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# SOBOLEV INEQUALITIES: SYMMETRIZATION AND SELF IMPROVEMENT VIA TRUNCATION

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ABSTRACT. We develop a new method to obtain symmetrization inequalities of Sobolev type. Our approach leads to new inequalities and considerable simplification in the theory of embeddings of Sobolev spaces based on rearrangement invariant spaces.

## 1. INTRODUCTION

A well known basic principle in the theory of Sobolev embeddings, due to Maz'ya, and Federer and Fleming (cf. [15] and the references therein), is the equivalence<sup>1</sup> between the isoperimetric inequality and the Gagliardo-Nirenberg inequality

$$(1.1) \quad \|f\|_{L^{n/(n-1)}} \leq c_n \|\nabla f\|_{L^1}, \forall f \in C_0^\infty(\mathbb{R}^n).$$

A second, somewhat less well known principle, which is often rediscovered in the literature<sup>2</sup>, and is also apparently due to Maz'ya [15], states that, roughly speaking, under rather general circumstances a weak type Sobolev inequality implies a strong type Sobolev inequality. We refer to [1], [22] and [6]. In particular, the first two quoted papers show how weak  $L^p$  Sobolev inequalities self improve by truncation to  $L(p, q)$  inequalities, while [6] provides a nice survey and a unified treatment of the cases  $p = 1$  and  $1 < p < n$ , of the Sobolev embedding.

It is also known that Sobolev inequalities have an in-built \*reiteration\* property which is due to a combination of the chain rule and Hölder's inequalities. For example, since for any  $\alpha > 1$  we have  $|\nabla |f|^\alpha| = \alpha |f|^{\alpha-1} |\nabla f|$ , it follows that if we pick  $p \in (1, n)$ , and let  $q = \frac{np}{n-p}$ ,  $\alpha = \frac{(n-1)p}{n-p} = \frac{n-1}{n}q$ , we have  $(\alpha - 1)p' = q$ ,  $q(\frac{n-1}{n} - \frac{1}{p'}) = 1$ , and  $\|f\|_{L^q}^q = \| |f|^\alpha \|_{L^{n/(n-1)}}^{n/(n-1)}$ . Therefore, from (1.1) we thus have that, for  $f \in C_0^\infty(\mathbb{R}^n)$ ,

$$\begin{aligned} \|f\|_{L^q}^{q(n-1)/n} &= \| |f|^\alpha \|_{L^{n/(n-1)}} \leq c_n \left\| \alpha |f|^{\alpha-1} |\nabla f| \right\|_{L^1} \\ &\leq c_n \alpha \|f\|_{L^q}^{q/p'} \|\nabla f\|_{L^p}, \end{aligned}$$

which immediately yields the classical Sobolev inequality.

It follows from the discussion above that, roughly speaking, “all”  $L^p$  Sobolev inequalities follow the Gagliardo-Nirenberg inequality (1.1) or, equivalently, from

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<sup>1</sup>In fact, the equivalence is sharp all the way down to the constants.

<sup>2</sup>See [6].

the isoperimetric inequality. But one can go further. Talenti [21], using the isoperimetric inequality and the co-area formula, obtained a powerful rearrangement inequality, which is very close to the Polya-Szëgo principle (cf. (1.4) below)

$$(1.2) \quad s^{1-1/n} (-f^*)'(s) \leq c_n \frac{\partial}{\partial s} \int_{\{|f|>f^*(s)\}} |\nabla f(x)| dx,$$

where  $f^*$  denotes the non-increasing rearrangement of  $f$ .

In particular, Talenti's inequality can be used to prove Sobolev inequalities in the setting of rearrangement invariant spaces (cf. [21], [5]), where, in principle, the chain rule argument is not available. Moreover, given the precise information about the constant  $c_n$  in (1.2), Talenti's inequality allows one to obtain best possible constants for the classical Sobolev inequalities (cf. [21]).

A somewhat different rearrangement inequality<sup>3</sup> was used in [2] to study the borderline case  $p = n$ ,

$$(1.3) \quad f^{**}(t) - f^*(t) \leq c_n t^{1/n} |\nabla f|^{**}(t), f \in C_0^\infty(\mathbb{R}^n), t > 0,$$

where  $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$ . The proofs of (1.3) in [2] and in [11] use the symmetrization principle<sup>4</sup> of Polya-Szëgo,

$$(1.4) \quad |\nabla f^\circ|^{**}(t) \leq |\nabla f|^{**}(t), f \in C_0^\infty(\mathbb{R}^n).$$

The inequality (1.3) is further extended in [13] using both Talenti's inequality (1.2) and the isoperimetric inequality.

The sharpest form of the classical Sobolev inequalities, including the critical exponent  $p = n$ , follow from (1.3), namely, for  $1 < p \leq n, 1 \leq q \leq \infty$ , we have

$$(1.5) \quad \left\{ \int_0^\infty [(f^{**}(t) - f^*(t)) t^{1/p-1/n}]^q \frac{dt}{t} \right\}^{1/q} \leq c_{n,p} \left\{ \int_0^\infty [|\nabla f|^*(t) t^{1/p}]^q \frac{dt}{t} \right\}^{1/q}.$$

It turns out, however, that the important case  $p = 1$ , which is also valid, requires a separate argument since  $c_{n,p}$  blows up as  $p$  tends to 1. Indeed (1.5) for  $p = 1$  is the sharp form of the Gagliardo-Nirenberg inequality due to Poornima [17] (cf. (1.9) below).

Symmetrization inequalities imply Sobolev inequalities in the setting of rearrangement invariant spaces. Indeed, from (1.3) we obtain: for any r.i. space  $X$  with upper Boyd<sup>5</sup> index  $\beta_X < 1$ , we have (cf. [16])

$$\left\| t^{-1/n} (f^{**}(t) - f^*(t)) \right\|_X \leq c \|\nabla f\|_X, f \in C_0^\infty(\mathbb{R}^n),$$

where  $c = c(n, X)$ . Moreover, the inequality is sharp (cf. Section 4 below): if  $Y$  is any r.i. space then the validity of

$$\|f\|_Y \leq c \|\nabla f\|_X, f \in C_0^\infty(\mathbb{R}^n)$$

implies that

$$\|f\|_Y \leq \left\| t^{-1/n} (f^{**}(t) - f^*(t)) \right\|_X.$$

<sup>3</sup>A slightly different but equivalent inequality had been obtained earlier in [11].

<sup>4</sup> $f^\circ(x) = f^*(\gamma_n |x|^n)$ , is the symmetric decreasing rearrangement of  $f$ ,  $\gamma_n$  is the measure of the unit ball in  $\mathbb{R}^n$ .

