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VOLUME ENTROPY, WEIGHTED GIRTHS AND STABLE BALLS ON GRAPHS

FLORENT BALACHEFF

ABSTRACT. We prove new isoperimetric inequalities on graphs involving quantities linked with concepts from differential geometry. First we bound from above the product of the volume entropy (defined as the log of the exponential growth rate of the universal cover of the graph) and the girth of weighted graphs in terms of their cyclomatic number. In a second part, we study a natural polyhedron associated to a weighted graph: the stable ball. In particular we relate the volume of this polyhedron, the weight of the graph and its cyclomatic number.

1. INTRODUCTION

Certain isoperimetric inequalities valid on manifolds can sometimes be extended to graphs. For example, the asymptotic behaviour of the systolic volume of a surface in terms of its genus, investigated in [5] and [6], has a 1-dimensional analog valid on weighted graphs [4].

Namely, let (Γ, w) be a weighted graph that contains a cycle (see subsection 2.1 for a precise definition). The *weight* of (Γ, w) (or *1-volume*), denoted by $w(\Gamma)$, is the sum of the weight of its edges

$$w(\Gamma) = \sum_{e \in E(\Gamma)} w(e).$$

The *weighted girth* of (Γ, w) (or *systole*) is defined as

$$g_w(\Gamma) = \inf \{w(C) \mid C \text{ cycle of } \Gamma\},$$

where the *weight* (or *weighted length*) of a cycle C , denoted by $w(C)$, is the sum of the weights of its edges.

With this two quantities, we can define the following isoperimetric invariant of Γ

$$\sigma(\Gamma) = \inf_w \frac{w(\Gamma)}{g_w(\Gamma)}.$$

where the infimum is taken over all weight functions w on Γ .

Theorem (Bollobás & Szemerédi [4]) *Let Γ be a graph of cyclomatic number $b \geq 3$. Then*

$$\sigma(\Gamma) \geq \frac{3 \ln 2}{2} \frac{b-1}{\ln(b-1) + \ln \ln(b-1) + 4 \ln 2 - \ln \ln 2}.$$

This lower bound gives an asymptotic lower estimate in terms of the cyclomatic number b :

$$\sigma(\Gamma) \geq \frac{3 \ln 2}{2} \frac{b}{\ln b} + o\left(\frac{b}{\ln b}\right). \quad (1.1)$$

On the other hand, for each $b \geq 2$, we can construct a weighted graph (Γ, w) (see [1]) of cyclomatic number b such that

$$\frac{w(\Gamma)}{g_w(\Gamma)} \leq 8 \ln 2 \frac{b}{\ln b}.$$

Thus, the estimate (1.1) gives the asymptotic behaviour of this isoperimetric quantity in terms of the cyclomatic number. This is the analog of an estimate on surfaces, which states the following. Let S be a closed surface of genus ζ . The infimum ratio among all Riemannian metrics of the volume of the surface over the square of the systole (defined as the length of the shortest noncontractible loop) is denoted by $\sigma(S)$ and satisfies the following inequality (see [6]):

$$\sigma(S) \geq \pi \frac{\zeta}{(\ln \zeta)^2} + o\left(\frac{\zeta}{(\ln \zeta)^2}\right). \quad (1.2)$$

In [5], the authors prove that the order of the lower bound (1.2) is the good one.

The cyclomatic number of a graph and the genus of a surface play the same role in these inequalities: they measure the dimension of the first homology group. Recall that for a graph Γ , the first homology group coincides with the *real cycle space* (that is the real vector space spanned by a cycle basis). To see this, consider $E(\Gamma)$, the set of edges of Γ . For each edge, choose an arbitrary orientation and denote by $\mathbb{E}(\Gamma) = \{e_1, \dots, e_k\}$ the set of oriented edges ($k = |E(\Gamma)|$ where $|\cdot|$ denotes the cardinality of a set). The vector space spanned by $\mathbb{E}(\Gamma)$

$$\mathcal{C}(\Gamma, \mathbb{R}) = \left\{ \sum_{i=1}^k a_i \cdot e_i \mid a_i \in \mathbb{R} \right\}$$

is the space of simplicial chains of Γ viewed as a simplicial complex (see [11]). In this one dimensionnal case, the first homology group $H_1(\Gamma, \mathbb{R})$ embeds in $\mathcal{C}(\Gamma, \mathbb{R})$ as a subspace which is exactly the real cycle space $Z(\Gamma, \mathbb{R})$. The first Betti number (the dimension of $H_1(\Gamma, \mathbb{R})$) is thus equal to the dimension of $Z(\Gamma, \mathbb{R})$ which is the cyclomatic number.

