

Mittag-Leffler Conditions on Modules

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ABSTRACT. We study Mittag-Leffler conditions on modules providing relative versions of classical results by Raynaud and Gruson. We then apply our investigations to several contexts. First of all, we give a new argument for solving the Baer splitting problem. Moreover, we show that modules arising in cotorsion pairs satisfy certain Mittag-Leffler conditions. In particular, this implies that tilting modules satisfy a useful finiteness condition over their endomorphism ring. In the final section, we focus on a special tilting cotorsion pair related to the pure-semisimplicity conjecture.

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INTRODUCTION

In the last few years, Mittag-Leffler conditions on modules were successfully employed in a number of different problems ranging from tilting theory to commutative algebra, and to a conjecture originating in algebraic topology. Indeed, the translation of certain homological properties of modules into Mittag-Leffler conditions was a key step in solving the Baer splitting problem raised by Kaplansky in 1962 [3], as well as in proving that every tilting class is determined by a class of finitely presented modules [12, 14], and it is part of the strategy for tackling the telescope conjecture for module categories [6].

Motivated by these results, in this paper we undertake a systematic study of such conditions, and we give further applications of these tools. In fact, we give a new proof for the result in [3]. Moreover, using the theory of matrix subgroups, we provide a new interpretation of certain finiteness conditions of a module over its endomorphism ring, in particular of endofiniteness. Furthermore, we show that Mittag-Leffler conditions appear naturally in the theory of cotorsion pairs, that is, pairs of classes of modules that are orthogonal to each other with respect to the Ext functor. As a consequence, we discover a new finiteness condition satisfied by tilting modules. Finally, we employ our investigations to discuss the pure-semisimplicity conjecture, developing an idea from [2].

Further applications of our work to finite-dimensional hereditary algebras, and to cotorsion pairs given by modules of bounded projective dimension will appear in [5] and [13], respectively.

We now give some details on the conditions we are going to investigate. Raynaud and Gruson studied in [36] the right modules M over a ring R having the property that the canonical map

$$\rho: M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} (M \otimes_R Q_i)$$

is injective for any family of left R -modules $\{Q_i\}_{i \in I}$. They showed that this is the case if and only if M is the direct limit of a direct system $(F_\alpha, f_{\beta\alpha})_{\beta, \alpha \in \Lambda}$ of finitely presented modules such that the inverse system

$$(\mathrm{Hom}_R(F_\alpha, B), \mathrm{Hom}_R(f_{\beta\alpha}, B))_{\beta, \alpha \in \Lambda}$$

satisfies the Mittag-Leffler condition for any right R -module B . Therefore such modules M are said to be Mittag-Leffler modules.

In this paper, we study relative versions of these properties by restricting the choice of the family $\{Q_i\}_{i \in I}$ and of B . We thus consider the notions of a *Q-Mittag-Leffler module* and of a *B-stationary module*. Part of our work consists in developing these notions following closely [36].

While the definition of a *Q-Mittag-Leffler module* relies on the injectivity of the natural transformation ρ , the *B-stationary modules* are not “canonically” defined. We introduce the stronger notion of *strict B-stationary modules*. Again,

this notion is inspired by [36]. Indeed, if \mathcal{B} is the class of all right modules, then the strict \mathcal{B} -stationary modules are precisely the strict Mittag-Leffler modules introduced by Raynaud and Gruson, and later studied by Azumaya [10] and other authors under the name of locally projective modules. We characterize strict \mathcal{B} -stationary modules in terms of the injectivity of the natural transformation

$$\nu = \nu(M, B, V): M \otimes_R \operatorname{Hom}_S(B, V) \rightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), V).$$

This relates our investigations to results on matrix subgroups obtained by Zimmermann in [40].

As mentioned above, our original motivation are the results in [12], where it was made apparent that, for a countably presented module M , the vanishing of $\operatorname{Ext}_R^1(M, B)$ for all modules B belonging to a class \mathcal{B} closed under direct sums, can be characterized in terms of \mathcal{B} -stationarity, see Theorem 3.11 for a precise statement. Furthermore, also the vanishing of \varprojlim^1 , the first derived functor of the inverse limit \varprojlim , can be interpreted in this way, see [19] and [3]. We believe that a thorough understanding of \mathcal{B} -stationary modules and of their relationship with strict \mathcal{B} -stationary and \mathcal{Q} -Mittag-Leffler modules will provide a new insight in problems related to the vanishing of certain homological functors. The applications we present in this paper are oriented towards such developments.

Let us illustrate such applications by focussing on cotorsion pairs. Let S be a set of finitely presented modules, and let $(\mathcal{M}, \mathcal{L})$ be the cotorsion pair generated by S . In other words, \mathcal{L} is the class of modules defined by the vanishing of $\operatorname{Ext}_R^1(S, -)$, while \mathcal{M} is defined by the vanishing of $\operatorname{Ext}_R^1(-, \mathcal{L})$, see Definition 9.1. Denote further by \mathcal{C} the class defined by the vanishing of $\operatorname{Tor}_1^R(S, -)$. We prove in Theorem 9.5 that a module is \mathcal{L} -stationary if and only if it is \mathcal{C} -Mittag-Leffler. Moreover, it turns out that every module in \mathcal{M} is strict \mathcal{L} -stationary.

In particular, this applies to cotorsion pairs arising in tilting theory (Corollary 9.8), yielding that every tilting module T is strict T -stationary. The latter property can be interpreted in terms of matrix subgroups and allows us to show that certain tilting modules are noetherian over their endomorphism ring, see Proposition 10.1 and [5].

The paper is organized as follows. In Section 1 we introduce \mathcal{Q} -Mittag-Leffler modules, and we study the closure properties of the class \mathcal{Q} and of the class of \mathcal{Q} -Mittag-Leffler modules. For our applications to cotorsion pairs, it is relevant to note the good behavior of Mittag-Leffler modules with respect to filtrations established in Proposition 1.9. We revisit the topic of \mathcal{Q} -Mittag-Leffler modules in Section 5, where we characterize them in the spirit of [36]: since the map ρ is bijective when M is finitely presented, and since every module is a direct limit of finitely presented modules, one has to determine the “gluing” conditions on the canonical maps $u_\alpha, u_{\beta\alpha}$ in the direct limit presentation $(M, (u_\alpha)_{\alpha \in I}) = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ that imply the injectivity of ρ . These conditions are called *dominating with respect to \mathcal{Q}* . We introduce them in Section 4 where we also study their basic properties.

\mathcal{B} -stationary modules are introduced in Section 3. Hereby we adopt the language of H -subgroups from [43], which is the topic of Section 2. Our first aim is to give an intrinsic characterization of \mathcal{B} -stationarity. This characterization, obtained in Theorem 4.8, is also given in terms of *dominating maps*. It will allow us to study the interplay between the conditions \mathcal{B} -stationary and \mathcal{Q} -Mittag-Leffler in Section 6.

The interrelationship between the different conditions is further pursued in Section 9, after having introduced and characterized the strict \mathcal{B} -stationary modules in Section 8. Note that the condition strict \mathcal{B} -stationary has again a good behavior under filtrations, cf. Proposition 8.13. This intertwines our investigations with the theory of cotorsion pairs. Our main results in this context are Theorem 9.5 and its application to tilting cotorsion pairs in Corollary 9.8, which we have already described above. Moreover, we prove that a set of finitely presented modules S generates a cotorsion pair $(\mathcal{M}, \mathcal{L})$ with \mathcal{L} being definable if and only if also all first syzygies of modules in S are finitely presented.

A further important application is given in Section 7 which is devoted to Baer modules over domains. A module M over a commutative domain R is said to be a Baer module if

$$\mathrm{Ext}_R^1(M, T) = 0 \quad \text{for any torsion module } T.$$

Kaplansky in [32] raised the question whether Baer modules are projective. The last step in the positive solution of Kaplansky's problem was made in [3]. In the present work, we prove that a countably generated Baer module over an arbitrary commutative domain is always a Mittag-Leffler module. This yields another proof of the fact that Baer modules over commutative domains are projective.

Let us mention that the techniques introduced by Raynaud and Gruson have also been used by Drinfeld in [16]. We give in Corollary 5.5 a detailed proof of [16, Theorem 2.2].

Finally, as a last application, we consider left pure-semisimple hereditary rings in Section 10. In particular, we use Corollary 9.8 to study the tilting cotorsion pair generated by the preprojective right modules, following an idea from [2].

Notation. Let R be a ring. Denote by $\mathrm{Mod}\text{-}R$ the category of all right R -modules, and by $\mathrm{mod}\text{-}R$ the subcategory of all modules possessing a projective resolution consisting of finitely generated modules. $R\text{-Mod}$ and $R\text{-mod}$ are defined correspondingly.

For a right R -module M , we denote by

$$M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

its character module. Instead of the character module we can also consider another dual module, for example, for modules over an artin algebra Λ we can take $M^* = D(M)$ where D denotes the usual duality. If S is a class of modules, we denote by S^* the corresponding class of all duals B^* of modules $B \in S$.