<sup>5</sup>The restriction on the Boyd indices is only required to guarantee that the inequality  $\|g^{**}\|_X \leq c_X \|g\|_X$ , holds for all  $g \in X$ .

Note that for  $X = L^p$  the condition  $\beta_X < 1$  translates into  $p > 1$ . The fact that spaces near  $L^1$  cannot be treated using (1.3), and the previous discussion showing the central role of the Gagliardo-Nirenberg inequalities [cf. (1.1) and (1.9)], suggested that there could be another more powerful underlying rearrangement inequality that would allow for a unified treatment.

The purpose of this paper is to show that truncation can be actually used as a method to obtain symmetrization inequalities. In other words rather than show that a Sobolev inequality implies other Sobolev inequalities one case at a time, we prove that from a Sobolev inequality we can obtain a symmetrization inequality that “implies all the Sobolev inequalities”.

Our analysis leads indeed to new symmetrization inequalities that allow for a unified treatment of the Sobolev inequalities at both end points in the setting of r.i. spaces. Remarkably, our approach also provides a considerable simplification to the methods used to prove the classical symmetrization inequalities discussed above. This is important for the application of our methods to generalized settings like metric spaces (cf. [7]), fractional derivatives (cf. [14]), capacities, etc, which we hope to treat elsewhere.

The following is our main result. We could call it a “symmetrization by truncation principle”, and it is part of a family of similar inequalities, we consider here the most important case, namely the end point  $p = 1$  (cf. Section 2.3 below).

**Theorem 1.** *The following statements are equivalent*

(i)

$$(1.6) \quad W_0^{1,1}(\mathbb{R}^n) \subset L^{n/(n-1),\infty}(\mathbb{R}^n).$$

(ii)

$$(1.7) \quad \int_0^t s^{-\frac{1}{n}} [f^{**}(s) - f^*(s)] ds \leq cn \int_0^t |\nabla f|^*(s) ds, \quad f \in C_0^\infty(\mathbb{R}^n).$$

(iii) *For any rearrangement invariant space  $X$  with lower Boyd index<sup>6</sup>  $\alpha_X > 0$  we have*

$$(1.8) \quad \left\| s^{-1/n} (f^{**}(s) - f^*(s)) \right\|_X \preceq \| |\nabla f| \|_X, \quad f \in C_0^\infty(\mathbb{R}^n).$$

(iv)

$$(1.9) \quad W_0^{1,1}(\mathbb{R}^n) \subset L^{n/(n-1),1}(\mathbb{R}^n).$$

To understand how Theorem 1 represents an improvement over the known results, we note that the implication (1.6) $\Rightarrow$ (1.9) is the self improvement that follows by the usual method of truncation (cf. [1], [21], [6]). On the other hand, by “symmetrization by truncation” we obtain the new rearrangement inequality (1.7) which readily gives (1.8), and thus we have obtained the most general form of the Sobolev inequalities in the context of r.i. spaces. Moreover, in the process we have eliminated the restriction on the upper Boyd indices of [16] and we are able to treat spaces near  $L^1$  in a unified manner. In particular, we note that Theorem 1, and the discussion preceding it, shows that the symmetrization inequality<sup>7</sup> (1.7) is equivalent to the isoperimetric inequality.

<sup>6</sup>For  $X = L^p$ ,  $\alpha_{L^p} = 1/p > 0$  translates into  $p < \infty$ .

<sup>7</sup>We shall also refer sometimes to inequalities involving the quantity  $f^{**}(t) - f^*(t)$  as “oscillation inequalities”.

Furthermore, since we believe that it is methodologically important for further extensions, and in order to clarify the role of the assumptions that intervene in the proof of the basic inequalities, in Section 2 below we provide a simple direct proof that all the main rearrangement inequalities discussed here, namely (1.2), (1.3) (1.7), (1.4)<sup>8</sup> follow directly from the straightforward weak type Sobolev inequality (1.6) via truncation.

A complete discussion concerning Sobolev embeddings in the setting of r.i. spaces is then given in the final Section 4. Our approach treats all cases in a unified manner with optimal conditions, the optimal spaces are explicitly constructed and, moreover, we give a unified treatment of all the borderline cases as well (the reader should compare our approach with the ones that are currently available in the literature: cf. [5], [16], [8], [9], and the references quoted therein). We also show how our methods provide a considerable simplification to recent results on the compactness of Sobolev embeddings (cf. [10] and [19]).

We stress that in this paper we have not attempted to prove the most general results, but rather we aim to illustrate the power of our methods. In particular, in order not to obscure the simplicity of the arguments we work for the most part on  $\mathbb{R}^n$ , and we formulate our results as inequalities. This is justified since the extensions to regular domains can be obtained using well known techniques, while more sophisticated extensions would require a separate treatment.

As usual, the symbol  $f \simeq g$  will indicate the existence of a universal constant  $c > 0$  (independent of all parameters involved) so that  $(1/c)f \leq g \leq cf$ , while the symbol  $f \preceq g$  means that  $f \leq cg$ , and  $f \succeq g$  means that  $f \geq cg$ .

## 2. SYMMETRIZATION INEQUALITIES BY TRUNCATION

The purpose of this section is to show that all the symmetrization inequalities discussed in the Introduction follow from the Sobolev embedding

$$(2.1) \quad W_0^{1,1}(\mathbb{R}^n) \subset L^{n/(n-1),\infty}(\mathbb{R}^n),$$

by truncation.

Since it will be important for us to keep track of the constants of the embedding (2.1), and in order to provide a self contained presentation, we present a proof of (2.1) following [6], who in turn credits Santalo for the method of proof.

**Lemma 1.** *Let  $f \in W_0^{1,1}(\mathbb{R}^n)$ , then*

$$\sup_{t>0} t |\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{n-1}{n}} \leq \frac{1}{\gamma_n^{1/n}} \int_{\mathbb{R}^n} |\nabla f(x)| dx,$$

where  $\gamma_n = \text{measure of the unit ball in } \mathbb{R}^n$ .

*Proof.* Let  $f \in C_0^\infty(\mathbb{R}^n)$ , then as it is well known (see [20, Page 125]) we have the representation

$$f(x) = \frac{1}{n\gamma_n} \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x-y) \frac{y_j}{|y|^n} dy.$$

Thus,

$$|f(x)| \leq \frac{1}{n\gamma_n} \int_{\mathbb{R}^n} |\nabla f(y)| \frac{1}{|x-y|^{n-1}} dy.$$

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<sup>8</sup>Actually the version we prove of (1.4) is slightly weaker in as much as the constant  $n$  appears on the right hand side of the inequality.