In this article, we are first interested in the normalization of volume entropy by weighted girth. Let $(\tilde{\Gamma}, \tilde{w})$ be the universal (weighted) cover of (Γ, w) . Fix a vertex x_0 of Γ and \tilde{x}_0 a lift of this vertex to $\tilde{\Gamma}$. The *volume entropy* (or *asymptotic volume*) of (Γ, w) is defined as

$$h_{vol}(\Gamma, w) = \lim_{R \rightarrow +\infty} \frac{\ln \tilde{w}(B(\tilde{x}_0, R))}{R} \quad (1.3)$$

where $B(\tilde{x}_0, R)$ is the maximal subgraph of $(\tilde{\Gamma}, \tilde{w})$ whose vertices are at distance at most R from \tilde{x}_0 (for the weighted distance $d_{\tilde{w}}$ associated to the weight function \tilde{w}). Since the weighted graph (Γ, w) is compact, the limit in (1.3) exists and does not depend on the vertex x_0 of Γ and its lift (see [8]).

The product $h_{vol}(\Gamma, w) \cdot g_w(\Gamma)$ is invariant under scaling and its higher dimensional analog has been studied in [10] for manifolds. The author proves that this quantity is bounded from above for each surface and is not bounded for manifolds of dimension more than three. We will prove the following upper bound for weighted graphs:

Theorem 1 *Let (Γ, w) be a weighted graph of cyclomatic number $b \geq 1$. Then*

$$h_{vol}(\Gamma, w) \cdot g_w(\Gamma) \leq 2 \ln(8b^3 - 1).$$

The order of this bound is asymptotically optimal and can be improved to $3 \ln b$ for regular graphs with weight function $\equiv 1$ (see propositions 2.1 and 2.5).

In another direction, we can endow the real cycle space of a weighted graph (Γ, w) with a natural norm called the *stable norm* defined by

$$\|v\|_w = \sum_{i=1}^k |a_i| w(e_i)$$

for $v = \sum_{i=1}^k a_i e_i \in Z(\Gamma, \mathbb{R})$. The unit ball $\mathcal{B}_{st}(\Gamma, w)$ of this norm is called the *stable ball* of (Γ, w) and coincides with the polyhedron

$$\text{conv} \left\{ \frac{C_1}{w(C_1)}, \dots, \frac{C_{2m}}{w(C_{2m})} \right\} \subset Z(\Gamma, \mathbb{R})$$

where *conv* is the convex hull of a set and $\{C_j\}_{j=1}^{2m}$ is the set of all the oriented cycles of Γ (see [2]). Note that if C is an oriented cycle, $-C$ is a different oriented cycle. Examples of stable balls of elementary graphs are drawn in Figures 1 and 2. Observe that this polyhedron depends both on combinatorial properties of Γ and on the weight function w .

We now define a measure on $Z(\Gamma, \mathbb{R})$ in the following way. Let $\langle \cdot, \cdot \rangle_w$ be the scalar product defined on $\mathcal{C}(\Gamma, \mathbb{R})$ by

$$\langle e_i, e_j \rangle_w = \sqrt{w(e_i)w(e_j)} \delta_{ij},$$

where δ_{ij} denotes the Kronecker symbol and $e_i, e_j \in \mathbb{E}$. The restriction of this scalar product to the subspace $Z(\Gamma, \mathbb{R})$ is still a scalar product and we denote by μ_w the associated measure. This measure coincides with the restriction to $Z(\Gamma, \mathbb{R})$ of the Lebesgue measure with a specific normalisation depending on the weight function w .

We can naturally associate to a graph a weighted graph in which all the edges have weight 1. In this case $w(\Gamma) = |E(\Gamma)|$ and we omit the letter w in the expressions $\mathcal{B}_{st}(\Gamma, w)$ and μ_w . For graphs we obtain the following inequality.

Theorem 2 *Let Γ be a graph of cyclomatic number $b \geq 1$. Then*

$$\frac{2^b}{b!} \left(\frac{b}{|E(\Gamma)|} \right)^b \leq \mu(\mathcal{B}_{st}(\Gamma)) \leq \frac{2^b}{b!}.$$

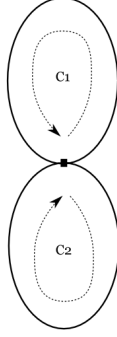
The two cases of equality are reached by the wedge of b circles $\bigvee_{i=1}^b S_i^1$.

We also obtain an inequality for weighted graphs.

