Martín i Pedret, Joaquim; Milman, Mario. «Higher-order symmetrization inequalities and applications». Journal of mathematical analysis and applications, Vol. 330, Issue 1 (June 2007), p. 91-113. DOI 10.1016/j.jmaa.2006.07.033

# HIGHER ORDER SYMMETRIZATION INEQUALITIES AND APPLICATIONS

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Abstract. We prove new extended forms of the Pólya-Szegö symmetrization principle. As a consequence new sharp embedding theorems for generalized Sobolev and Besov spaces are proved.

## 1. Introduction

Recently sharp forms of the Sobolev embedding theorem have been obtained using new symmetrization inequalities. In [1] it was shown that the oscillation of the decreasing rearrangement of $f$ given by the quantity $f^{**}(t)-f^{*}(t)$ can be estimated by

$$f^{**}(t)-f^{*}(t)\leq c_n t^{1/n}\left|\nabla f\right|^{**}(t),\ f\in C_0^{\infty}(\mathbb{R}^n), \tag{1.1}$$

where $f^{**}(t)=\frac{1}{t}\int_0^t f^{*}(s)ds$, and $f^{*}$ is the non-increasing rearrangement of $f$.

While variants of (1.1) had been known before (cf. [18], [10]), the formulation of inequalities in terms of the oscillation $f^{**}(t)-f^{*}(t)$ leads to general forms of the Sobolev embedding theorem that are sharp up to the end points. Moreover, (1.1) has also proven to be particularly useful in the study of higher order Sobolev inequalities (cf. [1], [14], [15]).

The fractional case was treated in [12] where the following estimate for moduli of continuity was obtained: Let $X(\mathbb{R}^n)$ be a r.i. space, $f\in X(\mathbb{R}^n)$, then

$$f^{**}(t)-f^{*}(t)\leq c\frac{\omega_X(t^{1/n},f)_1}{\phi_X(t)}, \tag{1.2}$$

where $\phi_X(t)$ is the fundamental function[1] of $X(\mathbb{R}^n)$. Using (1.2), sharp embeddings for generalized Besov spaces of order $s\leq 1$ were derived in [12].

Higher order derivatives pose a challenge for symmetrization methods since the Pólya-Szegö symmetrization principle,

$$\left|\nabla f^{\circ}\right|^{**}(t)\leq\left|\nabla f\right|^{**}(t), \tag{1.3}$$

where $f^{\circ}(x)=f^{*}\left(\gamma_n\left|x\right|^n\right)$ ($x\in\mathbb{R}^n,\ \gamma_n=$ measure of the unit ball in $\mathbb{R}^n$) denotes the spherical decreasing rearrangement of $f$, which underlies the validity of (1.1) and (1.2), fails for higher order derivatives. Nevertheless, in [14] it was shown that starting from the embedding theorem implied by (1.1) one could develop an iteration argument that leads to sharp higher order Sobolev estimates.

*Key words and phrases.* Symmetrization, Besov spaces, Sobolev spaces, rearrangement invariant spaces.

2000 Mathematics Subject Classification Primary: 46E30, 26D10.

* Supported by "programa Ramón y Cajal (MCYT)", and supported in part by MTM2004-02299 and by 2005SGR00556.

[1] See Section 2 below.

The method of [14] is indirect, based on certain inequalities and constructions for r.i. spaces. For further analysis it is of interest to have a pointwise inequality of the type (1.1) for higher order derivatives. Observe that (1.1) readily implies the somewhat weaker pointwise estimate

$$f^{**}(t)-f^{*}(t)\leq C_n t^{1/n}\int_t^\infty \left|\nabla f\right|^{**}(s)\frac{ds}{s},\ \ f\in C_0^1(\mathbb{R}^n), \tag{1.4}$$

from which we can iterate (cf. Corollary 2 below) an estimate involving higher order derivatives of all orders

$$f^{**}(t)-f^{*}(t)\leq c(n,k)t^{1/n}\sum_{|\alpha|=k}\int_t^\infty s^{\frac{k-1}{n}}\left(D^\alpha f\right)^{**}(s)\frac{ds}{s},\ \ f\in C_0^k(\mathbb{R}^n). \tag{1.5}$$

This simple formula leads efficiently to sharp higher order Sobolev embeddings and clarifies the role of the assumptions in the embedding theorems in [14]. We remark that using

$$-\frac{d}{dt}f^{**}(t)=\frac{f^{**}(t)-f^{*}(t)}{t},\text{ and } f^{\circ **}=f^{**},$$

we can rewrite (1.1) as

$$\frac{d}{dt}(-f^{\circ **}(t))\leq c_n t^{1/n-1}\left|\nabla f\right|^{**}(t). \tag{1.6}$$

In comparing (1.6) with the classical Pólya-Szegö principle (1.3) we note that the expression that appears on the left hand side of (1.6) involves a derivative associated with the spherical nonincreasing rearrangement of $f$, but the order in which we take the operations $**$ and $\frac{d}{dt}$ has been reversed. Nevertheless, one feels that (1.6), which is a consequence of (1.3), can be considered as a form of the Pólya-Szegö principle from which it is a consequence. A similar comment applies to (1.5), which could then be considered as a "higher order inequality of Pólya-Szegö type".

For the higher order fractional case we extend (1.2) as follows (cf. section 6): for all $f\in C_0^k(\mathbb{R}^n)$ we have

$$f^{**}(t)-f^{*}(t)\leq ct^{1/n}\int_t^\infty \frac{s^{\frac{k-1}{n}}}{\phi_X(s)}\left(\int_0^s\frac{\omega_X(f,z^{\frac{1}{n}})_{k+1}}{z^{\frac{k}{n}}}\frac{dz}{z}\right)\frac{ds}{s}, k\geq 2. \tag{1.7}$$

Here $\omega_X(f,t)_r$ is the $r-$modulus of continuity of $f\in X\left(\mathbb{R}^n\right)$, defined by

$$\omega_X(f,t)_r=\sup_{|h|\leq t}\left\|\Delta_h^r f\right\|_X \qquad (t>0),$$

with $\Delta_h^1 f(x)=f(x+h)-f(x)$ and $\Delta_h^{r+1}f(x)=\Delta_h^1(\Delta_h^r)f(x)$, and $\phi_X$ is the fundamental function of $X$ (see section 2 below).

Using (1.7) will allows us to extend the embedding results obtained in [12] to higher order generalized Besov spaces.

When working with functions defined on domains, the Pólya-Szegö principle (1.3), which underlies the validity of (1.1)-(1.5), requires that the Sobolev functions vanish at the boundary and therefore the extension by approximation requires strong conditions on the boundary of the domains. In this direction we note that Rakotoson [16] obtained independently an inequality closely connected to (1.1) on domains with boundary satisfying a Lipschitz condition and without assuming that the Sobolev functions vanish at the boundary.

In this paper we prove versions of (1.1) and (1.5) for Maz'ya domains and without assuming that the functions vanish at the boundary. For example, in Theorem 2 below we show that if $\Omega$ is bounded open domain in $\mathbb{R}^n$ (for the sake of definiteness we fix $|\Omega| = 1$) which belongs to the Maz'ya class $\mathcal{J}_\alpha$, $1 - \frac{1}{n} \leq \alpha < 1$, then for all $f \in W^{1,1}(\Omega)$ we have

$$f^{**}(t) - f^*(t) \leq ct^{1-\alpha}(|\nabla f|)^{**}(t),\ t \in (0, 1/2), \tag{1.8}$$

and

$$\left(f - \int_\Omega f\right)^{**}(t) - \left(f - \int_\Omega f\right)^{*}(t) \leq ct^{1-\alpha}(|\nabla f|)^{**}(t),\ t \in (0, 1).$$

In particular, we see that for Maz'ya domains in the class $\mathcal{J}_\alpha$, $1 - \frac{1}{n} \leq \alpha < 1$, (1.8) holds for all $t \in (0, 1)$, whenever $\int_\Omega f = 0$. In fact, as we shall see in Theorem 3 below, the last inequality characterizes the domains in Maz'ya's class $\mathcal{J}_\alpha$. For higher order derivatives (see Lemma 1 below) we get that for all $k \geq 1$, for all $f \in W^{k,1}(\Omega)$, and for all $0 < t < 1/2$, we have

$$f^{**}(t) - f^*(t) \leq c(k,n)t^{1-\alpha}\left(\sum_{|\beta|=k}\int_t^1 s^{(k-1)(1-\alpha)}\left(D^\beta f\right)^{**}(s)\frac{ds}{s} + \sum_{1\leq|\beta|\leq k-1}\left\|D^\beta f\right\|_{L^1}\right)$$

In particular, Lipschitz domains correspond to $\alpha = 1 - \frac{1}{n}$ (cf. Example 1 below).