Let  $H = \{x : |f(x)| > t\}$ , then, combining the previous inequality with Chebyshev's inequality and Fubini, we find that

$$t |H| \leq \int_H |f(x)| dx \leq \frac{1}{n\gamma_n} \int_{\mathbb{R}^n} |\nabla f(y)| \int_H \frac{dx}{|x-y|^{n-1}} dy.$$

For a fixed  $y$  let  $B = B(y, r)$  be a ball such that  $|B| = |H|$ . Then by symmetrization

$$\int_H \frac{dx}{|x-y|^{n-1}} \leq \int_B \frac{dx}{|x-y|^{n-1}} = n\gamma_n r = n\gamma_n^{1-1/n} |H|^{1/n}.$$

Summarizing

$$t |H| \leq \frac{|H|^{1/n}}{\gamma_n^{1/n}} \int_{\mathbb{R}^n} |\nabla f(y)| dy.$$

□

**2.1. Talenti's inequality.** Our starting point is the weak type inequality

$$(2.2) \quad \sup_{t>0} t |\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{n-1}{n}} \leq \gamma_n^{-1/n} \int_{\mathbb{R}^n} |\nabla f(x)| dx, f \in W_0^{1,1}(\mathbb{R}^n).$$

Let  $0 < t_1 < t_2 < \infty$ , the truncations of  $f$  are defined by

$$f_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1 & \text{if } |f(x)| > t_2, \\ |f(x)| - t_1 & \text{if } t_1 < |f(x)| \leq t_2, \\ 0 & \text{if } |f(x)| \leq t_1. \end{cases}$$

Observe that if  $f \in W_0^{1,1}(\mathbb{R}^n)$  then  $f_{t_1}^{t_2} \in W_0^{1,1}(\mathbb{R}^n)$ , therefore replacing  $f$  by  $f_{t_1}^{t_2}$  in (2.2) we obtain

$$\sup_{t>0} t |\{x \in \mathbb{R}^n : |f_{t_1}^{t_2}(x)| > t\}|^{\frac{n-1}{n}} \leq \gamma_n^{-1/n} \int_{\mathbb{R}^n} |\nabla f_{t_1}^{t_2}(x)| dx.$$

We obviously have

$$\sup_{t>0} t |\{x \in \mathbb{R}^n : |f_{t_1}^{t_2}(x)| > t\}|^{\frac{n-1}{n}} \geq (t_2 - t_1) |\{x \in \mathbb{R}^n : |f(x)| \geq t_2\}|^{\frac{n-1}{n}},$$

and

$$|\nabla f_{t_1}^{t_2}| = |\nabla f| \chi_{\{t_1 < |f| \leq t_2\}}.$$

Therefore,

$$(t_2 - t_1) |\{x \in \mathbb{R}^n : |f(x)| \geq t_2\}|^{1-1/n} \leq \gamma_n^{-1/n} \int_{\{t_1 < |f| \leq t_2\}} |\nabla f(x)| dx.$$

Let  $0 \leq a < b$ , and consider  $t_1 = f^*(b)$ ,  $t_2 = f^*(a)$ . Then

$$\begin{aligned} (2.3) \quad (f^*(a) - f^*(b))a^{1-1/n} &\leq (f^*(a) - f^*(b)) |\{x \in \mathbb{R}^n : |f(x)| \geq f^*(a)\}|^{1-1/n} \\ &\leq \gamma_n^{-1/n} \int_{\{f^*(b) < |f| \leq f^*(a)\}} |\nabla f(x)| dx \\ &\leq \gamma_n^{-1/n} \int_0^{b-a} |\nabla f|^*(s) ds, \end{aligned}$$

whence  $f^*$  is locally absolutely continuous.

Let  $s > 0$  and  $h > 0$ ; the previous considerations with  $t_1 = f^*(s + h)$  and  $t_2 = f^*(s)$  yield

$$(f^*(s) - f^*(s + h)) s^{1-1/n} \leq \gamma_n^{-1/n} \int_{\{f^*(s+h) < |f| \leq f^*(s)\}} |\nabla f(x)| dx.$$

Thus,

$$\frac{(f^*(s) - f^*(s + h))}{h} s^{1-1/n} \leq \frac{\gamma_n^{-1/n}}{h} \int_{\{f^*(s+h) < |f| \leq f^*(s)\}} |\nabla f(x)| dx.$$

Letting  $h \rightarrow 0$  we obtain (1.2).

**2.2. The Oscillation Inequality.** We now prove the oscillation inequality (1.3). We will integrate by parts, so let us note first that using (2.3) we have, for  $0 < s < t$ ,

$$(2.4) \quad s(f^*(s) - f^*(t)) \leq \gamma_n^{-1/n} s^{1/n} \int_0^{t-s} |\nabla f|^*(s) ds.$$

Now,

$$(2.5) \quad \begin{aligned} f^{**}(t) - f^*(t) &= \frac{1}{t} \int_0^t (f^*(s) - f^*(t)) ds \\ &= \frac{1}{t} \left\{ [s(f^*(s) - f^*(t))]_0^t + \int_0^t s(-f^*)'(s) ds \right\} \\ &= \frac{1}{t} \int_0^t s(-f^*)'(s) ds, \end{aligned}$$

where the integrated term  $[s(f^*(s) - f^*(t))]_0^t$  vanishes on account of (2.4).

Now, starting from (2.5) we readily get

$$\begin{aligned} f^{**}(t) - f^*(t) &= \frac{1}{t} \int_0^t s(-f^*)'(s) ds = \frac{1}{t} \int_0^t s^{1/n} s^{1-1/n} (-f^*)'(s) ds \\ &\leq \frac{t^{1/n}}{t} \int_0^t s^{1-1/n} (-f^*)'(s) ds \\ &\leq \gamma_n^{-1/n} \frac{t^{1/n}}{t} \int_0^t \left( \frac{\partial}{\partial s} \int_{\{|f| > f^*(s)\}} |\nabla f(x)| dx \right) ds \\ &\leq \gamma_n^{-1/n} t^{1/n} |\nabla f|^{**}(t), \end{aligned}$$

where in the third step we used (1.2).

