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Extrapolation methods and Rubio de Francia's extrapolation Theorem

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Abstract. We develop a general framework to study extrapolation of inequalities.

To the memory of Jose Luis Rubio de Francia

Contents

1. Introduction

The starting point of our research is the celebrated extrapolation theorem of Rubio de Francia (cf. [**22**], [**11**] and the references therein) which, roughly speaking, asserts that if an operator satisfies sufficiently many weighted L^p inequalities for a fixed p then structurally similar weighted L^q, $q \neq p$, estimates follow as well. More precisely, Rubio de Francia's classical extrapolation theorem asserts that if T is a linear operator such that for a given $p_0 \in (1, \infty)$, and for all weights $w \in A_{p_0}$(the Muckenhoupt class A_{p_0}) we have

$$T : L^{p_0}(w) \to L^{p_0}(w), \text{ boundedly,} \tag{1.1}$$

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then it follows that for all $p \in (1,\infty)$ we have

$$T : L^p(w) \to L^p(w), \forall w \in A_p, \text{ boundedly.}$$

Rubio de Francia's theorem has become a fundamental tool in the theory of weighted norm inequalities (cf. [**11**] and [**12**] for detailed accounts).

In his paper [**23**] (cf. also [**24**]) Rubio de Francia develops his ideas further and shows how L^p spaces and other lattices can be constructed by means of unions or intersections of other L^q spaces. This provides a beautiful explanation of why if sufficient weighted norm estimates are known for one fixed index p then other inequalities follow for other indices "by extrapolation".

A different extrapolation process, which originated with Yano's extrapolation theorem (cf. [**25**] and [**26**]), was developed by Jawerth and Milman (cf. [**13**], [**19**]), and deals with operators acting on interpolation scales in such a way that the corresponding norm estimates deteriorate at the end points. This extrapolation theory aims to obtain information about the operator from the speed of the norm blow ups and, in particular, to derive alternative end point estimates. It turns out that in the theory of [**13**] representing spaces as sums or intersections also plays a fundamental role.

This led, many years ago, to the question if there was a connection between these two disparate theories. In this paper we show that generalizing slightly the setting of the theory of extrapolation spaces of Jawerth and Milman (or the setting of the theory of Rubio de Francia!) we can create a framework that is general enough to unify it with the theory of Rubio de Francia. Thus, in our framework, one can prove Yano's type extrapolation theorems via Rubio de Francia's method, which in our setting corresponds to taking limits in the estimates, or one can show that the classical theory of weak interpolation of Calderón [**9**] follows from the extrapolation of weighted norm inequalities for the Calderón operator. In particular, the celebrated interpolation theorem of Boyd [**6**] follows by extrapolation. Likewise, in our setting Rubio de Francia's theorem can be obtained in a suitable functorial fashion using the Δ or $\sum$ methods of [**13**]. This allows, for example, to prove versions of Rubio de Francia's theorem for many different classes of weights, with a unified method. Our approach can be also used to obtain extrapolation of weighted norm inequalities for other types of spaces, etc. In short, the $\sum$ and Δ methods of extrapolation developed by [**13**] are naturally connected with $\bigcup$ and $\bigcap$ of spaces studied by Rubio de Francia and thus connected with the geometry of Banach spaces and factorization.

As it often happens in mathematics the added generality helps to clarify the proofs and new connections emerge. The idea of factorization in our setting is almost trivial but it is remarkably powerful. Let us consider informally the factorization of operators mapping intersections of spaces. Suppose that $T : X \to X$ is a bounded operator where $X = \bigcap_{\nu \in I} X_\nu$. We ask: What can be said about estimates on individual spaces X_ν? A factorization in this setting is simply the statement that for each $\nu \in I$ there exists $\eta \in I$ such that $T : X_\eta \to X_\nu$ with norm control. Suppose that T is a fixed factorizable operator for which there are "enough" individual norm estimates of the type $T : X_\eta \to X_\nu$ (think of T as a linearized version of a maximal operator for example) and let G be some operator whose continuity on the space X is in question. Suppose that individual estimates of the type $T : X_\eta \to X_\nu$ imply the same type of estimates for G. Then we can extrapolate $G : X \to X$ (cf. Section 5 and Section 6 below)

In spite of it's length the paper is just an invitation for readers to formulate a more general theory. For example, we do not consider vector valued inequalities, duality, etc. Hopefully this will be accomplished in the not too distant future, but after so many years of delay it is time for us to publish our results. On the other hand, in the tradition of this field, we present throughout the text concrete examples and applications including the connection to the theory of Beurling algebras.

The paper is organized as follows. In Section 2 we introduce the basic constructions or extrapolation methods, in Section 3 we discuss some of the basic examples including rearrangement invariant spaces and connections to the geometry of Banach spaces as well as Beurling algebras, the connection with the theory of Jawerth and Milman is discussed in 4, in Section 5 we introduce the idea of factorization as a method to construct and deconstruct inequalities, these concepts are then applied to prove extrapolation theorems in a very general setting in Section 6, the classical extrapolation theorems of Rubio de Francia are discussed in Section 7. Section 8 is devoted to Hardy type operators acting on rearrangement invariant spaces, we use these results in Section 9 to show an approach to weak interpolation theory, including Boyd's interpolation theorem, using extrapolation of weighted norm inequalities for Hardy operators.

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2. Intersections, Sums and Unions of Banach spaces.

2.1. Scales. It will be useful to establish some notation. Let E, F be a couple of normed spaces such that $E \subset F$. Let $c > 0$, we shall write $E \overset{c}{\subset} F$ if $\|x\|_F \leq c\,\|x\|_E$ $(x \in E)$.

Given E, F normed spaces, we let $\mathcal{L}(E, F)$ be the space of linear bounded operators $T : E \to F$, provided with its usual operator norm.

A **scale** is an indexed family of **Banach** spaces $\{X_\alpha\}_{\alpha \in A}$. We assume that the index set A has been (partially) ordered in the following way:

$$\alpha \preceq \beta \Leftrightarrow X_\beta \overset{1}{\subset} X_\alpha, \qquad (\alpha, \beta \in A). \tag{2.1}$$

We shall say that a **scale** is **incomparable** if

$$\alpha \preceq \beta \Leftrightarrow \alpha = \beta.$$

[1]During his soujourn to Madrid in 1985

A scale $\{X_\alpha\}_{\alpha\in A}$ is said to be **compatible** if there exists a Hausdorff topological vector space $\mathcal{U}$ such that each X_α is algebraically and topologically embedded in $\mathcal{U}$. A scale $\{X_\alpha\}_{\alpha\in A}$ is **strongly compatible** if there exists a Banach space $\widetilde{X}$ such that $X_\alpha \overset{1}{\subset} \widetilde{X}, \alpha \in A$.

Given a scale $\{X_\alpha\}_{\alpha\in A}$, the norm of each space X_α is usually denoted by $\|.\|_\alpha$.

2.2. Δ and $\bigcap$ Methods. Let $\{X_\alpha\}_{\alpha\in A}$ be a compatible scale. Let

$$\Delta\{X_\alpha\}_{\alpha\in A} = \left\{ x \in \bigcap_{\alpha\in A} X_\alpha : \|x\|_{\Delta\{X_\alpha\}_{\alpha\in A}} := \sup_{\alpha\in A} \|x\|_\alpha < \infty \right\}.$$

It is easy to see that $\left(\Delta\{X_\alpha\}_{\alpha\in A}, \|.\|_{\Delta\{X_\alpha\}_{\alpha\in A}}\right)$ is a Banach space with the following properties:

(a) $\Delta\{X_\alpha\}_{\alpha\in A} \overset{1}{\subset} X_\alpha$ for any $\alpha \in A$.

(b) If F is a Banach space such that $F \overset{1}{\subset} A_\alpha$ $(\forall \alpha \in A)$, then

$$F \overset{1}{\subset} \Delta\{X_\alpha\}_{\alpha\in A}.$$

A scale $\{X_\alpha\}_{\alpha\in A}$ is said to be $\Delta-$**total** if

$$\Delta\left\{X_\alpha\right\}_{\alpha\in A} = \bigcap_{\alpha\in A} X_\alpha$$

as linear spaces.

Remark 1. *It is easy to construct scales that are not $\Delta-$total. Consider the scale $\left\{L^p[0,1]\right\}_{1<p<\infty}$, then $\Delta\left\{L^p[0,1]\right\}_{1<p<\infty} = L^\infty[0,1] \subsetneq \bigcap_{1<p<\infty} L^p[0,1]$, since $\ln t \in \bigcap_{1<p<\infty} L^p[0,1]$.*

2.3. $\sum$ and $\bigcup$ Methods. Let $\{X_\alpha\}_{\alpha\in A}$ be a scale. Let

$$\bigcup\{X_\alpha\}_{\alpha\in A} = \{x : x \in X_\alpha \text{ for some } \alpha \in A\}.$$

There is a natural homogenous functional that can be defined on $\bigcup\{X_\alpha\}_{\alpha\in A}$:

$$\|x\|_{\cup\{X_\alpha\}_{\alpha\in A}} := \inf_{\alpha\in A} \|x\|_\alpha.$$

We will say that the scale $\{X_\alpha\}_{\alpha\in A}$ is $\bigcup-$**complete** if $\left(\bigcup\{X_\alpha\}_{\alpha\in A}, \|.\|_{\cup\{X_\alpha\}_{\alpha\in A}}\right)$ is a Banach space.

Remark 2. *It is easy to see that, in general, $\left(\bigcup\{X_\alpha\}_{\alpha\in A}, \|.\|_{\cup\{X_\alpha\}_{\alpha\in A}}\right)$ is not a Banach space, in fact it may not even have a linear structure. For example, for the scale $\left\{L^p(0,\infty)\right\}_{1\le p<\infty}$, the set $\bigcup\left\{L^p(0,\infty)\right\}_{1\le p<\infty}$ does not have a linear structure. To see this pick $f \geq 0$ such that $f \in L^2$ but $f \notin L^p$ if $p \neq 2$. Then f and $f^2 \in \bigcup\left\{L^p(0,\infty)\right\}_{1\le p<\infty}$ but $f + f^2 \notin \bigcup\left\{L^p(0,\infty)\right\}_{1\le p<\infty}$ since*

$$\left\|f+f^2\right\|_p \leq \|f\|_p + \left\|f^2\right\|_p \leq 2\left\|f+f^2\right\|_p.$$

Example 1. *The scale $\left\{L^p(0,1)\right\}_{1<p<\infty}$ is not $\bigcup-$**complete**, $\bigcup\left\{L^p(0,1)\right\}_{1<p<\infty}$ is a normed space but it is not complete. It is easy to establish the triangle inequality for $\bigcup\left\{L^p(0,1)\right\}_{1<p<\infty}$. Indeed, let $f_0, f_1 \in \bigcup\left\{L^p(0,1)\right\}_{1<p<\infty}$, and let $\varepsilon > 0$. Then* *there exist* $p_0, p_1 > 1$, $f_i^0 \in L^{p_i}, i = 0, 1$, *such that* $\|f_i\|_{p_i} \leq \|f_i\|_{\cup\{L^p(0,1)\}_{1<p<\infty}} + \varepsilon$. *Moreover, since* $f_0, f_1 \in L^{\min(p_0,p_1)}(0,1)$, *we have*

$$\begin{aligned}\|f_0 + f_1\|_{\cup\{L^p(0,1)\}_{1<p<\infty}} &\leq \|f_0 + f_1\|_{\min(p_0,p_1)} \\ &\leq \|f_0\|_{p_0} + \|f_1\|_{p_1} \\ &\leq \|f_0\|_{\cup\{L^p(0,1)\}_{1<p<\infty}} + \|f_1\|_{\cup\{L^p(0,1)\}_{1<p<\infty}} + 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ *we obtain the triangle inequality. To see that* $\bigcup\{L^p(0,1)\}_{1<p<\infty}$ *is not complete pick* $f \in L^1(0,1)$ *such that* $f \notin L^p(0,1)$ *for* $p > 1$. *We can write* $f = \sum_{i=1}^{\infty} f_i$ *(in* L^1*), with* $f_i \in L^{p_i}, p_i \downarrow 1$, *and* $\sum_{i=1}^{\infty} \|f_i\|_{p_i} < \infty$. *Let* $s_n = \sum_{i=1}^{n} f_i$. *Given* $\varepsilon > 0$, *select* $n < m$ *so large that* $\sum_{i=n}^{m} \|f_i\|_{p_i} < \varepsilon$, *and pick* $1 < p < p_m$. *Then*

$$\begin{aligned}\|s_m - s_n\|_{\cup\{L^p(0,1)\}_{1<p<\infty}} &\leq \|s_m - s_n\|_p \leq \sum_{i=n}^{m} \|f_i\|_p \\ &\leq \sum_{i=n}^{m} \|f_i\|_{p_i} < \varepsilon.\end{aligned}$$

But $f \notin \bigcup\{L^p(0,1)\}_{1<p<\infty}$.

Example 2. *Let* $0 < \alpha < 1$, $I = \left\{u \in L^{\frac{\alpha}{1-\alpha}} : u > 0, \|u\|_{L^{\frac{\alpha}{1-\alpha}}} = 1\right\}$. *Consider the scale* $\left\{L^1(u^{-1})\right\}_{u\in I}$. *It follows that* I *is ordered by*

$$u, v \in I, \quad u \preceq v \Leftrightarrow v \leq u \quad a.e.$$

By Hölder's inequality and its converse,

$$\|f\|_{L^\alpha} = \inf\left\{\int |f|\, u^{-1} : u > 0, \|u\|_{L^{\frac{\alpha}{1-\alpha}}} = 1\right\}. \tag{2.2}$$

Therefore,

$$L^\alpha = \bigcup\left\{L^1(u^{-1})\right\}_{u\in I}.$$

Thus, $\bigcup\left\{L^1(u^{-1})\right\}_{u\in I}$ *is not a normed space. The scale* $\left\{L^1(u^{-1}d\mu)\right\}_{u\in I}$ *is compatible since* $L^1(u^{-1}) \subset L^0$, *where* L^0 *is the space of all real, almost everywhere finite valued, Lebesgue−measurable functions. But* $\left\{L^1(u^{-1}d\mu)\right\}_{u\in I}$ *is not strongly compatible. In fact, if for some Banach space* X *we have* $L^1(u^{-1}) \overset{1}{\subset} X$ $\forall u \in I$, *then*

$$\|f\|_X \leq \inf_{u\in I} \|f\|_{L^1(u^{-1})} = \|f\|_{L^\alpha}.$$

Therefore $L^\alpha \overset{1}{\subset} X$, *which is not possible since otherwise* $(L^\alpha)' \neq \{0\}$.

Suppose that $\{X_\alpha\}_{\alpha\in A}$ is a strongly compatible scale. Let[2]

$$\sum\{X_\alpha\}_{\alpha\in A} = \left\{\sum_{\alpha\in A} x_\alpha;\ \text{absolutely in } \widetilde{X},\ x_\alpha \in X_\alpha \text{ for some } \alpha \in A\right\}.$$

We endow $\sum\{X_\alpha\}_{\alpha\in A}$ with the norm

$$\|x\|_{\sum\{X_\alpha\}_{\alpha\in A}} := \inf\left\{\sum_{\alpha\in A} \|x_\alpha\|_\alpha : x = \sum_{\alpha\in A} x_\alpha, \quad x_\alpha \in X_\alpha \text{ for some } \alpha \in A\right\}.$$

[2]We use the usual notation of unordered sums (cf. [**13**]).

Then $\left(\sum\{X_\alpha\}_{\alpha\in A}, \|.\|_{\sum\{X_\alpha\}_{\alpha\in A}}\right)$ is the smallest Banach space with the property

$$X_\beta \overset{1}{\subset} \sum\{X_\alpha\}_{\alpha\in A}\ (\beta\in A).$$

The next theorem gives necessary and sufficient conditions for a strongly compatible scale to be $\bigcup$−complete.

Theorem 1. *Let $\{X_\alpha\}_{\alpha\in A}$ be a strongly compatible scale, then*

(i) $\left(\bigcup\{X_\alpha\}_{\alpha\in A}, \|.\|_{\cup\{X_\alpha\}_{\alpha\in A}}\right)$ is a normed space if and only if $\forall\alpha,\beta\in A$, $X_\alpha+X_\beta \overset{1}{\subset} \bigcup\{X_\alpha\}_{\alpha\in A}$.

(ii) The following conditions are equivalent

a. $\left(\bigcup\{X_\alpha\}_{\alpha\in A}, \|.\|_{\cup\{X_\alpha\}_{\alpha\in A}}\right)$ is a Banach space.

b. $\bigcup\{X_\alpha\}_{\alpha\in A} = \sum\{X_\alpha\}_{\alpha\in A}$.

c. $\forall\,\{\alpha_n\}_{n\in N}\subset A$, $\sum\{X_{\alpha_n}\}_{n\in N}\subset\bigcup\{X_\alpha\}_{\alpha\in A}$.