For a class $\mathcal{M} \subset \text{Mod-}R$ and a class $\mathcal{N} \subset R\text{-Mod}$, we set

$$\begin{aligned}\mathcal{M}^\circ &= \{X \in \text{Mod-}R \mid \text{Hom}_R(M, X) = 0 \text{ for all } M \in \mathcal{M}\}, \\ {}^\circ\mathcal{M} &= \{X \in \text{Mod-}R \mid \text{Hom}_R(X, M) = 0 \text{ for all } M \in \mathcal{M}\}, \\ \mathcal{M}^\perp &= \{X \in \text{Mod-}R \mid \text{Ext}_R^1(M, X) = 0 \text{ for all } M \in \mathcal{M}\}, \\ {}^\perp\mathcal{M} &= \{X \in \text{Mod-}R \mid \text{Ext}_R^1(X, M) = 0 \text{ for all } M \in \mathcal{M}\}, \\ \mathcal{M}^\tau &= \{X \in R\text{-Mod} \mid \text{Tor}_1^R(M, X) = 0 \text{ for all } M \in \mathcal{M}\}, \\ {}^\tau\mathcal{N} &= \{X \in \text{Mod-}R \mid \text{Tor}_1^R(X, N) = 0 \text{ for all } N \in \mathcal{N}\}.\end{aligned}$$

Moreover, we denote by $\text{Add } \mathcal{M}$ (respectively, $\text{add } \mathcal{M}$) the class consisting of all modules isomorphic to direct summands of (finite) direct sums of modules of \mathcal{M} . The class consisting of all modules isomorphic to direct summands of products of modules of \mathcal{M} is denoted by $\text{Prod } \mathcal{M}$. Further, $\text{Gen } \mathcal{M}$ and $\text{Cogen } \mathcal{M}$ denote the class of modules generated, respectively cogenerated, by modules of \mathcal{M} . If \mathcal{M} consists of a single module M , we just write M^\perp , $\text{Add } M$, $\text{Prod } M$, etc. Finally, we write $\varinjlim \mathcal{M}$ for the class of all modules D such that $D = \varinjlim_{i \in I} M_i$ where $\{M_i \mid i \in I\}$ is a direct system of modules from \mathcal{M} .

We will say that a module M_R with endomorphism ring S is *endonoetherian* if M is noetherian when viewed as a left S -module. If ${}_S M$ has finite length, then we say that M is *endofinite*.

1. \mathcal{Q} -MITTAG-LEFFLER MODULES

Definition 1.1 ([38]). Let M be a right module over a ring R , and let \mathcal{Q} be a class of left R -modules. We say that M is a *\mathcal{Q} -Mittag-Leffler module* if the canonical map

$$\rho: M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} (M \otimes_R Q_i)$$

is injective for any family $\{Q_i\}_{i \in I}$ of modules in \mathcal{Q} . If \mathcal{Q} just consists of a single module Q , then we say that M is *Q -Mittag-Leffler*.

We will need the following lemma.

Lemma 1.2. Let M_R and ${}_R Q$ be a right and a left R -module, respectively. Assume that $Q = \varinjlim (K_\alpha, f_{\beta\alpha})_{\beta, \alpha \in I}$. For any $\alpha \in I$, let $f_\alpha: K_\alpha \rightarrow Q$ be the induced map.

Let $q_1, \dots, q_n \in Q$ and $x_1, \dots, x_n \in M$ be such that $\sum_{i=1}^n x_i \otimes q_i$ is the zero element of $M \otimes_R Q$. Then there exist $\alpha_0 \in I$ and $k_1, \dots, k_n \in K_{\alpha_0}$ such that $\sum_{i=1}^n x_i \otimes k_i$ is the zero element of $M \otimes_R K_{\alpha_0}$ and $f_{\alpha_0}(k_i) = q_i$ for every $i = 1, \dots, n$.

Proof. Choose β such that $\{q_1, \dots, q_n\} \subseteq f_\beta(K_\beta)$. For every $i \in \{1, \dots, n\}$, let $k'_i \in K_\beta$ be such that $f_\beta(k'_i) = q_i$. Since $\sum_{i=1}^n x_i \otimes q_i = 0$ in $M \otimes_R Q \cong$

$\varinjlim (M \otimes_R K_\alpha)$, there exists $\alpha_0 \geq \beta$ such that

$$0 = (M \otimes f_{\alpha_0\beta}) \left(\sum_{i=1}^n x_i \otimes k'_i \right) = \sum_{i=1}^n x_i \otimes f_{\alpha_0\beta}(k'_i).$$

Now α_0 and $k_i = f_{\alpha_0\beta}(k'_i)$, $i = 1, \dots, n$, satisfy the desired properties. \square

Here are some closure properties of the class \mathcal{Q} . Related results can be found in work of Rothmaler [38, Theorem 2.2, Remark 2.3] and Zimmermann [40, 2.2].

Theorem 1.3. *Let R be a ring, and $\mathcal{Q} \subseteq R\text{-Mod}$. Assume that $M \in \text{Mod-}R$ is \mathcal{Q} -Mittag-Leffler. Then the following statements hold true.*

- (i) M is \mathcal{Q}' -Mittag-Leffler where \mathcal{Q}' is the class of all pure submodules of modules in \mathcal{Q} .
- (ii) M is $\text{Prod } \mathcal{Q}$ -Mittag-Leffler.
- (iii) M is $\varinjlim \mathcal{Q}$ -Mittag-Leffler.

Proof.

- (i) Let $\{Q_i\}_{i \in I}$ be a family of modules in \mathcal{Q} , and let $\{Q'_i\}_{i \in I}$ be a family of left R -modules such that for any $i \in I$ the module Q'_i is a pure submodule of Q_i . For any $i \in I$, denote by $\varepsilon_i: Q'_i \rightarrow Q_i$ the inclusion. As every ε_i is a pure monomorphism, so is $\prod_{i \in I} \varepsilon_i$. Then we have the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M \otimes \prod_{i \in I} Q'_i & \xrightarrow{M \otimes \prod_{i \in I} \varepsilon_i} & M \otimes \prod_{i \in I} Q_i \\ & & \rho' \downarrow & & \rho \downarrow \\ 0 & \longrightarrow & \prod_{i \in I} (M \otimes_R Q'_i) & \xrightarrow{\prod_{i \in I} (M \otimes \varepsilon_i)} & \prod_{i \in I} (M \otimes_R Q_i). \end{array}$$

As $\rho(M \otimes \prod_{i \in I} \varepsilon_i)$ is injective, so is ρ' .

- (ii) is proved in [38, p. 39].
- (iii) We follow the argument in [21, Lemma 3.1].

Let $\{Q_i\}_{i \in I}$ be a family of modules such that, for any $i \in I$,

$$Q_i = \varinjlim (K_\alpha^i, f_{\beta\alpha}^i)_{\beta, \alpha \in I_i}$$

and $K_\alpha^i \in \mathcal{Q}$ for any $\alpha \in I_i$. For any $i \in I$ and $\alpha \in I_i$, let $f_\alpha^i: K_\alpha^i \rightarrow Q_i$ denote the canonical morphism.

We want to show that $\rho: M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} (M \otimes_R Q_i)$ is injective. Let

$$y = \sum_{j=1}^n x_j \otimes (q_j^i)_{i \in I}$$

be an element in the kernel of ρ . This means that, for any $i \in I$, $\sum_{j=1}^n x_j \otimes q_j^i$ is the zero element of $M \otimes_R Q_i$. By Lemma 1.2, for each $i \in I$, there exists $\alpha_i \in I_i$ and $k_1^i, \dots, k_n^i \in K_{\alpha_i}^i$ such that $\sum_{j=1}^n x_j \otimes k_j^i$ is the zero element of $M \otimes_R K_{\alpha_i}^i$ and $f_{\alpha_i}^i(k_j^i) = q_j^i$ for every $j = 1, \dots, n$.

Consider the commutative diagram

$$\begin{array}{ccc} M \otimes \prod_{i \in I} K_{\alpha_i}^i & \xrightarrow{M \otimes \prod_{i \in I} f_{\alpha_i}^i} & M \otimes \prod_{i \in I} Q_i \\ \rho' \downarrow & & \rho \downarrow \\ \prod_{i \in I} (M \otimes_R K_{\alpha_i}^i) & \xrightarrow{\prod_{i \in I} (M \otimes f_{\alpha_i}^i)} & \prod_{i \in I} (M \otimes_R Q_i) \end{array}.$$

By construction,

$$\gamma = \sum_{j=1}^n x_j \otimes (q_j^i)_{i \in I} = \left(M \otimes \prod_{i \in I} f_{\alpha_i}^i \right) \left(\sum_{j=1}^n x_j \otimes (k_j^i)_{i \in I} \right)$$

and

$$\rho' \left(\sum_{j=1}^n x_j \otimes (k_j^i)_{i \in I} \right) = \left(\sum_{j=1}^n x_j \otimes k_j^i \right)_{i \in I} = 0.$$

Note that ρ' is injective because $K_{\alpha_i} \in \mathcal{Q}$ for any $i \in I$. This shows that $\gamma = 0$. \square

Proposition 1.4. *Let R be a ring. The following statements hold true for $M \in \text{Mod-}R$.*

- (i) *Let $\mathcal{Q}_1, \dots, \mathcal{Q}_n \subseteq R\text{-Mod}$, and let $\mathcal{Q} = \bigcup_{i=1}^n \mathcal{Q}_i$. If M is \mathcal{Q}_i -Mittag-Leffler for all $1 \leq i \leq n$, then M is \mathcal{Q} -Mittag-Leffler.*
- (ii) *Let \mathcal{Q}_1 and \mathcal{Q}_2 be two classes in $R\text{-Mod}$, and let \mathcal{Q} be the class consisting of all extensions of modules in \mathcal{Q}_1 by modules in \mathcal{Q}_2 . Suppose that M is \mathcal{Q}_i -Mittag-Leffler for $i = 1, 2$, and that the functor $M \otimes -$ is exact on any short exact sequence with first term in \mathcal{Q}_1 and end-term in \mathcal{Q}_2 . Then M is \mathcal{Q} -Mittag-Leffler.*

Proof.

- (i) Let $\{Q_i\}_{i \in I}$ be a family of modules in \mathcal{Q} . For any $j = 1, \dots, n$, set

$$I_j = \{i \in I \mid Q_i \in \mathcal{Q}_j \text{ and } Q_i \notin \mathcal{Q}_k \text{ for } k < j\}.$$

Then $\prod_{i \in I} Q_i = \bigoplus_{j=1}^n (\prod_{i \in I_j} Q_i)$. As $\rho_j: M \otimes \prod_{i \in I_j} Q_i \rightarrow \prod_{i \in I_j} (M \otimes Q_i)$ is injective for any $j = 1, \dots, n$, it follows that ρ is also injective.