**Remark 1.** Since it will be useful below we observe that in an intermediate step of the previous derivation we implicitly obtained the inequality

$$(2.6) \quad \int_0^t s^{1-1/n} (-f^*)'(s) ds \leq \gamma_n^{-1/n} \int_0^t |\nabla f|^*(s) ds.$$

**2.3. Integrated Oscillation Inequality.** We prove (1.7). Starting from (2.5) and integrating by parts we have

$$\begin{aligned} \int_0^t s^{-\frac{1}{n}} [f^{**}(s) - f^*(s)] ds &= \int_0^t s^{-1-\frac{1}{n}} \int_0^s u (-f^*)' (u) du ds \\ &= -n \int_0^t \int_0^s u (-f^*)' (u) du ds^{-1/n} \\ &= -ns^{-1/n} \int_0^s u (-f^*)' (u) du \Big|_0^t + \int_0^t s^{1-1/n} (-f^*)' (s) ds. \end{aligned}$$

Since by (2.4) and (2.5) it follows that

$$s^{-1/n} \int_0^s u (-f^*)' (u) du = s^{1-1/n} (f^{**}(s) - f^*(s)) \leq \int_0^s |\nabla f|^* (s) ds,$$

the integrated term vanishes at  $t = 0$ . Consequently, in view of (1.3) and (2.6), we can continue our estimates with

$$\begin{aligned} \int_0^t s^{-\frac{1}{n}} [f^{**}(s) - f^*(s)] ds &= -nt^{1-1/n} (f^{**}(t) - f^*(t)) + \gamma_n^{-1/n} n \int_0^t |\nabla f|^* (s) ds \\ &\leq \gamma_n^{-1/n} n \int_0^t |\nabla f|^* (s) ds, \end{aligned}$$

as we wished to show.

**Remark 2.** Using a standard limiting argument we may extend the validity of (1.3) and (1.7) from functions in  $C_0^\infty(\mathbb{R}^n)$  to all functions in  $W_0^{1,1}(\mathbb{R}^n)$ . For example, suppose that (1.7) holds for functions in  $C_0^\infty(\mathbb{R}^n)$ . Then, given  $f \in W_0^{1,1}(\mathbb{R}^n)$  select  $f_k \in C_0^\infty(\mathbb{R}^n)$  such that

$$f_k(x) \rightarrow f(x) \text{ a.e. and } f_k \rightarrow f \text{ in } W_0^{1,1}(\mathbb{R}^n).$$

Since  $f_k^*(t) \rightarrow f^*(t)$  a.e. we can use Fatou's lemma

$$\begin{aligned} \int_0^t s^{-\frac{1}{n}} [f^{**}(s) - f^*(s)] ds &\leq \liminf \int_0^t s^{-\frac{1}{n}} [f_k^{**}(s) - f_k^*(s)] ds \leq \liminf \int_0^t |\nabla f_k|^* (s) ds \\ &= \liminf \int_0^t |\nabla(f_k + f - f)|^* (s) ds \\ &\leq \liminf \int_0^t |\nabla(f_k - f)|^* (s) ds + \int_0^t |\nabla f|^* (s) ds \\ &\leq \liminf_n \| |\nabla(f_k - f)| \|_{L^1} + \int_0^t |\nabla f|^* (s) ds \\ &= \int_0^t |\nabla f|^* (s) ds, \end{aligned}$$

as we wished to prove. The extension of (1.3) is proved similarly.

**2.4. An elementary proof of the Polya-Szëgo principle.** We will actually prove a slightly weaker form of the Polya-Szëgo principle, namely

$$|\nabla f^\circ|^{**} (s) \leq n |\nabla f|^{**} (s).$$



Our starting point is Talenti's inequality (cf. Section 2.1 above): if  $f \in W_0^{1,1}(\mathbb{R}^n)$  then

$$s^{1-1/n} (-f^*)' (s) \leq \gamma_n^{-1/n} \frac{\partial}{\partial s} \int_{\{|f| > f^*(s)\}} |\nabla f(x)| dx.$$

We claim that if  $\Phi$  is a positive Young's function, then

$$(2.7) \quad \Phi \left( n \gamma_n^{1/n} s^{1-1/n} (-f^*)' (s) \right) \leq \frac{\partial}{\partial s} \int_{\{|f| > f^*(s)\}} \Phi(n |\nabla f(x)|) dx.$$

Assuming momentarily the validity of (2.7) we get

$$\int_0^\infty \Phi \left( n \gamma_n^{1/n} s^{1-1/n} (-f^*)' (s) \right) ds \leq \int_{\mathbb{R}^n} \Phi(n |\nabla f(x)|) dx,$$

and since,

$$\int_0^\infty \Phi \left( n \gamma_n^{1/n} s^{1-1/n} (-f^*)' (s) \right) ds = \int_{\mathbb{R}^n} \Phi(|\nabla f^\circ(x)|) dx$$

it follows that for all Young functions  $\Phi$  we have

$$\int_{\mathbb{R}^n} \Phi(|\nabla f^\circ(x)|) dx \leq \int_{\mathbb{R}^n} \Phi(n |\nabla f(x)|) dx.$$

The last inequality implies, by a well known result of Hardy-Littlewood-Polya (cf. [3, Page 88]),

$$\int_0^t |\nabla f^\circ|^*(s) ds \leq n \int_0^t |\nabla f|^*(s) ds,$$

as we wished to show.

It remains to prove (2.7). Here we follow Talenti's argument (it is important for our purposes to note that at this point in the argument we are not using the isoperimetric inequality or the co-area formula). Let  $s > 0$ , then we have three different alternatives: (i)  $s$  belongs to some exceptional set of measure zero, (ii)  $(f^*)'(s) = 0$ , or (iii) there is a neighborhood of  $s$  such that  $(f^*)'(u)$  is not zero, i.e.  $f^*$  is strictly decreasing. In the two first cases there is nothing to prove. In case alternative (iii) holds then it follows immediately from the properties of the rearrangement that for a suitable small  $h_0 > 0$  we can write

$$h = |\{f^*(s+h) < |f| \leq f^*(s)\}|, \quad 0 < h < h_0.$$

Therefore for sufficiently small  $h$  we can apply Jensen's inequality to obtain,

$$\frac{1}{h} \int_{\{f^*(s+h) < |f| \leq f^*(s)\}} \Phi(|\nabla f(x)|) dx \geq \Phi \left( \frac{1}{h} \int_{\{f^*(s+h) < |f| \leq f^*(s)\}} |\nabla f(x)| dx \right).$$

Arguing like Talenti [21] we thus get

$$\begin{aligned} \frac{\partial}{\partial s} \int_{\{|f| > f^*(s)\}} \Phi(|\nabla f(x)|) dx &\geq \Phi \left( \frac{\partial}{\partial s} \int_{\{|f| > f^*(s)\}} |\nabla f(x)| dx \right) \\ &\geq \Phi \left( n \gamma_n^{1/n} s^{1-1/n} (-f^*)' (s) \right), \end{aligned}$$

as we wished to show.