Proof. (i) Suppose that $\left(\bigcup\{X_\alpha\}_{\alpha\in A}, \|.\|_{\cup\{X_\alpha\}_{\alpha\in A}}\right)$ is a normed space. Given $x\in X_\alpha+X_\beta$, $\varepsilon>0$, we can find a decomposition $x=x_0+x_1$ such that

$$\|x_0\|_\alpha + \|x_1\|_\beta \le \|x\|_{X_\alpha+X_\beta}+\varepsilon.$$

Moreover,

$$\begin{aligned}\|x\|_{\cup\{X_\alpha\}_{\alpha\in A}} &= \|x_0+x_1\|_{\cup\{X_\alpha\}_{\alpha\in A}}\\ &\le \|x_0\|_{\cup\{X_\alpha\}_{\alpha\in A}} + \|x_1\|_{\cup\{X_\alpha\}_{\alpha\in A}}\\ &\le \|x_0\|_\alpha + \|x_1\|_\beta .\end{aligned}$$

Combining these two inequalities and letting $\varepsilon\to0$ proves that $X_\alpha+X_\beta\overset{1}{\subset}\bigcup\{X_\alpha\}_{\alpha\in A}$. Assume now that $X_\alpha+X_\beta\overset{1}{\subset}\bigcup\{X_\alpha\}_{\alpha\in A}$. We shall verify that $\|.\|_{\cup\{X_\alpha\}_{\alpha\in A}}$ defines a norm on $\bigcup\{X_\alpha\}_{\alpha\in A}$. It is plain that

$$\inf_{\alpha\in A}\|\lambda x\|_\alpha = |\lambda|\inf_{\alpha\in A}\|x\|_\alpha .$$

Suppose that for a given $x\in\bigcup\{X_\alpha\}_{\alpha\in A}$ we have $\inf_{\alpha\in A}\|x\|_\alpha=0$. Then

$$\begin{aligned}0 &= \inf_{\alpha\in A}\|x\|_\alpha \ge \|x\|_{\sum\{X_\alpha\}_{\alpha\in A}}\\ &\Rightarrow x=0 \ \ (\text{since } \|.\|_{\sum\{X_\alpha\}_{\alpha\in A}} \text{ is a norm}).\end{aligned}$$

Finally, given $x,y\in\bigcup\{X_\alpha\}_{\alpha\in A}$, and $\varepsilon>0$, $\exists\alpha,\beta\in A$ such that $x\in X_\alpha$, $y\in X_\beta$, and $\|x\|_\alpha\le\|x\|_{\cup\{X_\alpha\}_{\alpha\in A}}+\varepsilon$, $\|y\|_\beta\le\|y\|_{\cup\{X_\alpha\}_{\alpha\in A}}+\varepsilon$. From $x+y\in X_\alpha+X_\beta\subset\bigcup\{X_\alpha\}_{\alpha\in A}$, it follows that

$$\begin{aligned}\|x+y\|_{\cup\{X_\alpha\}_{\alpha\in A}} &\le \|x+y\|_{X_\alpha+X_\beta}\le\|x\|_\alpha+\|y\|_\beta\\ &\le \|x\|_{\cup\{X_\alpha\}_{\alpha\in A}}+\|y\|_{\cup\{X_\alpha\}_{\alpha\in A}}+2\varepsilon.\end{aligned}$$

$ii)$ a→b) If $\left(\bigcup\{X_\alpha\}_{\alpha\in A}, \|.\|_{\cup\{X_\alpha\}_{\alpha\in A}}\right)$ is a Banach space, then the inclusion $\sum\{X_\alpha\}_{\alpha\in A}\overset{1}{\subset}\bigcup\{X_\alpha\}_{\alpha\in A}$ follows directly from the fact $\sum\{X_\alpha\}_{\alpha\in A}$ is the smallest Banach space with the property $X_\alpha\overset{1}{\subset}\bigcup\{X_\alpha\}_{\alpha\in A}$. Conversely, given

$x \in \bigcup\{X_\alpha\}_{\alpha\in A}$, and $\varepsilon > 0$, there exists $\alpha \in A$ such that $x \in X_\alpha$, and $\|x\|_\alpha \leq \|x\|_{\cup\{X_\alpha\}_{\alpha\in A}} + \varepsilon$. It follows that

$$\|x\|_{\sum\{X_\alpha\}_{\alpha\in A}} \leq \|x\|_\alpha \leq \|x\|_{\cup\{X_\alpha\}_{\alpha\in A}} + \varepsilon.$$

We conclude letting $\varepsilon \to 0$.

b→c) trivial.

c→a) By part (i) of this theorem $\bigcup\{X_\alpha\}_{\alpha\in A}$ is a normed space. We prove the completeness. Suppose that $\sum \|x_n\|_{\cup\{X_\alpha\}_{\alpha\in A}} < \infty$, and let $\varepsilon > 0$. There exist $\alpha_n \in A$ such that

$$\|x_n\|_{\alpha_n} \leq \|x_n\|_{\cup\{X_\alpha\}_{\alpha\in A}} + \varepsilon/2^n,\ n = 1, ...$$

Therefore

$$\|x_n\|_{\sum\{X_{\alpha_n}\}_{n\in N}} \leq \|x_n\|_{\alpha_n} \leq \|x_n\|_{\cup\{X_\alpha\}_{\alpha\in A}} + \varepsilon/2^n,$$

$$\sum_n \|x_n\|_{\sum\{X_{\alpha_n}\}_{n\in N}} < \infty.$$

By the completeness of $\sum\{X_{\alpha_n}\}_{n\in N}$ there exists $x \in \sum\{X_{\alpha_n}\}_{n\in N}$ such that $x = \sum_n x_n$ (in $\sum\{X_{\alpha_n}\}_{n\in N}$ sense). Since by hypothesis,

$$\sum\{X_{\alpha_n}\}_{n\in N} \overset{1}{\subset} \cup\{X_\alpha\}_{\alpha\in A}.$$

It follows that

$$x = \sum_n x_n \text{ (in } \bigcup\{X_\alpha\}_{\alpha\in A}).$$

□

Remark 3. *It is plain that the condition (i) in the previous Theorem is equivalent to:* $\forall\alpha_1, \cdots, \alpha_n \in A$, $X_{\alpha_1} + \cdots + X_{\alpha_n} \subset \cup\{X_\alpha\}_{\alpha\in A}$.

Remark 4. *Combining (i) and (ii) in the previous theorem we see that if A is finite then $\bigcup\{X_\alpha\}_{\alpha\in A}$ is a normed space if and only if $\bigcup\{X_\alpha\}_{\alpha\in A}$ is a Banach space.*

The next corollary follows immediately from Theorem 1:

Corollary 1. *Let $\{X_\alpha\}_{\alpha\in A}$ be a strongly compatible scale. Then*

(i) Suppose that for all $\alpha_0, \alpha_1 \in A$, $\exists\alpha \in A$ such that $\alpha_i \preceq \alpha$. Then $\bigcup\{X_\alpha\}_{\alpha\in A}$ is a normed space.

(ii) Suppose that $\forall\{\alpha_n\}_{n\in N} \subset A$, $\exists\alpha \in A$ such that $\alpha_n \preceq \alpha$. Then $\bigcup\{X_\alpha\}_{\alpha\in A}$ is a Banach space.

Remark 5. *The converse to the Theorem 1 is in general not true. For example, we claim that the scale $\left\{\frac{p}{p-1}L^p[0,1]\right\}_{1<p\leq 2}$ is strongly compatible and condition (i) of Theorem 1 is satisfied but $\left\{\frac{p}{p-1}L^p[0,1]\right\}_{1<p\leq 2}$ is incomparable. To prove this*

claim we first note that by Hardy's inequality,

$$\begin{aligned}\|f\|_{L\log L} &= \int_0^1 f^*(s)\log\frac{1}{s}ds = \int_0^1 f^{**}(s)ds \\ &\leq \left(\int_0^1 f^{**}(s)^p ds\right)^{\frac{1}{p}} \\ &\leq \frac{p}{p-1}\|f\|_{L^p}.\end{aligned}$$

Therefore

$$\frac{p}{p-1}L^p[0,1] \overset{1}{\subset} L\log L.$$

Thus $\left\{\frac{p}{p-1}L^p[0,1]\right\}_{1<p\leq 2}$ *is strongly compatible. To show that* $\left\{\frac{p}{p-1}L^p[0,1]\right\}_{1<p\leq 2}$ *is incomparable let us suppose that* $p \leq q$ *and, moreover, that*

$$\frac{q}{q-1}L^q[0,1] \overset{1}{\subset} \frac{p}{p-1}L^p[0,1]. \tag{2.3}$$

Applying the norm inequality implied by (2.3) to the function $\chi_{[0,1]}$ *gives* $\frac{p}{p-1} \leq \frac{q}{q-1}$. *Therefore we get* $q \leq p$. *Finally, we prove condition (i): if* $f = f_0 + f_1$ *with* $f_0 \in \frac{q}{q-1}L^q[0,1], f_1 \in \frac{p}{p-1}L^p[0,1]$, *then*

$$\begin{aligned}\|f\|_{L\log L} &= \int_0^1 (f_0+f_1)^*(s)\log\frac{1}{s}ds = \int_0^1 (f_0+f_1)^{**}(s)ds \\ &\leq \int_0^1 f_0^{**}(s)ds + \int_0^1 f_1^{**}(s)ds \\ &\leq \frac{q}{q-1}\|f_0\|_{L^q} + \frac{p}{p-1}\|f_1\|_{L^p}.\end{aligned}$$

This shows that

$$\frac{q}{q-1}L^q[0,1] + \frac{p}{p-1}L^p[0,1] \overset{1}{\subset} L\log L.$$

We shall now prove that $\sum\{X_\alpha\}_{\alpha\in A}$ is in a suitable sense the completition of $\bigcup\{X_\alpha\}_{\alpha\in A}$.

THEOREM 2. *Let* $\{X_\alpha\}_{\alpha\in A}$ *be a strongly compatible scale and let*

$$Y = \left\{\sum\nolimits' x_i : x_i \in \bigcup\{X_\alpha\}_{\alpha\in A}\right\}$$

(here the sum with a dash ($\sum'$) *is used to indicate that we only consider finite sums). Equip* Y *with*

$$\|x\|_Y = \inf\sum\nolimits' \|x_i\|_{\cup\{X_\alpha\}_{\alpha\in A}}.$$

Then $(Y, \|.\|)_Y$ *is a normed space, and for any normed space* Z *such that* $\bigcup\{X_\alpha\}_{\alpha\in A} \overset{1}{\subset} Z$ *we have*

$$Y \overset{1}{\subset} Z.$$

Furthermore, if $\widehat{Y}$ *denotes the completition of* Y *then*

$$\widehat{Y} = \sum\{X_\alpha\}_{\alpha\in A}.$$

Proof. By Theorem 1-(i), it follows readily that $\|.\|_Y$ defines a norm on Y. Let Z be a normed space such that

$$\bigcup\{X_\alpha\}_{\alpha\in A} \overset{1}{\subset} Z.$$

Then given $y\in Y$, there exist $\alpha_1,\cdots,\alpha_n\in A$ such that $y=\sum_1^n y_{\alpha_i}$, and

$$\|y\|_Z \leq \sum_1^n \|y_{\alpha_i}\|_Z \leq \sum_1^n \|y_{\alpha_i}\|_{\alpha_i} \leq (1+\varepsilon)\,\|y\|_Y\,.$$

On the other hand, since $X_\alpha \overset{1}{\subset} Y \overset{1}{\subset} \widehat{Y}$, we have

$$\sum\{X_\alpha\}_{\alpha\in A} \overset{1}{\subset} \widehat{Y}.$$

Conversely, since $Y \overset{1}{\subset} \sum\{X_\alpha\}_{\alpha\in A}$ and $\sum\{X_\alpha\}_{\alpha\in A}$ is a Banach space it follows that

$$\widehat{Y} \overset{1}{\subset} \sum\{X_\alpha\}_{\alpha\in A}.$$

□

Corollary 2. *Let $\{X_\alpha\}_{\alpha\in A}$ be a strongly compatible scale such that $\bigcup\{X_\alpha\}_{\alpha\in A}$ is a normed space. Let Z be a subspace which is dense in each $X_\alpha,\alpha\in A$. Then, for all $x\in Z$, we have*

$$\|x\|_{\sum\{X_\alpha\}_{\alpha\in A}} = \inf_{\alpha\in A}\ \|x\|_\alpha\,.$$

3. Examples

For future reference and, moreover, in order to give the reader a better idea of the scope of the theory we are developing in this paper, we now illustrate with concrete examples how the constructions given in the previous section are connected with familiar mathematical objects studied in functional and harmonic analysis.

3.0.1. ***Banach Lattices***. Our first example deals with $\sum-\bigcup$ constructions in the setting of lattices. Our basic reference here is [**15**].

Let (Ω,μ) be a $\sigma-$finite measure space. Let $L^0=L^0(\Omega,\mu)$ denote the space of all real $\mu-$almost everywhere finite valued $\mu-$measurable functions on Ω, with the usual identification of $\mu-$almost equal functions. We shall say that a linear subspace $X=X(\Omega)\subset L^0$ is a **Banach lattice** if the following properties are satisfied

(3.1)

1) If $|x|\leq|y|\,,\ x\in L^0, y\in X \Rightarrow x\in Y$ and $\|x\|_X\leq\|y\|_X$ *(Lattice property)*
2) $0\leq x_n\uparrow x\ \ \mu-a.e. \Rightarrow \|x_n\|_X\leq\|x\|_X$ *(Fatou property)*
3) $\exists x\in X$ such that $x(\omega)>0$ a.e. $\omega\in\Omega$.
4) For all $A\subset\Omega$ s.t. $\mu(A)<\infty\Rightarrow\chi_A\in X$.

The associate space X' is the Banach lattice defined by

$$X'=\left\{w\in L^0:\|w\|_{X'}=\sup_{\left\{x\in X:\|x\|_X\leq1\right\}}\int_\Omega|xw|\,d\mu<\infty\right\}.$$

It is well known that $X''=X$, in particular

$$\|x\|_X=\sup\left\{\int_\Omega|xw|\,d\mu : w\in X', \|w\|_{X'}\leq1\right\}. \tag{3.2}$$

We summarize our findings using the language of scales:

LEMMA 1. *Let* $I = \{w \in X' : w \geq 0, \|w\|_{X'} \leq 1\}$, *and consider the scale* $\left\{L^1(w)\right\}_{w\in I}$. *Then*

(i) $X = \Delta \left\{L^1(w)\right\}_{w\in I}$.

(ii) $\left\{L^1(w)\right\}_{w\in I}$ *is a* $\Delta-$*total scale.*

(iii) The order in I *is given by* $w \preceq u \Leftrightarrow w \leq u$, $\mu-$*a. e.*

PROOF. (i) Follows from (3.2).

(ii) From $X \overset{1}{\subset} L^1(w)$, $\forall w \in I$, we get $X \subset \bigcap_{w\in I} L^1(w)$. Conversely,

$$\begin{aligned} x \in \bigcap_{w\in I} L^1(w) &\Rightarrow x \in L^1(w), \ \forall w \in I \\ &\Rightarrow |x|\, w \in L^1, \ \forall w \in I \\ &\Rightarrow x \in X'' = X. \end{aligned}$$

(iii) Remark that given $w, u \in I$ such that $w \preceq u$, then for all measurable sets A we have

$$\int_A w \leq \int_A u.$$

This implies $w \leq u$. Since the converse is trivial we have shown that $w \preceq u \Leftrightarrow w \leq u$, $\mu-$a.e. □

3.1. $p-$Convex and $q-$Concave spaces. Our basic references in what follows are [**23**] and [**15**]. In this section given a Banach lattice X we let

$$X^+ := \{w \in X : w > 0 \ \mu - \text{a.e.}\}.$$

3.1.1. *$p-$Convex spaces.* Let $p \geq 1$. Recall that a Banach lattice X is said to be $p-$convex if there exists a positive constant $C > 0$ such that $\forall x_1, ..., x_n \in X$ we have

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_X \leq C \left(\sum_{i=1}^n \|x_i\|_X^p \right)^{1/p}.$$

Since the constant C will play no role in what follows, we assume without loss of generality that $C = 1$.

Let X be a Banach lattice and let $p \geq 1$. Recall the following well known constructions

$$\begin{aligned} X^p &= \{x : |x|^{1/p} \in X\}, \\ \|x\|_{X^p} &= \left\| |x|^{1/p} \right\|_X^p. \end{aligned}$$

Note that

$$X \text{ is } p - \text{convex} \Leftrightarrow X^p \text{ is a Banach lattice.}$$

Moreover, the associate space $(X^p)'$ is also a Banach lattice.

EXAMPLE 3. *If* $X = L^p(w)$ *with* $p > 2$, *then* X *is* $2-$*convex and*

$$\left(X^2\right)' = L^{p^*}(w^{1-p^*}), \text{ where } p^* = \frac{p}{p-2}.$$

The import of these concepts for our purposes here stems from the following elementary fact:

Theorem 3. *Let $1 \le p < \infty$, and X be a $p-$convex space, let $I = \left\{ w \in \left((X^p)' \right)^+, \|w\|_{(X^p)'} \le 1 \right\}$. Then*

(i) $\{L^p(w)\}_{w\in I}$ is a $\Delta-$total scale such that $w \preceq u \Leftrightarrow w \le u, \mu - a.e.$

(ii) $X = \Delta\{L^p(w)\}_{w\in I}$.

Proof. (i) Since $X \subset L^p(w)$, $\forall w \in I$, it follows that $X \subset \bigcap_{w\in I} L^p(w)$. Conversely, if $y \in L^p(w)$ then $\forall w \in \left((X^p)' \right)^+$ we have $|y|^p w \in L^1$ and therefore $|y|^p \in (X^p)'' = X^p$, which implies that $y \in X$. Finally, the fact that $w \preceq u \Leftrightarrow w \le u$, can be proved as in Lemma 1.