This leads to the following general form of the Sobolev embedding theorem (cf. Theorem 6 below):

**Theorem 1.** *(see [14]) Let $Y(\Omega)$ be a r.i. space, the Sobolev space $W^{k,Y}(\Omega)$ is defined[2] by*

$$W^{k,Y}(\Omega) = \left\{f : D^\beta f \in Y, \text{ for all } \beta,\ |\beta| \leq k\right\},$$
$$\|f\|_{W^{k,Y}(\Omega)} = \sum_{0\leq|\beta|\leq k}\left\|D^\beta f\right\|_{Y(\Omega)}.$$

*Let $\Omega \in \mathcal{J}_\alpha, 1 - \frac{1}{n} \leq \alpha < 1$. Let $k \in \mathbb{N}$, $1 \leq k < n$, and let $Y(\Omega)$ be a r.i. space with Boyd indices[3] such that $(k-1)(1-\alpha) < \underline{\alpha}_Y \leq \overline{\alpha}_Y < 1$. Then*

$$W^{k,Y}(\Omega) \subset Y_{(1-\alpha)k}(\Omega),$$

*where $Y_{(1-\alpha)k}(\Omega)$ denotes the rearrangement invariant set (which with a different notation was introduced in [14]) defined by*

$$Y_{(1-\alpha)k}(\Omega) := \left\{f : t^{-k(1-\alpha)}(f^{**}(t) - f^*(t)) \in Y(\Omega)\right\},$$
$$\|f\|_{Y_{(1-\alpha)k}} = \left\|t^{-k(1-\alpha)}(f^{**}(t) - f^*(t))\right\|_Y.$$

In particular, we obtain the following sharp version of the Sobolev embedding theorem for Maz'ya's domains $\Omega \in \mathcal{J}_\alpha, 1 - \frac{1}{n} \leq \alpha < 1$,

$$W^{k,p}(\Omega) \subset L^{p^*,p}(\Omega), \tag{1.9}$$

where $1 < p \leq \frac{1}{k(1-\alpha)}$, $\frac{1}{p^*} = \frac{1}{p} - (1-\alpha)k$, with the convention that $p^* = \infty$ when $p = \frac{1}{k(1-\alpha)}$, in which case $L^{\infty,p}(\Omega) = \left\{f : \|f\|^p_{L^{\infty,p}(\Omega)} = \int_0^1 (f^{**}(t) - f^*(t))^p \frac{dt}{t} < \infty\right\}$.

Theorem 1 (and in particular (1.9)) extends the results of [14] to Sobolev spaces with rough domains and without requiring the Sobolev functions to vanish at the boundary. We note that, for Lipschitz domains and r.i. spaces, somewhat related

[2]When $Y = L^p$ we use the classical notation $W^{k,Y}(\Omega) = W^{k,p}(\Omega)$.

[3]See Section 2 below.

results were recorded in [9] but with an indirect formulation that only concerns with Banach spaces, while the sharp form of the Sobolev embedding theorem (1.9) uses spaces that are not necessarily linear spaces. In this direction we should also note the dissertation of Kalis [8] where results of this type are extended to Sobolev spaces of vector fields through the use of a connection to Poincaré inequalities.

As usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq c\, f$, while the symbol $f \preceq g$ means that $f \leq c\, g$, and $f \succeq g$ means that $f \geq c\, g$.

## 2. Preliminaries

We rather briefly collect some definitions, notations and properties about functions and function spaces which are used in this paper.

In what follows, given a vector $u \in \mathbb{R}^m$, we denote by $|u|$ its Euclidean norm.

Let $g$ be a locally integrable function having weak derivatives of all orders up to $r \in \mathbb{N}$, we denote by $d^r g$ the vector $\left(D^\beta g\right)_{|\beta|=r}$ of all derivatives of order $|\beta| = r$. It is well-know and easy to see that

$$\left|\nabla \left|d^{k-1}g\right|\right| \preceq \left|d^k g\right|, \quad k = 1, 2, \cdots, r. \tag{2.1}$$

A rearrangement invariant space (r.i. space), $Y = Y(\Omega)$, is a Banach function space of Lebesgue measurable functions on $\Omega \subset \mathbb{R}^n$ endowed with a norm $\|\cdot\|_Y$ that satisfies the Fatou property and is such that, if $f \in Y$ and $g^* = f^*$, then $g \in Y$ and $\|g\|_Y = \|f\|_Y$.

Every r.i. space $Y$ has a representation as a function space on $Y\hat{}(0, |\Omega|)$ such that

$$\|f\|_{Y(\Omega)} = \|f^*\|_{Y\hat{}(0,|\Omega|)}.$$

Since the measure space will be always clear from the context it is convenient to "drop the hat" and use the same letter $Y$ to indicate the different versions of the space $Y$ that we use.

The upper and lower Boyd indices associated with a r.i. space $Y$ are defined by

$$\overline{\alpha}_Y = \inf_{s>1} \frac{\ln h_Y(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_Y = \sup_{s<1} \frac{\ln h_Y(s)}{\ln s}, \tag{2.2}$$

where $h_Y(s)$ denotes the norm on $Y\hat{}(0, |\Omega|)$ of the dilation operator $E_s$, $s > 0$, defined by

$$E_s f(t) = \begin{cases} f^*(\frac{t}{s}) & 0 < t < s|\Omega|, \\ 0 & s|\Omega| < t < |\Omega| \end{cases}.$$

It is also useful sometimes to consider a slightly different set of indices obtained by means of replacing $h_Y(s)$ in (2.2) by

$$M_Y(s) = \sup_{t \in (0, \min(1, \frac{1}{s})|\Omega|)} \frac{\phi_Y(ts)}{\phi_Y(t)}, \ s > 0,$$

where $\phi_Y(s)$ is the fundamental function of $X$ :

$$\phi_Y(s) = \|\chi_E\|_Y,$$

where $E$ is any measurable subset of $\Omega$ with $|E| = s$.

The corresponding indices are denoted $\overline{\beta}_Y$, $\underline{\beta}_Y$, and will be referred to as the upper and lower fundamental indices of $Y$. Actually, we have (cf. [2])

$$0 \leq \underline{\alpha}_Y \leq \underline{\beta}_Y \leq \overline{\beta}_Y \leq \overline{\alpha}_Y \leq 1.$$

We shall usually formulate conditions on r.i spaces in terms of the Hardy operators defined by

$$Pf(t)=\frac{1}{t}\int_0^t f(s)ds;\qquad Q_a f(t)=\frac{1}{t^a}\int_t^\infty s^a f(s)\frac{ds}{s},\quad 0\le a<1.$$

In particular, it is well known that if $Y$ is a r.i. space, $P$ (resp. $Q_a$) is bounded on $Y$ if and only if $\overline{\alpha}_Y<1$ (resp. $a<\underline{\alpha}_Y$) (see for example [2, Chapter 3]). If $a=0$, we shall write $Q$ instead of $Q_0$.

## 3. Rearrangement Inequality and isoperimetric Inequality

In this section we show how to derive the basic rearrangement inequality (1.1), without requiring the Sobolev functions to vanish at the boundary, on rough domains. For example, combining Theorem 2 and Example 1 below, it follows that for a bounded domain $\Omega\subset\mathbb{R}^n$ with Lipschitz boundary we have the following version of (1.1)

$$f^{**}(t)-f^{*}(t)\le c_n t^{1/n}\left|\nabla f\right|^{**}(t),\ t\in(0,\left|\Omega\right|/2),\ f\in W^{1,1}(\Omega). \tag{3.1}$$

More generally, for domains $\Omega$ in Maz'ya's class $\mathcal{J}_\alpha$ $(1-1/n\le\alpha<1)$ (see Definition 1 below) we need to replace $t^{1/n}$ by $t^{1-\alpha}$ on the right hand side of (3.1). Interestingly, as we shall soon see, this leads to Sobolev embeddings that depend on $\alpha$ (see Example 2 bellow for domains with cusps connected with the exponent $\alpha$)

In what follows we consider bounded domains.

**Definition 1.** *(See* [13, page 162]*) A domain $\Omega$ belongs to the class $\mathcal{J}_\alpha$ $(1-1/n\le\alpha<1)$ if there exists a constant $M\in(0,\left|\Omega\right|)$ such that*

$$U_\alpha(M)=\sup\frac{\left|\mathcal{S}\right|^\alpha}{P_\Omega(\mathcal{S})}<\infty$$

*where the sup is taken over all $\mathcal{S}$ open bounded subsets of $\Omega$ such that $\Omega\cap\partial\mathcal{S}$ is a manifold of class $C^\infty$ and $\left|\mathcal{S}\right|\le M$, (in which case we will say that $\mathcal{S}$ is an admissible subset) and where for a measurable set $E\subset\Omega$, $P_\Omega(E)$ is the De Giorgi perimeter of $E$ in $\Omega$ defined by*

$$P_\Omega(E)=\sup\left\{\int_E div\varphi\ dx:\varphi\in[C_0^1(\Omega)]^n,\ \ \left\|\varphi\right\|_{L^\infty(\Omega)}\le1\right\}.$$

By an approximation process it follows that if $\Omega$ is a bounded domain in $\mathcal{J}_\alpha$, then for any $0<M<\left|\Omega\right|$, there is a constant $c_M>0$ such that, for all measurable set $E\subset\Omega$ with $\left|E\right|\le M$, we have

$$P_\Omega(E)\ge c_M\left|E\right|^\alpha. \tag{3.2}$$

Indeed, this was already observed in [17, Lemma 1.12] and follows as a direct consequence of [13, Corollary 3.2.4 and Theorem 6.1.3][4].