## 3. PROOF OF THE MAIN THEOREM 1

For the proof we need a slight extension of the following well known fact (probably due to Hardy and Calderón): if  $g$  and  $h$  are positive and decreasing and such that

$$\int_0^t g(s)ds \preceq \int_0^t h(s)ds, \forall t > 0,$$

then for any r.i. norm  $X$  we have

$$\|g\|_X \preceq \|h\|_X.$$

We extend this result as follows

**Lemma 2.** *Let  $f$  and  $g$  be two positive functions on the half line. Moreover, suppose that there exists a real number  $\alpha$  such that the function  $t^\alpha f(t)$  is monotone (increasing or decreasing). Then, for any r.i. space  $X$  with lower Boyd index  $\alpha_X > 0$ , there exists a constant  $C = C(\alpha, X)$  such that if  $\int_0^t f(s)ds \leq \int_0^t g(s)ds$ , holds for all  $t > 0$ , then*

$$\|f\|_X \leq C\|g\|_X.$$

*Proof.* Let  $Pg(t) = \frac{1}{t} \int_0^t g(s)ds$  and its adjoint  $Qg(t) = \int_t^\infty g(s) \frac{ds}{s}$  be the usual Hardy operators (notice that  $Q$  is a positive operator and that  $Qg(t)$  is a decreasing function). Then, applying the operator  $Q$  to the inequality  $Pf(t) \leq Pg(t)$ , and using the fact that  $Q \circ P = P \circ Q$ , we obtain

$$\int_0^t Qf(s)ds \leq \int_0^t Qg(s)ds, \text{ for all } t > 0.$$

Since the integrated functions are decreasing we can apply the usual Hardy-Calderón Lemma (see the discussion preceeding this lemma) to obtain

$$\|Qf\|_X \leq \|Qg\|_X.$$

Moreover, since  $\alpha_X > 0$ , we can continue with

$$(3.1) \quad \|Qf\|_X \leq c_X \|Q\|_{X \rightarrow X} \|g\|_X.$$

To estimate the left hand side of (3.1) from below we assume first that the function  $t^\alpha f(t)$  is increasing. If  $\alpha \neq 0$ , then

$$Qf(t) \geq \int_t^{2t} s^\alpha f(s) s^{-\alpha} \frac{ds}{s} \geq t^\alpha f(t) \int_t^{2t} s^{-\alpha-1} ds = \frac{1-2^{-\alpha}}{\alpha} f(t).$$

While if  $\alpha = 0$  then we readily see that  $Qf(t) \geq \frac{1}{2} f(t)$ . Similarly, if the function  $t^\alpha f(t)$  is decreasing,  $\alpha \neq 0$ , then

$$Qf(t) \geq \int_t^{2t} s^\alpha f(s) s^{-\alpha} \frac{ds}{s} \geq (2t)^\alpha f(2t) \int_t^{2t} s^{-\alpha-1} ds = \frac{2^\alpha - 1}{\alpha} f(2t).$$

While if  $\alpha = 0$  then we readily see that  $Qf(t) \geq \frac{1}{2} f(2t)$ . Thus, if  $t^\alpha f(t)$  is monotone, we have

$$(3.2) \quad \|f\|_X \leq C(\alpha) \|Qf\|_X.$$

Combining (3.2) and (3.1) the desired result follows.  $\square$

We may now proceed with the proof of Theorem 1

*Proof.* In Section 2.3 we have proved the implication (i)  $\rightarrow$  (ii).  
(ii)  $\rightarrow$  (iii). Using the Hardy operator  $P$  we rewrite (1.7) as

$$P(s^{-\frac{1}{n}}[f^{**}(s) - f^*(s)])(t) \preceq P(|\nabla f|^*(s))(t).$$

Let  $h(s) = s^{-\frac{1}{n}}[f^{**}(s) - f^*(s)]$ , and  $g(s) = |\nabla f|^*(s)$ , and note that  $s^{1+1/n}h(s) = s[f^{**}(s) - f^*(s)] = \int_{f^*(s)}^{\infty} \lambda_f(u) du$  (draw a picture!) is increasing. Therefore, by lemma 2, we find that

$$\left\| s^{-1/n}(f^{**}(s) - f^*(s)) \right\|_X \preceq \| |\nabla f| \|_X,$$

as we wished to show.

(iii)  $\rightarrow$  (iv) Let  $X = L^1$ , then (1.8) reads

$$\int_0^\infty s^{1-\frac{1}{n}}[f^{**}(s) - f^*(s)] \frac{ds}{s} \preceq \| |\nabla f| \|_1,$$

and the result follows since formally<sup>9</sup> integrating by parts yields

$$\begin{aligned} \int_0^\infty s^{1-\frac{1}{n}}[f^{**}(s) - f^*(s)] \frac{ds}{s} &= [1 - 1/n] \int_0^\infty f^{**}(s) s^{1-1/n} \frac{ds}{s} \\ &= [1 - 1/n] \|f\|_{L^{n/(n-1),1}}. \end{aligned}$$

(iv)  $\rightarrow$  (i) This is of course trivial since

$$W_0^{1,1}(\mathbb{R}^n) \subset L^{n/(n-1),1}(\mathbb{R}^n) \subset L^{n/(n-1),\infty}(\mathbb{R}^n).$$

□

#### 4. SOBOLEV INEQUALITIES IN R.I. SPACES

In this section we give a self contained approach to the theory of Sobolev inequalities in the setting of r.i. spaces. Our results provide optimal results all the way to the borderline cases.

We recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces and refer the reader to [3] for a complete treatment.

Let  $\Omega$  be a domain in  $\mathbb{R}^n$ . A Banach function space  $X(\Omega)$  is called a r.i. space if  $g \in X(\Omega)$  implies that all functions  $f$  with the same decreasing rearrangement,  $f^* = g^*$ , also belong to  $X(\Omega)$ , and, moreover,  $\|f\|_{X(\Omega)} = \|g\|_{X(\Omega)}$ . Let us assume that we define  $f(x) = 0$  whenever  $x \in \mathbb{R}^n \setminus \Omega$ , then any r.i. space  $X(\Omega)$  can be “reduced” to one-dimensional space (which by abuse of notation we will still denote by  $X$ ),  $X = X(0, |\Omega|)$  consisting of all  $g : (0, |\Omega|) \mapsto \mathbb{R}$  such that  $g^*(t) = f^*(t)$  for some function  $f \in X(\Omega)$ . We shall further assume that our r.i. spaces satisfy the so-called *Fatou property*, i.e., for any sequence of functions  $f_k \rightarrow f$  a.e.,  $f_k \in X$ , and such that  $\sup_k \|f_k\|_X \leq M$ , it follows that  $f \in X$  and  $\|f\|_X \leq \liminf \|f_k\|_X$ .