(ii) Let $x \in X$, then $|x|^p \in X^p$, and

$$\|x\|_X^p = \| |x|^p \|_{X^p}.$$

Therefore,

$$\|x\|_X^p = \sup_{\left\{ w\in((X^p)')^+ : \|w\|_{(X^p)'} \le 1 \right\}} \int_\Omega |x|^p \, w d\mu.$$

$\square$

3.1.2. *$q-$Concave spaces.* Let $q \ge 1$. A Banach lattice X is said to be $q-$concave if there exists a positive constant $C > 0$ such that $\forall x_1, ..., x_n \in X$ we have

$$\left(\sum_{i=1}^n \|x_i\|_X^q \right)^{1/q} \le C \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{1/q} \right\|_X.$$

Since the constant C will play no role in what follows, we assume without loss of generality that $C = 1$.

Let $1 < p, q < \infty$, $1/p + 1/q = 1$. Let $\widehat{X}_q$ be defined by

$$\widehat{X}_q = \left\{ x \in L^0 : |x|^{1/p} |g|^{1/q} \in X, \ \forall g \in L^1 \right\}$$

endowed with the norm

$$\|x\|_{\widehat{X}_q} = \sup_{\|g\|_{L^1} \le 1} \left\| |x|^{1/p} |g|^{1/q} \right\|_X.$$

The next result follows easily from the definitions, but we provide the proof for the sake of completeness.

Theorem 4. *Let $1 < q \le \infty$, and let X be a $q-$concave space, then $\left(\widehat{X}_q, \|x\|_{\widehat{X}_q} \right)$ is a Banach lattice such that*

$$\left(\widehat{X}_q \right)' = (X')^p.$$

(Notice that $\widehat{X}_\infty = X$).

Proof. Since

$$X \text{ is } q-\text{concave} \Leftrightarrow X' \text{ is } p-\text{convex},$$

it follows that $(X')^p$ and $Z = \left((X')^p \right)'$ are both Banach lattices. Moreover,

$$Z' = \left((X')^p \right)'' = (X')^p.$$

Therefore we only need to prove that $Z = \widehat{X}_q$. But

$$\begin{aligned} x \in \widehat{X}_q &\Leftrightarrow |x|^{1/p} |g|^{1/q} \in X,\ \forall g \in L^1 \\ &\Leftrightarrow |x|^{1/p} g \in X,\ \forall g \in L^q \\ &\Leftrightarrow |x|^{1/p} gy \in L^1,\ \forall y \in X',\ \forall g \in L^q \\ &\Leftrightarrow |x|^{1/p} y \in L^p,\ \forall y \in X' \\ &\Leftrightarrow x |y|^p \in L^1,\ \forall y \in X' \\ &\Leftrightarrow x \in \left((X')^p\right)' = Z. \end{aligned}$$

The equality $\|.\|_{\widehat{X}_q.} = \|.\|_Z$ follows by taking supremum over all g, y, such that $\|g\|_{L^1} \leq 1$ and $\|y\|_{X'} \leq 1$. □

Example 4. *If $X = L^p(w)$ with $1 < p < 2$, then X is $2-$concave and*

$$\widehat{X}_2 = L^{p^*}(w^{1+p^*}),\ \textit{where } p^* = \frac{p}{2-p}.$$

If X is $q-$concave then X' is $p-$convex, therefore combining Theorem 3 and Theorem 4, we have

$$\begin{aligned} \|x\|_{X'} &= \sup_{\left\{w \in (((X')^p)')^+ : \|w\|_{((X')^p)'} \leq 1\right\}} \left(\int_\Omega |x|^p\, w d\mu\right)^{\frac{1}{p}} \\ &= \sup_{\left\{w \in \left(\widehat{X}_q\right)^+ : \|w\|_{(\widehat{X}_q)^+} \leq 1\right\}} \left(\int_\Omega |x|^p\, w d\mu\right)^{\frac{1}{p}}. \end{aligned}$$

By duality we have

i) if $\|w\|_{\widehat{X}_q} = 1 \ \Rightarrow\ L^q(w^{1-q}) \overset{1}{\subset} X,$

ii) $\forall x \in X,\ \exists w \in \left(\widehat{X}_q\right)^+$ with $\|w\|_{\widehat{X}q} = 1$ such that $\|x\|_X = \|x\|_{L^q(w^{1-q})}$.

To see the second claim take $y \in (X')^+$ with $\|y\|_{X'} = 1$ and such that $\|x\|_X = \int xyd\mu$, then let u be defined by $x = \|x\|_X\, y^{\frac{1}{q-1}} u$.

We have thus proved the following

Theorem 5. *Let X be a $q-$concave space, let $I = \left\{w \in \left(\widehat{X}_q\right)^+, \|w\|_{\widehat{X}_q} \leq 1\right\}$. Then*

(i) $\left\{L^q(w^{1-q})\right\}_{w \in I}$ is a $\bigcup-$complete scale, such that $w \preceq u \Leftrightarrow u \leq w$, $\mu-a.e.$

(ii) $X = \bigcup \left\{L^q(w^{1-q})\right\}_{w \in I}$.

Remark 6. *Note that in this case if $w \preceq u$ then for all measurable sets A*

$$\int_A w^{1-q} \leq \int_A u^{1-q} \Leftrightarrow w^{1-q} \leq u^{1-q} \Leftrightarrow u \leq w.$$

3.2. Beurling Spaces. Our basic reference for this section is [**5**].

Let Π be a locally compact Abelian group with Haar invariant measure dx. Let Ω be a subcone of the cone of strictly positive functions w on Π which are measurable, and summable respect to dx. Consider a norm $N(w)$ on Ω such that

$$0 < \int w dx \leq N(w) < \infty, \text{ for all } w \in \Omega, \tag{3.3}$$

and, moreover, such that Ω is complete in the following sense: for any sequence $\{w_n\}_n \subset \Omega$ such that $\sum_n N(w_n) < \infty$ it follows that $w = \sum_n w_n \in \Omega$, and $N(w) \leq \sum_n N(w_n)$. Let $\Omega_0 = \{w : w \in \Omega \text{ and } N(w) = 1\}$.

Let $1 < p, q < \infty$, $1/p + 1/q = 1$, Beurling's spaces are defined by

$$A^p = \left\{ F \in \bigcup_{w \in \Omega_0} L^p(w^{1-p}) : \|F\|_{A^p} = \inf_{w \in \Omega_0} \|F\|_{L^p(w^{1-p})} < \infty \right\},$$

$$B^q = \left\{ G \in \bigcap_{w \in \Omega_0} L^q(w) : \|G\|_{B^p} = \sup_{w \in \Omega_0} \|G\|_{L^q(w)} < \infty \right\},$$

where $\|f\|_{L^r(w)} = \left(\int |f|^r w\right)^{1/r}$. It follows readily from the definitions that B^q is a Banach space. The same is true of $(A^p, \|.\|_{A^p})$, but, as it often happens with $\bigcup$ constructions, the proof that A^p is a Banach space requires more effort. Indeed, Beurling's approach (cf. [**5**]-Theorem 1) to this problem is to find a new expression for $\|\cdot\|_{A^p}$:

$$\|F\|_{A^p} = \inf_{w \in \Omega} W(F, w)$$

where

$$W(F, w) = \frac{1}{p} \|F\|^p_{L^p(w^{1-p})} + \frac{1}{q} N(w).$$

This allows Beurling to see that $(A^p, \|.\|_{A^p})$ is a Banach space and in fact Beurling proves that

$$(B^q)' = A^p.$$

REMARK 7. *In our context Beurling's results can be restated as follows: the scale* $\{L^q(w)\}_{w \in \Omega_0}$ *is* $\Delta-$*total, and the scale* $L^p(w^{1-p})_{w \in \Omega_0}$ *is* $\bigcup -$*complete.*

REMARK 8. *If we assume that* Ω *is closed under convolution, and the norm* N *satisfies*

$$N(w_1 * w_2) \leq N(w_1) N(w_2), w_1, w_2 \in \Omega,$$

then A^p *is a Banach algebra under addition and convolution.*

REMARK 9. *Although in our discussion we assumed for convenience that* $1 < p < \infty$, *it is also possible to consider in a similar fashion the limiting cases* $p = 1$ *and* $p = \infty$ *(cf.* [**5**]*).*

3.3. Rearrangement invariant Banach Lattices. We return to the study of Banach lattices (cf. Section 3.0.1 above) but here we assume additionally that our spaces are rearrangement invariant. Our basic references for this section are [**4**] and [**14**].

A Banach lattice X over $(\mathbf{R}^+, dx)$ (dx =Lebesgue measure) will be called rearrangement invariant if the following property is satisfied:

$$f \in X \Leftrightarrow f^* \in X \text{ and } \|f\|_X = \|f^*\|_X .$$

(Here $f^*(t) = \inf\ \{\lambda > 0 : m\{x \in \mathbf{R}^+ : |f(x)| > \lambda\} \leq t\}$ is the so-called **non-increasing rearrangement** of f).

We let X^d denote the cone of all non-negative and non-increasing functions (briefly decreasing functions) of X.

The associate space X' is given by

$$X' = \left\{ f \in L_0 : \|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int_0^\infty f^*(x) g^*(x) dx < \infty \right\}. \tag{3.4}$$

X' is also a rearrangement invariant Banach lattice and

$$X'' = X. \tag{3.5}$$

Given a decreasing function w, the **Lorentz space** associated to w is the rearrangement invariant Banach lattice defined by (cf. [**17**]):

$$\Lambda(w) = \left\{ f \in L_0 : \|f\|_{\Lambda(w)} = \int_0^\infty f^*(x) w(x) dx < \infty \right\}.$$

It follows readily from (3.4) that

Lemma 2. *Let X be a rearrangement invariant Banach lattice, and let $I = \left\{ w \in (X')^d : \|w\|_{X'} \leq 1 \right\}$. Then $\{\Lambda(w)\}_{w \in I}$ is a $\Delta-$total scale, $X = \Delta\{\Lambda(w)\}_{w \in I}$, and $w \preceq u \Leftrightarrow w \prec u$, where $\prec$ is the Hardy-Littlewood order (i.e. $w \prec u \Leftrightarrow \int_0^r w \leq \int_0^r u$, $\forall r > 0$).*

Proof. Let $w, u \in I$, then $w \preceq u$ implies that

$$\int_0^r w = \left\| \chi_{[0,r)} \right\|_{\Lambda(w)} \leq \left\| \chi_{[0,r)} \right\|_{\Lambda(u)} = \int_0^r u, \forall r > 0. \tag{3.6}$$

But since w, u are decreasing (3.6) is equivalent to $w \prec u$ in the Hardy-Littlewod order.

To prove that $\{\Lambda(w)\}_{w \in I}$ is a $\Delta-$total scale note that

$$X \subset \Lambda(w) \ \ \forall w \in I \Rightarrow X \subset \bigcap_{w \in I} \Lambda(w).$$

Conversely, suppose that $y \in \bigcap_{w \in I} \Lambda(w)$, then $y \in \Lambda(w)$ $\forall w \in (X')^d \Leftrightarrow y^* \in \Lambda(w)$ $\forall w \in (X')^d$. This means that

$$y^* w \in L^1 \ \forall w \in (X')^d$$

thus

$$y^* \in X'' = X.$$

Therefore, since X is a rearrangement invariant Banach lattice, we have $y \in X$. $\square$

Let w be a decreasing function. The **Marcinkiewicz space** $M(w)$ associated to w is the rearrangement invariant Banach lattice defined by:

$$M(w) = \left\{ f \in L_0 : \|f\|_{M(w)} = \sup_{x > 0} \frac{\int_0^x f^*(t) dt}{\int_0^x w(t) dt} < \infty \right\}.$$

In particular, given $f \in X$ we can consider the Marcinkiewicz space $M(f^*)$. Then

$$\|f\|_{M(f^*)} = \sup_{x>0} \frac{\int_0^x f^*(t)dt}{\int_0^x f^*(t)dt} = 1.$$

Consequently $f \in M(f^*)$. Moreover, if $g \in M(f^*)$ then

$$\int_0^x g^*(t)dt \leq \|g\|_{M(f^*)} \int_0^x f^*(t)dt. \tag{3.7}$$

Therefore (cf. [**9**]) $g \in X$. Thus, $M(f^*) \overset{1}{\subset} X$ for all $f \in X$. But from (3.7) we have

$$\|g\|_X \leq \|g\|_{M(f^*)} \|f\|_X .$$

If we set $f = g/ \|g\|_{M(f^*)}$ the inequality above becomes an equality. Therefore we have

$$\|g\|_X = \inf_{\left\{f \in X: \|f\|_X = 1\right\}} \|g\|_{M(f^*)} .$$

Hence we have proved that

Theorem 6. *Let X be a rearrangement invariant Banach lattice, and let $I = \left\{w \in X^d, \|w\|_X \leq 1\right\}$. Then,*

$$X = \bigcup \{M(w)\}_{w \in I} .$$

$\{M(w)\}_{w\in I}$ is a $\bigcup-$complete scale such that $w \preceq u \Leftrightarrow u \prec w$ (Hardy-Littlewood order).

4. The Jawerth-Milman Theory

In the extrapolation theory developed in [**13**] and [**19**] and the references therein one also finds two basic functors Δ and $\sum$. In this section we show how our setting unifies the theory of Jawerth-Milman with the theory of Rubio de Francia.

Our basic references in this section are [**13**] and [**19**].

We need the following technical Lemma (cf. [**16**] Example 23.3 (iv)), whose proof we include for the sake of completeness.

Lemma 3. *Let (Ω, μ) be a finite measure space. Let $\{f_\alpha\}_{\alpha\in A} \subset L^1(\Omega,\mu)$ be a family of non-negative functions. The following statements hold:*

(i) If for each $\alpha_0, \alpha_1 \in A$, $\exists \beta \in A$ such that $f_{\alpha_0} \leq f_\beta$ and $f_{\alpha_1} \leq f_\beta$ ($\mu-$a.e.). Then

$$\sup_{\alpha\in A} \int f_\alpha d\mu = \int \sup_{\alpha \in A} f_\alpha d\mu.$$

(ii) If for each $\alpha_0, \alpha_1 \in A$, $\exists \beta \in A$ such that $f_{\alpha_0} \geq f_\beta$ and $f_{\alpha_1} \geq f_\beta$ ($\mu-$a.e.). Then

$$\inf_{\alpha\in A} \int f_\alpha d\mu = \int \inf_{\alpha \in A} f_\alpha d\mu.$$

Proof. (i) First assume that the functions f_α are uniformly bounded; $0 \leq f_\alpha \leq M$, $\mu-$a.e. on Ω. Then the set of numbers

$$\left\{\int f_\alpha d\mu : \alpha \in A\right\}$$

is bounded by $M\mu(\Omega)$ and therefore $P = \sup_{\alpha\in A} \int f_\alpha d\mu$ is a finite number. Pick an increasing sequence $\{f_{\alpha_n}\}_{n\in N}$ such that

$$\int f_{\alpha_n} d\mu \uparrow P \quad \text{as } n \to \infty.$$

The pointwise supremum $f_0(x) = \sup f_{\alpha_n}(x)$ is a $\mu-$measurable function such that $\int f_0 d\mu = P$, moreover every f_α satisfies $f_\alpha(x) \le f_0(x)$ $\mu-$a.e. This shows that $f_0 = \sup_{\alpha\in A} f_\alpha$. If the $f_\alpha' s$ are not necessarily bounded on Ω, then for $n = 1, 2. \dots$ we consider the functions $f_{\alpha,n} = \inf(f_\alpha, n)$. Then $f_n = \sup\{f_{\alpha,n} : \alpha \in A\}$ exists in $L^0(\Omega, \mu)$, and it follows readily that

$$f_0 = \sup\{f_{\alpha,n} : \alpha \in A,\ n = 1, 2. \dots\} = \sup_{\alpha\in A} f_\alpha.$$

(ii) If $f_{\alpha_0} \in \{f_\alpha\}_{\alpha\in A}$, then the set $\{f_\alpha : f_\alpha \le f_{\alpha_0}\}$ has the same lower bounds as the set $\{f_\alpha\}_{\alpha\in A}$. Therefore it is sufficient to prove that $\{f_\alpha : f_\alpha \le f_{\alpha_0}\}$ has an infimum. To this end note that the set

$$\{f_{\alpha_0} - f_\alpha : f_\alpha \le f_{\alpha_0}\}$$

satisfies $0 \le f_{\alpha_0} - f_\alpha \le f_{\alpha_0}$ and therefore $w = \sup\{f_{\alpha_0} - f_\alpha\}$ exists by part (i) of the theorem. But then we have

$$f_{\alpha_0} - w = \inf_{\alpha\in A} f_\alpha.$$

□

Remark 10. *Note that the supremum (resp. the infimum) of the set $\{f_\alpha\}_{\alpha\in A}$ is obtained as the supremum (resp. the infimum) of an appropriate countable subset. Note also that the lemma above can be immediately extended to the case that (Ω, μ) is $\sigma-$finite.*