- (ii) Let $\{Q_i\}_{i \in I}$ be a family of left modules such that, for any $i \in I$, there is an exact sequence

$$0 \rightarrow Q_i^1 \rightarrow Q_i \rightarrow Q_i^2 \rightarrow 0$$

with $Q_i^1 \in \mathcal{Q}_1$ and $Q_i^2 \in \mathcal{Q}_2$. Then we have the commutative diagram

$$\begin{array}{ccccccc} M \otimes \prod_{i \in I} Q_i^1 & \rightarrow & M \otimes \prod_{i \in I} Q_i & \rightarrow & M \otimes \prod_{i \in I} Q_i^2 & \rightarrow & 0 \\ \rho_1 \downarrow & & \rho \downarrow & & \rho_2 \downarrow & & \\ 0 \rightarrow \prod_{i \in I} (M \otimes_R Q_i^1) & \rightarrow & \prod_{i \in I} (M \otimes_R Q_i) & \rightarrow & \prod_{i \in I} (M \otimes_R Q_i^2) & \rightarrow & 0 \end{array}$$

where the bottom row is exact by assumption on $M \otimes -$. As ρ_1 and ρ_2 are injective, ρ is also injective. \square

Corollary 1.5. *Let R be a ring, and $M \in \text{Mod-}R$. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$ such that M is \mathcal{T} -Mittag-Leffler and \mathcal{F} -Mittag-Leffler. Assume further that the functor $M \otimes -$ is exact on any short exact sequence with first term in \mathcal{T} and end-term in \mathcal{F} . Then M is a Mittag-Leffler module.*

- Examples 1.6.** (1) Let R be a commutative domain and denote by \mathcal{T} and \mathcal{F} the classes of torsion and torsionfree modules, respectively. Any flat R -module M which is \mathcal{T} -Mittag-Leffler and \mathcal{F} -Mittag-Leffler is a Mittag-Leffler module.
- (2) Let Λ be a tame hereditary finite dimensional algebra over an algebraically closed field k , and let \mathbf{t} be the class of all finitely generated indecomposable regular modules. It was shown by Ringel in [37, 4.1] that the classes $(\mathcal{F}, \text{Gen } \mathbf{t})$ with $\mathcal{F} = \mathbf{t}^\circ = {}^\perp \mathbf{t}$ form a torsion pair, and for every module $X \in \text{Mod } \Lambda$ there is a pure-exact sequence

$$0 \rightarrow \mathbf{t}X \rightarrow X \rightarrow X/\mathbf{t}X \rightarrow 0$$

where $\mathbf{t}X = \sum_{f \in \text{Hom}(Y, X), Y \in \mathbf{t}} \text{Im } f \in \text{Gen } \mathbf{t}$ is the trace of \mathbf{t} in X , and $X/\mathbf{t}X \in \mathcal{F}$. Thus a module $M \in \text{Mod } \Lambda$ is Mittag-Leffler provided it is \mathcal{T} -Mittag-Leffler and \mathcal{F} -Mittag-Leffler.

- (3) [40, 2.5] If Q is a left R -module satisfying the maximum condition for finite matrix subgroups (see Definition 8.6), for example an endonoetherian module, then every right R -module is Q -Mittag-Leffler.
- (4) [38, 2.4], [40, 2.1, 2.4] Let $\mathcal{Q} \subseteq R\text{-Mod}$. The class of \mathcal{Q} -Mittag-Leffler modules is closed under pure submodules and pure extensions. A direct sum of modules is \mathcal{Q} -Mittag-Leffler if and only if so are all direct summands. If N is a finitely generated submodule of a \mathcal{Q} -Mittag-Leffler module M , then M/N is \mathcal{Q} -Mittag-Leffler.

Further examples are provided by the following results.

Proposition 1.7. *Let $R \rightarrow S$ be a ring epimorphism. Let M_S be a finitely presented right S -module, and let N be a finitely generated R -submodule of M . Then M_R and M/N are Mittag-Leffler with respect to the class $S\text{-Mod}$.*

Proof. Let $\{Q_i\}_{i \in I}$ be a family of left S -modules. Since $R \rightarrow S$ is a ring epimorphism

$$M \otimes_R \prod_{i \in I} Q_i \cong M \otimes_R \left(S \otimes_S \prod_{i \in I} Q_i \right) = (M \otimes_R S) \otimes_S \prod_{i \in I} Q_i = M \otimes_S \prod_{i \in I} Q_i.$$

As M_S is finitely presented, this is isomorphic to

$$\prod_{i \in I} M \otimes_S Q_i \cong \prod_{i \in I} M \otimes_R S \otimes_S Q_i \cong \prod_{i \in I} M \otimes_R Q_i.$$

This yields that the canonical map $\rho: M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M \otimes_R Q_i$ is in fact an isomorphism.

It follows from Example 1.6(4) that M/N is also a Mittag-Leffler module with respect to the class $S\text{-Mod}$. \square

Definition 1.8. Let M be a right R -module, and let τ denote an ordinal. An increasing chain $(M_\alpha \mid \alpha \leq \tau)$ of submodules of M is a *filtration* of M provided that $M_0 = 0$, $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for all limit ordinals $\alpha \leq \tau$, and $M_\tau = M$.

Given a class C , a filtration $(M_\alpha \mid \alpha \leq \tau)$ is a *C -filtration* provided that $M_{\alpha+1}/M_\alpha \in C$ for any $\alpha < \tau$. We say also that M is a *C -filtered module*.

We have the following result about the behavior of the Mittag-Leffler property with respect to filtrations.

Proposition 1.9. *Let S be a class of right R -modules that are Mittag-Leffler with respect to a class $\mathcal{Q} \subseteq S^\tau$. Then any module isomorphic to a direct summand of an $S \cup \text{Add } R$ -filtered module is \mathcal{Q} -Mittag-Leffler.*

Proof. As projective modules are Mittag-Leffler and $(S \cup \text{Add } R)^\tau = S^\tau$, we can assume that S contains $\text{Add } R$. Moreover, since the class of \mathcal{Q} -Mittag-Leffler modules is closed by direct summands, we only need to prove the statement for S -filtered modules.

Let M be an S -filtered right R -module. Let τ be an ordinal such that there exists an S -filtration $(M_\alpha)_{\alpha \leq \tau}$ of M . We shall show that M is \mathcal{Q} -Mittag-Leffler proving by induction that M_α is \mathcal{Q} -Mittag-Leffler for any $\alpha \leq \tau$. Observe that for any $\beta \leq \alpha \leq \tau$, M_α and M_α/M_β are S -filtered modules, so they belong to ${}^\tau \mathcal{Q}$.

As $M_0 = 0$, the claim is true for $\alpha = 0$. If $\alpha < \tau$ then, as $\mathcal{Q} \subseteq S^\tau$, we can apply an argument similar to the one used in Proposition 1.4 to the exact sequence

$$0 \rightarrow M_\alpha \rightarrow M_{\alpha+1} \rightarrow M_{\alpha+1}/M_\alpha \rightarrow 0$$

to conclude that if M_α is \mathcal{Q} -Mittag-Leffler then $M_{\alpha+1}$ is \mathcal{Q} -Mittag-Leffler.

Let $\alpha \leq \tau$ be a limit ordinal, and assume that M_β is \mathcal{Q} -Mittag-Leffler for any $\beta < \alpha$. We shall prove that $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ is \mathcal{Q} -Mittag-Leffler. Let $\{Q_i\}_{i \in I}$ be a family of modules in \mathcal{Q} , and let

$$x \in \text{Ker} \left(M_\alpha \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M_\alpha \otimes_R Q_i \right).$$

There exists $\beta < \alpha$ and $y \in M_\beta \otimes_R \prod_{i \in I} Q_i$ such that $x = (\varepsilon_\beta \otimes_R \prod_{i \in I} Q_i)(y)$, where $\varepsilon_\beta: M_\beta \rightarrow M_\alpha$ denotes the canonical inclusion. Considering the commutative diagram

$$\begin{array}{ccc} M_\beta \otimes \prod_{i \in I} Q_i & \xrightarrow{\varepsilon_\beta \otimes \prod_{i \in I} Q_i} & M_\alpha \otimes \prod_{i \in I} Q_i \\ \rho' \downarrow & & \downarrow \rho \\ \prod_{i \in I} (M_\beta \otimes Q_i) & \xrightarrow{\prod_{i \in I} (\varepsilon_\beta \otimes Q_i)} & \prod_{i \in I} (M_\alpha \otimes Q_i) \end{array}$$

we see that $0 = \prod_{i \in I} (\varepsilon_\beta \otimes Q_i) \rho'(y)$. As ρ' is injective because M_β is \mathcal{Q} -Mittag-Leffler and, for any $i \in I$, $\varepsilon_\beta \otimes Q_i$ is also injective because $\text{Tor}_1^R(M_\alpha/M_\beta, Q_i) = 0$, we deduce that $y = 0$. Therefore $x = 0$, and ρ is injective. \square

For any $n \geq 1$, the natural transformation

$$\rho: M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M \otimes_R Q_i$$

induces a natural transformation

$$\rho_n: \text{Tor}_n^R \left(M, \prod_{i \in I} Q_i \right) \rightarrow \prod_{i \in I} \text{Tor}_n^R (M, Q_i).$$

We note the following characterization of the injectivity of ρ_n .

Proposition 1.10. *Let R be a ring, and $\mathcal{Q} \subseteq R\text{-Mod}$. Let $M \in \text{Mod-}R$ and $n \geq 1$. Then*

$$\rho_n: \text{Tor}_n^R \left(M, \prod_{i \in I} Q_i \right) \rightarrow \prod_{i \in I} \text{Tor}_n^R (M, Q_i)$$

is injective for any family $\{Q_i\}_{i \in I}$ of modules in \mathcal{Q} if and only if the n -th syzygy $\Omega^n(M)$ of M in a projective presentation is \mathcal{Q} -Mittag-Leffler.