[4]We include a proof for the sake of completeness. Let $0<\varepsilon<\left|\Omega\right|-M$ and let $c_M=\sup\left\{\frac{\left|\mathcal{S}\right|^\alpha}{P_\Omega(\mathcal{S})},\ \mathcal{S}\subset\Omega\text{ admissible},\ \left|\mathcal{S}\right|\le M+\varepsilon\right\}$. From [13, Corolary 3.2.4], $0<c_M<\infty$ and we can find an admissible sequence $E_m\subset\Omega$ such that, $\left|E_m\right|\to\left|E\right|$ and $P_\Omega(E_m)\to P_\Omega(E)$. Then $\left|E_m\right|\le\left|E\right|+\varepsilon\le M+\varepsilon$ for $m$ large. Since by the definition of $c_M$, $\left|E_m\right|^\alpha\le c_M P_\Omega(E_m)$, and (3.2) follows.

**Example 1.** *If $\Omega$ is a bounded domain, starshaped with respect to a ball, or a bounded domain having the cone property, or a Lipschitz domain, then $\Omega$ belongs to the class $\mathcal{J}_{1-1/n}$ (see* [13]*).*

**Example 2.** *If*

$$\Omega=\left\{x\in\mathbb{R}^n:\sum_{i=1}^{N-1}x_i^2<x_n^{2\beta},\ 0<x_n<a,\ \beta\geq 1\right\}$$

*then $\Omega\in\mathcal{J}_{\alpha'}$, $\alpha'=\frac{\beta(n-1)}{\beta(n-1)+1}$ and $\Omega\notin\mathcal{J}_{\alpha}$, for $\alpha<\alpha'$ (see* [13, page 176]*).*

**Example 3.** *If $\Omega$ is a John domain then $\Omega\in\mathcal{J}_{1-1/n}$ (see* [3]*).*

In what follows it will be convenient to normalize our domains so that $|\Omega|=1$.

**Theorem 2.** *Let $\Omega$ be a domain in $\mathcal{J}_{\alpha}$. Then there exists a constant $c>0$ such that for all $f\in W^{1,1}(\Omega)$ we have*

$$f^{**}(t)-f^{*}(t)\leq ct^{1-\alpha}\left|\nabla f\right|^{**}(t),\ t\in(0,1/2). \tag{3.3}$$

*If $\int_{\Omega}f=0$, then (3.3) holds up to $t=1$, more precisely there exists a constant $c>0$ such that for all $f\in W^{1,1}(\Omega)$ we have*

$$\left(f-\int_{\Omega}f\right)^{**}(t)-\left(f-\int_{\Omega}f\right)^{*}(t)\leq ct^{1-\alpha}\left|\nabla f\right|^{**}(t),\ t\in(0,1). \tag{3.4}$$

*Proof.* We first establish (3.3) assuming that $f\in W^{1,1}(\Omega)\cap C^{\infty}(\Omega)$. Then, since $f$ is smooth, by Federer's co-area formula (cf. [5]), we have that

$$I(t)=\int_{f^*(t)<|f|\leq f^*(t/2)}\left|\nabla f(x)\right|dx=\int_{f^*(t)}^{f^*(t/2)}H_{n-1}(\{x:|f(x)|=r\})dr, \tag{3.5}$$

where $H_{n-1}$ denotes the $(n-1)$-Hausdorff measure. Since for any measurable set $E$ one has

$$P_{\Omega}(E)\leq H_{n-1}(\partial E\cap\Omega),$$

it follows that if $E=\{x:|f(x)|\geq r\}$, we have $\partial E\subset\{x:|f(x)|=r\}$ (in fact by regularity of $f$ it follows from Sard's lemma that $\partial E=\{x:|f(x)|=r\}$ a.e. $r$). Consequently,

$$I(t)\geq\int_{f^*(t)}^{f^*(t/2)}P_{\Omega}(\{x:|f(x)|\geq r\})dr.$$

Now since for $r\in(f^*(t),f^*(t/2))$ we have

$$|\{x:|f(x)|\geq r\}|\leq|\{x:|f(x)|>f^*(t)\}|\leq t<\frac{1}{2},$$

it follows from (3.2) that

$$P_{\Omega}(\{x:|f(x)|\geq r\})\geq c_{1/2}\left|\{x:|f(x)|\geq r\}\right|^{\alpha}.$$

Therefore for all $0<t<\frac{1}{2}$ we have

$$\begin{aligned}I(t)&=\int_{f^*(t)}^{f^*(t/2)}P_{\Omega}(\{x:|f(x)|\geq r\})dr\geq c_{1/2}\int_{f^*(t)}^{f^*(t/2)}\left|\{x:|f(x)|\geq r\}\right|^{\alpha}dr\\&\succeq\left|\{x:|f(x)|\geq f^*(t/2)\}\right|^{\alpha}(f^*(t/2)-f^*(t))\\&\succeq\left(\frac{t}{2}\right)^{\alpha}(f^*(t/2)-f^*(t)).\end{aligned}$$

On the other hand

$$\begin{aligned} I(t) &= \int_{f^*(t)<|f|\leq f^*(t/2)} |\nabla f(x)|\, dx \leq \int_0^{t/2} |\nabla f|^* (s)ds \\ &\leq \int_0^t |\nabla f|^* (s)ds = t\, |\nabla f|^{**}(t). \end{aligned}$$

So all together we have obtained,

$$[f^*(t/2) - f^*(t)] \preceq t^{1-\alpha} |\nabla f|^{**}(t),\ t \in (0,1/2).$$

Consider now an arbitrary $f \in W^{1,1}(\Omega)$. Select $f_n \in W^{1,1}(\Omega)\cap C^\infty(\Omega)$ such that $f_n^* \to f^*$ a.e. and $f_n \to f$ in $W^{1,1}(\Omega)$. Then, by the first part of the proof,

$$[f_n^*(t/2) - f_n^*(t)] \preceq t^{1-\alpha} |\nabla f_n|^{**}(t),\ t \in (0,1/2),$$

but

$$\begin{aligned} t\, |\nabla f_n|^{**}(t) &\leq \int_0^t |\nabla (f_n - f)|^* (s)ds + \int_0^t |\nabla f|^* (s)ds \\ &\leq \| |\nabla (f_n - f)| \|_{L^1(\Omega)} + t\, |\nabla f|^{**}(t), \end{aligned}$$

therefore

$$\begin{aligned} [f^*(t/2) - f^*(t)] &= \lim_{n\to\infty} [f_n^*(t/2) - f_n^*(t)] \\ &\preceq \lim_{n\to\infty} t^{1-\alpha} |\nabla f_n|^{**}(t) \\ &\preceq t^{1-\alpha} |\nabla f|^{**}(t),\ t \in (0,1/2). \end{aligned}$$

Finally to prove (3.3), simply note that the previous inequality yields

$$\begin{aligned} f^{**}(t) - f^*(t) &\leq \frac{1}{t}\int_0^t (f^*(s/2) - f^*(s))ds + (f^*(t/2) - f^*(t)) \\ &\preceq \left( \frac{1}{t}\int_0^t s^{1-\alpha} |\nabla f|^{**}(s)ds + t^{1-\alpha} |\nabla f|^{**}(t) \right) \\ &\preceq \left( |\nabla f|^{**}(t) \frac{1}{\alpha}\int_0^t s^{1-\alpha} \frac{ds}{s} + t^{1-\alpha} |\nabla f|^{**}(t) \right) \\ &\preceq t^{1-\alpha} |\nabla f|^{**}(t). \end{aligned}$$

If $\int_\Omega f = 0$, and $1/2 \leq t < 1$, then

$$f^{**}(t) - f^*(t) \leq f^{**}(1/2) \leq 2\int_0^1 f^*(s)ds = 2\, \|f\|_{L^1(\Omega)}.$$

Since $\Omega \in \mathcal{J}_\alpha$, the following Sobolev-Poincaré inequality holds (see [13])

$$\|f\|_{L^1(\Omega)} = \left\| f - \int_\Omega f \right\|_{L^1(\Omega)} \preceq \| |\nabla f| \|_{L^1(\Omega)}.$$

Finally, since $1/2 \leq t < 1$

$$\| |\nabla f| \|_{L^1(\Omega)} \preceq t^{1-\alpha} \| |\nabla f| \|_{L^1(\Omega)} = t^{1-\alpha} |\nabla f|^{**}(1) \leq t^{1-\alpha} |\nabla f|^{**}(t).$$

□

**Remark 1.** *(See Talenti [18]) Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain and let $f \in C_0^\infty(\Omega)$. Then by the isoperimetric inequality we have,*

$$H_{n-1}(\{x : |f(x)| = r\}) \geq n\beta_n^{1/n} \left|\{x : |f(x)| \geq r\}\right|^{1-1/n},$$

*where $\beta_n$ is the measure of the unit ball. Inserting this in (3.5) we find*

$$\begin{aligned}\int_{f^*(t)<|f|\leq f^*(t/2)} |\nabla f(x)|\, dx &\geq n\beta_n^{1/n} \int_{f^*(t)}^{f^*(t/2)} \left|\{x : |f(x)| \geq r\}\right|^{1-1/n} dr \\ &\geq n\beta_n^{1/n} \left|\{x : |f(x)| \geq f^*(t)\}\right|^{1-1/n} [f^*(t/2) - f^*(t)] \\ &\geq n\beta_n^{1/n} t^{1-1/n} [f^*(t/2) - f^*(t)].\end{aligned}$$

*Therefore by an easy argument, which actually is contained in the proof of the previous theorem, we can recover the fundamental inequality*

$$f^{**}(t) - f^*(t) \leq c_n t^{1/n} \left|\nabla f\right|^{**}(t),\ f \in C_0^\infty(\Omega).$$

*One could also note that this last inequality follows from Theorem 2 on a ball together with a scaling argument.*

We finish this section showing that the converse of Theorem 2 holds.