Consider the following example

Example 5. *(Yano's Extrapolation Theorem (cf. [**25**])). Let $T : L^p[0,1] \to L^p[0,1]$ be a bounded linear operator such that $\|T\|_{\mathcal{L}(L^p[0,1],L^p[0,1])} \le \frac{c}{p-1}$, $p > 1$. Then*

$$T : LLogL[0,1] \to L^1[0,1] \textit{ is bounded.}$$

Proof. We consider the following strongly compatible scales $\left\{\frac{1}{p-1}L^p[0,1]\right\}_{p>1}$ and $\{L^p[0,1]\}_{p>1}$. By Theorem 1-1 $\bigcup\left\{\frac{1}{p-1}L^p[0,1]\right\}_{p>1}$, and $\bigcup\{L^p[0,1]\}_{p>1}$ are normed spaces. By Corollary 2, if $f \in \bigcap\{L^p[0,1]\}_{p>1}$,

$$\|f\|_{\sum\left\{\frac{1}{p-1}L^p[0,1]\right\}_{p>1}} = \inf_{p>1} \frac{1}{p-1}\|f\|_p.$$

Now, since $L^{p,1}[0,1] \overset{p^{1-1/p}}{\subset} L^p[0,1]$, we have

$$\inf_{p>1} \frac{1}{p-1}\|f\|_p \le \inf_{p>1} \frac{p^{1-1/p}}{p-1}\|f\|_{p,1}.$$

Thus

$$\frac{p^{1-1/p}}{p-1}\|f\|_{p,1} = \frac{p^{1-1/p}}{p-1}\int_0^1 t^{1/p-1} f^*(t)dt \le \frac{p}{p-1}\int_0^1 t^{1/p-1} f^*(t)dt.$$

But the family $\left\{\frac{pt^{1/p-1}}{p-1};(0<t<1)\right\}_{p>1}$ satisfies the hypothesis of Lemma 3, hence

$$\inf_{p>1}\int_0^1 \frac{pt^{1/p-1}}{p-1}f^*(t)dt=\int_0^1 \inf_{p>1}\frac{pt^{1/p-1}}{p-1}f^*(t)dt.$$

On the other hand, since $0<t<1$,

$$\inf_{p>1}\frac{p}{p-1}t^{1/p-1}=\inf_{0<\theta<1}\frac{1}{\theta t^{\theta}}=\log\frac{1}{t},$$

(where $\theta=1-1/p$). Hence

$$\inf_{p>1}\frac{p}{p-1}\left\|f\right\|_p\le\int_0^1 f^*(t)\log\frac{1}{t}dt.$$

Let P be Hardy's operator (cf. 7.3) then it is well-known that

$$\left\|Pf\right\|_p\le\frac{p}{p-1}\left\|f\right\|_p.$$

Now

$$\left\|Pf^*\right\|_1=\inf_{p>1}\left\|Pf^*\right\|_p\le\int_0^1 f^*(t)\log\frac{1}{t}dt,$$

and by Fubini

$$\left\|Pf^*\right\|_1=\int_0^1\frac{1}{t}\int_0^t f(s)dsdt=\int_0^1 f^*(t)\log\frac{1}{t}dt.$$

So we have proved that if $f\in\bigcap\left\{L^p[0,1]\right\}_{p>1}$ then

$$\left\|f\right\|_{\sum\left\{\frac{1}{p-1}L^p[0,1]\right\}_{p>1}}=\int_0^1 f^*(t)\log\frac{1}{t}dt=\left\|f\right\|_{L\log L}.$$

Now using that $\bigcap\left\{L^p[0,1]\right\}_{p>1}$ is dense in each $L^p[0,1]$, we conclude that

$$T:LLogL[0,1]\to L^1[0,1]\text{ is bounded.}$$

□

Let $\overline{A}$ be a Banach pair, and let ρ be a quasi-concave function, the space $\overline{A}_{\rho,1,J}$ consists of all $a\in\sum(\overline{A})$ such that

$$\left\|a\right\|_{\overline{A}_{\rho,1,J}}=\inf_u\int_0^\infty\frac{J(t,u(t);\overline{A})}{\rho(t)}\frac{dt}{t}<\infty,$$

where the infimum is taken over all representations $a=\int_0^\infty u(t)\frac{dt}{t}$ (with convergence in $\sum(\overline{A})$, $u(t):(0,\infty)\to\Delta(\overline{A})$ strongly measurable), and where the J-functional is defined by

$$J(t,a;\overline{A})=\max\left\{\left\|a\right\|_{A_0},t\left\|a\right\|_{A_1}\right\}.$$

Theorem 7. *(cf. [**13**] Theorem 3.1) Let* $\{\rho_\theta\}_{\theta\in(0,1)}$ *a family of quasi-concave functions. If* $\sup_\theta\rho_\theta(t)=\rho(t)<\infty$, *then*

$$\sum\nolimits_\theta\overline{A}_{\rho_\theta,1,J}=\overline{A}_{\rho,1,J}. \tag{4.1}$$

In order to recover this result using unions instead of sums, consider for each $\{\theta_1, \ldots, \theta_n\} \subset (0,1)$ the set

$$A = \{\phi = \sup\{\rho_{\theta_1}, \ldots, \rho_{\theta n}\} : n \in N\}. \tag{4.2}$$

Note that A is a set of quasi-concave functions such that $\{\rho_\theta\}_{\theta\in(0,1)} \subset A$. Moreover, by Theorem 2, $\bigcup \left\{\overline{A}_{\phi,1,J}\right\}_{\phi\in A}$ is a normed space. Now, $A_0 \bigcap A_1$ is dense in each $\overline{A}_{\phi,1,J}$, therefore, by Corollary 2, if $x \in A_0 \bigcap A_1$,

$$\begin{aligned}
\|x\|_{\sum\left\{\overline{A}_{\rho_\theta,1,J}\right\}_{\theta\in(0,1)}} &= \|x\|_{\sum\left\{\overline{A}_{\phi,1,J}\right\}_{\phi\in A}} \\
&= \inf_{\phi\in A} \|x\|_\phi \\
&= \inf_{\phi\in A} \inf_u \int_0^\infty \frac{J(t,u(t);\overline{A})}{\phi(t)} \frac{dt}{t} \\
&= \inf_u \inf_{\phi\in A} \int_0^\infty \frac{J(t,u(t);\overline{A})}{\phi(t)} \frac{dt}{t} \\
&= \inf_u \int_0^\infty \inf_{\phi\in A} \frac{J(t,u(t);\overline{A})}{\phi(t)} \frac{dt}{t} \text{ (by Lemma 3)} \\
&= \inf_u \int_0^\infty \frac{J(t,u(t);\overline{A})}{\sup_{\phi\in A}\phi(t)} \frac{dt}{t}.
\end{aligned}$$

Since $\phi = \sup\{\rho_{\theta_1}, \ldots, \rho_{\theta n}\}$, it is plain that $\sup_\theta \rho_\theta = \sup_{\phi\in A} \phi$. Thus

$$\begin{aligned}
\int_0^\infty \frac{J(t,u(t);\overline{A})}{\sup_{\phi\in A}\phi(t)} \frac{dt}{t} &= \int_0^\infty \frac{J(t,u(t);\overline{A})}{\sup_\theta \rho_\theta(t)} \frac{dt}{t} \\
&= \int_0^\infty \inf_\theta \frac{J(t,u(t);\overline{A})}{\rho_\theta(t)} \frac{dt}{t}.
\end{aligned}$$

So we have proved that

$$\inf_{\phi\in A} \|x\|_{\overline{A}_{\phi,1,J}} = \|x\|_{\overline{A}_{\rho,1,J}}.$$

Since $A_0 \bigcap A_1$ is dense in each $\overline{A}_{\phi,1,J}$ we obtain

$$\bigcup \left\{\overline{A}_{\phi,1,J}\right\}_{\phi\in A} = \sum\nolimits_\theta \overline{A}_{\rho_\theta,1,J} = \overline{A}_{\rho,1,J}.$$

It is also known (cf. [**13**]) that if we work with the classical real interpolation spaces $(A_0, A_1)_{\theta,q,K}$, and M is a tame function in the sense that $M(\theta) \simeq M(2\theta)$, for θ close to 0, and $M(\theta) \simeq M(1-2(1-\theta))$, for θ close to 1, then for all $1 \le q \le \infty$,

$$\sum \left\{M(\theta)\,(A_0, A_1)_{\theta,q;K}\right\}_{0<\theta<1} = \sum \left\{M(\theta)\,(A_0, A_1)_{\theta,1;J}\right\}_{0<\theta<1}.$$

Since $A_0 \bigcap A_1$ is dense in each $\overline{A}_{\theta,1,J}$, we have that if $x \in A_0 \bigcap A_1$,

$$\|x\|_{\sum\left\{M(\theta)(A_0,A_1)_{\theta,q;K}\right\}_{0<\theta<1}} = \inf_{\phi\in A} \|x\|_{\overline{A}_{\phi,1,J}}$$

where the set A is defined as in (4.2).

5. Operators Δ and $\sum$−factorizable: basic concepts

In this section we initiate the study of extrapolation of operators acting on families of spaces. This study can be considerably simplified for operations that satisfy suitable conditions which we shall call "factorizations". The import of these notions for extrapolation of inequalities should become clear in the next section. For this reason, rather than to read through all the definitions of all the different types of factorizations at once, we suggest to the reader that after going through Definition 1 she/he should go directly to Theorem 8 below and section 6 and, in particular, to the proof of the extrapolation Theorem 10 in that section (and return back and forward to this section as needed).

Let $\{X_w\}_{w\in I_0}$ and $\{Y_\nu\}_{\nu\in I_1}$ two scales. Suppose that $T : \Delta\{X_w\}_{w\in I_0} \to \Delta\{Y_\nu\}_{\nu\in I_1}$ (resp. $T : \bigcup\{X_w\}_{w\in I_0} \to \bigcup\{Y_\nu\}_{\nu\in I_1}$), is a bounded operator. We then ask what can be said about the action of T on each of the spaces of the family $\{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1}$? Conversely, how many estimates on members of the family are needed to guarantee that an operator is bounded on Δ (resp $\bigcup$)?

In this context the following definition is very natural. Let $\{X_w\}_{w\in I}$ be a scale. A subset $J \subset I$ will be called $\Delta-$**abundant** if $\Delta\{X_w\}_{w\in J} \simeq \Delta\{X_w\}_{w\in I}$. Analogously $J \subset I$ will be called $\bigcup-$**abundant**[3] if $\bigcup\{X_w\}_{w\in J} \simeq \bigcup\{X_w\}_{w\in I}$.

Example 6. *Let X be a Banach Lattice and let $I = \{w \in X' : w \geq 0, \|w\|_{X'} \leq 1\}$. By Lemma 1) $\{L^1(w)\}_{w\in I}$ is a $\Delta-$total scale and $X = \Delta\{L^1(w)\}_{w\in I}$. Then the sets*

$$J_0 = \{w \in X' : w > 0, \ \ \|w\|_{X'} \leq 1\} \ \text{ and } J_1 = \{w \in X' : w \geq 0, \ \ \|w\|_{X'} \leq 2\}$$

*are $\Delta-$**abundant.***

Proof. J_0 is abundant: Given $\varepsilon > 0$, and $x \in X$, $\exists w \geq 0$, $\|w\|_{X'} \leq 1$ such that

$$\int_\Omega |x|\, w d\mu + \varepsilon \geq \|x\|_X .$$

By (3.1) (3) there exists $u \in X'$, $u > 0$, such that $\|u\|_{X'} \leq 1$. Therefore

$$\begin{aligned} \|x\|_X &\leq \int_\Omega |x|\, w d\mu + \varepsilon \leq \int_\Omega |x|\,(w + \varepsilon u) d\mu + \varepsilon \\ &\leq \|(w+\varepsilon u)\|_{X'} \sup_{\left\{w\in X' : w>0 \ \|w\|_{X'}\leq 1\right\}} \|x\|_{L^1(w)} + \varepsilon \\ &\leq (1+\varepsilon) \sup_{\left\{w\in X' : w>0 \ \|w\|_{X'}\leq 1\right\}} \|x\|_{L^1(w)} + \varepsilon. \end{aligned}$$

It is plain that J_1 is abundant. □

We now give several definitions of factorizations. We suggest to the reader that right after reading Definition 1 she/he goes directly to Theorem 8 and to Section 6, and returns here for more definitions as needed.

Definition 1. *Let $\{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1}$ be two compatible scales and let T be a bounded linear operator*

$$T : \Delta\{X_w\}_{w\in I_0} \to \Delta\{Y_\nu\}_{\nu\in I_1}.$$

[3] If the operation Δ or $\bigcup$ is clear from the context we simply say that a set J is "abundant".

We shall say that T is ***strongly $\Delta-$factorizable*** *if $\exists C(T)>0$ such that $\forall \nu \in I_1$, $\exists w \in I_0$ such that*

(i) T can be extended to a bounded operator $T : X_w \to Y_\nu$ with norm $\leq C(T)$.

(ii) $\|T\|_{\mathcal{L}\left(\Delta\{X_w\}_{w\in I_0},\Delta\{Y_\nu\}_{\nu\in I_1}\right)} \leq C(T)$.

(iii) The set

$$J = \{w \in I_0 : (\nu, w) \text{ satisfies } (i) \text{ above for some } \nu \in I_1\}$$

is $\Delta-$abundant.

We shall say that T is ***$\Delta-$factorizable*** *if $\forall \nu \in I_1$, $\exists w \in I_0$ such that*

(i) T can be extended to a bounded operator $T : X_w \to Y_\nu$.

(ii) The set

$$J = \{w \in I_0 : (\nu, w) \text{ satisfies } (i) \text{ above for some } \nu \in I_1\}$$

is $\Delta-$abundant.

Definition 2. *Suppose that T is a bounded linear operator*

$$T : \Delta\{X_w\}_{w\in I_0} \to \Delta\{X_w\}_{w\in I_0}.$$

We shall say that T is ***strongly $\Delta-$diagonal factorizable*** *if $\exists C(T)>0$ such that $\forall w \in I_0$, $\exists \widetilde{w} \in I_0$ such that*

(i) T can be extended to a bounded operator $T : X_{\widetilde{w}} \to X_{\widetilde{w}}$ with norm $\leq C(T)$.

(ii) Moreover $\|T\|_{\mathcal{L}\left(\Delta\{X_w\}_{w\in I_0},\Delta\{X_w\}_{w\in I_0}\right)} \leq C(T)$.

(iii) The set

$$J = \{\widetilde{w} \in I_0 : w \text{ is associated with some } w \in I_0 \text{ as in } (i)\}$$

is $\Delta-$abundant.

Definition 3. *Let $\{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1}$ be two $\bigcup-$complete scales, and let T be a bounded linear operator,*

$$T : \bigcup\{X_w\}_{w\in I_0} \to \bigcup\{Y_\nu\}_{\nu\in I_1}.$$

We shall say that T is ***strongly $\sum-$factorizable*** *if $\exists C(T)>0$ such that $\forall w \in I_0$, $\exists \nu \in I_1$ such that*

(i) $T : X_w \to Y_\nu$, with $\|T\|_{X_w\to Y_\nu} \leq C(T)$.

(ii) $\|T\|_{\mathcal{L}\left(\cup\{X_w\}_{w\in I_0},\cup\{Y_\nu\}_{\nu\in I_1}\right)} \approx C(T)$.

(iii) The set

$$J = \{\nu \in I_1 : (w, \nu) \text{ satisfies } (i) \text{ for some } w \in I_0\}$$

is $\bigcup-$abundant.

We shall say that T is ***$\sum-$factorizable*** *if $\forall w \in I_0$, $\exists \nu \in I_1$ such that*

(i) $T : X_w \to Y_\nu$ is bounded.

(ii) The set $J = \{\nu \in I_1 : (w,\nu) \text{ satisfies } (i) \text{ above for some } w \in I_0\}$ is $\bigcup-$abundant.

Definition 4. *Suppose that T is a bounded linear operator*

$$T : \bigcup\{X_w\}_{w\in I_0} \to \bigcup\{X_w\}_{w\in I_0}.$$

We shall say that T is ***strongly $\sum-$diagonal factorizable*** *if $\exists C(T)>0$ such that $\forall w \in I_0$, $\exists \widetilde{w} \in I_0$ such that*

(i) $T : X_{\widetilde{w}} \to X_{\widetilde{w}}, \|T\|_{X_{\widetilde{w}}\to X_{\widetilde{w}}} \leq C(T)$.

(ii) $\|T\|_{\mathcal{L}\left(\cup\{X_w\}_{w\in I_0},\cup\{X_w\}_{w\in I_0}\right)} \approx C(T)$.

(iii) The set $J = \{\widetilde{w} \in I_0 :$ *associated with some index* $w \in I_0$ *as in* $(i)\}$ *is* $\bigcup-$*abundant.*

We shall say that T *is* $\sum -$***diagonal factorizable*** *if* $\forall w \in I_0$, $\exists \widetilde{w} \in I_0$ *such that*

(i) $\forall w \in I_0$, $\exists \widetilde{w} \in I_0$ *such that* $T : X_{\widetilde{w}} \to X_{\widetilde{w}}$ *is bounded.*

(ii) The set $J = \{\widetilde{w} \in I_0 :$ *associated with some index* $w \in I_0$ *as in* $(i)\}$ *is* $\bigcup-$*abundant.*

Remark 11. *When we work with scales of Banach lattices the definitions above also make sense when dealing with quasi-linear operators.*

In the next section we show that for strongly factorizable operators it is possible to reconstruct the "indexed" norm inequalities for T from the estimates of T on extrapolation spaces and conversely. We shall exploit this idea to prove "extrapolation theorems".