Proof. By dimension shifting we may assume that $n = 1$. Fix a projective presentation of M

$$0 \rightarrow \Omega^1(M) \rightarrow F \rightarrow M \rightarrow 0$$

with F a free module. The claim follows by considering the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Tor}_1^R\left(M, \prod_{i \in I} Q_i\right) & \rightarrow & \Omega^1(M) \otimes \prod_{i \in I} Q_i & \rightarrow & F \otimes \prod_{i \in I} Q_i \\ & & \rho_1 \downarrow & & \rho \downarrow & & \rho' \downarrow \\ 0 & \rightarrow & \prod_{i \in I} \text{Tor}_1^R(M, Q_i) & \rightarrow & \prod_{i \in I} (\Omega^1(M) \otimes_R Q_i) & \rightarrow & \prod_{i \in I} (F \otimes_R Q_i) \end{array}$$

in which ρ' is injective because the free module F is Mittag-Leffler. Then ρ_1 is injective if and only if so is ρ . \square

2. H -SUBGROUPS

We recall a notion from [43] which will be very useful in the sequel.

Definition 2.1. Let M , M' and B be right R -modules, and let $v \in \text{Hom}_R(M, M')$. The subgroup (and $\text{End}_R B$ -submodule) of $\text{Hom}_R(M, B)$ consisting of all compositions of v with maps in $\text{Hom}_R(M', B)$ is denoted by

$$H_v(B) = \text{Hom}_R(M', B)v$$

and is called an H -subgroup of $\text{Hom}_R(M, B)$.

Remark 2.2 ([43, 2.10]). Let M , M' and B be right R -modules, and let $v \in \text{Hom}_R(M, M')$.

- (1) H_v is a subfunctor of $\text{Hom}_R(M, -)$ commuting with direct products. If M is finitely generated, then H_v also commutes with direct sums.
- (2) An $\text{End } B$ -submodule U of $\text{Hom}_R(M, B)$ is an H -subgroup if and only if there are a set I and a homomorphism $u \in \text{Hom}_R(M, B^I)$ such that $U = H_u(B)$.

For the following discussion it is important to keep in mind the following easy observations.

Lemma 2.3. Let M , M' , N be right R -modules, $u \in \text{Hom}_R(M, N)$, $v \in \text{Hom}_R(M, M')$.

- (1) If there is $h \in \text{Hom}_R(M', N)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{v} & M' \\ u \downarrow & \swarrow h & \\ N & & \end{array}$$

commutes, then $H_u(B) \subseteq H_v(B)$ for any right R -module B .

- (2) If B is a right R -module such that $H_u(B) \subseteq H_v(B)$, then $H_{ut}(B) \subseteq H_{vt}(B)$ for all $t \in \text{Hom}_R(X, M)$, $X \in \text{Mod-}R$.

Recall that a homomorphism $\pi : B \rightarrow B''$ is a *locally split epimorphism* if for each finite subset $X \subseteq B''$ there is a map $\varphi = \varphi_X : B'' \rightarrow B$ such that $x = \pi\varphi(x)$ for all $x \in X$. Observe that every split epimorphism is locally split, and every locally split epimorphism is a pure epimorphism. *Locally split monomorphisms* are defined dually. Moreover, a submodule B' of a module B is said to be a *locally split* (or strongly pure [42]) *submodule* if the embedding $B' \subseteq B$ is locally split.

Lemma 2.4. *Let M and M' be right R -modules, and let $v \in \text{Hom}_R(M, M')$. Assume that M is finitely generated. Let further $\varepsilon : B' \rightarrow B$ be a pure monomorphism. If M' is finitely presented or ε is a locally split monomorphism, then*

$$\varepsilon H_v(B') = H_v(B) \cap \varepsilon \text{Hom}_R(M, B').$$

Proof. The first case is treated in [12, Lemma 4.1] or [3, Lemma 2.8]. For the second case, we assume that ε is a locally split monomorphism. We show the inclusion \supseteq . Consider $f \in \text{Hom}_R(M, B')$ such that $\varepsilon f = h v$ for some $h \in \text{Hom}_R(M', B)$. Choose a generating set x_1, \dots, x_n of M together with a map $\varphi : B \rightarrow B'$ such that $f(x_i) = \varphi \varepsilon f(x_i)$ for all $1 \leq i \leq n$. Then the composition $h' = \varphi h$ satisfies $f = h' v \in H_v(B')$. The inclusion \subseteq is clear. \square

Lemma 2.5. *Assume that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{v} & M' \\ u \downarrow & \nearrow h & \\ N & & \end{array}$$

of right R -modules and module homomorphisms commutes. Assume further that B is a right R -module such that $H_u(B) = H_v(B)$.

- (1) *If h factors through a homomorphism $m \in \text{Hom}_R(M', M'')$, then $H_u(B) = H_{mv}(B)$.*
- (2) *Assume that M is finitely generated and N is finitely presented. If $B' \subseteq B$ is a pure submodule, then $H_u(B') = H_v(B')$.*
- (3) *Assume that M is finitely generated and M' is finitely presented. If $B \xrightarrow{\pi} B''$ is a pure-epimorphism, then $H_u(B'') = H_v(B'')$.*
- (4) *If M is finitely generated and $B' \subseteq B$ is a locally split submodule, then $H_u(B') = H_v(B')$.*
- (5) *If M is finitely generated and $B \xrightarrow{\pi} B''$ is a locally split epimorphism, then $H_u(B'') = H_v(B'')$.*

Proof. (1) is left to the reader. For the remaining statements, note first that by Lemma 2.3 it suffices to show $H_v(B') \subseteq H_u(B')$ and $H_v(B'') \subseteq H_u(B'')$, respectively.

For (2) and (4), observe that Lemma 2.4 yields

$$\varepsilon H_u(B') = H_u(B) \cap \varepsilon \text{Hom}_R(M, B'),$$

where $\varepsilon : B' \rightarrow B$ denotes the canonical embedding. Then $\varepsilon H_v(B') \subseteq H_v(B) \cap \varepsilon \operatorname{Hom}_R(M, B') = \varepsilon H_u(B')$, and since ε is a monomorphism, we deduce that $H_v(B') \subseteq H_u(B')$.

In statement (3), we have that $\pi : B \rightarrow B''$ is a pure epimorphism and M' is finitely presented, so

$$\operatorname{Hom}_R(M', \pi) : \operatorname{Hom}_R(M', B) \rightarrow \operatorname{Hom}_R(M', B'')$$

is also an epimorphism. Thus, if $fv \in H_v(B'')$, then there exists $g \in \operatorname{Hom}_R(M', B)$ such that $\pi g = f$. By hypothesis $gv \in H_u(B)$, so $fv = \pi gv \in \pi H_u(B) \subseteq H_u(B'')$.

For statement (5), we consider $fv \in H_v(B'')$, and choose a generating set x_1, \dots, x_n of M together with a map $\varphi : B'' \rightarrow B$ such that $fv(x_i) = \pi \varphi fv(x_i)$ for all $1 \leq i \leq n$. Then the composition $h = \varphi f$ satisfies $fv = \pi hv$. Moreover, $h v \in H_u(B)$, so there is $h' \in \operatorname{Hom}_R(N, B)$ such that $h v = h' u$. Thus $fv = \pi h' u \in H_u(B'')$. \square

3. \mathcal{B} -STATIONARY MODULES

Definition 3.1. An inverse system of sets $(H_\alpha, h_{\alpha\gamma})_{\alpha, \gamma \in I}$ is said to satisfy the *Mittag-Leffler condition* if for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that $h_{\alpha\gamma}(H_\gamma) = h_{\alpha\beta}(H_\beta)$ for any $\gamma \geq \beta$.

Let us specify the Mittag-Leffler condition for the case $I = \mathbb{N}$.

Example 3.2. An inverse system of the form

$$\cdots H_{n+1} \xrightarrow{h_n} H_n \cdots H_2 \xrightarrow{h_1} H_1$$

satisfies the Mittag-Leffler condition if and only if for any $n \in \mathbb{N}$ the chain of subsets of H_n

$$h_n(H_{n+1}) \supseteq \cdots \supseteq h_n \cdots h_{n+k}(H_{n+k+1}) \supseteq \cdots$$

is stationary.

According to Raynaud and Gruson [36, p. 74] the following characterization of Mittag-Leffler inverse systems is due to Grothendieck as it is implicit in [28, 13.2.2]. We give a proof for completeness' sake.

Lemma 3.3. Consider an inverse system of the form

$$\mathcal{H} : \cdots H_{n+1} \xrightarrow{h_n} H_n \cdots H_2 \xrightarrow{h_1} H_1.$$

For any $m > n \geq 1$ set $h_{nm} = h_n \cdots h_{m-1}$, and, for any $n \geq 1$ let $g_n : \varprojlim H_i \rightarrow H_n$ denote the canonical map.

The inverse system \mathcal{H} satisfies the Mittag-Leffler condition if and only if for any $n \geq 1$ there exists $\ell(n) > n$ such that

$$g_n(\varprojlim H_i) = h_{n\ell(n)}(H_{\ell(n)}) = h_n \cdots h_{\ell(n)-1}(H_{\ell(n)}).$$

Proof. Observe that since, for any $m > n \geq 1$, $g_n = h_{nm}g_m$, always

$$g_n(\varprojlim H_i) \subseteq \bigcap_{m>n} h_{nm}(H_m).$$

Assume now that \mathcal{H} satisfies the Mittag-Leffler condition. We only need to show that for any $n \geq 1$ there exists $\ell(n) > n$ such that $h_{n\ell(n)}(H_{\ell(n)}) \subseteq g_n(\varprojlim H_i)$. To this aim fix $n \geq 1$.

Applying repeatedly that \mathcal{H} satisfies the Mittag-Leffler condition we find a sequence of elements in \mathbb{N}

$$(*) \quad n = n_0 < n_1 < \cdots < n_i < \cdots$$

such that $h_{n_i n_{i+1}}(H_{n_{i+1}}) = h_{n_i m}(H_m)$ for all $i \geq 0$ and $m \geq n_{i+1}$. Now we show that $\ell(n)$ can be taken to be n_1 .

Let $a \in h_{n_0 n_1}(H_{n_1})$. Then $a \in h_{n_0 n_2}(H_{n_2})$, and there is $a_1 \in h_{n_1 n_2}(H_{n_2}) \subseteq H_{n_1}$ such that $a = h_{n_0 n_1}(a_1)$. In this fashion, the properties of the sequence $(*)$ allow us to find a sequence $a_0 = a, a_1, \dots, a_i, \dots$ such that $a_i \in H_{n_i}$ and $h_{n_i n_{i+1}}(a_{i+1}) = a_i$ for any $i \geq 0$. Hence $b = (a_i) \in \varprojlim H_{n_i} = \varprojlim H_j$ and $g_n(b) = a_0 = a$ as desired.