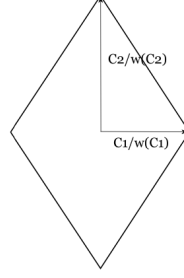
Theorem 3 *Let (Γ, w) be a weighted graph of cyclomatic number $b \geq 1$. Then*

$$(\mu_w(\mathcal{B}_{st}(\Gamma, w)))^{2/b} \cdot w(\Gamma) \geq \omega_b^{2/b}$$

where ω_b is the volume of the Euclidean unit ball of \mathbb{R}^b .

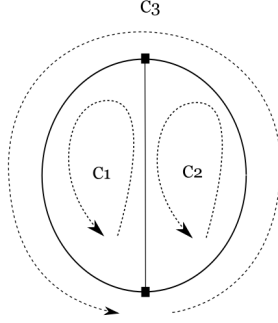


The graph 81

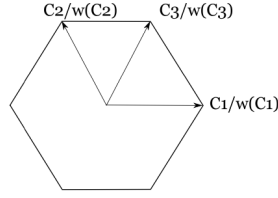


The corresponding stable ball

FIGURE 1



The graph 82



The corresponding stable ball

FIGURE 2

As $\omega_b^{2/b} \simeq (2\pi e)/b$ when b goes to infinity, the order of this estimate is optimal because for the wedge of b circles $\bigvee_{i=1}^b S_i^1$ endowed with the constant weight function $w \equiv 1$ this product is equivalent to $(2e)^2/b$.

The first part of this paper is dedicated to the study of volume entropy and contain the proof of Theorem 2.4. In the second part, we prove Theorems 3.1 and 3.2.

2. VOLUME ENTROPY, GIRTHS AND SCALES OF GRAPHS

2.1. Preliminaries. By a graph we mean a finite non-oriented multigraph (we allow multiple edges and loops). For a graph Γ we denote by $V(\Gamma)$ the set of its vertices and by $E(\Gamma)$ the set of its edges. A *weighted graph* is a pair (Γ, w) , where Γ is a graph and $w : E(\Gamma) \rightarrow \mathbb{R}_+^*$ is a weight function. For $e \in E(\Gamma)$, we call $w(e)$ the weight of e .

We recall the definition of the exponential growth rate of a group (see [7]): we will use it in the proof of Theorem 2.4, and in the proof of proposition 2.5 and 2.8.

Let G be a group of finite presentation and Σ be a finite generating set. We define the *algebraic length* of an element α of G with respect to Σ as the smallest integer l such that $\alpha = \alpha_1 \dots \alpha_l$, where $\alpha_i \in \Sigma \cup \Sigma^{-1}$. It is denoted by $|\alpha|_\Sigma$.

The *exponential growth rate* of G with respect to the system Σ is defined as

$$\omega(G, \Sigma) = \lim_{R \rightarrow +\infty} N_\Sigma(R)^{1/R}$$

where $N_\Sigma(R) = \text{card}\{\alpha \in G \mid |\alpha|_\Sigma \leq R\}$ is the cardinal of the ball of radius R of $(G, |\cdot|_\Sigma)$ centered at its origin. For a group of finite presentation G and a finite generating set Σ of G ,

$$\omega(G, \Sigma) \leq 2 \cdot \text{card}(\Sigma) - 1. \quad (1.3)$$

2.2. Volume entropy and girths of regular graphs. In this subsection, we establish an upper bound of the product volume entropy times the girth for regular graphs which is better than the bound of Theorem 2.4. The *valence* of a vertex of a graph is defined as the number of incident edges at this vertex. Let $v \geq 2$ be an integer. We say that a graph Γ is *regular* of valence v if the valence of each vertex is constant equal to v . Recall that we can naturally associate to a graph a weighted graph in which all the edges have weight 1. In this case $w(\Gamma) = |E(\Gamma)|$ and we denote $g(\Gamma)$ and $h_{vol}(\Gamma)$ the corresponding girth and volume entropy.

We will show the following result.

Proposition 2.1. *Let Γ be a regular graph of cyclomatic number $b \geq 1$. Then*

$$h_{vol}(\Gamma) \cdot g(\Gamma) \leq 3 \ln b. \quad (1.4)$$

Proof. Let v be the valence of Γ . We have

$$h_{vol}(\Gamma) = \ln(v - 1). \quad (1.5)$$

To see this, fix some vertex \tilde{x}_0 in the universal cover $\tilde{\Gamma}$. As $\tilde{\Gamma}$ is an infinite regular tree of valence v ,

$$\begin{aligned} |B(\tilde{x}_0, R)| &= v(1 + (v - 1) + \dots + (v - 1)^{R-1}) \\ &= v \frac{(v - 1)^R - 1}{v - 2} \end{aligned}$$

for each positive integer R . We deduce (1.5).