Remark 12. *If an operator* T *satisfies that* $\forall \nu \in I_1$, $\exists w \in I_0$ *such that* $T : X_w \to Y_\nu$ *is bounded, and the set*

$$J = \{w \in I_1 : \textit{such that } T : X_w \to Y_\nu \textit{ is bounded for some } \nu \in I_1\}$$

is $\Delta-$*abundant, then it is not necessarily true that* T *is bounded from* $\Delta\{X_w\}_{w\in I_0}$ *to* $\Delta\{Y_\nu\}_{\nu\in I_1}$. *For example, consider the scale* $\{L^p[0,1]\}_{1\leq p<\infty}$ *and the operator* $Qf(x) = \int_x^1 f(s)\frac{ds}{s}$, *then* $Q : L^p[0,1] \to L^p[0,1]$ *is bounded for all* $1 \leq p < \infty$, *thus* Q *is defined from the linear space* $\bigcap_{1\leq p<\infty} L^p[0,1]$ *to the linear space* $\bigcap_{1\leq p<\infty} L^p[0,1]$. *However* Q *is not bounded if we endow* $\bigcap_{1\leq p<\infty} L^p[0,1]$ *with the* $\Delta-$*norm* $\|f\|_{\Delta\{L^p[0,1]\}_{1\leq p<\infty}} := \sup\limits_{1\leq p<\infty} \|f\|_p$, *since* $\sup\limits_{1\leq p<\infty} \left\|Q\chi_{[0,1]}\right\|_p = \infty$. *The point here is that* $\{L^p[0,1]\}_{1\leq p<\infty}$ *is not a* $\Delta-$*total scale (see Remark 1).*

In other words we need some type of control of the norm to be able to extrapolate when we work with non $\Delta-$*total scales, notice that in the previous example the norm of* Q *blows up like* p *as* $p \to \infty$.

For $\Delta-$total scales on the other hand we have the following

Theorem 8. *Let* $\{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1}$ *two* $\Delta-$*total scales. If* T *is a linear operator which satisfies that* $\forall \nu \in I_1$, $\exists w \in I_0$ *such that*

(i) $T : X_w \to Y_\nu$ *is bounded.*

(ii) The set $\{w \in I_0 : (\nu, w)$ *satisfy* (i) *above for some* $\nu \in I_1\}$ *is* $\Delta-$*abundant.*

Then

$$T : \Delta\{X_w\}_{w\in I_0} \to \Delta\{Y_\nu\}_{\nu\in I_1} \textit{ is bounded.}$$

Proof. It is clear that T is defined from the linear space $\bigcap_{w\in I_0} X_w$ to the linear space $\bigcap_{\nu\in I_1} Y_\nu$. Since the scales are $\Delta-$total, if we consider on $\bigcap_{w\in I_0} X_w$ (resp. on $\bigcap_{\nu\in I_1} Y_\nu$) the usual norm, then

$$\bigcap_{w\in I_0} X_w = \Delta\{X_w\}_{w\in I_0} \text{ and } \bigcap_{\nu\in I_1} Y_\nu = \Delta\{Y_\nu\}_{\nu\in I_1}.$$

On the other hand, since by hypothesis $\forall \nu \in I_1$, the set $J(\nu) = \{w \in I_0$ such that $T : X_w \to Y_\nu$ is bounded $\}$ is not empty and $J = \bigcup_{\nu\in I_1} J(\nu)$ is $\Delta-$abundant, we have

$$\Delta\{X_w\}_{w\in I_0} = \Delta\{X_w\}_{w\in J}.$$

To prove that T is continuous, we use the closed graph theorem. Suppose that $\|x_n\|_{\Delta\{X_w\}_{w\in J}}. \to 0$ and $\|Tx_n - y\|_{\Delta\{Y_\nu\}_{\nu\in I_1}} \to 0$ for some $y \in \Delta\{Y_\nu\}_{\nu\in I_1}$. For every $\nu \in I_1$ select $w \in J$, then since $\Delta\{X_w\}_{w\in J} \overset{1}{\subset} X_w$ we have $\|x_n\|_{X_w} \to 0$ and $\|Tx_n - y\|_{Y_\nu} \to 0$, which implies that $y = 0$ since $T : X_w \to Y_\nu$ is bounded. □

For the next result we shall consider strongly compatible scales $\{Z_w\}_{w\in I}$ such that each Z_w is reflexive, $\Delta\{Z_w\}_{w\in I}$ is dense in each Z_w, (this ensures that the scale $\{Z_w^*\}_{w\in I}$ is strongly compatible). Let us also assume that $\Delta\{Z_w^*\}_{w\in I}$ is dense in each Z_w^*. Then we have the following:

Theorem 9. *Let $\{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1}$ be two $\bigcup -$total scales, as above. If T is a linear operator which satisfies that $\forall w \in I_0$, $\exists \nu \in I_1$, such that*
(i) $T : X_w \to Y_\nu$ is bounded.
(ii) The set $\{\nu \in I_1 : (\nu, w)$ satisfy (i) above for some $w \in I_0\}$ is $\bigcup -$abundant.
Then

$$T : \bigcup\{X_w\}_{w\in I_0} \to \bigcup\{Y_\nu\}_{\nu\in I_1} \text{ is bounded.}$$

Proof. Since $J(w) = \{\nu \in I_1$ such that $T : X_w \to Y_\nu$ is bounded$\}$ is not empty and $J = \bigcup_{w\in I_0} J(w)$ is $\bigcup -$abundant we get

$$\bigcup\{Y_\nu\}_{\nu\in I_1} = \bigcup\{Y_\nu\}_{\nu\in J} = \sum_{\nu\in J} Y_\nu.$$

Since there is a common dense subset (cf. [**13**] chapter 5)

$$\left(\bigcup\{Y_\nu\}_{\nu\in I_1}\right)^* = \left(\sum_{\nu\in J} Y_\nu\right)^* = \Delta\{Y_\nu^*\}_{\nu\in J}.$$

Thus $\{Y_\nu^*\}_{\nu\in J}$ is a $\Delta-$total. Similarly we also have that

$$\left(\bigcup\{X_w\}_{w\in I_0}\right)^* = \Delta\{X_w^*\}_{w\in I_0}.$$

Considering now the scales $\{Y_\nu^*\}_{\nu\in J}$ and $\{X_w^*\}_{w\in I_0}$, it follows readily that the operator T^* satisfies the hypothesis of Theorem 8, hence

$$T^* : \Delta\{Y_\nu^*\}_{\nu\in J} \to \Delta\{X_w^*\}_{w\in I_0}.$$

is bounded. Since each X_w (resp. Y_ν) has a common dense subset, we have that (cf. [**10**]) $(\Delta\{X_w^*\}_{w\in I_0}.)^* = \sum_{w\in I_0} X_w^{**} = \sum_{w\in I_0} X_w$ (resp. $(\Delta\{Y_\nu^*\}_{\nu\in J})^* = \sum_{\nu\in J} Y_\nu$), thus

$$T : \bigcup\{X_w\}_{w\in I_0} \to \bigcup\{Y_\nu\}_{\nu\in I_1} \text{ is bounded.}$$

□

Remark 13. *If we work with scales of Banach lattices, then $X^* = X', Y^* = Y'$. The adjoint operator T^* $X' \to Y'$ is well defined. Moreover, since $X'' = X$, the previous result remains true without the reflexivity assumption.*

6. Extrapolation Theorems of Rubio de Francia type

The purpose of this section is to show how factorizations can be used to extrapolate estimates from one operator to another operator. The motivation for this type of extrapolation comes from the fact that, in classical analysis and elsewhere, a few basic operators control the norm estimates of large families of operators. For example, a good deal of the theory of weighted norm inequalities for singular integral operators can be reduced to the study of the weighted norm inequalities of (the simpler) maximal operator of Hardy-Littlewood or other related maximal operators. Likewise, in interpolation theory, many interpolation estimates can be reduced to estimates for the so called Calderón operators, etc. This explains the fundamental importance of extrapolation in the applications.

In the general context we have developed in this paper the extrapolation process takes a very simple form. The main ideas are an extension of Rubio de Francia's beautiful papers [**23**], [**24**], where on can also find applications of the theory. In connection with applications we refer the reader to the monograph [**11**].

Let T be an operator acting on certain spaces of two given scales $\{X_w\}_{w\in I_0}$ and $\{Y_v\}_{v\in I_1}$. We associate to T a set of indices which we call the **signature** of T on $(\{X_w\}_{w\in I_0}, \{Y_v\}_{v\in I_1})$:

$$S(T, \{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1}) = S(T) = \{(w,\nu) \in I_0 \times I_1 : T : X_w \to Y_\nu \text{ is bounded}\}.$$

In case the domain and range scales are the same, $\{X_w\}_{w\in I_0} = \{Y_\nu\}_{\nu\in I_1}$, it will be also useful to consider what we shall call the **diagonal signature** of T:

$$S^d(T) = \{w \in I : (w,w) \in S(T)\}.$$

Given H, T two operators acting on the same scale, we write $S(H) \subset S(T)$ to indicate that

$$H : X_w \to Y_\nu \text{ bounded} \Rightarrow T : X_w \to Y_\nu \text{ bounded},$$

while $S^d(H) \subset S^d(T)$ means that

$$H : X_w \to X_w \text{ bounded} \Rightarrow T : X_w \to X_w \text{ bounded}.$$

In this context we shall use the symbol $\Subset$ to indicate that an inclusion holds with norm estimates. Thus, $S(H) \Subset S(T)$ means that there is a universal constant $c > 0$ such that

$$H : X_w \to Y_\nu \text{ with } \|H\|_{\mathcal{L}(X_w,Y_\nu)} \leq C \Rightarrow T : X_w \to Y_\nu \text{ with } \|T\|_{\mathcal{L}(X_w,Y_\nu)} \leq cC.$$

An analogous interpretation stands for the notation $S^d(H) \Subset S^d(T)$.

Theorem 10. *(cf. [**23**]) Let $\{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1}$ be two scales (resp. $\bigcup$–complete scales) and let $X = \Delta\{X_w\}_{w\in I}$, $Y = \Delta\{Y_\nu\}_{\nu\in I_1}$ (resp. $X = \bigcup\{X_w\}_{w\in I}$, $Y = \bigcup\{Y_\nu\}_{\nu\in I_1}$). Then*

(i) Let $H : X \to Y$ be a strongly $\Delta-$factorizable (resp. strongly $\sum-$factorizable) linear operator. Suppose that T is a linear operator acting on some spaces of the family $(\{X_w\}_{w\in I_0}, \{Y_\nu\}_{\nu\in I_1})$. The following extrapolation result holds:

$$S(H) \Subset S(T) \Rightarrow T : X \to Y \text{ is bounded.}$$

(ii) Let $H : X \to X$ be a strongly $\Delta-$diagonal factorizable (resp. strongly $\sum-$diagonal factorizable) linear operator. Suppose that T is a linear operator acting on some spaces of the family $\{X_w\}_{w\in I}$. The following extrapolation holds

$$S^d(H) \Subset S^d(T) \Rightarrow T : X \to X \text{ is bounded.}$$

Proof. (i). Suppose that $H : X \to Y$ is strongly $\Delta-$factorizable and $S(H) \Subset S(T)$. It follows that there exists $C(T) > cC(H) > 0$ such that $\forall \nu \in I_1$, the set $J(\nu) = \{w \in I_0$ such that $T : X_w \to Y_\nu$ is bounded with norm $\leq C(T)\}$ is not empty and in fact $J = \bigcup_{\nu \in I_1} J(\nu)$ is $\Delta-$abundant. Thus,

$$\|Tx\|_{Y_\nu} \leq C(T) \sup_{w \in J(\nu)} \|x\|_{X_w}, \text{ for each } \nu \in I_1$$

$$\begin{aligned}\sup_{\nu \in I_1} \|Tx\|_{Y_\nu} &\leq C(T) \sup_{\nu \in I_1} \sup_{w \in J(\nu)} \|x\|_{X_w} \\ &\leq C(T) \sup_{w \in I_0} \|x\|_{X_w} \text{ (since } J \text{ is abundant).}\end{aligned}$$

Therefore,

$$\|T\|_{\mathcal{L}(\Delta\{X_w\}_{w \in I_0}, \Delta\{Y_\nu\}_{\nu \in I_1})} \leq C(T).$$

Suppose now that $H : X \to Y$ is strongly $\sum-$factorizable and $S(H) \Subset S(T)$. Then T satisfies that $\exists C(T) > 0$ such that $\forall w \in I_0$, $\exists \nu \in I_1$ such that $T : X_w \to Y_\nu$ is bounded with norm $\leq C(T)$, and the set $J = \bigcup_{w \in I_0} J(w) = \bigcup_{w \in I_0} \left\{\nu \in I_1 : T : X_w \to Y_\nu, \|T\|_{\mathcal{L}(X_w, Y_\nu)} \leq C(T)\right\}$ is $\bigcup-$abundant. Therefore

$$\|Tx\|_{Y_\nu} \leq C(T) \|x\|_{X_w} \text{ for all } \nu \in J(w).$$

Thus

$$\inf_{\nu \in J(w)} \|Tx\|_{Y_\nu} \leq C(T) \|x\|_{X_w}, \text{ for all } w \in I_0.$$

Taking infimum we get

$$\begin{aligned}\inf_{w \in I_0} \inf_{\nu \in J(w)} \|Tx\|_{Y_\nu} &\leq C(T) \inf_{w \in I_0} \|x\|_{X_w} \\ \inf_{\nu \in J} \|Tx\|_{Y_\nu} &\leq C(T) \inf_{w \in I_0} \|x\|_{X_w}.\end{aligned}$$

But since J is abundant we get

$$\inf_{v \in I_1} \|Tx\|_{Y_\nu} \simeq \inf_{\nu \in J} \|Tx\|_{Y_\nu} \leq C(T) \inf_{w \in I_0} \|x\|_{X_w}.$$

In other words,

$$T : \bigcup \{X_w\}_{w \in I_0} \to \bigcup \{Y_\nu\}_{\nu \in I_1}.$$

Note that since our scales were assumed to be $\bigcup-$ complete the last statement is equivalent to

$$T : \sum \{X_w\}_{w \in I_0} \to \sum \{Y_\nu\}_{\nu \in I_1}$$

with

$$\|T\|_{\mathcal{L}(\sum\{X_w\}_{w \in I_0}, \sum\{Y_\nu\}_{\nu \in I_1})} \leq C(T).$$

The proof of (ii) for the diagonal case is obtained *mutatis-mutandis.* □

Remark 14. *When we work with scales of Banach lattices the previous result remains true for quasi-linear operators*

Theorem 11. *(cf.* [**23**]*) Let* $(\{X_w\}_{w \in I_0}, \{Y_\nu\}_{\nu \in I_1})$ *be two* $\Delta-$*total scales, and let* $X = \Delta\{X_w\}_{w \in I_0}$, $Y = \Delta\{Y_\nu\}_{\nu \in I_1}$.

(i) Let $H : X \to Y$ *be a* $\Delta-$*factorizable linear operator. Suppose that* T *is a linear operator acting on some spaces of the family* $(\{X_w\}_{w \in I_0}, \{Y_\nu\}_{\nu \in I_1})$ *in such a way that* $S(H) \subset S(T)$. *Then we can extrapolate* $T : X \to Y$.

(ii) Let $H : X \to X$ be a diagonal $\Delta-$ factorizable linear operator. Suppose that T is a linear operator acting on some spaces of the family $\{X_w\}_{w\in I}$ in such a way that $S^d(H) \subset S^d(T)$. Then we can extrapolate $T : X \to X$.

Proof. (i) If $H : X \to Y$ is $\Delta-$factorizable then $\forall \nu \in I_1$, $\exists w \in I_0$ such that H can be extended to a bounded operator $H : X_w \to Y_\nu$, and the set $J = \{w \in I_0 : H : X_w \to Y_\nu \text{ for some } \nu \in I_1\}$ is $\Delta-$abundant. But the assumption $S(H) \subset S(T)$ allow us to transfer the estimates for the operator H to the operator T. Therefore by Theorem 8 $T : X \to Y$ is bounded.

(ii) Can be proved in the same way. □

Note that, in general, the definitions of strongly factorizable and strongly diagonal factorizable are not equivalent. We shall say that a scale has the **Rubio de Francia property** if every $\Delta-$strongly factorizable operator (resp. $\Sigma-$strongly factorizable) on the scale is strongly diagonal factorizable (resp. $\Sigma-$strongly diagonal factorizable).

To prove that a given scale $\{X_w\}_{w\in I}$ has the **Rubio de Francia property** usually is related with some "extra" properties of the index set I, for example if the index set is a convex subset of a Banach lattice (cf. Section 3.0.1 and Section 3.3 above). In such cases we can implement an abstract version of the so called "Rubio de Francia Algorithm" to prove that these scales have the Rubio de Francia property.