The converse implication is clear because of the remarks at the beginning of the proof. \square

The characterization above does not extend to uncountable inverse limits; an example where this fails is implicit in Example 9.11. We will be interested in inverse systems arising by applying the functor $\text{Hom}_R(-, B)$ on a direct system.

Remark 3.4. Let $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of right R -modules, B a right R -module, and $\beta \geq \alpha \in I$. Then

$$(\text{Hom}_R(F_\alpha, B), \text{Hom}_R(u_{\beta\alpha}, B))_{\beta, \alpha \in I}$$

is an inverse system of left modules over the endomorphism ring of B , and

$$\text{Hom}_R(u_{\beta\alpha}, B)(\text{Hom}_R(F_\beta, B)) = \text{Hom}_R(F_\beta, B)u_{\beta\alpha} = H_{u_{\beta\alpha}}(B).$$

Applying Lemma 2.3(1) to the situation of Remark 3.4 we obtain the following.

Lemma 3.5. Let $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of right R -modules with direct limit M , and denote by $u_\alpha: F_\alpha \rightarrow M$ the canonical map. Let B be a right R -module.

- (i) If $\gamma \geq \beta \geq \alpha$, then $H_{u_{\gamma\alpha}}(B) \subseteq H_{u_{\beta\alpha}}(B)$.
- (ii) $H_{u_{\alpha}}(B) \subseteq H_{u_{\beta\alpha}}(B)$ for any $\beta \geq \alpha$.

This allows to interpret the Mittag-Leffler condition on inverse systems as in Remark 3.4 in terms of H -subgroups.

Lemma 3.6. Let $(F_{\alpha}, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of right R -modules. Let $\alpha, \beta \in I$ with $\beta \geq \alpha$, and let B be a right R -module. The following statements are equivalent.

- (1) For any $\gamma \geq \alpha$, the inclusion $H_{u_{\beta\alpha}}(B) \supseteq H_{u_{\gamma\alpha}}(B)$ implies $H_{u_{\beta\alpha}}(B) = H_{u_{\gamma\alpha}}(B)$.
- (2) $H_{u_{\beta\alpha}}(B) \subseteq \bigcap_{\gamma \geq \beta} H_{u_{\gamma\alpha}}(B)$.
- (2') $H_{u_{\beta\alpha}}(B) = \bigcap_{\gamma \geq \beta} H_{u_{\gamma\alpha}}(B)$.
- (3) $H_{u_{\beta\alpha}}(B) \subseteq \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B)$.
- (3') $H_{u_{\beta\alpha}}(B) = \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B)$.

Proof. By Lemma 3.5 (i) it follows immediately that (1) implies (2), and that (2) and (2'), as well as (3) and (3') are equivalent statements. Further, it is clear that (3) \Rightarrow (1).

We show (2) \Rightarrow (3). Let $\gamma \geq \alpha$ and choose $\gamma_1 \in I$ such that $\gamma_1 \geq \gamma$ and $\gamma_1 \geq \beta$. By (2), $H_{u_{\beta\alpha}}(B) \subseteq H_{u_{\gamma_1\alpha}}(B)$ and, by Lemma 3.5(i), $H_{u_{\gamma_1\alpha}}(B) \subseteq H_{u_{\gamma\alpha}}(B)$. Hence, $H_{u_{\beta\alpha}}(B) \subseteq \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B)$ as we wanted to proof. \square

We adopt the following definition inspired by the terminology in [29].

Definition 3.7. Let B be a right R -module.

- (1) A direct system $(F_{\alpha}, u_{\beta\alpha})_{\beta, \alpha \in I}$ of right R -modules is said to be B -stationary provided that the inverse system $(\text{Hom}_R(F_{\alpha}, B), \text{Hom}_R(u_{\beta\alpha}, B))_{\beta, \alpha \in I}$ satisfies the Mittag-Leffler condition, in other words, provided for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that the equivalent conditions in Lemma 3.6 are satisfied.
- (2) A right R -module M is said to be B -stationary if there exists a B -stationary direct system of finitely presented modules $(F_{\alpha}, u_{\beta\alpha})_{\beta, \alpha \in I}$ such that $M = \varinjlim F_{\alpha}$.
- (3) Let \mathcal{B} be a class of right R -modules. We say that a direct system $(F_{\alpha}, u_{\beta\alpha})_{\beta, \alpha \in I}$ or a right R -module M are \mathcal{B} -stationary if they are B -stationary for all $B \in \mathcal{B}$.

Let us start by discussing some closure properties of the class \mathcal{B} .

Proposition 3.8. Let $\{B_j\}_{j \in J}$ be a family of right R -modules. Let $(F_{\alpha}, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of right R -modules. Then the following statements are equivalent.

- (1) $(F_{\alpha}, u_{\beta\alpha})_{\beta, \alpha \in I}$ is $\prod_{j \in J} B_j$ -stationary.
- (2) For any $\alpha \in I$ there exists $\beta \geq \alpha$ such that $H_{u_{\beta\alpha}}(B_j) = \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B_j)$ for any $j \in J$.

If F_{α} is finitely generated for any $\alpha \in I$, then the statements above are further equivalent to the following one:

- (3) $(F_{\alpha}, u_{\beta\alpha})_{\beta, \alpha \in I}$ is $\bigoplus_{j \in J} B_j$ -stationary.

Proof. We use the same arguments as in [3, 2.6].

(1) \iff (2). Statement (1) holds if and only if, for any $\alpha \in I$ there exists β such that $H_{u_{\beta\alpha}}(\prod_{j \in J} B_j) = \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(\prod_{j \in J} B_j)$ if and only if

$$\prod_{j \in J} H_{u_{\beta\alpha}}(B_j) = \bigcap_{\gamma \geq \alpha} \prod_{j \in J} H_{u_{\gamma\alpha}}(B_j) = \prod_{j \in J} \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B_j).$$

Equivalently, if and only if (2) holds.

The proof of (2) \iff (3) follows in a similar way by observing that

$$H_{u_{\beta\alpha}}\left(\bigoplus_{j \in J} B_j\right) = \bigoplus_{j \in J} H_{u_{\beta\alpha}}(B_j).$$

provided all F_α are finitely generated. \square

Corollary 3.9. *Let \mathcal{B} be a class of right R -modules. Let M be a \mathcal{B} -stationary right R -module. Then the following statements hold true.*

- (i) *M is \mathcal{B}' -stationary where \mathcal{B}' denotes the class of all modules isomorphic either to a pure submodule or to a pure quotient of a module in \mathcal{B} .*
- (ii) *M is Add \mathcal{B} -stationary if and only if it is Prod \mathcal{B} -stationary if and only if there exists a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $\varinjlim F_\alpha \cong M$ having the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that $\overline{H}_{u_{\beta\alpha}}(B) = \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B)$ for any $B \in \mathcal{B}$.*
- (iii) *M is Add \mathcal{B} - and Prod \mathcal{B} -stationary for every $B \in \mathcal{B}$.*

Proof. The statements in Lemma 2.5(2) and (3) imply statement (i). Statement (ii) is a direct consequence of Proposition 3.8 combined with (i), and (iii) is a special case of (ii). \square

Proposition 3.10. *Let $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of right R -modules, and let B be a right R -module. Consider the following statements.*

- (1) *For any infinite chain $\alpha_1 \leq \alpha_2 \leq \dots \in I$ the direct system $(F_{\alpha_n}, u_{\alpha_{n+1}\alpha_n})_{n \in \mathbb{N}}$ is \mathcal{B} -stationary.*
- (1') *For any infinite chain $\alpha_1 \leq \alpha_2 \leq \dots \in I$ the chain of subgroups*

$$H_{u_{\alpha_2\alpha_1}}(B) \supseteq H_{u_{\alpha_3\alpha_1}}(B) \supseteq \dots$$

is stationary.

- (2) *The direct system $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ is \mathcal{B} -stationary.*

Then (1) and (1') are equivalent statements which imply (2).

Proof. The fact that (1) and (1') are equivalent statements follows directly from the definitions taking into account Example 3.2 and Remark 3.4.

We prove now (1) \Leftarrow (2). Assume for a contradiction that there exists α such that for any $\beta \geq \alpha$ condition (1) in Lemma 3.6 fails. Now we construct a countable chain in I such that condition (1) fails.

Set $\alpha_1 = \alpha$. Let $n \geq 1$, and assume we have constructed $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ such that

$$H_{u_{\alpha_2 \alpha_1}}(B) \not\supseteq H_{u_{\alpha_3 \alpha_1}}(B) \cdots \not\supseteq H_{u_{\alpha_n \alpha_1}}(B).$$

As Lemma 3.6 (1) fails for $\alpha_n \geq \alpha_1$, there exists $\gamma \geq \alpha_1$ such that

$$H_{u_{\alpha_n \alpha_1}}(B) \not\supseteq H_{u_\gamma \alpha_1}(B).$$

Let $\alpha_{n+1} \in I$ be such that $\alpha_{n+1} \geq \gamma$ and $\alpha_{n+1} \geq \alpha_n$. By Lemma 3.5 (i),

$$H_{u_{\alpha_{n+1} \alpha_1}}(B) \subseteq H_{u_\gamma \alpha_1}(B) \subsetneq H_{u_{\alpha_n \alpha_1}}(B)$$

as wanted. □

For later reference, we recall the following result.