In the case $b = 1$, we obtain $v = 2$ and so $h_{vol}(\Gamma) = 0$. The inequality (1.4) is then trivial.

In the case $g(\Gamma) = 1$ we have $b = |E(\Gamma)| - |V(\Gamma)| + 1$. By elementary considerations $2 \cdot |E(\Gamma)| = v \cdot |V(\Gamma)|$ and $|V(\Gamma)| \geq 1$, so we get $b \geq v/2$. Therefore,

$$\begin{aligned} h_{vol}(\Gamma) &= \ln(v - 1) \\ &\leq \ln(2b - 1) \\ &\leq 3 \ln b, \end{aligned}$$

and (1.4) follows in this case.

Now suppose that $b > 1$ and $g(\Gamma) > 1$. We will show the following lemma.

Lemme 2.2.

$$g(\Gamma) \leq \frac{3 \ln b}{\ln(v - 1)}. \quad (1.6)$$

Proof of Lemma 2.2. For all $R < g(\Gamma)/2$, the ball centered at any vertex x of Γ of radius R is a tree. Thus the calcul of the weight of a ball centered in a vertex of radius $[g(\Gamma)/2]$ gives the following estimate (compare with [3], p.14)

$$|E(\Gamma)| \geq v \frac{(v-1)^{[g(\Gamma)/2]} - 1}{v-2}.$$

With $|E(\Gamma)| = v(b-1)/(v-2)$, we deduce

$$[g(\Gamma)/2] \leq \frac{\ln b}{\ln(v-1)}.$$

Then

$$g(\Gamma) \leq 1 + 2 \frac{\ln b}{\ln(v-1)} \leq 3 \frac{\ln b}{\ln(v-1)},$$

and we are done.

Now we combine inequalities (1.5) and (1.6) to get the inequality (1.4). \square

The asymptotic behaviour of (1.4) is optimal when b goes to infinity. It is realized by the wedge of b circles (the graph with one vertex and b loops of weight 1).

In the case of a graph provided with some control on the valence of its vertices, we easily obtain a lower and an upper bound for the volume entropy. We denote by $v(s)$ the valence of a vertex s .

Proposition 2.3. *Let Γ be a graph. Suppose that there exists two integers $2 \leq \delta \leq \Delta$ such that for all $s \in V(\Gamma)$, $\delta \leq v(s) \leq \Delta$. Then*

$$\ln(\delta - 1) \leq h_{vol}(\Gamma) \leq \ln(\Delta - 1).$$

2.3. Volume entropy and girths of weighted graphs. We now generalize (1.4) to weighted graphs.

Theorem 2.4. *Let (Γ, w) be a weighted graph of cyclomatic number b . Then*

$$h_{vol}(\Gamma, w) \cdot g_w(\Gamma) \leq 2 \ln(8b^3 - 1). \quad (1.7)$$

Proof. The proof involves techniques inspired by [12]. Denote by $p : \tilde{\Gamma} \rightarrow \Gamma$ the universal covering map. We call *essential domain* of the universal cover $\tilde{\Gamma}$ a maximal (for inclusion) connected subgraph $D \subset \tilde{\Gamma}$ such that the restriction of the projection p to D is injective. So $p(D)$ is a spanning tree. Given a vertex x_0 of Γ , we can define the *fundamental group* of Γ at x_0 denoted by $\pi_1(\Gamma, x_0)$ as the quotient of the set of closed walks based at x_0 by the following equivalence relation. Let P_1 and P_2 be two closed walks. Each of these walks write as a word whose letters are elements of $\mathbb{E}(\Gamma)$ (the set of oriented edges). To each word we can associate a reduced word which is obtained by deleting sequences ee^{-1} or $e^{-1}e$ where $e \in \mathbb{E}(\Gamma)$. The walks P_1 and P_2 are said equivalent if they reduce to the same word. The fundamental group is then a free group with b generators for the concatenation operation. Observe that $\pi_1(\Gamma, x_0)$ acts transitively and freely on the set $p^{-1}(x_0)$. Furthermore, given an essential domain $D \subset \tilde{\Gamma}$ and a vertex $\tilde{x}_0 \in D$, each element γ of the fundamental group $\pi_1(\Gamma, p(\tilde{x}_0))$ defines an unique essential domain $\gamma \cdot D$ such that $p(\gamma \cdot D) = p(D)$ and $\gamma \cdot \tilde{x}_0 \in \gamma \cdot D$.

