}_X}{p}}\|g\|_{X^{(p)}} + C(X,p,c)\,\|h\|_{X^{(p)}}\,. \tag{5.1}$$

Now by Lemma 6-(2) we see that

$$\|D_\alpha\|_{X^{(p)}} = h_{X^{(p)}}(1/\alpha) \leq (1/\alpha)^{\underline{\gamma}_{X^{(p)}}} = (1/\alpha)^{\frac{\underline{\gamma}_X}{p}}\,.$$

Thus

$$\begin{aligned}\sup_{\alpha>1}\left(\left(\frac{\ln\alpha+1}{\alpha}\right)^{1/p}\frac{\|g\|_{X^{(p)}}}{\|D_\alpha\|_{X^{(p)}}}\right) &\geq \sup_{\alpha>1}\left(\frac{\ln\alpha+1}{\alpha^{1-\underline{\gamma}_X}}\right)^{1/p}\|g\|_{X^{(p)}} \\ &= \left(\frac{e^{-\underline{\gamma}_X}}{1-\underline{\gamma}_X}\right)^{1/p}\|g\|_{X^{(p)}}.\end{aligned} \tag{5.2}$$

Combining (5.2) with (5.1) we get

$$\left(\left(\frac{e^{-\underline{\gamma}_X}}{1-\underline{\gamma}_X}\right)^{1/p} - 2c\left(\frac{1}{1-\frac{\overline{\gamma}_X}{p}}\right)\right)\|g\|_{X^{(p)}} \leq C(X,p,c)\,\|h\|_{X^{(p)}}\,. \tag{5.3}$$

Thus we see that if $\underline{\gamma}_X$ is sufficiently close to 1, the left hand side of (5.3) is bigger than 0, and we obtain

$$\|g\|_{X^{(p)}} \leq C\,\|h\|_{X^{(p)}}\,.$$

□

Remark 9. *If X is a rearrangement invariant Banach space such that $1-\varepsilon < \underline{\gamma}_X \leq \overline{\gamma}_X < 1$, where ε is taken such that the left hand side of (5.3) is bigger than zero, then there exists $\delta > 0$ such that $X^{(1+\delta)}$ satisfies that $1-\varepsilon \leq \underline{\gamma}_{X^{(1+\delta)}} \leq \overline{\gamma}_{X^{(1+\delta)}} < 1$ and (5.3) is bigger than zero for $X^{(1+\delta)}$, so in this case the Theorem says that $\|g\|_{X^{(p(1+\delta))}} \leq C\,\|h\|_{X^{(p(1+\delta))}}$. Observe that for example when $X = L^{1+\epsilon}$ then $\|g\|_{L^{p(1+\varepsilon)}} \leq C\,\|h\|_{L^{p(1+\varepsilon)}}$ (cf. [**29**].)*

Remark 10. *If X a is a rearrangement invariant Banach space satisfying*

$$h_X(\alpha) = \alpha^{\overline{\beta}_X} \quad \text{if} \quad \alpha > 1 \quad \text{and } h_X(\alpha) = \alpha^{\underline{\beta}_X} \quad \text{if} \quad \alpha < 1,$$

then is it possible to formulate the result above in terms of Boyd indices.

Remark 11. *The assumption that $g \in X^{(p)}$ can be now removed by an approximation argument as indicated in* [**24**].

Remark 12. *For a version of the Gehring Lemma using Boyd indices in the case that X is a Orlicz space see* [**28**].

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