Theorem 3.11 ([12, Theorem 5.1]). *Let \mathcal{B} be a class of right R -modules such that if $B \in \mathcal{B}$, then $B^{(\mathbb{N})} \in \mathcal{B}$, and let $\mathcal{A} = {}^\perp \mathcal{B}$. Let moreover*

$$F_1 \xrightarrow{u_1} F_2 \xrightarrow{u_2} F_3 \rightarrow \cdots \rightarrow F_n \xrightarrow{u_n} F_{n+1} \rightarrow \cdots$$

be a countable direct system of finitely presented right R -modules, and consider the pure exact sequence

$$(*) \quad 0 \rightarrow \bigoplus_{n \in \mathbb{N}} F_n \xrightarrow{\varphi} \bigoplus_{n \in \mathbb{N}} F_n \rightarrow \varinjlim F_n \rightarrow 0$$

where $\varphi \varepsilon_n = \varepsilon_n - \varepsilon_{n+1} u_n$ and $\varepsilon_n: F_n \rightarrow \bigoplus_{n \in \mathbb{N}} F_n$ denotes the canonical morphism for every $n \in \mathbb{N}$. Then the following statements are equivalent.

- (1) *The direct system $(F_n, u_n)_{n \in \mathbb{N}}$ is \mathcal{B} -stationary.*
- (2) *$\text{Hom}_R(\varphi, B)$ is surjective for all $B \in \mathcal{B}$.*
- (3) *$\varprojlim^1 \text{Hom}_R(F_n, B) = 0$ for all $B \in \mathcal{B}$.*

If F_n belongs to \mathcal{A} for all $n \in \mathbb{N}$, then the following statement is further equivalent.

- (4) *$\varinjlim F_n \in \mathcal{A}$.*

Corollary 3.12. *Let \mathcal{B} be a class of right R -modules such that if $B \in \mathcal{B}$, then $B^{(\mathbb{N})} \in \mathcal{B}$, and let $\mathcal{A} = {}^\perp \mathcal{B}$. Then the following statements are equivalent.*

- (1) *Every countable direct system of finitely presented modules in \mathcal{A} has limit in \mathcal{A} .*
- (2) *Every countable direct system of finitely presented modules in \mathcal{A} is \mathcal{B} -stationary.*
- (3) *Every direct system of finitely presented modules in \mathcal{A} is \mathcal{B} -stationary.*

Proof. Assume (1). Let $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of finitely presented right R -modules such that $F_\alpha \in \mathcal{A}$ for any $\alpha \in I$. Let $\alpha_1 \leq \alpha_2 \leq \dots$ be a chain in I . By (1), $\varinjlim (F_{\alpha_n}, u_{\alpha_{n+1}\alpha_n}) \in \mathcal{A}$. Then $(F_{\alpha_n}, u_{\alpha_{n+1}\alpha_n})_{n \in \mathbb{N}}$ is \mathcal{B} -stationary by Theorem 3.11, hence condition (3) follows by Proposition 3.10.

Obviously (3) implies (2). To see that (2) implies (1), let $A = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ be such that I is countable and F_α are finitely presented modules in \mathcal{A} . Taking a cofinal set of I if necessary we may assume that $I = \mathbb{N}$. Our hypothesis allows us to use Theorem 3.11 to conclude that $A \in \mathcal{A}$. \square

Examples 3.13.

- (1) Let \mathcal{B} be a class of right R -modules such that if $B \in \mathcal{B}$, then $B^{(\mathbb{N})} \in \mathcal{B}$, and let $\mathcal{A} = {}^\perp \mathcal{B}$. If $M \in \mathcal{A}$ is countably presented, then M is \mathcal{B} -stationary.

In fact, M can be written as direct limit of a countable direct system as in Theorem 3.11, and for all modules $B \in \mathcal{B}$ the map $\text{Hom}_R(\varphi, B)$ is surjective because $\text{Ext}_R^1(\varinjlim F_n, B) = 0$.

- (2) Let \mathcal{B} and \mathcal{A} be as in (1). Assume that R is a right noetherian ring and \mathcal{B} consists of modules of injective dimension at most one. Then every $M \in \mathcal{A}$ is \mathcal{B} -stationary.

In fact, the additional assumption on \mathcal{B} means that \mathcal{A} is closed by submodules: Let $N \leq M \in \mathcal{A}$. For any $B \in \mathcal{B}$, if we apply $\text{Hom}_R(-, B)$ to the exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0,$$

we obtain the exact sequence

$$\text{Ext}_R^1(M/N, B) \rightarrow \text{Ext}_R^1(M, B) = 0 \rightarrow \text{Ext}_R^1(N, B) \rightarrow \text{Ext}_R^2(M/N, B) = 0.$$

Hence, $\text{Ext}_R^1(N, B) = 0$.

As R_R is noetherian, any finitely generated submodule of M is finitely presented. Let I denote the directed set of all finitely generated submodules of M ; then $M = \bigcup_{F \in I} F$. If $F_1 \leq F_2 \leq \dots \leq F_n \leq \dots$ is a chain in I , then $N = \bigcup_{n \in \mathbb{N}} F_n$ is a submodule of M and it is in \mathcal{A} . By Theorem 3.11, N is \mathcal{B} -stationary. By Proposition 3.10, M is \mathcal{B} -stationary.

- (3) Let M be a module with a perfect decomposition in the sense of [7], for example M a Σ -pure-injective module, or M a finitely generated module with perfect endomorphism ring. Let \mathcal{M} be a class of finitely presented modules in $\text{Add } M$. Then every $N \in \varinjlim \mathcal{M}$ is $\text{Mod-}R$ -stationary.

In fact, we can write $N = \varinjlim F_\alpha$ where $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ is a direct system of finitely presented modules in $\overline{\mathcal{M}}$. If we take a chain $\alpha_1 \leq \alpha_2 \leq \dots$ in I , then $(F_{\alpha_n}, u_{\alpha_{n+1}\alpha_n})_{n \in \mathbb{N}}$ is a direct system in $\text{Add } M$ with a totally ordered index set, so it follows from [7, 1.4] that the pure exact sequence $(*)$ considered in Theorem

3.11 is split exact. In particular, $\text{Hom}_R(\varphi, B)$ is surjective for all modules B , hence $(F_{\alpha_n}, u_{\alpha_{n+1}\alpha_n})_{n \in \mathbb{N}}$ is $\text{Mod-}R$ -stationary by Theorem 3.11. Now the claim follows from Proposition 3.10.

(4) Let B be a Σ -pure-injective module. Then every right R -module M is $\text{Add } B$ -stationary.

To see this, write $M = \varinjlim F_\alpha$ where $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ is a direct system of finitely presented modules. If we take a chain $\alpha_1 \leq \alpha_2 \leq \dots$ in I and consider the direct system $(F_{\alpha_n}, u_{\alpha_{n+1}\alpha_n})_{n \in \mathbb{N}}$, then for any $B' \in \text{Add } B$ we know that $\text{Hom}_R(-, B')$ is exact on the pure exact sequence $(*)$ considered in Theorem 3.11. So $\text{Hom}_R(\varphi, B')$ is surjective for all modules $B' \in \text{Add } B$, and the claim follows again by combining Theorem 3.11 and Proposition 3.10.

4. DOMINATING MAPS

From the characterization of Mittag-Leffler modules in [36], we know that a right module is Q -Mittag-Leffler for any left module Q if and only if it is B -stationary for any right module B . We will now investigate the relationship between the properties Q -Mittag-Leffler and B -stationary when we restrict our choice of Q and B to subclasses of $R\text{-Mod}$ and $\text{Mod-}R$, respectively.

As a first step, in Theorem 4.8 we provide a characterization of when a module M is B -stationary which is independent from the direct limit presentation of M . To this end, we need the following notion which is inspired by the corresponding notion from [36].

Definition 4.1. Let Q be a left R -module, and let B be a right R -module. Let moreover $u: M \rightarrow N$ and $v: M \rightarrow M'$ be right R -module homomorphisms. We say that v *B-dominates* u with respect to Q if

$$\ker(u \otimes_R Q) \subseteq \bigcap_{h \in H_v(B)} \ker(h \otimes_R Q).$$

For classes of modules \mathcal{Q} and \mathcal{B} in $R\text{-Mod}$ and $\text{Mod-}R$, respectively, we say that v *B-dominates* u with respect to \mathcal{Q} if v B -dominates u with respect to Q for any $Q \in \mathcal{Q}$ and any $B \in \mathcal{B}$.

If $\mathcal{Q} = R\text{-Mod}$, we simply say that v \mathcal{B} -dominates u .

If $\mathcal{B} = \text{Mod-}R$, we say that v dominates u with respect to \mathcal{Q} , and of course, this means that $\ker(u \otimes_R Q) \subseteq \ker(v \otimes_R Q)$ for all left modules $Q \in \mathcal{Q}$.

Finally, if $\mathcal{Q} = R\text{-Mod}$ and $\mathcal{B} = \text{Mod-}R$, then we are in the case treated in [36, 2.1.1], and we say that v dominates u .

We note some properties of dominating maps.

Lemma 4.2. Let $u: M \rightarrow N$ and $v: M \rightarrow M'$ be right R -module homomorphisms, and let B be a right R -module and Q a left R -module.

- (1) “ B -dominating with respect to Q ” is translation invariant on the right. That is, if v B -dominates u with respect to Q and $t: X \rightarrow M$ is a homomorphism, then vt B -dominates ut with respect to Q .
- (2) “ B -dominating with respect to Q ” is stable by composition on the left. More precisely, if v B -dominates u with respect to Q and $m: M' \rightarrow M''$ is a homomorphism, then mv B -dominates u with respect to Q .

Proof. (1) By hypothesis, $\ker(ut \otimes Q) = \ker(u \otimes Q)(t \otimes Q)$ is contained in

$$\bigcap_{h \in H_v(B)} \ker(h \otimes_R Q)(t \otimes Q) = \bigcap_{ht \in H_{vt}(B)} \ker(ht \otimes_R Q) = \bigcap_{h \in H_{vt}(B)} \ker(h \otimes_R Q).$$

(2) As $H_{mv}(B) \subseteq H_v(B)$,

$$\bigcap_{h \in H_v(B)} \ker(h \otimes_R Q) \subseteq \bigcap_{h \in H_{mv}(B)} \ker(h \otimes_R Q).$$

Hence, if v B -dominates u with respect to Q , we deduce that also mv B -dominates u with respect to Q . \square

We recall the following property of direct limits.