<sup>9</sup>The fact that the integrated term vanishes can be easily justified by a familiar limiting argument.

The upper and lower Boyd indices<sup>10</sup> associated with a r.i. space  $X$  are defined by

$$\beta_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \alpha_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s},$$

where  $h_X(s)$  denotes the norm of the dilation operator, i.e.

$$h_X(s) = \sup_{f \in X} \frac{\|f^*(\frac{s}{\cdot})\|_{X(0,|\Omega|)}}{\|f^*\|_{X(0,|\Omega|)}}, s > 0.$$

Furthermore we shall assume, essentially without loss, that the spaces we consider are separable, and unless otherwise specified we shall also assume that we work on  $\mathbb{R}^n$ . However, whenever appropriate, we shall briefly indicate the necessary modifications to treat more general regular domains.

The results and the proofs of this section are similar to those of the papers [16] and [18], however in our present treatment we have no restrictions on the upper Boyd index  $\beta_X$ .

We record the following elementary result for Hardy operators (cf. [16]).

**Lemma 3.** *Let  $X$  be a r.i. space with the lower Boyd index  $\alpha_X > \alpha \geq 0$ . Then*

(i)

$$\|t^{-\alpha} Qf(t)\|_X \leq C(\alpha, X) \|t^{-\alpha} f(t)\|_X.$$

(ii) *If  $f^{**}(\infty) = 0$ , then*

$$\|t^{-\alpha} f^{**}(t)\|_X \leq C(\alpha, X) \|t^{-\alpha} [f^{**}(t) - f^*(t)]\|_X.$$

*Proof.* Both assertions can be found in [16]. For example see [[16], Lemma 2.5] for (i). To prove (ii) use the Fundamental theorem of Calculus to write  $f^{**}(t) = \int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s}$  and apply (i).  $\square$

We use the notation

$$|D^k f| = \left( \sum_{|\alpha|=k} |D^\alpha f|^2 \right)^{1/2}.$$

**Theorem 2.** *Let  $X$  be a r.i. space with  $\alpha_X > \frac{k-1}{n}$  for some  $k \in \mathbb{N}$ ,  $k < n$ . Then there exists a constant  $C > 0$ , such that*

$$(4.1) \quad \|t^{-k/n} [f^{**}(t) - f^*(t)]\|_X \leq C \| |D^k f| \|_X, \quad f \in C_0^\infty(\mathbb{R}^n).$$

*Proof.* When  $k = 1$  the condition on  $\alpha_X$  is simply  $\alpha_X > 0$ , therefore (4.1) for  $k = 1$  was proved in Theorem 1 (iii). We prove the case  $k > 1$  by induction. Consider first the case  $k = 2$ , in which case may assume that  $\alpha_X > 1/n$ . Using (1.3) we get

$$\|t^{-2/n} [f^{**}(t) - f^*(t)]\|_X \preceq \|t^{-1/n} |\nabla f|^{**}(t)\|_X.$$

Applying Lemma 3 with  $\alpha = 1/n$  we can continue with

$$\|t^{-1/n} |\nabla f|^{**}(t)\|_X \preceq \|t^{-1/n} [|\nabla f|^{**}(t) - |\nabla f|^*(t)]\|_X.$$

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<sup>10</sup>In terms of the Hardy operators defined by

$$Pf(t) = \frac{1}{t} \int_0^t f(s) ds; \quad Q_a f(t) = \frac{1}{t^a} \int_t^\infty s^a f(s) \frac{ds}{s}, \quad 0 \leq a < 1;$$

$P$  (resp.  $Q_a$ ) is bounded on  $X$  if and only if  $\beta_X < 1$  (resp.  $a < \alpha_X$ ) (see for example [3, Chapter 3]). Notice that if  $a = 0$ ,  $Q_0 = Q$ .

At this point we apply the case  $k = 1$  to the right hand side to obtain

$$\|t^{-1/n}[\nabla f^{**}(t) - |\nabla f|^*(t)]\|_X \preceq \| |\nabla f| \|_X \preceq \|D^2 f\|_X.$$

Combining these inequalities thus proves the desired result for the case  $k = 2$ . The general case is obtained with the same argument. Indeed, assuming the inequality is valid for  $k - 1$ , we can write

$$\begin{aligned} \|t^{-k/n}[f^{**}(t) - f^*(t)]\|_X &\preceq \|t^{-(k-1)/n}|\nabla f|^{**}(t)\|_X \\ &\leq C_{k-1} \|t^{-(k-1)/n}[\nabla f|^{**}(t) - |\nabla f|^*(t)]\|_X \\ &\leq C_k \| |\nabla f|^{k-1} \|_X \\ &\leq C \|D^k f\|_X, \end{aligned}$$

and the result follows.  $\square$

To formulate necessary conditions we consider the linear integral operators<sup>11</sup> (cf. [4], [5], [16], [13], [8] and the references therein)

$$H_{k/n}g(t) = \int_t^\infty s^{k/n}g(s)\frac{ds}{s}.$$

The next result was recorded in [18] but with the restriction  $\beta_X < 1$ , the restriction was later removed in [9] but with a rather complicated proof. Our proof provides a considerable simplification.