Fix an essential domain D of $\tilde{\Gamma}$ and a vertex \tilde{x}_0 in D . D is a tree and denote by $\{\tilde{e}_1, \dots, \tilde{e}_m\} \subset E(\tilde{\Gamma})$ the boundary edges of D (that is the set of edges $\tilde{e} \in$

$E(\tilde{\Gamma}) \setminus E(D)$ such that one of its vertex $\tilde{v} \in V(D)$). We have $m = 2b$. Denote by $\{e_1, \dots, e_b\}$ the image of $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ under p , by x_0 the projection of \tilde{x}_0 and by \tilde{v}_i the unique vertex of the edge \tilde{e}_i belonging to $V(D)$.

Let $s = g_w(\Gamma)/2$ and $p(s)$ be the minimal number q of elements $\{\gamma_i\}_{i=1}^q$ of $\pi_1(\Gamma, x_0)$ such that each vertex not belonging to $V(D) \cup V(\gamma_1 \cdot D) \cup \dots \cup V(\gamma_q \cdot D)$ is at a weighted distance more than s from D . We will find an upper bound of $p(s)$. Fix an edge \tilde{e}_i in the boundary of D and enumerate the paths of weighted length s starting with the sequence $[\tilde{v}_i, \tilde{e}_i]$. The number of these paths is equal to the number of paths starting with $[p(\tilde{v}_i), p(\tilde{e}_i)]$ of weighted length s , which is less than $2b$. It is clear that each of these paths passes at most once through each element of $\{e_1, \dots, e_b\}$ (as $s = g_w(\Gamma)/2$). If we consider a path \tilde{c} of $\tilde{\Gamma}$ of weighted length s starting with $[\tilde{v}_i, \tilde{e}_i]$, the number of essential domains of the form $\gamma \cdot D$ that it passes through is exactly the number of edges in $\{e_1, \dots, e_b\}$ that belongs to $p(\tilde{c})$. Thus, this number is bounded from above by b . The number of essential domains of the form $\gamma \cdot D$ which intersect a path of weighted length s starting with the sequence $[\tilde{v}_i, \tilde{e}_i]$ is then bounded from above by $2b^2$.

As there are m elements in the boundary of D , and $m = 2b$, we get $p(s) \leq 4b^3$.

Denote by $\Sigma(s) = \{\gamma_i\}_{i=1}^{p(s)}$ a minimal set of $\pi_1(\Gamma, x_0)$ such that for each vertex $x \notin \cup_{i=0}^{p(s)} V(\gamma_i \cdot D)$ we have $d_{\tilde{w}}(x, D) \geq s$ (here γ_0 is by convention the neutral element). This set is a generating set of $\pi_1(\Gamma, x_0)$.

We will estimate the weight of the ball centered at $\tilde{x}_0 \in D$ with radius ns . We see that

$$B^{\tilde{\Gamma}}(x_0, ns) \subset H(D; \{\gamma \in \pi_1(\Gamma, x_0) \mid |\gamma|_{\Sigma(s)} \leq n\}),$$

where for a finite sequence $\gamma_1, \dots, \gamma_q \in \pi_1(\Gamma, x_0)$, $H(D; \gamma_1, \dots, \gamma_q)$ is defined as the maximal subgraph with vertex set $V(D) \cup V(\gamma_1 \cdot D) \cup \dots \cup V(\gamma_q \cdot D)$.

We deduce

$$\tilde{w}(B^{\tilde{\Gamma}}(\tilde{x}_0, ns)) \leq \tilde{w}(\Gamma) \cdot N_{\Sigma(s)}(n).$$

So by (1.3)

$$s \cdot h_{vol}(\Gamma, w) \leq \ln(2 \cdot p(s) - 1),$$

and we are done. \square

Inequality (1.7) yields the asymptotic behaviour

$$h_{vol}(\Gamma, w) \cdot g_w(\Gamma) \lesssim 6 \ln b.$$

We say that a weighted graph (Γ, w) has a *systolic basis* if there exists $x_0 \in V(\Gamma)$ and a generating set $\Sigma_0 = \{\gamma_i\}_{i=1}^b$ of $\pi_1(\Gamma, x_0)$ such that $w(\gamma_i) = g_w(\Gamma)$ for $i = 1, \dots, b$ (recall that the girth is also called systole). For weighted graph with systolic basis, we have the following lower bound.

Proposition 2.5. *Let (Γ, w) be a weighted graph of cyclomatic number b . Suppose that (Γ, w) has a systolic basis. Then*

$$h_{vol}(\Gamma, w) \cdot g_w(\Gamma) \geq \ln(2b - 1), \quad (1.8)$$

with equality for the wedge of b circles $\bigvee_{i=1}^b S_i^1$.