Lemma 4.3. *Let M be a right R -module, and let S be a class of finitely presented modules. Then $M \in \varinjlim S$ if and only if for any finitely presented module F and any map $u: F \rightarrow M$ there exists $S \in S$ and $v: F \rightarrow S$ such that u factors through v .*

Proof. Assume $M = \varinjlim S_Y$ where $(S_Y, u_{\delta_Y})_{\delta_Y \in I}$. Let F be a finitely presented module and $u \in \text{Hom}_R(F, M)$. Since $\text{Hom}_R(F, M)$ is canonically isomorphic to $\varinjlim \text{Hom}_R(F, S_Y)$, there exist $\gamma \in I$ and $v: F \rightarrow S_\gamma$ such that $u = u_\gamma v$ where $u_\gamma: S_\gamma \rightarrow M$ denotes the canonical morphism.

To prove the converse, write $M = \varinjlim F_\alpha$ where $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ is a direct system of finitely presented right R -modules. By hypothesis, for each $\alpha \in I$ there exists $S_\alpha \in S$, $v_\alpha: F_\alpha \rightarrow S_\alpha$ and $t_\alpha: S_\alpha \rightarrow M$ such that the canonical map $u_\alpha: F_\alpha \rightarrow M$ satisfies $u_\alpha = t_\alpha v_\alpha$. Fix $\alpha \in I$. As $\text{Hom}_R(S_\alpha, \varinjlim F_\gamma)$ is canonically isomorphic to $\varinjlim \text{Hom}_R(S_\alpha, F_\gamma)$, there exists $\beta \geq \alpha$ and a commutative diagram:

$$\begin{array}{ccc} S_\alpha & \xrightarrow{v'_{\beta\alpha}} & F_\beta \\ t_\alpha \downarrow & \swarrow u_\beta & \\ & M & \end{array}$$

Set $u'_{\beta\alpha} = v_\beta v'_{\beta\alpha}$.

It is not difficult to see that $(S_\alpha, u'_{\beta\alpha})_{\beta, \alpha \in I}$ is a direct system of modules in S such that $M = \varinjlim S_\alpha$. \square

The next result will provide us with a tool for comparing the relative Mittag-Leffler conditions. In fact, we will see in Theorem 4.8 that the \mathcal{B} -stationary modules are the modules satisfying the equivalent conditions in Proposition 4.4 for every $B \in \mathcal{B}$ and every $Q \in R\text{-Mod}$, while the \mathcal{Q} -Mittag-Leffler modules are the modules satisfying the equivalent conditions in Proposition 4.4 for every $B \in \text{Mod-}R$ and every $Q \in \mathcal{Q}$, see Theorem 5.1.

Proposition 4.4. *Let B be a right R -module, let Q be a left R -module, and let S be a class of finitely presented right R -modules. For a right R -module $M \in \varinjlim S$ the following statements are equivalent.*

- (1) *There is a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ having the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that $u_{\beta\alpha}$ B -dominates the canonical map $u_\alpha: F_\alpha \rightarrow M$ with respect to Q .*
- (2) *Every direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ has the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that $u_{\beta\alpha}$ B -dominates the canonical map $u_\alpha: F_\alpha \rightarrow M$ with respect to Q .*
- (3) *For any finitely presented module F (belonging to S) and any homomorphism $u: F \rightarrow M$ there exist a module $S \in S$ and a homomorphism $v: F \rightarrow S$ such that u factors through v , and v B -dominates u with respect to Q .*

Proof. (1) \Rightarrow (3). Let F be a finitely presented module and $u: F \rightarrow M$ a homomorphism. Since $\text{Hom}_R(F, M)$ is canonically isomorphic to $\varinjlim \text{Hom}_R(F, F_\alpha)$, there exists $\alpha_0 \in I$ and $t: F \rightarrow F_{\alpha_0}$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{t} & F_{\alpha_0} \\ u \downarrow & \swarrow u_{\alpha_0} & \\ M & & \end{array}$$

is commutative. By assumption there exists $\beta \geq \alpha_0$ such that $u_{\beta\alpha_0}$ B -dominates u_{α_0} with respect to Q . Set $v' = u_{\beta\alpha_0}t$. As $u = u_\beta u_{\beta\alpha_0}t = u_\beta v'$, we have $u \in H_{v'}(M)$. Moreover, since $u_{\beta\alpha_0}$ B -dominates u_{α_0} , it follows from Lemma 4.2(1) that $v' = u_{\beta\alpha_0}t$ B -dominates $u = u_{\alpha_0}t$ with respect to Q .

By hypothesis, $M = \varinjlim S_\gamma$ for a directed system $(S_\gamma, u'_{\delta\gamma})_{\delta, \gamma \in J}$ of modules in S . As F_β is finitely presented, there exist γ in J and $m: F_\beta \rightarrow S_\gamma$ such that the diagram

$$\begin{array}{ccc} F_\beta & \xrightarrow{m} & S_\gamma \\ u_\beta \downarrow & \swarrow u'_\gamma & \\ M & & \end{array}$$

commutes. Set $v = mv'$. Then u factors through v and, by Lemma 4.2(2), v B -dominates u with respect to Q .

(3) \Rightarrow (2). Consider a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$. Fix $\alpha_0 \in I$. We have to verify the existence of $\beta \geq \alpha_0$ such that $u_{\beta\alpha_0}$ B -dominates u_{α_0} with respect to Q . Applying the hypothesis with $u = u_{\alpha_0} : F_{\alpha_0} \rightarrow M$ we deduce that there exist a module $S \in S$, $v : F_{\alpha_0} \rightarrow S$ and $t : S \rightarrow M$ such that the diagram

$$\begin{array}{ccc} F_{\alpha_0} & \xrightarrow{v} & S \\ u_{\alpha_0} \downarrow & \nearrow t & \\ M & & \end{array}$$

is commutative and v B -dominates u_{α_0} with respect to Q . As S is finitely presented, $\text{Hom}_R(S, M)$ is canonically isomorphic to $\varinjlim \text{Hom}_R(S, F_\alpha)$. Hence there exist $\beta' \geq \alpha_0$ and $t' : S \rightarrow F_{\beta'}$ such that the diagram

$$\begin{array}{ccc} F_{\alpha_0} & \xrightarrow{v} & S \\ u_{\alpha_0} \downarrow & \nearrow t & \downarrow t' \\ M & \xleftarrow{u_{\beta'}} & F_{\beta'} \end{array}$$

is commutative. Since $u_{\beta'} u_{\beta'\alpha_0} = u_{\beta'} t' v$, there exists $\beta \geq \beta'$ such that $u_{\beta\beta'} u_{\beta'\alpha_0} = u_{\beta\beta'} t' v$, that is, $u_{\beta\alpha_0} = m v$ where $m = u_{\beta\beta'} t'$. By Lemma 4.2(2), $u_{\beta\alpha_0}$ B -dominates u_{α_0} with respect to Q .

Similarly, to see that condition (3) restricted to modules F belonging to S implies (1), we proceed as in (3) \Rightarrow (2) but considering a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ such that all $F_\alpha \in S$. \square

Observe that the condition $M \in \varinjlim S$ in the hypothesis of Proposition 4.4 is also necessary. This can be deduced from condition (3) by employing Lemma 4.3.

We will need the following result.

Proposition 4.5 ([36, Proposition 2.1.1]). *Let $u : M \rightarrow N$ and $h : M \rightarrow B$ be right R -module homomorphisms. The following statements are equivalent.*

- (i) $\ker(u \otimes_R Q) \subseteq \ker(h \otimes_R Q)$ for all left R -modules Q .
- (ii) $\ker(u \otimes_R B^*) \subseteq \ker(h \otimes_R B^*)$.

If $\text{coker}(u)$ is finitely presented, the following statement is further equivalent.

- (iii) h factors through u .

We can now interpret the property “ B -dominates” in terms of H -subgroups.

Proposition 4.6. *Let B be a right R -module. Let $u : M \rightarrow N$ and $v : M \rightarrow M'$ be right R -module homomorphisms. If $H_v(B) \subseteq H_u(B)$, then v B -dominates u . The converse implication holds true provided $\text{coker}(u)$ is finitely presented.*

Proof. Let $h: M \rightarrow B \in H_v(B)$. By hypothesis, there exists $h': N \rightarrow B$ such that $h = h'u$. Hence, for any left R -module Q

$$\ker(u \otimes_R Q) \subseteq \ker(h' \otimes Q)(u \otimes_R Q) \subseteq \ker(h \otimes_R Q).$$

This shows the claim.

For the converse implication, assume that v B -dominates u and $\text{coker}(u)$ is finitely presented. Let $h \in H_v(B)$. Then $\ker(u \otimes_R Q) \subseteq \ker(h \otimes_R Q)$ for any left R -module Q . By Proposition 4.5 this means that $h \in H_u(B)$. \square

Lemma 4.7. *Let B be a right R -module and let Q be a left R -module. Let further $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of finitely presented right R -modules with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$. For $\alpha, \beta \in I$ with $\beta \geq \alpha$, the following statements hold true.*

- (1) $u_{\beta\alpha}$ B -dominates the canonical map $u_\alpha: F_\alpha \rightarrow M$ with respect to Q if and only if $u_{\beta\alpha}$ B -dominates $u_{\gamma\alpha}$ with respect to Q for any $\gamma \geq \alpha$.
- (2) $u_{\beta\alpha}$ B -dominates the canonical map $u_\alpha: F_\alpha \rightarrow M$ if and only if $H_{u_{\beta\alpha}}(B) = \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B)$.

Proof.

- (1) To show the only-if-part, fix $\gamma \geq \alpha$. As $u_\alpha = u_\gamma u_{\gamma\alpha}$, for any left R -module Q

$$\ker(u_{\gamma\alpha} \otimes_R Q) \subseteq \ker(u_\alpha \otimes_R Q) \subseteq \bigcap_{h \in H_{u_{\beta\alpha}}(B)} \ker(h \otimes_R Q).$$

Therefore $u_{\beta\alpha}$ B -dominates $u_{\gamma\alpha}$ with respect to Q . The converse implication is clear from the properties of direct limits.