**Theorem 3.** *Let  $k \in \mathbb{N}$ ,  $k < n$ , and let  $X$  be a r.i. space such that  $\alpha_X > \frac{k-1}{n}$ ; and let  $Y$  be another r.i. space. Then there exists a constant  $C > 0$  such that  $\|f\|_Y \leq C \|D^k f\|_X$  for all  $f \in C_0^\infty(\mathbb{R}^n)$  if and only if  $H_{k/n}$  is a bounded operator from  $X \rightarrow Y$ .*

*Proof.* Suppose that  $H_{k/n}$  is a bounded operator,  $H_{k/n} : X \rightarrow Y$ . Let  $f \in C_0^\infty(\mathbb{R}^n)$ , and define  $g(t) = t^{-k/n}[f^{**}(t) - f^*(t)]$ , then

$$H_{k/n}g(t) = \int_t^\infty [f^{**}(s) - f^*(s)]\frac{ds}{s} = Q(f^{**} - f^*)(t) = f^{**}(t).$$

Therefore,

$$\begin{aligned} \|f\|_Y &\leq \|f^{**}\|_Y = \|H_{k/n}g\|_Y \\ &\leq \|H_{k/n}\|_{X \rightarrow Y} \|t^{-k/n}[f^{**}(t) - f^*(t)]\|_X \\ &\preceq \|D^k f\|_X \text{ (by (4.1)).} \end{aligned}$$

To prove the converse we consider first the case  $k = 1$ . Suppose that  $Y$  is a r.i. space such that  $\|f\|_Y \leq C \| |\nabla f| \|_X$  for all admissible  $f$ . Let  $g$  be an arbitrary non-negative function from  $X$ ; we must show that the function  $u$  defined by

$$u(t) = H_{1/n}g(t) = \int_t^\infty s^{1/n}g(s)\frac{ds}{s},$$

belongs to  $Y$ . Note that  $u'(t) = \frac{1}{n}t^{1/n-1}g(t)$ , therefore if we define  $f(x) = u(t)$  with  $t = |x|^n$ , we see that  $|\nabla f(x)| = nt^{1-1/n}|u'(t)|$ , whence  $|\nabla f(x)| = ng(t)$ . It follows

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<sup>11</sup>In the case of domains  $\Omega$  one needs to consider likewise the operators  $\tilde{H}_{k/n}g(t) = \int_t^{|\Omega|} s^{k/n}g(s)\frac{ds}{s}$ .

that

$$\begin{aligned}\|u\|_Y &\simeq \|f\|_Y \preceq \|\nabla f\|_X \text{ (by hypothesis)} \\ &= Cn\|g\|_X,\end{aligned}$$

as we wished to show. Suppose now that  $k > 1$ . Repeating the previous argument  $k$  times leads to the conclusion that the operators  $(H_{1/n})^k$  are bounded,  $(H_{1/n})^k : X \rightarrow Y$ . In particular, there exists an absolute constant  $c > 0$  such that  $\|(H_{1/n})^k g\|_Y \leq c\|g\|_X$ . To prove that  $H_{k/n}$  is a bounded operator  $H_{k/n} : X \rightarrow Y$ , we compare  $H_{k/n}$  with  $(H_{1/n})^k$ . By induction we find

$$(H_{1/n})^k g(t) = n^{k-1} \int_t^\infty s^{1/n} (s^{1/n} - t^{1/n})^{k-1} g(s) \frac{ds}{s}.$$

It follows by direct calculations that there exist constants  $c_m, a_n$  such that

$$(4.2) \quad \begin{aligned} H_{k/n} g(t) &= \sum_{m=0}^{k-1} c_m t^{m/n} (H_{1/n})^{k-m} g(t) \\ (H_{1/n})^k g\left(\frac{t}{2}\right) &\geq (a_n)^m t^{m/n} (H_{1/n})^{k-m} g(t), \quad m = 1, 2, \dots, k-1. \end{aligned}$$

Since the operators  $(H_{1/n})^k$  are bounded and the dilation operator is bounded on any r.i. space, it follows that

$$\begin{aligned}\left\| t^{m/n} (H_{1/n})^{k-m} g(t) \right\|_Y &\preceq \|(H_{1/n})^k g\|_Y \\ &\preceq \|g\|_X.\end{aligned}$$

Whence from (4.2) we obtain that

$$\|H_{k/n} g\|_Y \preceq \|g\|_X,$$

as we wished to show.

**Remark 3.** A similar proof of the necessity part is given in [9].

□

**Corollary 1.** Let  $k \in \mathbb{N}$ ,  $k < n$ , and let  $X$  be a r.i. space such that  $\alpha_X > \frac{k-1}{n}$ ; and let  $Y$  be another r.i. space. Then there exists a constant  $C > 0$  such that  $\|f\|_Y \leq C \| |D^k f| \|_X$  for all  $f \in C_0^\infty(\mathbb{R}^n)$  if and only if

$$\|f\|_Y \preceq \|t^{-k/n} [f^{**}(t) - f^*(t)]\|_X, \quad f \in C_0^\infty(\mathbb{R}^n).$$

*Proof.* Suppose that  $\|f\|_Y \leq C \| |D^k f| \|_X$  for all  $f \in C_0^\infty(\mathbb{R}^n)$ . Let  $f \in C_0^\infty(\mathbb{R}^n)$ , then by (4.1)  $t^{-k/n} [f^{**}(t) - f^*(t)] \in X$  and consequently by Theorem 3 we get,

$$\|H_{k/n}(t^{-k/n} [f^{**}(t) - f^*(t)])\|_Y \preceq \|t^{-k/n} [f^{**}(t) - f^*(t)]\|_X.$$

On the other hand, since

$$H_{k/n}(t^{-k/n} [f^{**}(t) - f^*(t)]) = Q(f^{**} - f^*) = f^{**},$$

we see that

$$\|f\|_Y \leq \|f^{**}\|_Y \preceq \|t^{-k/n} [f^{**}(t) - f^*(t)]\|_X$$

as we wished to show.

□

The previous discussion provides a method to construct the optimal range space for a Sobolev inequality. Indeed, let  $X$  be a r.i. space with  $\alpha_X > \frac{k-1}{n}$ , and let the Sobolev space  $W_0^{k,X} = W_0^{k,X}(\mathbb{R}^n)$  be defined to be the closure of  $C_0^\infty(\mathbb{R}^n)$  under the norm  $\| |D^k f| \|_X$ . Then the optimal target space  $Y$  for the embedding  $W_0^{k,X} \subset Y$  is given by the condition

$$(4.3) \quad \|f\|_Y = \|t^{-k/n}[f^{**}(t) - f^*(t)]\|_X < \infty.$$

However, the space  $Y$  defined by (4.3) may not give a linear function space. For example, if  $X = L^{n/k}$ ,  $k < n$ , then the optimal range space for Sobolev's inequality is given by the condition (cf. [2], [16])

$$\|f\|_Y = \left\{ \int_0^\infty (t^{-k/n}[f^{**}(t) - f^*(t)])^{n/k} dt \right\}^{k/n} = \|f\|_{L(\infty, n/k)} < \infty,$$

which is not a linear space. On the other hand, away from the borderline case (i.e. with a more restrictive condition on the lower Boyd index) it is easy to see that (4.3) is equivalent to a r.i. Banach space.

In what follows it will be useful to formally define when a Sobolev embedding is optimal.