**Theorem 3.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $|\Omega| = 1$. Assume that there exists $1 - 1/n \leq \alpha < 1$ such that for all $f \in W^{1,1}(\Omega)$ with $\int_\Omega f = 0$ the rearrangement inequality*

$$f^{**}(t) - f^*(t) \leq ct^{1-\alpha} \left|\nabla f\right|^{**}(t),\ t \in (0,1), \tag{3.6}$$

*holds. Then $\Omega \in \mathcal{J}_\alpha$.*

*Proof.* Let $f \in W^{1,1}(\Omega)$, set $g = f - \int_\Omega f$. Then

$$\begin{aligned}\|g\|_{L^{1/\alpha,\infty}(\Omega)} &\leq \sup_{0<t<1} t^\alpha g^{**}(t) = \sup_{0<t<1} t^\alpha \Big(\int_t^1 (g^{**}(s) - g^*(s))\frac{ds}{s} + \int_0^1 g^*(s)ds\Big) \\ &\preceq \sup_{0<t<1} t^\alpha \left(g^{**}(t) - g^*(t)\right) + \|g\|_{L^1(\Omega)} \\ &\preceq \sup_{0<t<1} \int_0^t \left|\nabla f\right|^*(s)ds + \|g\|_{L^1(\Omega).} \qquad \text{(by (3.6))} \\ &\preceq \||\nabla f|\|_{L^1(\Omega)} + \left\|f - \int_\Omega f\right\|_{L^1(\Omega)}.\end{aligned}$$

Therefore

$$\inf_{c\in\mathbb{R}} \|f - c\|_{L^{1/\alpha,\infty}(\Omega)} \leq \||\nabla f|\|_{L^1(\Omega)} + \left\|f - \int_\Omega f\right\|_{L^1(\Omega)}. \tag{3.7}$$

Let us see now that (3.6) implies the Sobolev-Poincaré inequality

$$\left\|f - \int_\Omega f\right\|_{L^1(\Omega)} \preceq \||\nabla f|\|_{L^1(\Omega)}, \quad \forall f \in W^{1,1}(\Omega). \tag{3.8}$$

By approximation it is enough to prove this claim assuming that $f \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$. Then $g = f - \int_\Omega f \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$. By (3.6),

$$(g^{**}(t) - g^*(t)) \preceq t^{-\alpha} \int_0^t \left|\nabla f\right|^*(s) \leq t^{-\alpha} \||\nabla f|\|_1,\ t \in (0,1).$$

Thus

$$t\left(g^{**}(t)-g^{*}(t)\right)\preceq t^{1-\alpha}\left\||\nabla f|\right\|_{1}\leq\left\||\nabla f|\right\|_{1}.$$

Since $t\left(g^{**}(t)-g^{*}(t)\right)=\int_{g^{*}(t)}^{\infty}|\{x:|g(x)|>s\}|ds$, and the last function is obviously increasing, we have

$$\sup_{0<t<1}t\left(g^{**}(t)-g^{*}(t)\right)=\lim_{t\to1^{-}}t\left(g^{**}(t)-g^{*}(t)\right)=\int_{0}^{1}g^{*}(s)ds-g^{*}(1^{-}).$$

Now,

$$g^{*}(1^{-})=\inf_{x\in\Omega}|g(x)|,$$

and since $\left|f-\int_{\Omega}f\right|\in C(\Omega)$,

$$\inf_{x\in\Omega}\left|f(x)-\int_{\Omega}f\right|=0.$$

Combining these observations we see that for all $f\in W^{1,1}(\Omega)\cap C^{\infty}(\Omega)$,

$$\int_{0}^{1}g^{*}(s)ds=\left\|f-\int_{\Omega}f\right\|_{L^{1}(\Omega)}\preceq\left\||\nabla f|\right\|_{L^{1}(\Omega)}.$$

Therefore we can write (3.7) as

$$\inf_{c\in\mathbb{R}}\|f-c\|_{L^{1/\alpha,\infty}(\Omega)}\preceq\left\||\nabla f|\right\|_{L^{1}(\Omega)}.$$

By Maz'ya's truncation principle (cf. [6, Theorem 4]), this inequality implies the strong type inequality,

$$\inf_{c\in\mathbb{R}}\|f-c\|_{L^{1/\alpha}(\Omega)}\preceq\left\||\nabla f|\right\|_{L^{1}(\Omega)}.$$

We conclude using the fact that Sobolev-Poincaré type inequalities imply the validity of Maz'ya's $\mathcal{J}_{\alpha}$ conditions (see [13, Lemma 2 and Corollary, page 169]). □

## 4. Inequalities for higher order derivatives

In this section we shall obtain a higher order version of (3.3). Throughout this section $\Omega$ will be a bounded domain with $|\Omega|=1$.

**Lemma 1.** *Let $\Omega\in\mathcal{J}_{\alpha}$, $k\geq1$. Then for all $f\in W^{k,1}(\Omega)$, and all $t\in(0,1/2)$,*

$$f^{**}(t)-f^{*}(t)\leq ct^{1-\alpha}\left(\int_{t}^{1}s^{(k-1)(1-\alpha)}\left|d^{k}f\right|^{**}(s)\frac{ds}{s}+\sum_{j=1}^{k-1}\left\|\left|d^{j}f\right|\right\|_{L^{1}(\Omega)}\right)\tag{4.1}$$

*where $c:=c(n,k)>0$ is a constant independent of $f$.*

*Proof.* (By induction). If $k=1$, then, by Theorem 2, we have that for all $f\in W^{1,1}(\Omega)$,

$$f^{**}(t)-f^{*}(t)\leq ct^{1-\alpha}|\nabla f|^{**}(t),\ t\in(0,1/2).$$

Thus, for all $t\in(0,1/2)$,

$$\begin{aligned}t^{\alpha-1}\left(f^{**}(t)-f^{*}(t)\right)&\preceq(|\nabla f|)^{**}(t)=\frac{1}{t}\int_{0}^{t}|\nabla f|^{*}(z)\,dz\leq2\int_{t}^{1}\frac{ds}{s^{2}}\int_{0}^{t}|\nabla f|^{*}(z)\,dz\\&\leq2\int_{t}^{1}\frac{ds}{s^{2}}\int_{0}^{s}|\nabla f|^{*}(z)\,dz=2\int_{t}^{1}|\nabla f|^{**}(s)\,\frac{ds}{s}.\end{aligned}$$

Assume now that (4.1) holds for $1, 2, \ldots, k-1$, then, if $f \in W^{k,1}(\Omega) \subset W^{k-1,1}(\Omega)$, we can write

(4.2)
$$f^{**}(t) - f^{*}(t) \leq ct^{1-\alpha}\left(\int_t^1 s^{(k-2)(1-\alpha)}\left|d^{k-1}f\right|^{**}(s)\frac{ds}{s} + \sum_{j=1}^{k-2}\left\|\left|d^j f\right|\right\|_{L^1}\right)$$
$$\preceq t^{1-\alpha}\left(\int_t^{1/2} s^{(k-2)(1-\alpha)}\left|d^{k-1}f\right|^{**}(s)\frac{ds}{s} + \sum_{j=1}^{k-1}\left\|\left|d^j f\right|\right\|_{L^1}\right).$$

Since $\left|d^{k-1}f\right| \in W^{1,1}(\Omega)$, we have that for $t \in (0, 1/2)$

$$\left|d^{k-1}f\right|^{**}(t) - \left|d^{k-1}f\right|^{*}(t) \leq ct^{1-\alpha}\left|\nabla\left|d^{k-1}f\right|\right|^{**}(t). \tag{4.3}$$

But

$$\begin{aligned}\left|d^{k-1}f\right|^{**}(s) &= \int_s^{1/2}\left(\left|d^{k-1}f\right|^{**}(z) - \left|d^{k-1}f\right|^{*}(z)\right)\frac{dz}{z} \\ &\quad + \left|d^{k-1}f\right|^{**}(1/2) \\ &= I_0(s) + \left|d^{k-1}f\right|^{**}(1/2) \\ &\preceq I_0(s) + \left\|\left|d^{k-1}f\right|\right\|_{L^1}.\end{aligned}$$

We estimate $I_0(s)$ using (4.3) and (2.1) to get

$$I_0(s) \leq c\int_s^1 z^{1-\alpha}\left|\nabla\left|d^{k-1}f\right|\right|^{**}(z) \leq c\int_s^1 z^{1-\alpha}\left|d^k f\right|^{**}(z)\frac{dz}{z}.$$

Now, inserting this estimate in (4.2) and a short argument involving Fubini's theorem yields (4.1). □

**Corollary 1.** *Let $\Omega \in \mathcal{J}_\alpha$. Then for all $f \in W^{k,1}(\Omega)$ $(k \geq 1)$, $t \in (0,1)$, we have*

$$f^{**}(t) \preceq \int_t^1 s^{k(1-\alpha)}\left|d^k f\right|^{**}(s)\frac{ds}{s} + \sum_{j=0}^{k-1}\left\|\left|d^j f\right|\right\|_{L^1(\Omega)}. \tag{4.4}$$