A prototype of the results we can prove is the following

Theorem 12. *Let Z be a Banach lattice, and let $I \subset \{w : w > 0,\ \|w\|_Z \le 1\}$. Let $\{X_w\}_{w\in I}$ be a scale such that*

(i)

$$\Delta\{X_w\}_{w\in I} \text{ is dense in each } X_w.$$

(ii) There exists $p \ge 1$ such that

$$\forall\lambda > 0,\ x \in X_w \Leftrightarrow x \in X_{\lambda w} \text{ and } \|x\|^p_{X_{\lambda w}} = \lambda \|x\|^p_{X_w} \tag{6.1}$$

and $\forall \{w_j\}_{j\in N} \subset I$ such $\sum_j \|w_j\|_Z \le 1$ then $\sum_j w_j \in I$ and

$$\sum_j \|x\|^p_{X_{w_j}} = \|x\|^p_{X_{\sum w_j}}. \tag{6.2}$$

Then any linear operator T is strongly $\Delta-$ factorizable if and only if it is strongly $\Delta-$diagonal factorizable.

Proof. Suppose that T is strongly $\Delta-$factorizable. Let $w \in I$, and define inductively the sequence $\{w_j\}_{j=0}^{\infty}$ in I so that: $w_0 = w$ and $T : X_{w_{j+1}} \to X_{w_j}$, with $\|T\|_{\mathcal{L}(X_{w_{j+1}}, X_{w_j})} \le C(T) \le c$. Since

$$\sum_{j=0}^{\infty} \left\|2^{-j-1} w_j\right\|_Z = \sum_{j=0}^{\infty} 2^{-j-1} \|w_j\|_Z \le \sum_{j=0}^{\infty} 2^{-j-1} = 1,$$

we see that

$$\widetilde{w} = \sum_{j=0}^{\infty} 2^{-j-1} w_j \in I.$$

Let $p \geq 1$ such that (6.1) and (6.2) are satisfied, and let $x \in \Delta\{X_w\}_{w\in I}$ then

$$\begin{aligned}
\|Tx\|_{X_{\widetilde{w}}}^p &= \sum_{j\geq 0} \|Tx\|_{X_{2^{-j-1}w_j}}^p \qquad \text{(by (6.2))} \\
&= \sum_{j\geq 0} 2^{-j-1} \|Tx\|_{X_{w_j}}^p \quad \text{(by (6.1))} \\
&\leq C(T)^p \sum_{j\geq 0} 2^{-j-1} \|x\|_{X_{w_{j+1}}}^p \quad (\text{since } \|T\|_{\mathcal{L}\left(X_{w_{j+1}}, X_{w_j}\right)} \leq C(T)) \\
&= 2C(T) \sum_{j\geq 0} 2^{-j-2} \|x\|_{X_{w_{j+1}}}^p \\
&\leq 2C(T) \left(\sum_{j\geq 0} \|x\|_{X_{2^{-j-2}w_{j+1}}}^p + \|x\|_{X_{2^{-1}w_0}}^p \right) \\
&= 2C(T) \|x\|_{X_{\sum_{j\geq 0} 2^{-j-1}w_j}}^p = 2C(T) \|x\|_{X_{\widetilde{w}}}^p .
\end{aligned}$$

Since $\Delta\{X_w\}_{w\in I}$ is densely embedded in $X_{\widetilde{w}}$, T can be extended to a bounded operator $T : X_{\widetilde{w}} \to X_{\widetilde{w}}$. It remains to prove that $J = \{\widetilde{w} : w \in I\}$ is abundant. We obviously have

$$\Delta\{X_w\}_{w\in I} \subset \Delta\{X_{\widetilde{w}}\}_{\widetilde{w}\in J}.$$

Conversely, since

$$\|x\|_{X_{\widetilde{w}}}^p = \sum_{j\geq 0} 2^{-j-1} \|x\|_{X_{w_j}}^p \geq 2^{-1} \|x\|_{X_{w_0}}^p = 2^{-1} \|x\|_{X_w}^p$$

we have

$$\Delta\{X_{\widetilde{w}}\}_{\widetilde{w}\in J} \overset{2^{1/p}}{\subset} \Delta\{X_w\}_{w\in I}.$$

□

Remark 15. *Conditions (6.1) and (6.2) are satisfied by the $\Delta-$scales that appear in Banach lattices, $p-$convex spaces, Beurling spaces and rearrangement invariant Banach lattices (cf. the examples in Section 3).*

To prove that on given scale an operator is factorizable is usually a difficult step that may involve a deep theorem on the structure of the spaces involved.

For example, let X, Y be $2-$convex spaces. By Theorem 3[4]

$$X = \Delta\{L^2(w)\}_{w\in I_0}, Y = \Delta\{L^2(\nu)\}_{\nu\in I_1}.$$

In our language Rubio de Francia's Theorems A and A' in [**23**] can be stated as follows

Theorem 13. *(Rubio de Francia [**23**]) Let X, Y be $2-$convex spaces. Then every bounded linear operator $T : X \to Y$ is $\Delta-$strongly factorizable and every bounded linear map $T : X \to X$ is $\Delta-$strongly diagonal factorizable*

We comment briefly on the proof of Theorem 13 given in [**23**]. The fact that any bounded linear operator

$$T : X = \Delta\{L^2(w)\}_{w\in I_0} \to Y = \Delta\{L^2(\nu)\}_{\nu\in I_1}$$

[4]Recall that the index sets in the situation at hand are given by $I_0 = \left\{w \in (X^2)'_+, \|w\|_{(X^2)'} \leq 1\right\}$ and $I_1 = \left\{\nu \in (Y^2)'_+, \|\nu\|_{(Y^2)'} \leq 1\right\}$.

is $\Delta-$strongly factorizable is a deep result. Rubio de Francia's proof uses a mini max theorem and an extension of Grothendieck's theorem due to Krivine: If X, Y are $2-$convex, then any bounded linear operator $T : X \to Y$ can be extended to a bounded operator $T : X(l^2) \to Y(l^2)$. The Rubio de Francia algorithm then is used to prove the $\Delta-$strong diagonal factorizability.

Likewise, since by duality:

$$X \text{ is } 2 - \text{concave} \Leftrightarrow X' \text{ is } 2 - \text{convex},$$

then Rubio de Francia's Theorems B and B' in [**23**]) state

THEOREM 14. *(Rubio de Francia* [**23**]*) Let X, Y be $2-$concave spaces. By Theorem 4*

$$X = \bigcup \{L^2(\frac{1}{w})\}_{w\in I_0}, Y = \bigcup \{L^2(\frac{1}{\nu})\}_{\nu \in I_1}.$$

Then every linear operator $T : X \to Y$ is strongly $\sum-$strongly factorizable. Moreover, every linear operator $T : X \to X$ is $\sum-$strongly diagonal factorizable.

EXAMPLE 7. *(cf.* [**23**]*) If X, Y are $p-$convex, (resp. $q-$concave) then any positive linear operator $T : X \to Y$ is strongly $\Delta-$factorizable. (resp. strongly $\sum-$factorizable). This is a consequence of the fact that in this case T can be extended to a bounded operator $T : X(l^p) \to Y(l^p)$. (cf.* [**15**]*-Proposition 1.d.9).*

EXAMPLE 8. *Closely related to these results is the so called Maurey-Pisier extrapolation theorem. We refer to* [**3**] *p. 22 for a statement and a proof using the language of extrapolation, in particular the proof uses the Rubio de Francia algorithm. Interestingly, the Maurey-Pisier extrapolation implies a form of Grothendieck's inequality formulated in terms of $p-$summing operators. We review the main points of this connection since extrapolation techniques could have other applications in this area. Let $X, Y,$ be Banach spaces, $0 < p < \infty$. We say that an operator $T : X \to Y$ is $p-$ summing if there exists a positive constant c, such that for every finite sequence $\{x_i\}_{i=1}^n \subset X$, we have*

$$\left\{ \sum_{i=1}^{n} (\|T(x_i)\|_Y)^p \right\}^{1/p} \leq c \sup_{\|x^*\|_{X^*}\leq 1} \left\{ \sum_{i=1}^{n} |x^*(x_i)|^p \right\}^{1/p}. \tag{6.3}$$

In this case we write $T \in \Pi_p(X, Y)$, and let

$$\|T\|_{\Pi_p(X,Y)} = \inf\{c : \ \textit{(6.3) holds}\}$$

The limiting case $p = \infty$, is simply a reformulation that T is bounded, in other words $\Pi_\infty(X, Y) = B(X, Y) =$ bounded operators. A basic fact concerning $p-$summing operators is the Pietsch factorization theorem which states that if $T \in \Pi_p(X, Y)$, then there exists a Radon measure μ, on B_{X^*}, the unit ball of X^*, with the $\sigma(X^*, X)$ topology, such that

$$\|Tx\|_Y \leq \|T\|_{\Pi_p(X,Y)} \left\{ \int_{B_{X^*}} |f(x)|^p \, d\mu(f) \right\}^{1/p}. \tag{6.4}$$

Conversely if there exists a measure μ such that (6.4) holds (with $\|T\|_{\Pi_p(X,Y)}$ replaced by some constant c) then it follows that $T \in \Pi_p(X, Y)$ and $\|T\|_{\Pi_p(X,Y)} \leq c$. An immediate consequence of this representation is the fact that the spaces $\Pi_p(X, Y)$ are ordered,

$$\Pi_p(X, Y) \subset \Pi_q(X, Y), \tag{6.5}$$

in fact

$$\|T\|_{\Pi_q(X,Y)} \leq \|T\|_{\Pi_p(X,Y)}.$$

As is well known an equivalent form of Grothendieck's theorem can be stated as follows. Let H be a Hilbert space and for a measure space (Ω, μ), let $L^1 = L^1(\Omega, \mu)$, then

$$\Pi_1(L^1, H) = B(L^1, H). \tag{6.6}$$

Let us review a route to prove (6.6) following Maurey-Pisier (cf. [**21**]*). One first establishes that*

$$\Pi_p(L^1, H) = B(L^1, H),\ 1 < p < \infty.$$

In view of (6.5) we therefore conclude that

$$\Pi_p(L^1, H) = \Pi_2(L^1, H),\ 1 < p < 2 \tag{6.7}$$

At this point an "extrapolation" argument is invoked which allows to establish (6.7) also for $p = 1$. More precisely it is shown that if for some $0 < p < q$ we have

$$\Pi_q(L^1, H) = \Pi_p(L^1, H) \tag{6.8}$$

then it also holds for all $r \leq p < q$ that

$$\Pi_q(L^1, H) = \Pi_r(L^1, H). \tag{6.9}$$

The connection with the extrapolation method discussed in this section is via the Pietsch factorization theorem. In fact in view of Pietsch's result we can rephrase (6.8) as saying $\exists\ C > 0$ such that $\forall\ \lambda \in P(B_{X^*})$ =probability measures on B_{X^*}, there exists $\mu \in P(B_{X^*})$ such that $\forall\ x \in X$, we have

$$\left\{\int_{B_{X^*}} |f(x)|^q \, d\lambda(f)\right\}^{1/q} \leq C \left\{\int_{B_{X^*}} |f(x)|^p \, d\mu(f)\right\}^{1/p}. \tag{6.10}$$

Thus (6.9) will follow by establishing that (6.10) implies that $\exists\ c > 0$ such that for each probability measure $\lambda \in P(B_{X^*})$, there exists $\mu \in P(B_{X^*})$ such that for $0 < r < p$ we have $\forall x \in X$,

$$\left\{\int_{B_{X^*}} |f(x)|^q \, d\lambda(f)\right\}^{1/q} \leq c \left\{\int_{B_{X^*}} |f(x)|^r \, d\mu(f)\right\}^{1/r}. \tag{6.11}$$

This follows by Maurey-Pisier extrapolation. The proof of this fact given in [**3**] *(cf. Lemma 5.1) emphasizes the role of the Rubio de Francia algorithm.*

7. The Classical Extrapolation Theorem of Rubio de Francia

In this section we give a streamlined argument to prove Rubio de Francia's celebrated extrapolation theorem for A_p weights. We also show that the same argument can be used to prove extrapolation theorems for other classes of weights.

Let us start by recalling some basic definitions.

Definition 5. *A weight $w > 0$ belongs to $A_p = A_p(\mathbb{R}^n)$, ($1 < p < \infty$ and $1/p + 1/q = 1$) if*

$$\|w\|_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(x)^{q-1}}dx\right)^{p-1} < \infty.$$

Here the supremum is taken over all cubes Q on R^n with sides parallel to the coordinate axes and where $|Q|$ =measure of Q.

The celebrated extrapolation theorem of Rubio de Francia for A_p-weights states (cf. [**11**]-Theorem 5.19)

THEOREM 15. *Let T be a quasi-linear operator. Let $1 < r < \infty$, and suppose that T is bounded in $L^r(w)$ for every weight $w \in A_r$, so that its norm as an operator on $L^r(w)$ depends only on $\|w\|_{A_r}$. Then, for every $w \in A_p$ $(1 < p < \infty)$, T is bounded on $L^p(w)$.*

We organize the proof of Theorem 15 in a "functorial" fashion using the Δ and $\sum$ methods of [**13**]. The proof consists of two steps. The first step is familiar: to represent L^p spaces as suitable Δ or $\bigcup$ extrapolation spaces (Lemma 4). This representation is then combined with a factorization theorem for the A_p classes of weights (Lemma 5) and further application of the Δ and $\sum$ methods to finally obtain Rubio de Francia's theorem. A similar argument yields an extrapolation theorem with the so called Calderón weights replacing the A_p weights (cf. Theorem 16 below).

LEMMA 4. *Let $1 < p \neq r < \infty$, then*

$$L^p(\mu) = \left\{ \begin{array}{ll} \Delta \left\{ L^r(ud\mu) \right\}_{u\in I} & \text{if } p < r \\ \bigcup \left\{ L^r(u^{-1}d\mu) \right\}_{u\in I} & \text{if } p > r \end{array} \right. .$$

where

$$I = \left\{ u \in L^s : u > 0, \ \ \|u\|_{L^s} = 1 \right\}.$$

($s = \left|\frac{p}{p-r}\right|$). *In other words: if $p < r$, then $\left\{L^r(ud\mu)\right\}_{u\in I}$ is a $\Delta-$total scale and if $p > r$, $\left\{L^r(u^{-1}d\mu)\right\}_{u\in I}$ is a $\bigcup-$complete scale.*

PROOF. Suppose that $r < p$, then $\frac{p}{r} > 1$, and

$$\|f\|_{L^p(d\mu)}^r = \left(\int (|f|^r)^{\frac{p}{r}} \, d\mu \right)^{r/p} = \left\| |f|^r \right\|_{L^{\frac{p}{r}}(d\mu)}.$$

By duality

$$\left\| |f|^r \right\|_{L^{\frac{p}{r}}(d\mu)} = \sup \left\{ \int |f|^r \, ud\mu : u > 0, \ \ \|u\|_{L^{\frac{p}{p-r}}(d\mu)} = 1 \right\}.$$

If $p < r$, then (use (2.2) with $\alpha = \frac{p}{r} < 1$)

$$\|f\|_{L^p(d\mu)}^r = \left\| |f|^r \right\|_{L^{\frac{p}{r}}(d\mu)} = \inf \left\{ \int |f|^r \, u^{-1}d\mu : u > 0, \ \ \|u\|_{L^{\frac{p}{r-p}}(d\mu)} = 1 \right\}.$$

$\square$

LEMMA 5. *(cf. [**11**]-Lemma 5.18) Let $1 < p, r < \infty$. Let s be defined by $\frac{1}{s} = \left|1 - \frac{r}{p}\right|$. Let $w \in A_p$. Then for every $u \geq 0$ in $L^s(w)$ there exists $\nu \geq 0$ in $L^s(w)$, such that*

$$\begin{array}{l} 1) \ u \leq \nu, \text{ a.e.} \\ 2) \ \|\nu\|_s \leq C \|u\|_s \\ 3) \left\{ \begin{array}{l} \nu w \in A_r \ \text{ if } r < p \\ \frac{w}{\nu} \in A_r \ \ \text{if } r > p \end{array} \right. \end{array}$$

Moreover, in either alternative of case 3) above, $\|\nu w\|_{A_r}$ (resp. $\left\|\frac{w}{v}\right\|_{A_r}$) depends only on $\|w\|_{A_p}$.