Proof. Let x_0 be the vertex of Γ which is the base-point of the systolic basis. We fix a lift $\tilde{x}_0 \in V(\tilde{\Gamma})$ of this vertex. The geometric length of an element $\alpha \in \pi_1(\Gamma, x_0)$ is defined by

$$\begin{aligned} |\alpha|_w &= \min\{w(\gamma) \mid \gamma \text{ path based at } x_0 \text{ representing } \alpha\} \\ &= d_{\tilde{w}}(\tilde{x}_0, \alpha.\tilde{x}_0). \end{aligned}$$

This geometric length does not depend on the lift \tilde{x}_0 .

We then have (see [10])

$$h_{vol}(\Gamma, w) = \lim_{R \rightarrow +\infty} \frac{\ln(N_w(R))}{R}$$

where $N_w(R) = \text{card}\{\alpha \in \pi_1(\Gamma, x_0) \mid |\alpha|_w \leq R\}$ is the cardinal of the ball of $(\pi_1(\Gamma, x_0), |\cdot|_w)$ with radius R centered at the origin.

We can easily prove (see [10]) the

Lemme 2.6. *Let (Γ, w) be a weighted graph and Σ be a generating set of $G = \pi_1(\Gamma, x_0)$. If there exist $\lambda, \mu > 0$ such that*

$$\lambda \cdot |\cdot|_\Sigma \leq |\cdot|_w \leq \mu \cdot |\cdot|_\Sigma,$$

then

$$\frac{1}{\mu} \ln \omega(G, \Sigma) \leq h_{vol}(\Gamma, w) \leq \frac{1}{\lambda} \ln \omega(G, \Sigma). \quad (1.9)$$

Now, as (Γ, w) has a systolic basis Σ_0 at x_0 , for every $\alpha \in \pi_1(\Gamma, x_0)$,

$$|\alpha|_w \leq |\alpha|_{\Sigma_0} \cdot g_w(\Gamma).$$

We immediately deduce (1.8).

The case of equality is realized by the wedge $\vee_{i=1}^b S_i^1$. □

2.4. Volume entropy and scales of graphs. Let (Γ, w) be a weighted graph. A *chain* of Γ is a path P such that the valence of each intermediate vertex is 2.

We define the *microscopic scale* of (Γ, w) as

$$C_{min}(\Gamma, w) = \min\{w(P) \mid P \text{ chain of } \Gamma\},$$

and the *macroscopic scale* of (Γ, w) as

$$C_{max}(\Gamma, w) = \max\{w(P) \mid P \text{ chain of } \Gamma\}.$$

The aim of this subsection is to prove isoperimetric inequalities involving the volume entropy and the scale of weighted graphs.

Proposition 2.7. *Let (Γ, w) be a weighted graph such that $v(s) \leq 3$ for every $s \in V$.*

Then

$$\frac{\ln 2}{C_{max}(\Gamma, w)} \leq h_{vol}(\Gamma, w) \leq \frac{\ln 2}{C_{min}(\Gamma, w)}. \quad (1.10)$$

Proof. Let T_3 be the infinite regular tree of valence 3. Fix a vertex v of T_3 and v' of Γ . We denote by w_{\min} (respectively w_{\max}) the constant weight function defined on T_3 equal to $C_{\min}(\Gamma, w)$ (respectively $C_{\max}(\Gamma, w)$). Then, for every $R > 0$,

$$w_{\max}(B_{w_{\max}}^{T_3}(v, R)) \leq w(B^{\tilde{\Gamma}}(v', R)) \leq w_{\min}(B_{w_{\min}}^{T_3}(v, R))$$

and so

$$3(2^{\lfloor R/C_{\max} \rfloor} - 1) \cdot C_{\max} \leq w(B^{\tilde{\Gamma}}(v', R)) \leq 3(2^{\lfloor R/C_{\min} \rfloor + 1} - 1) \cdot C_{\min}.$$

We deduce (1.10). \square

We can prove a stronger result for the normalization of the volume entropy by the minimum scale.

Proposition 2.8. *Let (Γ, w) be a weighted graph of cyclomatic number $b \geq 1$. Then*

$$h_{\text{vol}}(\Gamma, w) \cdot C_{\min}(\Gamma, w) \leq \ln(2b - 1). \quad (1.11)$$

The equality case is reached by the wedge of b circles $\bigvee_{i=1}^b S_i^1$.