*Proof.* We start with the familiar formula

$$f^{**}(t) = \int_t^{1/2}\left(f^{**}(s) - f^{*}(s)\right)\frac{ds}{s} + f^{**}(1/2),\ t \in (0, 1/2). \tag{4.5}$$

Now, we estimate the integrand in (4.5) using Lemma 1 and find that for all $t \in (0, 1/2)$

$$\begin{aligned}f^{**}(t) &\preceq \int_t^{1/2} s^{1-\alpha}\left(\int_s^1 z^{(k-1)(1-\alpha)}\left|d^k f\right|^{**}(z)\frac{dz}{z}\right)\frac{ds}{s} + \sum_{j=1}^{k-1}\left\|\left|d^j f\right|\right\|_{L^1(\Omega)} + f^{**}(1/2) \\ &\preceq \int_t^1 s^{1-\alpha}\left(\int_s^1 z^{(k-1)(1-\alpha)}\left|d^k f\right|^{**}(z)\frac{dz}{z}\right)\frac{ds}{s} + \sum_{j=0}^{k-1}\left\|\left|d^j f\right|\right\|_{L^1(\Omega)} \\ &\leq I(t) + \sum_{j=0}^{k-1}\left\|\left|d^j f\right|\right\|_{L^1(\Omega)}.\end{aligned}$$

Fubini's theorem yields

$$\begin{aligned} I(t) &= \int_t^1 z^{(k-1)(1-\alpha)} \left|d^k f\right|^{**}(z) \left(\int_t^z s^{1-\alpha}\frac{ds}{s}\right)\frac{dz}{z} \\ &\preceq \int_t^1 s^{k(1-\alpha)} \left|d^k f\right|^{**}(s)\frac{ds}{s}. \end{aligned}$$

Since for $1/2 < t < 1$ we obviously have

$$f^{**}(t) \leq f^{**}(\frac{1}{2}) \leq 2\left\|f\right\|_1$$

we get that (4.4) holds for all $t \in (0,1)$ . □

Using the same method given in the proof of Lemma 1 we obtain easily the following result

**Corollary 2.** *If $f \in C_0^k(\mathbb{R}^n)$ (or $f \in C_0^k(\Omega)$, where $\Omega$ an arbitrary domain in $\mathbb{R}^n$) and $k \geq 1$, then*

$$f^{**}(t) - f^{*}(t) \leq ct^{1/n}\int_t^\infty s^{\frac{k-1}{n}} \left|d^k f\right|^{**}(s)\frac{ds}{s},$$

*where $c := c(n,k) > 0$ is independent of $f$.*

## 5. Applications

In this section we extend and complement recent results in [14], [4], [9] and [12].

### 5.1. **Symmetrization inequalities and Sobolev-Poincaré type inequalities.**

In this section we show some sharp Sobolev-Poincaré type inequalities that follow from (3.3) and (3.4) in the context of $L^{p,q}$ spaces.

**Lemma 2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $|\Omega| = 1$. Assume that there exists $1 - 1/n \leq \alpha < 1$ such that for all $f \in W^{1,1}(\Omega)$ inequalities (3.3) and (3.4) hold. Then,*

*1. If $p > 1, 1-\alpha < 1/p$ and $r = \frac{p}{p(\alpha-1)+1}$ then $W^{1,p}(\Omega) \subset L^{r,p}(\Omega)$, moreover*

$$\inf_{c\in\mathbb{R}} \left\|f - c\right\|_{L^{r,p}(\Omega)} \preceq \left\||\nabla f|\right\|_{L^p(\Omega)}, \quad \forall f \in W^{1,p}(\Omega). \tag{5.1}$$

*(Notice that $L^{r,p}(\Omega) \subset L^r(\Omega)$, since $r > p$).*

*2. If $p > 1$ and $1 - \alpha = 1/p$, then $r = \frac{p}{p(\alpha-1)+1} = \infty$, and we have $W^{1,p}(\Omega) \subset L^{\infty,p}(\Omega)$ and*

$$\inf_{c\in\mathbb{R}} \left\|f - c\right\|_{L^{\infty,p}(\Omega)} \preceq \left\||\nabla f|\right\|_{L^p(\Omega)},$$

*where $L^{\infty,p}(\Omega) = \left\{f : \left\|f\right\|_{L^{\infty,p}(\Omega)}^p = \int_0^1 \left(f^{**}(t) - f^*(t)\right)^p \frac{dt}{t} < \infty\right\}$.*

*Proof.* 1. Let $f \in W^{1,p}(\Omega) \subset W^{1,1}(\Omega)$, and let $I = \left(\int_0^1 \left(\left(f^{**}(t) - f^*(t)\right)t^{1/r}\right)^p \frac{dt}{t}\right)^{1/p}$. Splitting the interval of integration $(0,1) = (0,\frac{1}{2}) \cup [\frac{1}{2},1)$ and using (3.3) we see that

$$\begin{aligned} I &\preceq \left\||\nabla f|^{**}\right\|_p + \left\|f\right\|_1 \\ &\preceq \left\|f\right\|_{W^{1,p}(\Omega)} \text{ (since } p > 1). \end{aligned}$$

By the fundamental theorem of Calculus we can write

$$f^{**}(t)=\int_t^1 \left(f^{**}(s)-f^{*}(s)\right)\frac{ds}{s}+\int_0^1 f^{*}(s)ds,$$

therefore using that $Q:L^{r,p}\rightarrow L^{r,p}$ is bounded, and $p>1$, we readily see that

$$\|f\|_{L^{r,p}(\Omega)}\preceq I+\|f\|_1\preceq \||\nabla f|\|_{L^p(\Omega)}+\|f\|_1\,. \tag{5.2}$$

Thus,

$$W^{1,p}(\Omega)\subset L^{r,p}(\Omega).$$

It follows from (5.2) that for any $c\in\mathbb{R}$ we have

$$\|f-c\|_{L^{r,p}(\Omega)}\preceq\left(\|f-c\|_{L^1(\Omega)}+\||\nabla f|\|_{L^p(\Omega)}\right).$$

Therefore

$$\begin{aligned}\inf_{c\in\mathbb{R}}\|f-c\|_{L^{r,p}(\Omega)}&\preceq\left(\inf_{c\in\mathbb{R}}\|f-c\|_{L^1(\Omega)}+\||\nabla f|\|_{L^p(\Omega)}\right)\\&\preceq\||\nabla f|\|_{L^1(\Omega)}+\||\nabla f|\|_{L^p(\Omega)}\ \text{(by (3.8))}\\&\leq\||\nabla f|\|_{L^p(\Omega)}\,,\end{aligned}$$

and (5.1) follows.

2. If $1-\alpha=1/p$, then $I=\|f\|_{L^{\infty,p}(\Omega)}$. We proceed as before, and we have

$$\begin{aligned}\|f\|_{L^{\infty,p}(\Omega)}&=\left(\int_0^1\left(f^{**}(t)-f^{*}(t)\right)^p\frac{dt}{t}\right)^{1/p}\\&\preceq\left\||\nabla f|^{**}\right\|_p+\|f\|_1\end{aligned}$$

and hence

$$\inf_{c\in\mathbb{R}}\|f-c\|_{L^{\infty,p}(\Omega)}\preceq\||\nabla f|\|_{L^p(\Omega)}\,.$$

□

**Remark 2.** *Consider the Brézis-Wainger spaces $BW^p(\Omega)$ defined by*

$$BW^p(\Omega)=\left\{f:\|f\|_{BW^p(\Omega)}=\left\{\int_0^1\left(\frac{f^{**}(s)}{1+\ln\frac{1}{s}}\right)^p\frac{ds}{s}\right\}^{1/p}<\infty\right\}.$$

*In the limiting case $1-\alpha=\frac{1}{p}$, the previous result implies a Poincaré-Sobolev inequality involving these spaces. Indeed, since $L(\infty,p)(\Omega)\subset BW^p(\Omega)$, and in fact (cf.* [1, Lemma 2]*)*

$$\|f\|_{BW^p(\Omega)}\preceq\|f\|_{L^{\infty,p}(\Omega)}+\|f\|_1\,,$$

*the second part of Lemma 2 gives*

$$\inf_{c\in\mathbb{R}}\|f-c\|_{BW^p(\Omega)}\preceq\||\nabla f|\|_{L^p(\Omega)}\,,\quad\forall f\in W^{1,p}(\Omega).$$

**Remark 3.** *In Lemma 2 we can also consider the case $p=1$, which corresponds to $r=\frac{p}{p(\alpha-1)+1}=\frac{1}{\alpha}$. We obtain*

$$\inf_{c\in\mathbb{R}}\|f-c\|_{L^{1/\alpha,\infty}(\Omega)}\leq\left\|f-\int_\Omega f\right\|_{L^{1/\alpha,\infty}(\Omega)}\preceq\||\nabla f|\|_{L^1(\Omega)}\,.$$

*For the details see the proof of Theorem 3 above.*

5.2. **Sobolev embeddings for domains $\Omega \subset \mathbb{R}^n$ of class $\mathcal{J}_\alpha$.** We extend the results of the previous section to the setting of r.i. spaces.