We are now ready for our proof of Theorem 15:

Proof. Suppose that $p > r$. Let $w \in A_p$ and let $L^p(w) = L^p(d\mu)$ (here $d\mu = w(x)dx$). By Lemma 4

$$L^p(d\mu) = \Delta \left\{L^r(ud\mu)\right\}_{u\in I}.$$

For $u \in I$ we let

$$J(u) = \left\{\nu \in L^s(d\mu) : (u,\nu) \text{ satisfies the conditions of Lemma 5}\right\},$$

and

$$J = \bigcup_{u\in I} J(u).$$

We claim that J is $\Delta-$abundant. Indeed, let $f \in L^p(d\mu)$ and $c > 1$, then there exists ν such that $(u,\nu) \in J$, and we have

$$\begin{aligned}
\|f\|_p^r &\leq c\int |f|^r \, ud\mu \leq c\int |f|^r \, \nu d\mu \quad \text{(by condition 1 Lemma 5)} \\
&\leq c\,\||f|^r\|_{\frac{p}{r}} \|\nu\|_s \qquad \text{(by Hölder's inequality)} \\
&\leq cC\,\||f|^r\|_{\frac{p}{r}} \|u\|_s \qquad \text{(by condition 2 Lemma 5)} \\
&\leq cC\,\||f|^r\|_{\frac{p}{r}} = cC\,\|f\|_p^r \qquad (\text{since } \|u\|_s = 1).
\end{aligned}$$

Therefore

$$L^p(w) \simeq \Delta \left\{L^r(\nu w)\right\}_{\nu\in J}. \tag{7.1}$$

By property 3, $\nu w \in A_r$, thus for all $\nu \in J$,

$$T : L^r(\nu w) \to L^r(\nu w)$$

with

$$\|T\|_{\mathcal{L}(L^r(\nu w),L^r(\nu w))} \leq c\,\|\nu w\|_{A^r} \leq \widetilde{c}\,\|w\|_{A^p} \text{ (by condition 3 Lemma 5).}$$

Extrapolating using the $\Delta-$method we obtain

$$T : \Delta \left\{L^r(\nu w)\right\}_{\nu\in J} \to \Delta \left\{L^r(\nu w)\right\}_{\nu\in J},$$

and by (7.1) we thus get

$$T : L^p(w) \to L^p(w) \text{ with } \|T\|_{\mathcal{L}(L^p(w),L^p(w))} \leq \widetilde{c}\,\|w\|_{A^p}.$$

Now consider the case $p < r$. Let $w \in A_p$, $L^p(w) = L^p(d\mu)$ ($d\mu = w(x)dx$). By Lemma 4,

$$L^p(d\mu) = \bigcup \left\{L^r(u^{-1}d\mu)\right\}_{u\in I}.$$

In this case the set J is $\bigcup-$abundant. Indeed, let $f \in L^p(d\mu)$, $0 < c < 1$. Then there exists $u \in I$ such that

$$\begin{aligned}
\|f\|_p^r &\geq c\int |f|^r \frac{d\mu}{u} \geq c\int |f|^r \frac{d\mu}{\nu} \quad \text{(by condition 1 Lemma 5)} \\
&\geq c\frac{\||f|^r\|_{\frac{p}{r}}}{\|\nu\|_s} \quad \left(\text{by Hölder's inequality, recall that } \frac{p}{r} < 1\right) \\
&\geq \frac{c\,\||f|^r\|_{\frac{p}{r}}}{C\,\|u\|_s} \qquad \text{(by condition 2 Lemma 5)} \\
&\geq \frac{c}{C}\,\||f|^r\|_{\frac{p}{r}} = \frac{c}{C}\,\|f\|_p^r \quad (\text{since } \|u\|_s = 1).
\end{aligned}$$

Therefore,

$$L^p(d\mu) \simeq \bigcup \left\{L^r(\nu^{-1}d\mu)\right\}_{\nu\in J}. \tag{7.2}$$

By condition 3 of Lemma 5, $\frac{w}{\nu} \in A_r$, thus

$$T : L^r(\frac{w}{\nu}) \to L^r(\frac{w}{\nu}),$$

is bounded with

$$\|T\|_{\mathcal{L}\left(L^r(\frac{w}{\nu}),L^r(\frac{w}{\nu})\right)} \le c\left\|\frac{w}{\nu}\right\|_{A^r} \le \widetilde{c}\,\|w\|_{A^p} \text{ (by condition 3 Lemma 5).}$$

Extrapolating using the $\sum -$method

$$T : \sum\left\{L^r(\frac{w}{\nu})\right\}_{\nu\in J} \to \sum\left\{L^r(\frac{w}{\nu})\right\}_{\nu\in J}.$$

It follows from (7.2) that $\bigcup\left\{L^r(\nu^{-1}d\mu)\right\}_{\nu\in J}$ is a Banach space, hence by Theorem 1-2,

$$\sum\left\{L^r(\frac{w}{\nu})\right\}_{\nu\in J} = \bigcup\left\{L^r(\nu^{-1}d\mu)\right\}_{\nu\in J}$$

i.e.

$$T : L^p(w) \to L^p(w) \text{ is bounded with } \|T\|_{\mathcal{L}(L^p(w),L^p(w))} \le \widetilde{c}\,\|w\|_{A^p}.$$

□

Remark 16. *We have actually proved that if $r < p$, then T is a strongly $\Delta-$factorizable operator respect to the scale $\{L^r(\nu w)\}_{v\in I}$, and if $r > p$ then is a strongly $\sum -$factorizable operator respect to the scale $\left\{L^r(\nu^{-1}d\mu)\right\}_{\nu\in I}$.*

We now show an analogous result for the so called Calderón weights C_p-weights (cf. [**2**]).

Definition 6. *A weight $w > 0$ belongs C_p,$1 \le p < \infty$ (a **Calderón weight**) if $w \in M_p \bigcap M^p$, where*

$$w \in M_p \text{ i.e. } \sup_{t>0}\left(\int_t^\infty \frac{w(x)}{x^p}dx\right)^{1/p}\left(\int_0^t w(x)^{1-p'}\right)^{1/p'} = \|w\|_{M_p} < \infty.$$

$$w \in M^p \text{ i.e. } \sup_{t>0}\left(\int_0^t w(x)\right)^{1/p}\left(\int_t^\infty \frac{w(x)^{1-p'}}{x^{p'}}dx\right)^{1/p} = \|w\|_{M^p} < \infty.$$

Let

$$\|w\|_{C_p} := \max\left(\|w\|_{M_p}, \|w\|_{M^p}\right).$$

Remark 17. *The M_p-condition is equivalent to the boundedness of the Hardy operator*

$$Pf(t) = \frac{1}{t}\int_0^t f(x)dx \tag{7.3}$$

on $L^p(w)$. The M^p-condition is equivalent to the boundedness of the conjugate Hardy operator

$$Qf(t) = \int_t^\infty f(x)\frac{dx}{x} \tag{7.4}$$

*on $L^p(w)$ (cf. [**20**]).*

Then we have the following extrapolation theorem

THEOREM 16. *(cf. [**2**] Proposition 2.7) Let T be a quasi-linear operator. Let $1 < r < \infty$, and suppose that T is bounded in $L^r(w)$ for every weight $w \in C_r$, with norm that depends only on $\|w\|_{C_r}$. Then, for every $w \in C_p$ $(1 < p < \infty)$, T is bounded on $L^p(w)$.*

We replace Lemma 5 by the following Lemma (cf. [**2**] Lemma 2.6)

LEMMA 6. *Let $1 < r, p < \infty$. Denote by s the exponent given by $\frac{1}{s} = \left|1 - \frac{r}{p}\right|$. Let $w \in C_p$. Then for every $u \geq 0$ in $L^s(w)$ there exists $\nu \geq 0$ in $L^s(w)$, such that*

$$\begin{array}{l} 1)\ u \leq \nu,\ a.e. \\ 2)\ \|\nu\|_{L^s(w)} \leq C\|u\|_{L^s(w)} \\ 3) \begin{cases} \nu w \in C_r & \text{if } r < p \\ \frac{w}{\nu} \in C_r & \text{if } r > p \end{cases} \end{array}$$

Moreover, in either alternative of case 3) above, $\|\nu w\|_{C_r}$ (resp. $\left\|\frac{w}{v}\right\|_{C_r}$) depends only on $\|w\|_{C_p}$.

For linear operators we can easily give proofs of the extrapolation theorems stated in this section directly via factorization. We illustrate this with the following

THEOREM 17. *Let T be a linear operator. Let $1 < r < \infty$, and suppose that T is bounded on $L^2(w)$ for every weight $w \in C_2$. Then, for every $w \in C_p$ $(1 < p < \infty)$, T is bounded on $L^p(w)$.*

PROOF. Let $S = P + Q$ be the Calderón operator (cf. (7.3) and (7.4) above). Let $w \in C_p$, then S is bounded on $L^p(w)$. Suppose that $p > 2$, then $L^p(w)$ is $2-$convex and can be written as $\Delta\{L^2(\nu)\}_{\nu \in I \subset C_2}$. By Theorem 13 S is $\Delta-$strongly diagonal factorizable. By definition $S^d(S) = C_2 \subset S^d(T)$. Therefore, by Theorem 11, T is bounded on $L^p(w)$. Likewise if $p < 2$, $L^p(w)$ is $2-$concave and an analogous argument using the $\sum -$method, Theorem 14 and Theorem 9 allows us to conclude. □

REMARK 18. *In the case of Calderón operators or Hardy operators one can show the factorization properties directly and in an elementary fashion (i.e. without using Grothendieck's inequality) (cf Theorem 18 and Theorem 19 below).*

8. Hardy operators acting on Rearrangement invariant spaces

In this section we study operators acting on rearrangement invariant spaces using extrapolation methods. We focus our attention on Hardy type operators.

Throughout this section we shall work on the measure space $(\mathbf{R}^+, dx)$, (where $\mathbf{R}^+ = [0, \infty)$ and dx will denote the Lebesgue measure on $\mathbf{R}^+$). Given $f, g \in L^0$, $f \prec g$ means that f is less or equal than g in the Hardy-Littlewood order. The following well-known result will be useful in what follows

LEMMA 7. *(Hardy's Lemma (cf. [**4**] Chapter 2, Proposition 3.6)). Let w_0 and w_1 be two non-negative and measurable functions on $\mathbf{R}^+$ and suppose that*

$$\int_0^t w_0(x)dx \leq \int_0^t w_1(x)dx, \text{ for all } t > 0.$$

Then for any decreasing function f,

$$\int_0^\infty f(x)w_0(x)dx \leq \int_0^\infty f(x)w_1(x)dx.$$

Definition 7. *For $0 \leq \lambda < 1$ the Hardy operators $P_{1-\lambda}$ and their corresponding conjugate operators Q_λ, are defined by*

$$P_{1-\lambda}f(t) = \frac{1}{t^{1-\lambda}} \int_0^t f(x)\, \frac{dx}{x^\lambda}, \quad Q_\lambda f(t) = \frac{1}{t^\lambda} \int_t^\infty f(x)\, \frac{dx}{x^{1-\lambda}}, \quad t > 0.$$

Here the fact that $P_{1-\lambda}$ and Q_λ are adjoint operators means that

$$\int_0^\infty P_{1-\lambda}f(t)g(t)dt = \int_0^\infty f(t)Q_\lambda g(t)dt. \tag{8.1}$$

Therefore, if X, Y are rearrangement invariant Banach lattices we have

$$P_{1-\lambda} : X \to Y \ \text{ bounded} \Leftrightarrow Q_\lambda : Y' \to X' \ \text{ bounded.} \tag{8.2}$$

Moreover, since

$$|P_{1-\lambda}f(t)|^* \leq (P_{1-\lambda}\,|f|\,(t))^* \leq P_{1-\lambda}f^*(t),$$

and (cf. [**4**] III-Proposition 5.2)

$$|Q_\lambda f(t)| \prec Q_\lambda f^*(t),$$

It follows that $P_{1-\lambda}$ (resp. Q_λ) is bounded on X, if and only if $P_{1-\lambda}$ (resp. Q_λ) is bounded on decreasing functions (i.e. $\|P_{1-\lambda}f\|_X \leq c\,\|f\|_X$, $\forall f \in X^d$).

For the next result recall that if Z is a rearrangement invariant Banach lattice, $Z = \Delta\{\Lambda(w)\}_{w\in I}$ where $\{\Lambda(w)\}_{w\in I}$ is a $\Delta-$ total scale with index given by $I = \left\{w \in (Z')^d, \|w\|_{Z'} \leq 1\right\}$.

Theorem 18. *Let X, Y be a couple of rearrangement invariant Banach lattices. Let T denote any of the operators $P_{1-\lambda}$ or Q_λ, and let T^* be the corresponding adjoint operator. Then following statements are equivalent:*

(i) $T : X \to Y$ is bounded.

(ii) T is $\Delta-$factorizable.

(iii) T is $\Delta-$strongly factorizable.

Proof. $(i) \Rightarrow (iii)$. Given $w \in I$, let

$$\widetilde{w} = \frac{T^*w}{\|T^*\|_{\mathcal{L}(Y',X')}}.$$

It is plain that $\widetilde{w} \in (X')^d$, $\|\widetilde{w}\|_{X'} \leq 1$, therefore $J = \left\{\widetilde{w} : w \in (Y')^d\right\}$ is $\Delta-$abundant. Moreover, for all decreasing functions f^*

$$\begin{aligned}\|Tf^*\|_{\Lambda(w)} &= \int_0^\infty Tf^*(t)w(t)dt = \int_0^\infty f^*(t)T^*w(t)dt \\ &= \|T^*\|_{\mathcal{L}(Y',X')} \int_0^\infty f^*(t)\widetilde{w}(t)dt.\end{aligned}$$

Therefore $T : \Lambda(\widetilde{w}) \to \Lambda(w)$ is bounded.

$(iii) \Rightarrow (ii)$. Is obvious.

$(ii) \Rightarrow (i)$. Since $X = \Delta\{\Lambda(w)\}_{w\in I_0}$ and $Y = \Delta\{\Lambda(\nu)\}_{\nu\in I_1}$, and both scales are $\Delta-$total, then Theorem 8 applies. □

In the case $X = Y$, we now show that the Hardy operators are $\Delta-$diagonal factorizable operators.

Theorem 19. *Let X be a rearrangement invariant Banach lattice. Let T denote any of the operators $P_{1-\lambda}$ or Q_λ, and let T^* be the corresponding adjoint operator. The following statements are equivalent*

(i) $T : X \to X$ is bounded.

(ii) $\exists C > 0$ and $C' > 0$ such that $\forall w \in (X')^d$ $\exists \widetilde{w} \in (X')^d$ such that $w \prec \widetilde{w}$, $\|\widetilde{w}\|_{X'} \leq C' \|w\|_{X'}$, and $T : \Lambda(\widetilde{w}) \to \Lambda(\widetilde{w})$ is bounded with $\|T\|_{\mathcal{L}(\Lambda(\widetilde{w}),\Lambda(\widetilde{w}))} \leq C$. Moreover,

$$\inf C = \rho_X(T).$$

(iii) $\exists C, C' > 0$ such that $\forall w \in X^d$ $\exists \widetilde{w} \in X^d$ such that $w \prec \widetilde{w}$, $\|\widetilde{w}\|_X \leq C' \|w\|_X$, and $T^* : \Lambda(\widetilde{w}) \to \Lambda(\widetilde{w})$ is bounded with $\|T^*\|_{\mathcal{L}(\Lambda(\widetilde{w}),\Lambda(\widetilde{w}))} \leq C$. Moreover,

$$\inf C = \rho_X(T^*).$$

In cases (ii) and (iii) the infimum is taken over all $C > 0$ such that claim (ii) (resp. (iii)) is satisfied. ($\rho_X(T)$ = spectral radius of T as operator from X to X).

Proof. (i) $\to$(ii) Let $C > \rho_X(T)$. Given $w \in (X')^d$ let us define

$$\widetilde{w} = \sum_{n=1}^{\infty} \frac{T^{*(n)} w}{C^{n-1}},$$

where $T^{*(n)} = T^* \circ \overset{n}{\cdots} \circ T^*$. Since $C > \rho_X(T) = \rho_{X'}(T^*)$, it follows from Gelfand's spectral radius formula that

$$C' = C \sum_{n=0}^{\infty} \frac{\left\|T^{*(n)}\right\|_{\mathcal{L}(X',X')}}{C^n} < \infty.$$

Thus

$$\|\widetilde{w}\|_{X'} \leq C \sum_{n=0}^{\infty} \frac{\left\|T^{*(n)}\right\|_{\mathcal{L}(X',X')}}{C^n} \|w\|_X = C' \|w\|_X .$$

Obviously $\widetilde{w}$ is decreasing and $w \prec \widetilde{w}$, since

$$\frac{1}{r}\int_0^r \widetilde{w}(x)dx \geq \frac{1}{r}\int_0^r T^* w(x)dx \geq \frac{1}{(1-\lambda)r}\int_0^r w(x)dx \geq \frac{1}{r}\int_0^r w(x)dx.$$

Moreover, since

$$T^*\widetilde{w} = \sum_{n=1}^{\infty} \frac{T^{*(n+1)} w}{C^{n-1}} = C\sum_{n=1}^{\infty} \frac{T^{*(n+1)} w}{C^{n}} \leq C\widetilde{w},$$

we have that

$$\int_0^\infty Tf(x)\widetilde{w}(x)dx = \int_0^\infty f(x)T^*\widetilde{w}(x)dx \leq C\int_0^\infty f(x)\widetilde{w}(x)dx,$$

that is

$$T : \Lambda(\widetilde{w}) \longmapsto \Lambda(\widetilde{w}) \text{ is bounded with } \|T\|_{\mathcal{L}(\Lambda_1(\widetilde{w}),\Lambda_1(\widetilde{w}))} \leq C.$$

The fact $\inf C = \rho_X(T)$ will follow readily from the previous computation if we see that for any $C < \rho_X(T)$ (ii) is not satisfied. Suppose not, i.e. then for some

$C < \rho_X(T)$ condition (ii) holds. Then, since for all $f \in X^d$, $T^{(n)}f$ is decreasing, we get

$$\begin{aligned}\int_0^\infty T^{(n)}f(x)w(x)dx &\le \int_0^\infty T^{(n)}f(x)\widetilde{w}(x)dx \quad (\text{since } w \prec \widetilde{w})\\ &\le C^n \int_0^\infty f(x)\widetilde{w}(x)dx\\ &\le C^n \|f\|_X \|\widetilde{w}\|_{X'} \quad (\text{by Hölder's inequality})\\ &\le C^n C' \|f\|_X \|w\|_{X'}.\end{aligned}$$

Thus

$$\left\|T^{(n)}\right\|_{\mathcal{L}(X,X)} \le C^n C'$$

which by Gelfand's formula for the spectral radius, implies that

$$\rho_X(T) = \inf_{n\ge1}\left(\left\|T^{(n}\right\|_{\mathcal{L}(X,X)}\right)^{1/n} \le C.$$

A contradiction since we are assuming that $C < \rho_X(T)$.