Proof. Fix a vertex x_0 and let $\Sigma = \{\gamma_1, \dots, \gamma_b\}$ be a minimal generating set of $\pi_1(\Gamma, x_0)$. For each element γ_i let P_i be the corresponding reduced walk. Choose a spanning tree T of Γ containing x_0 such that the edges of $\Gamma \setminus T$ denoted by $\{e_i\}_{i=1}^b$ verify $e_i \in P_i$ for $i = 1, \dots, b$ and $e_j \notin P_i$ for $j \neq i$. Denote by $p : \Sigma \cup \Sigma^{-1} \rightarrow \{e_1, \dots, e_b\}$ the map defined by $p(\gamma_i) = p(\gamma_i^{-1}) = e_i$. For every $\alpha \in \pi_1(\Gamma, x_0)$, we choose a reduced form $\alpha = \alpha_1 \dots \alpha_l$ where $\alpha_j \in \Sigma \cup \Sigma^{-1}$ and $l = |\gamma|_{\Sigma}$. If β is homotopic to α , β may be written as the concatenation of paths

$$[\beta_1, p(\alpha_1), \beta_2, p(\alpha_2), \dots, p(\alpha_l), \beta_{l+1}]$$

where β_i are paths of Γ . As $w(e_i) \geq C_{\min}(\Gamma, w)$ for $i = 1, \dots, l$, we get

$$w(\beta) \geq |\alpha|_{\Sigma} \cdot C_{\min}(\Gamma, w),$$

so

$$|\alpha|_w \geq |\alpha|_{\Sigma} \cdot C_{\min}(\Gamma, w).$$

From inequality (1.9), we deduce (1.11). \square

3. STABLE BALLS AND WEIGHTS OF GRAPHS

3.1. Inequalities for combinatorial graphs.

Theorem 3.1. *Let Γ be a graph of cyclomatic number b . Then*

$$\frac{2^b}{b!} \left(\frac{b}{|E(\Gamma)|} \right)^b \leq \mu(\mathcal{B}_{\text{st}}(\Gamma)) \leq \frac{2^b}{b!}. \quad (2.2)$$

The two cases of equality are reached by the wedge of b circles $\bigvee_{i=1}^b S_i^1$.

Proof. The stable norm agrees here with the norm $\|\cdot\|_1$ defined by $\|u\|_1 = \sum_{i=1}^k |u_i|$ for all $u = \sum_{i=1}^k u_i e_i \in Z(\Gamma, \mathbb{R})$ so we have

$$\mathcal{B}_{\text{st}}(\Gamma) = B_1^k \cap Z(\Gamma, \mathbb{R}), \quad (2.3)$$

where $k = |E(\Gamma)|$ and $B_1^k = \{(x_i) \in \mathbb{R}^k \mid \sum_{i=1}^k |x_i| \leq 1\}$ (we identify $C(\Gamma, \mathbb{R})$ with \mathbb{R}^k).

We can find in [9] the following estimate. For every b -plane P^b in \mathbb{R}^k

$$\mu_k(B_1^k \cap P^b) \geq \mu_k(B_1^k)^{b/k},$$

where μ_n is the canonical volume of \mathbb{R}^n . We deduce with (2.3) that

$$\mu(B_{st}(\Gamma)) \geq \mu_k(B_1^k)^{b/k}.$$

Since $\mu_n(B_1^n) = 2^n/n!$ for every positive integer n , we have

$$\mu(B_{st}(\Gamma)).k^b \geq \frac{2^b.k^b}{(k!)^{b/k}}.$$

With $k \geq b$, we get

$$\mu(B_{st}(\Gamma)).k^b \geq \frac{(2b)^b}{b!}.$$

As $k = |E(\Gamma)|$, the left inequality of (2.2) is then proved.

For the upper bound, we start with an other estimate obtained in [9]. For every b -plane P^b in \mathbb{R}^k ,

$$\mu_k(B_1^k \cap P^b) \leq \mu_b(B_1^b).$$

So

$$\mu(B_{st}(\Gamma)) \leq \mu_b(B_1^b),$$

which gives the right inequality of (2.2).

For the wedge of b circles

$$\mu(B_{st}(\bigvee_{i=1}^b S_i^1)) = \frac{2^b}{b!}$$

and $|E(\bigvee_{i=1}^b S_i^1)| = b$, so $\bigvee_{i=1}^b S_i^1$ realizes equality cases. □

Remark For a regular graph Γ of valence $v \geq 3$, we have

$$\left(\frac{v-2}{v}\right)^b \frac{2^b}{b!} \leq \mu_w(\mathcal{B}_{st}(\Gamma)) \leq \frac{2^b}{b!}.$$

3.2. Estimate for weighted graphs.

Theorem 3.2. *Let (Γ, w) be a weighted graph of cyclomatic number b . Then*

$$\mu_w(\mathcal{B}_{st}(\Gamma, w))^{2/b} \cdot w(\Gamma) \geq \omega_b^{2/b}, \quad (2.4)$$

where ω_b is the volume of the Euclidean unit ball of \mathbb{R}^b .