**Theorem 4.** *Let $\Omega$ be a bounded domain in $\mathcal{J}_\alpha$ with $|\Omega| = 1$. Let $1 \leq k < n$. Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Assume that*

$$\left\| \int_t^1 s^{k(1-\alpha)} g^{**}(s) \frac{ds}{s} \right\|_{X\hat{}(0,1)} \leq c \left\| g \right\|_{Y\hat{}(0,1)}, \qquad \forall g \in \mathcal{M}^+(0,1). \tag{5.3}$$

*Then the following statements hold:*

(1) *For $1 \leq k < n$ we have,*

$$W^{k,Y}(\Omega) \subset X(\Omega).$$

(2)

$$\inf_{\Lambda \in \mathcal{P}_{k-1}} \left\| f - \Lambda \right\|_{X(\Omega)} \preceq \left\| \left| d^k f \right| \right\|_{Y(\Omega)},$$

*where $\mathcal{P}_{k-1}$ is the set of polynomials of degree $k-1$.*

*Proof.* (1). By Corollary 1, we can write

$$f^{**}(t) \preceq \int_t^1 s^{k(1-\alpha)} \left| d^k f \right|^{**}(s) \frac{ds}{s} + \sum_{j=0}^{k-1} \left\| \left| d^j f \right| \right\|_{L^1(\Omega)}, \ t \in (0,1).$$

Applying the $X$ norm and then the triangle inequality we get

$$\left\| f \right\|_X \preceq \left\| \int_t^1 s^{k(1-\alpha)} \left| d^k f \right|^{**}(s) \frac{ds}{s} \right\|_X + \sum_{j=0}^{k-1} \left\| \left| d^j f \right| \right\|_{L^1(\Omega)}.$$

The first term on the right hand side can be estimated using (5.3). To estimate $\sum_{j=1}^{k-1} \left\| \left| d^j f \right| \right\|_{L^1(\Omega)}$, we simply use the fact that $Y(\Omega) \subset L^1(\Omega)$. It follows that

$$\left\| f \right\|_{X(\Omega)} \preceq \left\| f \right\|_{W^{k,Y}(\Omega)}.$$

(2). For simplicity we only consider the case $k = 2$. Let $f \in W^{2,Y}(\Omega)$ and let

$$p(x) = \int_\Omega f + \sum_{i=1}^n \left( \int_\Omega \frac{\partial f}{\partial x_i} \right) x_i \quad \text{and} \quad \Pi(x) = p(x) + \int_\Omega (f - p).$$

Let $g = f - \Pi = (f-p) - \int_\Omega (f-p)$. By Corollary 1

$$g^{**}(t) \preceq \int_t^1 s^{2(1-\alpha)} \left| d^2 f \right|^{**}(s) \frac{ds}{s} + \sum_{j=0}^{1} \left\| \left| d^j g \right| \right\|_{L^1(\Omega)}.$$

Since $W^{2,Y}(\Omega) \subset X(\Omega)$, we have

$$\begin{aligned} \left\| g \right\|_{X(\Omega)} &\preceq \left\| \int_t^1 s^{2(1-\alpha)} \left| d^2 f \right|^{**}(s) \frac{ds}{s} + \sum_{j=0}^{1} \left\| \left| d^j g \right| \right\|_{L^1(\Omega)} \right\|_{X(\Omega)} \\ &\preceq \left\| \left| d^2 f \right| \right\|_{Y(\Omega)} + \sum_{j=0}^{1} \left\| \left| d^j g \right| \right\|_{L^1(\Omega)} \quad \text{(by (5.3))}. \end{aligned}$$

Since $\Omega \in \mathcal{J}_\alpha$,

$$\begin{aligned}\left\||\nabla g|\right\|_{L^1(\Omega)} &\preceq \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} - \int_\Omega \frac{\partial f}{\partial x_i} \right\|_{L^1(\Omega)} \\ &\preceq \sum_{i=1}^n \left\| \left| \nabla \frac{\partial f}{\partial x_i} \right| \right\|_{L^1(\Omega)} \\ &\preceq \left\| |d^2 f| \right\|_{Y(\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|g\|_{L^1(\Omega)} &= \left\| (f-p) - \int_\Omega (f-p) \right\|_{L^1(\Omega)} \\ &\preceq \left\| |\nabla (f-p)| \right\|_{L^1(\Omega)} \\ &\preceq \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} - \int_\Omega \frac{\partial f}{\partial x_i} \right\|_{L^1(\Omega)} \\ &\preceq \sum_{i=1}^n \left\| \left| \nabla \frac{\partial f}{\partial x_i} \right| \right\|_{L^1(\Omega)} \\ &\preceq \left\| |d^2 f| \right\|_{Y(\Omega)}.\end{aligned}$$

Summarizing we have

$$\|g\|_{X(\Omega)} \preceq \left\| |d^2 f| \right\|_{Y(\Omega)}.$$

This concludes the proof since

$$\inf_{\Lambda \in \mathcal{P}_1} \|f - \Lambda\|_{X(\Omega)} \leq \|f - \Pi\|_{X(\Omega)} = \|g\|_{X(\Omega)}.$$

□

5.3. **Optimal Sobolev embeddings for domains $\Omega \subset \mathbb{R}^n$ of class $\mathcal{J}_{1-1/n}$.** In what follows $\Omega$ will be a bounded domain of class $\mathcal{J}_{1-1/n}$ with $|\Omega| = 1$. Let us also recall (see Examples 1, 2 and 3 above) that the class $\mathcal{J}_{1-1/n}$ includes several important examples, like domains with the cone property, domains with Lipschitz boundary or John domains. In fact if $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with $|\Omega| = 1$ then combining the results of [3] and Theorem 2 we have that

$$f^{**}(t) - f^*(t) \leq c t^{1/2} (|\nabla f|)^{**}(t), \quad t \in (0,1)$$

for all $f \in W^{11}(\Omega)$ such that $\int_\Omega f = 0$, if and only if $\Omega$ is a John domain.

**Theorem 5.** *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces such that $0 < \underline{\alpha}_Y$, $\overline{\alpha}_Y < 1$. The embedding*

$$W^{1,Y}(\Omega) \subset X(\Omega) \tag{5.4}$$

*holds for every $\Omega$ bounded domain of class $\mathcal{J}_{1-1/n}$, with $|\Omega| = 1$, if and only if*

$$\left\| \int_t^1 g^{**}(s) s^{\frac{1}{n}} \frac{ds}{s} \right\|_{X\hat{}(0,1)} \leq c \|g\|_{Y\hat{}(0,1)}, \qquad \forall g \in Y\hat{}(0,1). \tag{5.5}$$

*Moreover, if (5.4) holds,*

$$X(\Omega) \subset Y_{1/n}(\Omega). \tag{5.6}$$

*Proof.* Assume that condition (5.5) holds, then (5.4) follows by Theorem 4 with $\alpha = 1 - 1/n$ and $k = 1$.

Conversely, set

$$u(x) = \int_{\gamma_n |x|^n}^{1} f(s) s^{1/n-1} ds, \quad x \in B, \quad f \in \mathcal{M}^{+}(0,1).$$

($\gamma_n$ = measure of the unit ball in $\mathbb{R}^n$ and $B$ is the ball about the origin with radius $\gamma_n^{-n}$).

Observe that for $h \in Y\hat{}(0,1)$ we have that

$$|\{x \in B : h(\gamma_n |x|^n) > \lambda\}| = |\{t \in (0,1) : h(t) > \lambda\}| .$$

Consequently

$$u^{*}(t) = \int_t^1 s^{1/n} f(s) \frac{ds}{s}.$$

Moreover, an easy computation shows that

$$|\nabla u(x)| = n\gamma_n f(\gamma_n |x|^n).$$

Therefore (5.4) applied to the domain $B$ yields

$$\begin{aligned} \left\| \int_t^1 f(s) s^{1/n-1} ds \right\|_X &= \|u\|_X \preceq \|\nabla u\|_Y + \|u\|_Y \\ &\preceq \|f\|_Y + \|u\|_Y . \end{aligned}$$

We conclude observing that

$$\begin{aligned} \|u\|_Y &= \left\| \int_t^1 f(s) s^{1/n-1} ds \right\|_Y \leq \left\| \int_t^1 |f(s)| \frac{ds}{s} \right\|_{L^1} \\ &\leq c \|f\|_Y . \end{aligned}$$

The proof of (5.6) follows from [14, Theorem 3.6]. □

**Remark 4.** *Under the assumptions of Theorem 5 and starting from (5.5) and using a suitable modification of an argument in* [4] *it is possible to show a higher order version of (5.5), namely that if $W^{k,Y}(\Omega) \subset X(\Omega)$ holds for every bounded domain of class $\mathcal{J}_{1-1/n}$ then*

$$\left\| \int_t^1 g^{**}(s) s^{\frac{k}{n}} \frac{ds}{s} \right\|_{X\hat{}(0,1)} \leq c \|g\|_{Y\hat{}(0,1)} .$$

*We shall leave the details to the interested reader. Likewise, when dealing with necessary conditions for the embedding $W_0^{1,Y}(\Omega) \subset X(\Omega)$, where $\Omega$ is a ball with $|\Omega| = 1$, if we assume the density of functions in $L^1(0,1)$ that vanish in a neighborhood of* 1 *in $Y$ we can use the test functions $u(x) = \int_{\gamma_n |x|^n}^{1} f^{\circ}(s) s^{1/n-1} ds$, to prove the validity of (5.5) in this case as well.*

5.4. **Extensions to the results of Milman-Pustylnik.** Here we shall extend the results of [14] to Sobolev spaces on domains in the class $\mathcal{J}_\alpha$ and without boundary conditions.