(ii) $\to$ (i) is obvious.

The proof of $(i) \Leftrightarrow (iii)$ follows in the same way that the previous one applying that

$$T : X \to X \ \text{ bounded} \Leftrightarrow T^* : X' \to X' \ \text{ bounded.}$$

$\square$

REMARK 19. *There is an elegant result due to D. W. Boyd (see* [**6**] *and* [**7**]*) that gives the spectral radius of Hardy operators in terms of the growth of the operator norm of the dilation operators* $D_\alpha f(t) = f\left(\frac{t}{\alpha}\right)$ *as* $\alpha \to 0$ *and* $\alpha \to \infty$*. Boyd's result states that if* X *is a rearrangement invariant Banach lattice then*

$$\rho_X(P_{1-\lambda}) = \frac{1}{1-\lambda-\overline{\alpha}_X} \quad \text{and} \quad \rho_X(Q_\lambda) = \frac{1}{\lambda-\underline{\alpha}_X},$$

where

$$\overline{\alpha}_X = \lim_{\alpha\to\infty}\frac{\ln h_X(\alpha)}{\ln\alpha}, \ \underline{\alpha}_X = \lim_{\alpha\to0}\frac{\ln h_X(\alpha)}{\ln\alpha}$$

(here $h_X(\alpha)$ *is the norm of the dilatation operator* D_α *and* $\rho_X(T) =$ *spectral radius of* T *as operator from* X *to* X*).*

In particular

$$\begin{aligned}P_{1-\lambda} : X \to X \ \textit{is bounded} \ &\Leftrightarrow \overline{\alpha}_X < 1-\lambda\\ Q_\lambda : X \to X \ \textit{is bounded} \ &\Leftrightarrow \underline{\alpha}_X > \lambda\end{aligned}$$

9. Interpolation theory via Extrapolation

We consider the connection between the extrapolation theory developed in this paper and interpolation theory. Let us start by considering a special, but significant example. Let T be a quasi-linear operator such that T defines a bounded operator

$$T : L^1 \to L^{1,\infty} \text{ and } T : L^\infty \to L^\infty. \tag{9.1}$$

It is well-known that (9.1) is equivalent to the existence of a constant $C > 0$ such that

$$(Tf)^*(t) \le C\frac{1}{t}\int_0^t f^*(x)dx = CPf^*(t). \tag{9.2}$$

Hence, for any scale $\{\Lambda(w)\}_{w\in I}$ (9.2) implies that

$$S^d(P) \Subset S^d(T)$$

and since P is strongly $\Delta-$factorizable, $T : \Delta\{\Lambda(w)\}_{w\in I} \to \Delta\{\Lambda(w)\}_{w\in I}$ is bounded.

Let us now compute the signature of the operators $P_{1-\lambda}$ and Q_λ with respect to the scale $\{\Lambda(w)\}_{w\in I}$, where I is an index of decreasing functions. (By $w\downarrow$ we will denote that w is decreasing).

THEOREM 20. *The following statements hold:*
(i) $S(P_{1-\lambda}) = \{(w_0,w_1), w_i \downarrow \ ; P\circ Q_\lambda w_0(r) \le CPw_1(r) \text{ for all } r>0\}$.
(ii) $S(Q_\lambda) = \{(w_0,w_1), w_i \downarrow ; P_{1-\lambda}w_0(r) \le CPw_1(r) \text{ for all } r>0\}$.
(iii) $S(Q) = \{(w_0,w_1), w_i \downarrow ; P\circ Pw_0(r) \le CPw_1(r) \text{ for all } r>0\}$.
(iv) $S^d(P_{1-\lambda}) = \{w \downarrow ; Q_\lambda w(r) \le CPw(r) \text{ for all } r>0\}$.
(v) $S^d(Q_\lambda) = \{w\downarrow ; Q_\lambda w(r) \le CPw(r) \text{ for all } r>0\}$.
(vi) $S^d(Q) = \{w \downarrow ; P\circ Pw(r) \le CPw(r) \text{ for all } r>0\}$.

PROOF. (i) $w_0, w_1 \in S(P_{1-\lambda})$ if and only if

$$P_{1-\lambda} : \Lambda(w_0) \to \Lambda(w_1) \text{ is bounded.}$$

We need to show that

$$P_{1-\lambda} : \Lambda(w_0) \to \Lambda(w_1) \text{ bounded } \Leftrightarrow \exists C>0, P\circ Q_\lambda w_0(r) \le CPw_1(r),\ r>0.$$

($\Rightarrow$) Follows by testing with the functions $\chi_{[0,r]}$. Conversely, if for all $r>0$,

$$\int_0^r Q_\lambda w_0(s)ds \le C\int_0^r w_1(s)ds$$

then by Lemma 7

$$\int_0^\infty f^*(s)Q_\lambda w_0(s)ds \le C\int_0^\infty f^*(s)w_1(s)ds.$$

But $P_{1-\lambda}$ and Q_λ are adjoint operators:

$$\int_0^\infty P_{1-\lambda}f^*(s)w_0(s)ds \le C\int_0^\infty f^*(s)w_1(s)ds.$$

Thus,

$$P_{1-\lambda} : \Lambda(w_0) \to \Lambda(w_1), \text{ with } \|P_{1-\lambda}\|_{\mathcal{L}(\Lambda(w_0),\Lambda(w_1))} \le C.$$

(ii) and (iii) are proved in the same way.

(iv) Recall that $w \in S^d(P_{1-\lambda})$ if and only if $P_{1-\lambda} : \Lambda_1(w) \to \Lambda_1(w)$ is bounded. Therefore we need to prove the following statement

$$P_{1-\lambda} : \Lambda(w) \to \Lambda(w) \text{ bounded } \Leftrightarrow\ Q_\lambda w(r) \le cPw(r),\ r>0.$$

($\Rightarrow$) Follows by testing with the functions $\chi_{[0,r]}$. To prove the converse note that by Fubini's Theorem it follows that

$$P\circ Q_\lambda = \frac{P+Q}{1-\lambda},$$

which combined with $Q_\lambda w \le cPw$, yields

$$\int_0^r Q_\lambda w(s)ds \le \frac{c+1}{1-\lambda}\int_0^r w(s)ds.$$

Using Lemma 7 and the fact that $P_{1-\lambda}$ and Q_λ are adjoint operators gives

$$\int_0^\infty P_{1-\lambda} f^*(s) w(s) ds = \int_0^\infty f^*(s) Q_\lambda w(s) ds$$
$$\leq \frac{c+1}{1-\lambda} \int_0^\infty f^*(s) w(s) ds.$$

(v) and (vi) are proved in an analogous fashion. □

Following the notation introduced in [**1**] we will say that

$$w \in B_{1-\lambda} \Leftrightarrow Q_\lambda w \leq cPw.$$

By combining the previous Theorem and Boyd's theory (see Remark 19 above) we obtain

THEOREM 21. *Let T be a linear operator, and let X, Y be a couple of rearrangement invariant Banach lattices. Then*

(i) If $S(P_{1-\lambda}) \subset S(T)$ and $P_{1-\lambda} : X \to Y$ is bounded, then $T : X \to Y$, is bounded.

(ii) If $S(Q_\lambda) \subset S(T)$ and $Q_\lambda : X \to Y$ is bounded, then $T : X \to Y$, is bounded.

(iii) If $S(P_{1-\lambda_0}) \bigcap S(Q_{\lambda_1}) \subset S(T)$ and $P_{1-\lambda_0} \circ Q_{\lambda_1} : X \to Y$ is bounded, then $T : X \to Y$, is bounded.

(iv) If $S^d(P_{1-\lambda}) \subset S^d(T)$, then $T : X \to X$ is bounded for all rearrangement invariant Banach lattices X such that $\overline{\alpha}_X < 1 - \lambda$.

(v) If $S^d(Q_\lambda) \subset S^d(T)$, then $T : X \to X$, is bounded on all rearrangement invariant Banach lattices X such that $\underline{\alpha}_X > \lambda$.

(vi) If $S^d(P_{1-\lambda_0}) \bigcap S^d(Q_{\lambda_1}) \subset S^d(T)$, then $T : X \to X$ is bounded on all rearrangement invariant Banach lattices X such that $\overline{\alpha}_X < 1 - \lambda_0$ and $\underline{\alpha}_X > \lambda_1$.

PROOF. $(i), (ii), (iii)$ follow from the fact that (see Lemma 2)

$$X = \Delta\{\Lambda(w)\}_{w \in I_0} \text{ and } Y = \Delta\{\Lambda(\nu)\}_{w \in I_1},$$

where $I_0 = \left\{ w \in (X')^d, \|w\|_{X'} \leq 1 \right\}$ and $I_1 = \left\{ \nu \in (Y')^d, \|\nu\|_{Y'} \leq 1 \right\}$. Moreover, since both scales are $\Delta-$total and $T, P_{1-\lambda}$ and Q_λ are linear operators, Theorem 8 applies.

(iv) If X has upper Boyd index $\overline{\alpha}_X < 1-\lambda$ then (cf. Remark 19) $P_{1-\lambda} : X \to X$ is bounded and by hypothesis $S(P_{1-\lambda}) \subset S(T)$. Thus Theorem 8 applies.

(v) and (vi) are proved in the same way. □

If T is a quasi-linear operator we have the following result

THEOREM 22. *Let T be a quasi-linear operator and let X, Y be a couple of rearrangement invariant Banach lattices. Then*

(i) $S(P_{1-\lambda}) \Subset S(T)$ and $P_{1-\lambda} : X \to Y$ is bounded, then $T : X \to Y$, is bounded.

(ii) $S(Q_\lambda) \Subset S(T)$ and $Q_\lambda : X \to Y$ is bounded, then $T : X \to Y$ is bounded.

(iii) $S(P_{1-\lambda_0}) \bigcap S(Q_{\lambda_1}) \Subset S(T)$ and $P_{1-\lambda_0} \circ Q_{\lambda_1} : X \to Y$ is bounded, then $T : X \to Y$ is bounded.

(iv) $S^d(P_{1-\lambda}) \Subset S^d(T)$, then $T : X \to X$ is bounded on all rearrangement invariant Banach lattices X such that $\overline{\alpha}_X < 1 - \lambda$.

(v) $S^d(Q_\lambda) \Subset S^d(T)$, then $T : X \to X$, is bounded on all rearrangement invariant Banach lattices X such that $\underline{\alpha}_X > \lambda$.

(vi) $S^d(P_{1-\lambda_0})\bigcap S^d(Q_{\lambda_1}) \Subset S^d(T)$, *then* $T: X \to X$ *is bounded on all rearrangement invariant Banach lattices* X *such that* $\overline{\alpha}_X < 1-\lambda_0$ *and* $\underline{\alpha}_X > \lambda_1$.

Proof. The results follows from the fact that $P_{1-\lambda}$ and Q_λ are strongly $\Delta-$factorizable operators, Remark 19 and Theorem 10. $\square$

Remark 20. *Note that statement* (iv) *gives us an extrapolation theorem for* $B_{1-\lambda}-$*weights since* (iv) *states that if* $T:\Lambda(w)\to\Lambda(w)$ *is bounded for all decreasing weights* $w \in B_{1-\lambda}$ *then* $T: X \longmapsto X$ *for all rearrangement invariant Banach lattices* X *such that* $\overline{\alpha}_X < 1-\lambda$.

An important class of rearrangement invariant spaces that has been widely studied in the last decade are the classical **Lorentz spaces** $\Lambda^p(w)$ $(1 \le p < \infty)$ defined by

$$\Lambda^p(w) = \left\{ f : \|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(x)^p w(x)dx\right)^{1/p} < \infty \right\}.$$

It is well known (cf. [**1**]) that $\Lambda^p(w)$ is a Banach space if and only if $w \in B_p$, i.e. if there exits $c>0$ such that for all $r>0$

$$Q_p w(r) := r^{p-1}\int_r^\infty w(x)\frac{dx}{x^p} \le cPw(r).$$

Moreover,

$$\|f\|_{\Lambda^p(w)} \approx \|f^{**}\|_{\Lambda^p(w)}.$$

Using Lemma 6 we obtain the following extrapolation theorem for operators acting in Lorentz spaces with C_p weights, which again combines features of the Rubio de Francia and the Jawerth-Milman methods (cf. [**2**] Theorem 5.2 for a related result)

Theorem 23. *Let* T *be a quasi-linear operator. Let* $1<r<\infty$, *and suppose that* T *is bounded in* $\Lambda^r(w)$ *for every weight* $w \in C_r$, *with norm that depends only on* $\|w\|_{C_r}$. *Then, for every* $w\in C_p$ $(1<p<\infty)$, T *is bounded on* $\Lambda^p(w)$.

Proof. Given $w \in C_p$, using Lemma 6 and with the same proof of Lemma 4 we get

$$L^p(w) = \begin{cases} \Delta\{L^r(w\nu)\}_{\nu\in I} & \text{if } p>r \\ \cup\{L^r(w/\nu)\}_{\nu\in I} & \text{if } p<r \end{cases}$$

(with $\|w\nu\|_{C_r} \le C\,\|w\|_{C_p}$ and $\|w/\nu\|_{C_r} \le C\,\|w\|_{C_p}$).

Obviously $f \in \Lambda^p(w) \Leftrightarrow f^* \in L^p(w)$, thus

$$\Lambda^p(w) = \begin{cases} \Delta\{\Lambda^r(w\nu)\}_{\nu\in I} & \text{if } p>r \\ \bigcup\{\Lambda^r(w/\nu)\}_{\nu\in I} & \text{if } p<r \end{cases}$$

moreover all the Lorentz spaces involved are Banach spaces since $C_s \subset B_s$, $s \ge 1$. Now we can finish the proof as in Theorem 15. $\square$

As was observed at the beginning of this section, the signature of the Calderón operator $S=P+Q$ plays a central role. It is well known (cf. [**2**] and the references quoted therein) that $S:\Lambda^p(w)\to\Lambda^p(w)$ is bounded iff $w\in B_p\cap B^\infty$, i.e. there is $c>0$ such that for all $r>0$

$$Q_p w(r) \le cPw(r) \text{ and } PPw(r) \le cPw.$$

Thus $S^d(S)$ with respect to the scale $\{\Lambda^p(w)\}_{w\in I}$ is given by

$$S^d(S) = \{w : w \in B_p \cap B^\infty\}.$$

It is easy to see $C_p \subset B_p \cap B^\infty$ and $B_p \cap B^\infty$ is strictly smaller that C_p.

We shall see now that given $w \in B_p \cap B^\infty$ there exists $\tilde{w} \in C_p$ such that $\Lambda^p(w) = \Lambda^p(\tilde{w})$ and hence

$$S^d(S) = \{w : w \in C_p\}.$$

To prove this claim, let us recall (cf. [**2**]) that $w \in B_p \cap B^\infty$ if and only if there are two positive constants, c_0 and c_1 such that

$$c_0 Pw(r) \le Q_p w(r) \le c_1 Pw(r), \quad \forall r > 0.$$

Now we use again the iteration method. Let $\varepsilon > 0$ such that $\varepsilon c_1 < 1$, and define

$$\tilde{w} = \sum_{n\ge 1} \varepsilon^{n-1} Q_p^{(n)} w.$$

A standard argument (see for example [**8**]) shows that with this choice of ε we have that

$$\sum_{n\ge 1} \varepsilon^{n-1} Q_p^{(n)} w = Q_{p-\varepsilon} w.$$

By Fubini's theorem,

$$\begin{aligned} P\tilde{w} = PQ_{p-\varepsilon} w &= \frac{1}{p-\varepsilon}(Pw + Q_{p-\varepsilon} w) \\ &\le \frac{1}{(p-\varepsilon)\, c_0}(Q_p w + Q_{p-\varepsilon} w) \\ &\le \frac{2}{(p-\varepsilon)\, c_0}\tilde{w}. \end{aligned}$$

And since

$$Q_p \tilde{w} = \sum_{n=1}^{\infty} \varepsilon^{n-1} Q_p^{(n+1)} w \le \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \varepsilon^{n-1} Q_p^{(n)} w \le \frac{1}{\varepsilon}\tilde{w},$$

we get

$$P\tilde{w} + Q_p \tilde{w} \le C\tilde{w}.$$

Therefore by Proposition 2.8 of [**2**] we get that $\tilde{w} \in C_p$.

Finally, since

$$P\widetilde{w} = \sum_{n=1}^{\infty} \varepsilon^{n-1} PQ_p^{(n)} w \le \sum_{n\ge 1} \varepsilon^{n-1} c_1^n Pw = CPw$$

and

$$P\tilde{w} = PQ_{p-\varepsilon} w \ge \frac{1}{p-\varepsilon} Pw$$

we get

$$Pw \approx P\widetilde{w}$$

which by Lemma 7 implies that

$$\Lambda^p(w) = \Lambda^p(\tilde{w}).$$

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