Proof. Suppose that the result holds for graphs. Let (Γ, w) be a weighted graph of cyclomatic number b . For each $\epsilon > 0$, we can find a weight function w_ϵ close enough to w in the C^0 -topology such that

$$|\mu_{w_\epsilon}(\mathcal{B}_{st}(\Gamma, w_\epsilon)) - \mu_w(\mathcal{B}_{st}(\Gamma, w))| < \epsilon,$$

$$|w_\epsilon(\Gamma) - w(\Gamma)| < \epsilon,$$

and such that $w_\epsilon(e)$ is rational for every $e \in E := E(\Gamma)$. Fix an integer λ such that $\lambda w_\epsilon(e) \in \mathbb{N}^*$ for all $e \in E$. We have

$$\mu_{\lambda w_\epsilon}(\mathcal{B}_{st}(\Gamma, \lambda w_\epsilon)).((\lambda w_\epsilon)(\Gamma))^{b/2} = \mu_{w_\epsilon}(\mathcal{B}_{st}(\Gamma, w_\epsilon)).w_\epsilon(\Gamma)^{b/2}.$$

Choose an enumeration $\{e_i\}_{i=1}^{|E|}$ of the edges of Γ . If we subdivide each edge e_i of Γ in $k_\epsilon(i) = \lambda w_\epsilon(e_i)$ edges denoted by $e'_{i,j}$ for $j = 1, \dots, k_\epsilon(i)$, we get a graph Γ'_ϵ with cyclomatic number b , which is isometric to $(\Gamma, \lambda w_\epsilon)$ when it is endowed with the trivial weight function $w \equiv 1$. Denote by f this isometry and by $E' = E(\Gamma'_\epsilon)$ the set of the edges of Γ'_ϵ . We have

$$\mu(\mathcal{B}_{st}(\Gamma'_\epsilon)) = \mu_{\lambda w_\epsilon}(\mathcal{B}_{st}(\Gamma, \lambda w_\epsilon)). \quad (2.5)$$

To see this, observe that the isometry f induces a linear homomorphism F from $C(\Gamma, \mathbb{R})$ to $C(\Gamma'_\epsilon, \mathbb{R})$ (and so an isomorphism between $Z(\Gamma, \mathbb{R})$ and $Z(\Gamma'_\epsilon, \mathbb{R})$) which satisfies

$$F(e_i) = \sum_{j=1}^{k_\epsilon(i)} e'_{i,j},$$

for $i = 1, \dots, |E|$. The family

$$\left\{ \frac{1}{\sqrt{k_\epsilon(i)}} e_i \right\}_{i=1}^{|E|}$$

is an orthonormal basis for the scalar product $\langle \cdot, \cdot \rangle_{\lambda w_\epsilon}$ of $\mathbb{R}^{|E|}$ and

$$\left\{ \frac{1}{\sqrt{k_\epsilon(i)}} \sum_{j=1}^{k_\epsilon(i)} e'_{i,j} \right\}_{i=1}^{|E|}.$$

an orthonormal basis of $I(C(\Gamma, \mathbb{R}))$ for $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^{|E'|}$. We then find that the map $F|_{Z(\Gamma, \mathbb{R})}$ expressed in these orthonormal basis is the identity map, so we get the claim (2.5).

This construction can be realized for every $\epsilon > 0$. So if the result holds for graphs, it holds for every weighted graph.

Let Γ' be a graph of cyclomatic number b and denote $k = |E(\Gamma')|$. For every $u \in \mathbb{R}^k$,

$$\|u\|_1 \leq k \cdot \|u\|_2^2$$

where $\|u\|_2 = \sqrt{\sum_{i=1}^k u_i^2}$ is the Euclidean norm. So

$$\mu(\mathcal{B}_{st}(\Gamma')) \geq \mu(B_2(\frac{1}{\sqrt{k}})),$$

where $B_2(R)$ is the ball with radius R for the norm $\|\cdot\|_2$ in $Z(\Gamma', \mathbb{R})$. We deduce the inequality (2.4) for graphs. \square

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FLORENT BALACHEFF, UNIVERSITÉ DE GENÈVE - SECTION DE MATHÉMATIQUES, RUE DU
LIÈVRE 2-4, C.P. 64, CH-1211 GENÈVE 4

Email address: florent.balacheff@math.unige.ch