**Definition 1.** Let  $X, Y$  be r.i. spaces such that we have a continuous embedding  $W_0^{k,X} \subset Y$ . We shall say that  $W_0^{k,X} \subset Y$  is optimal if given any other r.i.  $Z$  such that  $W_0^{k,X} \subset Z$ , it follows that  $Y \subset Z$  continuously.

**Corollary 2.** Let  $X$  be a r.i. space with  $\alpha_X > \frac{k}{n}$  for some  $k \in \mathbb{N}$ ,  $k < n$ , and let  $Y$  be the r.i. space defined by the norm  $\|f\|_Y = \|t^{-k/n}f^{**}(t)\|_X$ . Then  $W_0^{k,X} \subset Y$ , and the embedding is optimal.

*Proof.* By Lemma 3 with  $\alpha = k/n$ ,

$$\|t^{-k/n}f^{**}\|_X \preceq \|t^{-k/n}[f^{**}(t) - f^*(t)]\|_X.$$

The result now follows from the previous Corollary.  $\square$

We conclude discussing how our results can be applied to simplify the study of compactness of Sobolev embeddings in the setting of r.i. spaces. For the study of compactness it is natural to restrict oneself to bounded domains  $\Omega$ , and henceforth all spaces will be assumed to be based on a bounded domain  $\Omega$  with smooth boundary.

In the study of compactness we will use the following characterization of compact sets (cf. [19] and the references therein):

**Lemma 4.** Let  $Z$  be a r.i. space and let  $H \subset Z$  be a bounded set. Then  $H$  is compact in  $Z$  iff  $H$  is compact in measure and  $H$  has absolutely equicontinuous norm<sup>12</sup>.

In order to use the results of this paper we recall the connection between optimal embeddings and compactness. Indeed, it is known from the classical  $L^p$  theory that optimal Sobolev embeddings are not compact. Pustylnik [19], has recently extended this result and, most importantly for our purposes, quantified the lack of compactness of optimal embeddings. More precisely, we have the following (cf. [19])

<sup>12</sup>Recall that a set  $H \subset Z$  is absolutely equicontinuous in norm if  $\forall \varepsilon > 0 \exists \delta > 0$  such that if  $|D| < \delta$  then  $\|f\chi_D\|_Z < \varepsilon$ .

**Lemma 5.** *Suppose that  $W_0^{k,X} \subset Y$  is optimal, and let  $Z$  be a r.i. space such that  $W_0^{k,X} \subset Z$  is compact. Then the inclusion  $Y \subset Z$  is absolutely continuous<sup>13</sup>.*

We also note for future use that by an easy case of the Rellich-Kondrachov theorem, the embedding  $W_0^{1,L^1} \subset L^1$  is compact. Therefore, since for any r.i. space  $X$  we have  $W_0^{k,X} \subset W_0^{1,L^1} \subset L^1$ , we see that all bounded sets in  $W_0^{k,X}$  are compact in measure. Consequently to verify that an embedding  $W_0^{k,X} \subset Z$  is compact it is only necessary to verify that bounded sets in  $W_0^{k,X}$  have absolutely continuous norm in  $Z$ .

With these preliminaries at hand we shall now provide our proof of the compactness result recently obtained in [19] and [10] with different but long and complicated methods of proof.

**Theorem 4.** *Let  $X, Z$  be r.i. spaces with  $\alpha_X > \frac{k-1}{n}$  and such that  $W_0^{k,X} \subset Z$ . Then the embedding  $W_0^{k,X} \subset Z$  is compact if and only if  $\tilde{H}_{\frac{n}{k}}$  is a compact operator  $\tilde{H}_{\frac{n}{k}} : X \rightarrow Z$ , here  $\tilde{H}_{\frac{n}{k}} f(t) =: \int_t^{|\Omega|} s^{k/n} f(s) \frac{ds}{s}$ .*

*Proof.* Suppose first that the embedding  $W_0^{k,X} \subset Z$  is compact and consider the optimal embedding  $W_0^{k,X} \subset Y$  provided by (4.3) or by Corollary 2. It follows readily, by a suitable modified version of Theorem 3 for bounded domains, that  $\tilde{H}_{\frac{n}{k}} : X \rightarrow Y$  is bounded. It is easy to see that this implies that  $\tilde{H}_{\frac{n}{k}}$  sends bounded sets  $A \subset X$  into sets  $\tilde{H}_{\frac{n}{k}}(A)$  which are compact in measure. Moreover, by Pustynnik's Lemma 5, the embedding  $Y \subset Z$  is absolutely equicontinuous and since we obviously can factor  $\tilde{H}_{\frac{n}{k}} : X \rightarrow Y \subset Z$ , we see that  $\tilde{H}_{\frac{n}{k}} : X \rightarrow Z$  also maps bounded sets into sets that are absolutely equicontinuous. Therefore, from the compactness criteria given by Lemma 4, we find that  $\tilde{H}_{\frac{n}{k}} : X \rightarrow Z$  is a compact operator.

Conversely, suppose that  $\tilde{H}_{\frac{n}{k}} : X \rightarrow Z$  is a compact operator, and let  $A$  be a bounded set in  $W_0^{k,X}$ . By the definition of  $W_0^{k,X}$  we may assume without loss that  $A \subset C_0^\infty$ . As pointed out above  $A$  is automatically compact in measure, therefore, by Lemma 4, to prove that  $A$  is compact in  $Z$  it remains to verify that  $A$  has absolutely equicontinuous norm. Define  $\tilde{A} = \{\tilde{f} : \tilde{f}(t) = t^{-k/n}[f^{**}(t) - f^*(t)], f \in A\}$ . By (4.1),  $\tilde{A}$  is a bounded set in  $X = X(0, |\Omega|)$ , therefore  $\tilde{H}_{\frac{n}{k}}(\tilde{A})$  is compact in  $Z$ , in particular it has absolutely equicontinuous norm,  $\lim_{a \rightarrow 0} \sup_{f \in \tilde{A}} \left\| \tilde{H}_{\frac{n}{k}} \tilde{f} \chi_{(0,a)} \right\|_Z = 0$ . Moreover, since

$$\tilde{H}_{\frac{n}{k}} \tilde{f} \geq f^{**} \geq f^*$$

it follows that

$$\lim_{a \rightarrow 0} \sup_{f \in A} \|f \chi_{(0,a)}\|_Z = 0,$$

and consequently  $A$  has absolutely equicontinuous norm as we wished to show.  $\square$

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<sup>13</sup>This means that every bounded set  $H \subset Y$  is absolutely continuous in norm in  $Z$ .



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