First at all, notice that if that $\Omega$ is a bounded domain in $\mathcal{J}_\alpha$ with $|\Omega| = 1$ and $Y(\Omega)$ is a r.i. space such that $\overline{\alpha}_Y < 1$, then it follows from (3.3) that

$$W^{1,Y}(\Omega) \subset Y_{1-\alpha}(\Omega).$$

For the case $2 \leq k < n$ we have

**Theorem 6.** [5] *Let $\Omega$ be a bounded domain of class $\mathcal{J}_\alpha$ with $|\Omega|=1$. Let $k\in\mathbb{N}$, $2\leq k<n$, and let $Y(\Omega)$ be a r.i. space such that $(k-1)(1-\alpha)<\underline{\alpha}_Y\leq\overline{\alpha}_Y<1$. Then*

$$W^{k,Y}(\Omega)\subset Y_{(1-\alpha)k}(\Omega).$$

*Proof.* By Lemma 1, for all $t\in(0,1/2)$,

$$f^{**}(t)-f^{*}(t)\leq ct^{1-\alpha}\left(\int_t^1 s^{(k-1)(1-\alpha)}\left|d^kf\right|^{**}(s)\frac{ds}{s}+\sum_{j=1}^{k-1}\left\|\left|d^jf\right|\right\|_{L^1(\Omega)}\right).$$

Thus,

$$\begin{aligned} t^{-k(1-\alpha)}f^{**}(t)-f^{*}(t) &\preceq t^{(1-k)(1-\alpha)}\int_t^1 s^{(k-1)(1-\alpha)}|d^kf|^{**}(s)\frac{ds}{s}\\ &+t^{(1-k)(1-\alpha)}\sum_{j=1}^{k-1}\left\|\left|d^jf\right|\right\|_{L^1}\\ &=I_0(t)+I_1(t). \end{aligned}$$

By Fubini's theorem

$$\begin{aligned} I_0(t) &\preceq \frac{1}{t}\int_0^t|d^kf|^{*}(s)ds\\ &+t^{(1-k)(1-\alpha)}\int_t^1 s^{(k-1)(1-\alpha)}|d^kf|^{*}(s)\frac{ds}{s}\\ &=|d^kf|^{**}(t)+Q_{(k-1)(1-\alpha)}(|d^kf|^{*})(t),\ t\in(0,1/2). \end{aligned}\tag{5.7}$$

Moreover, for $0<t<1/2$

$$\begin{aligned} I_1(t) &= t^{(1-k)(1-\alpha)}\sum_{j=1}^{k-1}\left\|\left|d^jf\right|\right\|_{L^1}\\ &\preceq\left(t^{(1-k)(1-\alpha)}\int_t^1 s^{(k-1)(1-\alpha)}\frac{ds}{s}\right)\sum_{j=1}^{k-1}\left\|\left|d^jf\right|\right\|_{L^1}\\ &=Q_{(k-1)(1-\alpha)}\left(1\right)(t)(\sum_{j=1}^{k-1}\left\|\left|d^jf\right|\right\|_{L^1}). \end{aligned}\tag{5.8}$$

Likewise, if $1/2\leq t<1$,

$$\begin{aligned} t^{-k(1-\alpha)}\left(f^{**}(t)-f^{*}(t)\right) &\leq \sup_{1/2\leq t\leq 1}\left(t^{-k(1-\alpha)}f_o^{*}(t)\right)\\ &\preceq f^{**}(1/2)\preceq f^{**}(1)=\left\|f\right\|_{L_1}. \end{aligned}\tag{5.9}$$

Thus by (5.7), (5.8), and (5.9) we get that for $t\in(0,1)$

$$\begin{aligned} t^{-k(1-\alpha)}(f^{**}(t)-f^{*}(t)) &\preceq |d^kf|^{**}(t)+Q_{(k-1)(1-\alpha)}\left(|d^kf|^{*}\right)(t)\\ &+Q_{(k-1)(1-\alpha)}\left(1\right)(t)(\sum_{j=1}^{k-1}\left\|\left|d^jf\right|\right\|_{L^1})+\left\|f\right\|_{L_1}. \end{aligned}$$

[5] This Theorem coincides with Theorem 1 of the introduction

Therefore, by the conditions on indices, and the fact that $Y(\Omega) \subset L^1$, we get that $\|f\|_{Y_{(1-\alpha)k}(\Omega)}$ can be estimated by

$$\left\| |d^k f|^{**}(t) + Q_{(k-1)(1-\alpha)}\left(|d^k f|^*\right)(t) + Q_{(k-1)(1-\alpha)}(1)(t)\left(\sum_{j=1}^{k-1} \left\| |d^j f| \right\|_{L^1}\right) + \|f\|_{L_1} \right\|_Y$$

$$\preceq \left( \left\| |d^k f| \right\|_Y + \sum_{j=0}^{k-1} \left\| |d^j f| \right\|_{L^1} \right)$$

$$\leq \sum_{j=0}^{k} \left\| |d^k f| \right\|_Y .$$

□

**Proposition 1.** *Let $\Omega \in \mathcal{J}_\alpha$, let $p > 1$, and suppose that $kp(1-\alpha) \leq 1$. Then*

$$W^{k,p}(\Omega) \subset L^{q^*,p}(\Omega), \quad q^* = \frac{p}{1 - kp(1-\alpha)}.$$

*Note that if $kp(1-\alpha) = 1$, then*

$$W^{k,p}(\Omega) \subset L^{\infty,p}(\Omega) \subset BW^p(\Omega).$$

**Remark 5.** *Using Corollary 2, we can obtain an easy proof of the following result (see* [14, Theorem 1.2]*): Let $\Omega$ be an open domain in $\mathbb{R}^n$, let $k \in \mathbb{N}$, $2 \leq k < n$ and let $Y(\Omega)$ be a r.i. space such that $\frac{k-1}{n} < \underline{\alpha}_Y \leq \overline{\alpha}_Y < 1$. Then*

$$W_0^{k,X}(\Omega) \subset Y_{k/n}(\Omega). \tag{5.10}$$

*In fact, by Corollary 2*

$$f^{**}(t) - f^*(t) \leq ct^{1/n} \int_t^\infty s^{\frac{k-1}{n}} |d^k f|^{**}(s) \frac{ds}{s}.$$

*Thus*

$$t^{\frac{-k}{n}}(f^{**}(t) - f^*(t)) \leq ct^{\frac{1-k}{n}} \int_t^\infty s^{\frac{k-1}{n}} |d^k f|^{**}(s) \frac{ds}{s}.$$

*By Fubini*

$$\begin{aligned} t^{\frac{1-k}{n}} \int_t^\infty s^{\frac{k-1}{n}} g^{**}(s) \frac{ds}{s} &= \frac{n}{n-k+1} \left( \frac{1}{t} \int_0^t g^*(s) ds + t^{\frac{1-k}{n}} \int_t^\infty s^{\frac{k-1}{n}} g^*(s) \frac{ds}{s} \right) \\ &= \frac{n}{n-k+1} \left( g^{**}(t) + Q_{\frac{k-1}{n}}(g^*)(t) \right). \end{aligned}$$

*Therefore*

$$\|f\|_{Y_{k/n}} \preceq \left\| |d^k f|^{**} \right\|_Y + \left\| Q_{\frac{k-1}{n}}(|d^k f|^*) \right\|_Y .$$

*The conditions on the indices of the spaces imply that*

$$\left\| |d^k f|^{**} \right\|_Y \leq c \left\| |d^k f| \right\|_Y \quad \text{and} \quad \left\| Q_{\frac{k-1}{n}}(|d^k f|^*) \right\|_Y \leq c \left\| |d^k f| \right\|_Y .$$

*concluding the proof.*

*To compare this result with* [14]*, recall that in* [14, Theorem 1.2] *the conditions imposed on $Y(\Omega)$ are: $\overline{\alpha}_Y < 1$ and "$Y$ satisfies the $Q(k-1)$ condition", i.e.*

$$\int_1^\infty s^{\frac{k-1}{n}} h_Y\left(\frac{1}{s}\right) \frac{ds}{s} < \infty.$$

*If $Y'$ denotes the associate space of $Y$ then (cf.* [2, Chapter 3, Proposition 5.11]*) then $h_Y\left(\frac{1}{s}\right)=\frac{1}{s}h_{Y'}(s)$ and we have: $Y(\Omega)$ satisfies the $Q(k-1)$ condition iff*

$$\int_1^\infty s^{\frac{k-1}{n}-1}h_{Y'}(s)\frac{ds}{s}<\infty.$$

*In turn the last condition is equivalent to (cf.* [2, Chapter 3, Lemma 5.9]*)*

$$\overline{\alpha}_{Y'}<1-\frac{k-1}{n}.$$

*Now, as is well known $\overline{\alpha}_{Y'}=1-\underline{\alpha}_Y$, and therefore we see that $Y(\Omega)$ satisfies the $Q(k-1)$ condition if and only if $\frac{k-1}{n}<\underline{\alpha}_Y$.*

## 6. Sobolev embeddings on $\mathbb{R}^n$: The fractional case

In this section we deal with the higher order version of (1.2). (A detailed study of the fractional case on general domains will be considered in [11].)

**Lemma 3.** *If $f\in C_0^\infty(\mathbb{R}^n)$ then, for every $k\geq 2$*

$$f^{**}(t)-f^*(t)\leq ct^{1/n}\int_t^\infty \frac{s^{\frac{k-1}{n}}}{\phi_X(s)}\left(\int_0^s \frac{\omega_X(f,z^{\frac{1}{n}})_{k+1}}{z^{\frac{k}{n}}}\frac{dz}{z}\right)\frac{ds}{s}. \tag{6.1}$$

*Proof.* By Corollary 2,

$$f^{**}(t)-f^*(t)\preceq t^{1/n}\int_t^\infty s^{\frac{k-1}{n}}\left|d^kf\right|^{**}(s)\frac{ds}{s}. \tag{6.2}$$

On the other hand by (1.2)

$$\begin{aligned}\left|d^kf\right|_1^{**}(s)&=\int_s^\infty\left(\left|d^kf\right|^{**}(x)-\left|d^kf\right|^{*}(x)\right)\frac{dx}{x}\\ &\leq\int_s^\infty\left(\frac{\omega_X(\left|d^kf\right|,x^{1/n})}{\phi_X(x)}\right)\frac{dx}{x}.\end{aligned}$$

Inserting the last estimate in (6.2) and using Fubini we find

$$\begin{aligned}f^{**}(t)-f^*(t)&\preceq t^{1/n}\int_t^\infty\left(s^{\frac{k-1}{n}}\int_s^\infty\left(\frac{\omega_X(\left|d^kf\right|,x^{1/n})}{\phi_X(x)}\right)\frac{dx}{x}\right)\frac{ds}{s}\\ &\preceq t^{1/n}\sum_{|\alpha|=k}\int_t^\infty\frac{s^{\frac{k-1}{n}}\omega_X(D^\alpha f,s^{1/n})}{\phi_X(s)}\frac{ds}{s}.\end{aligned}$$

Finally, using the well known estimate (see for example [7])

$$\omega_X(D^\alpha f,s)\preceq\int_0^s\frac{\omega_X(f,z)_{k+1}}{z^k}\frac{dz}{z}\qquad(|\alpha|=k)$$

(6.1) follows readily. □

Let $X=X(\mathbb{R}^n)$ be a r.i. space, and let $Y$ be a r.i space over $(0,\infty)$ and let $s>0$. Set $r=[s]+1$ ($[s]$=integral part of $s$).

The Besov space $\mathring{\mathrm{B}}^s_{X,Y}(\mathbb{R}^n)$ is defined (see [12]) as the closure of $C_0^\infty(\mathbb{R}^n)$ under the seminorm

$$\|f\|_{\mathring{\mathrm{B}}^s_{X,Y}(\Omega)}=\left\|\frac{t^{-\frac{s}{n}}\omega_X\left(f,t^{1/n}\right)_r}{\phi_Y(t)}\right\|_Y.,$$

**Example 4.** *Let $X = L^p(\mathbb{R}^n)$ and $Y = L^q([0,\infty))$, then $\phi_Y(t) = t^{1/q}$ and*

$$\|f\|_{\mathring{B}^s_{X,Y}(\Omega)} \simeq \left(\int_0^\infty \left(t^{-s}\omega_p(f,t)_r\right)^q \frac{dt}{t}\right)^{1/q}.$$

*Thus $\mathring{B}^s_{L^p,L^q}(\mathbb{R}^n)$ coincides with the usual space $\mathring{B}^s_{p,q}(\mathbb{R}^n)$.*

**Theorem 7.** *(See [12] if $0 < s \le 1$) Let $X = X(\mathbb{R}^n)$, $Y(0,\infty)$ be r.i. spaces, and let $s > 1$. Moreover suppose that*

$$\frac{s-1}{n} < \underline{\alpha}_Y - \overline{\beta}_Y + \underline{\beta}_X \quad \text{and} \quad \overline{\alpha}_Y < \underline{\beta}_Y + \frac{s-[s]}{n}.$$

*Then*

$$\mathring{B}^s_{X,Y}(\mathbb{R}^n) \subset Y_{\frac{s}{n}}(X)(\mathbb{R}^n),$$

*where $Y_{\frac{s}{n}}(X)$ is the rearrangement invariant set introduced in [12], defined by*[6]

$$Y_s(X) = \left\{ f : \|f\|_{Y(\infty,s,X)} = \left\| t^{-s}\frac{\phi_X(t)}{\phi_Y(t)} f_0^*(t)\right\|_Y < \infty \right\}.$$

*Proof.* By Lemma 3 (with $k = [s]$) we have that

$$f^{**}(t) - f^*(t) \preceq t^{1/n}\int_t^\infty \frac{s^{\frac{k-1}{n}}}{\phi_X(s)}\left(\int_0^s \frac{\omega_X(f,z^{\frac{1}{n}})_{k+1}}{z^{\frac{k}{n}}}\frac{dz}{z}\right)\frac{ds}{s}.$$

Thus

$$\|f\|_{Y_{\frac{s}{n}}(X)} \preceq \left\| t^{\frac{1-s}{n}}\frac{\phi_X(t)}{\phi_Y(t)}\int_t^\infty \frac{s^{\frac{k-1}{n}}}{\phi_X(s)}\left(\int_0^s \frac{\omega_X(f,z^{\frac{1}{n}})_{k+1}}{z^{\frac{k}{n}}}\frac{dz}{z}\right)\frac{ds}{s}\right\|_Y = I.$$

The conditions on the indices (see [12, Lemma 1 and Remark 1]) ensure that

$$\left\| t^{\frac{1-s}{n}}\frac{\phi_X(t)}{\phi_Y(t)} Qh(t)\right\|_Y \preceq \left\| t^{\frac{1-s}{n}}\frac{\phi_X(t)}{\phi_Y(t)} h(t)\right\|_Y.$$

Thus

$$I \preceq \left\| \frac{t^{-\frac{s-k}{n}}}{\phi_Y(t)}\left(\int_0^s \frac{\omega_X(f,z^{\frac{1}{n}})_{k+1}}{z^{\frac{k}{n}}}\frac{dz}{z}\right)\right\|_Y.$$

We claim that

$$\left\| \frac{t^{-\frac{s-k}{n}+1}}{\phi_Y(t)} Ph(t)\right\|_Y \preceq \left\| \frac{t^{-\frac{s-k}{n}+1}}{\phi_Y(t)} h(t)\right\|_Y. \tag{6.3}$$

This given we have

$$\begin{aligned} I &\preceq \left\| \frac{t^{-\frac{s-k}{n}+1}}{\phi_Y(t)}\left(\frac{1}{s}\int_0^s \frac{\omega_X(f,z^{\frac{1}{n}})_{k+1}}{z^{\frac{k}{n}}}\frac{dz}{z}\right)\right\|_Y \\ &\preceq \left\| \frac{t^{-\frac{s}{n}}\omega_X(f,s^{\frac{1}{n}})_{k+1}}{\phi_Y(t)}\right\|_Y \\ &= \|f\|_{\mathring{B}^s_{X,Y}(\Omega)}. \end{aligned}$$

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[6]If $\frac{s}{n} < \underline{\alpha}_Y - \overline{\beta}_Y + \underline{\beta}_X$, then

$$\left\| t^{-s/n}\frac{\phi_X(t)}{\phi_Y(t)} f_0^*(t)\right\|_Y \simeq \left\| t^{-s/n}\frac{\phi_X(t)}{\phi_Y(t)} f^{**}(t)\right\|_Y$$

thus, in this case $Y_{\frac{s}{n}}(X)$ is a r.i. space.

It remains to prove (6.3). Let $\alpha = \frac{s-k}{n}$. Since $P$ is a positive operator we may assume without loss that $h \geq 0$. Now

$$\frac{t^{1-\alpha}}{\phi_Y(t)} Ph(t) = \int_0^1 \frac{h(st)(st)}{\phi_Y(st)}^{1-\alpha} s^{\alpha-1} \frac{\phi_Y(st)}{\phi_Y(t)} ds \leq \int_0^1 \frac{h(st)(st)}{\phi_Y(st)}^{1-\alpha} s^{\alpha-1} M_Y(s) ds.$$

Thus

$$\left\| \frac{t^{1-\alpha}}{\phi_Y(t)} Ph(t) \right\|_Y \leq \int_0^1 s^{\alpha-1} d_Y(\frac{1}{s}) M_Y(s) ds \left\| \frac{t^{1-\alpha}}{\phi_Y(t)} h(t) \right\|_Y ,$$

and by the definitions of indices we have

$$\int_0^1 s^{\alpha-1} d_Y(\frac{1}{s}) M_Y(s) ds < \infty \Leftrightarrow \ \overline{\alpha}_Y < \underline{\beta}_Y + \alpha.$$

□

**Corollary 3.** *Let $Y = L^q$, and let $X$ be a r.i. space such that $\frac{s-1}{n} < \underline{\beta}_X$. Then for all $f \in C_0^\infty(\mathbb{R}^n)$,*

$$\int_0^\infty \left( t^{-\frac{s}{n}} \phi_X(t)(f^{**}(t) - f^*(t)) \right)^q \frac{dt}{t} \leq c \int_0^\infty \left( t^{-s} \omega_X(f, t)_{[s]+1} \right)^q \frac{dt}{t}.$$

**Acknowledgement 1.** *We are grateful to the referee for useful remarks and suggestions to improve the paper.*

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