- (2) By (1), $u_{\beta\alpha}$ B -dominates u_α if and only if $u_{\beta\alpha}$ B -dominates $u_{\gamma\alpha}$ for any $\gamma \geq \alpha$. As $\text{coker } u_{\gamma\alpha}$ is finitely presented, we know from Proposition 4.6 that the latter is equivalent to $H_{u_{\beta\alpha}}(B) \subseteq H_{u_{\gamma\alpha}}(B)$ for any $\gamma \geq \alpha$. But this means $H_{u_{\beta\alpha}}(B) = \bigcap_{\gamma \geq \alpha} H_{u_{\gamma\alpha}}(B)$ by Lemma 3.6. \square

From Lemma 4.7 and Definition 3.7, we immediately obtain the announced characterization of B -stationary modules.

Theorem 4.8. *Let B be a right R -module, and let S be a class of finitely presented modules. For a right R -module $M \in \varinjlim S$, the following statements are equivalent.*

- (1) M is B -stationary.
- (2) There is a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ having the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that $u_{\beta\alpha}$ B -dominates the canonical map $u_\alpha: F_\alpha \rightarrow M$.
- (3) For any finitely presented module F (belonging to S) and any homomorphism $u: F \rightarrow M$ there exist a module $S \in S$ and a homomorphism $v: F \rightarrow S$ such that u factors through v , and v B -dominates u .

We close this section with some closure properties of the class \mathcal{B} in the definition of “ \mathcal{B} -dominating”. Let us first prove the following preliminary result.

Proposition 4.9. *Let B be a right R -module. Let $u: M \rightarrow N$ and $v: M \rightarrow M'$ be right R -module homomorphisms. Then the following statements are equivalent.*

- (1) v B -dominates u .
- (2) For any finitely presented left R -module Q

$$\ker(u \otimes_R Q) \subseteq \bigcap_{h \in H_v(B)} \ker(h \otimes_R Q).$$

- (3) For any (finitely presented) left R -module Q

$$\ker(u \otimes_R Q) \subseteq \ker(\tilde{h} \otimes_R Q).$$

where $\tilde{h}: M \rightarrow \prod_{H_v(B)} B$ is the product map induced by all $h \in H_v(B)$.

Proof. We follow the idea in the proof of [36, Proposition 2.1.1].
Fix $h \in H_v(B)$. Consider the push-out diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & B \\ u \downarrow & & \downarrow u' \\ N & \xrightarrow{h'} & N' \end{array}.$$

Recall that it will stay a push-out diagram when we apply the functor $- \otimes_R Q$ for any left module Q . Hence we have the exact sequence

$$0 \rightarrow \ker(u \otimes_R Q) \cap \ker(h \otimes_R Q) \rightarrow \ker(u \otimes_R Q) \xrightarrow{h \otimes_R Q} \ker(u' \otimes_R Q) \rightarrow 0.$$

This shows that, for any left module Q , $\ker(u \otimes_R Q) \subseteq \ker(h \otimes_R Q)$ if and only if $\ker(u' \otimes_R Q) = 0$, that is, if and only if u' is a pure monomorphism.

Since a morphism is a pure monomorphism if and only if it is a monomorphism when tensoring by finitely presented modules, we deduce that (1) and (2) are equivalent statements.

To prove that (2) and (3) are equivalent, note that $\bigcap_{h \in H_v(B)} \ker(h \otimes_R Q)$ is the kernel of the product map induced by all homomorphisms $h \otimes_R Q$ with $h \in H_v(B)$. When Q is finitely presented, the natural morphism $\rho: \prod_{H_v(B)} B \otimes_R Q \rightarrow \prod_{H_v(B)} (B \otimes_R Q)$ is an isomorphism. Hence

$$\bigcap_{h \in H_v(B)} \ker(h \otimes_R Q) = \ker(\tilde{h} \otimes_R Q)$$

and the statement is verified. To obtain the statement for arbitrary Q , proceed as in the proof of (1) \Rightarrow (2). \square

Proposition 4.10. *Let $u: M \rightarrow N$ and $v: M \rightarrow M'$ be right R -module homomorphisms. Let \mathcal{B} be a class of right R -modules such that v \mathcal{B} -dominates u . Then*

- (i) *v \mathcal{B}' -dominates u , where \mathcal{B}' denotes the class of all pure submodules of modules in \mathcal{B} .*
- (ii) *v $\text{Prod } \mathcal{B}$ -dominates u .*
- (iii) *v $\varinjlim \mathcal{B}$ -dominates u provided that M' is finitely presented.*

Proof.

- (i) Let $B \in \mathcal{B}$, and assume that the inclusion $\varepsilon: C \rightarrow B$ is a pure monomorphism. If $h \in H_v(C)$, then $\varepsilon h \in H_v(B)$, and $\ker(h \otimes Q) = \ker(\varepsilon h \otimes Q)$ contains $\ker(u \otimes Q)$.
- (ii) By (i) it is enough to consider modules of the form $\prod_{i \in I} B_i$ where $\{B_i\}_{i \in I}$ is a family of modules in \mathcal{B} . Let Q be a finitely presented module. As the canonical morphism $\rho: (\prod_{i \in I} B_i) \otimes Q \rightarrow \prod_{i \in I} B_i \otimes_R Q$ is an isomorphism and, as any $h \in H_v(\prod_{i \in I} B_i)$ is induced by a family $(h_i)_{i \in I}$ where $h_i \in H_v(B_i)$ for any $i \in I$, we deduce that

$$\bigcap_{h \in H_v(\prod_{i \in I} B_i)} \ker(h \otimes_R Q) = \bigcap_{h \in H_v(B_i), i \in I} \ker(h \otimes_R Q).$$

Then the claim follows from Proposition 4.9.

- (iii) Let $\{B_i, f_{ji}\}_{i \in I}$ be a direct system of modules in \mathcal{B} , and let $h \in H_v(\varinjlim B_i)$. Then $h = h'v$ for some $f: M' \rightarrow \varinjlim B_i$. As M' is finitely presented, there exists $j \in I$ such that h' factors through the canonical map $f_j: B_j \rightarrow \varinjlim B_i$. So, there exists $g: M' \rightarrow B_j$ such that $h' = f_j g$, thus $h = f_j g v$ with $g v \in H_v(B_j)$. Hence, for any left module Q , we have $\ker(u \otimes_R Q) \subseteq \ker(g v \otimes_R Q) \subseteq \ker(h \otimes_R Q)$. \square

5. \mathcal{Q} -MITTAG-LEFFLER MODULES REVISITED

As a next step towards establishing a relationship between \mathcal{Q} -Mittag-Leffler and \mathcal{B} -stationary modules, we provide a characterization of \mathcal{Q} -Mittag-Leffler modules in terms of dominating maps. It is inspired by work of Azumaya and Facchini [11, Theorem 6].

Theorem 5.1. *Let \mathcal{Q} be a class of left R -modules, and let S be a class of finitely presented right R -modules. For a right R -module $M \in \varinjlim S$, consider the following statements.*

- (1) *M is \mathcal{Q} -Mittag-Leffler.*
- (2) *Every direct system of finitely presented right R -modules $(F_\alpha, u_{\beta, \alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta, \alpha})_{\beta, \alpha \in I}$ has the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that $u_{\beta, \alpha}$ dominates the canonical map $u_\alpha: F_\alpha \rightarrow M$ with respect to \mathcal{Q} .*
- (3) *For every finitely presented module F (belonging to S) and every homomorphism $u: F \rightarrow M$ there are a module $S \in S$ and a homomorphism $v: F \rightarrow S$ such that u factors through v and $\ker(u \otimes_R Q) = \ker(v \otimes_R Q)$ for all $Q \in \mathcal{Q}$.*

- (4) For every countable (finite) subset X of M there are a countably presented \mathcal{Q} -Mittag-Leffler module $N \in \varinjlim S$ and a homomorphism $v: N \rightarrow M$ such that $X \subseteq v(N)$ and $v \otimes_R Q$ is a monomorphism for all $Q \in \mathcal{Q}$.
- (5) For every finitely generated submodule M_0 of M there are a finitely presented module $S \in S$ and a homomorphism $w: M_0 \rightarrow S$ such that the embedding $\varepsilon: M_0 \rightarrow M$ factors through w and $\ker(\varepsilon \otimes_R Q) = \ker(w \otimes_R Q)$ for all $Q \in \mathcal{Q}$.

Then (1), (2), (3), and (4) are equivalent statements, and (5) implies the other statements. Moreover, if $R \in \mathcal{Q}$, then all statements are equivalent.

Proof. (1) \Rightarrow (2). Assume for a contradiction that there is a direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ such that there exists $\beta \in I$ satisfying that for any $\beta \geq \alpha$ there exists $Q_\beta \in \mathcal{Q}$ such that

$$\ker(u_\alpha \otimes Q_\beta) \not\subseteq \ker(u_{\beta\alpha} \otimes Q_\beta).$$

If x_1, \dots, x_n is a generating set of F_α , then for each $\beta \geq \alpha$ we can choose

$$a_\beta = \sum_{i=1}^n x_i \otimes q_\beta^i \in \ker(u_\alpha \otimes Q_\beta) \setminus \ker(u_{\beta\alpha} \otimes Q_\beta).$$

Set $x = \sum_{i=1}^n x_i \otimes (q_\beta^i)_{\beta \geq \alpha} \in F_\alpha \otimes \prod_{\beta \geq \alpha} Q_\beta$. Consider the commutative diagram:

$$\begin{array}{ccc} F_\alpha \otimes \prod_{\beta \geq \alpha} Q_\beta & \xrightarrow{u_\alpha \otimes \prod_{\beta \geq \alpha} Q_\beta} & M \otimes \prod_{\beta \geq \alpha} Q_\beta \\ \rho' \downarrow & & \rho \downarrow \\ \prod_{\beta \geq \alpha} (F_\alpha \otimes Q_\beta) & \xrightarrow{\prod_{\beta \geq \alpha} (u_\alpha \otimes Q_\beta)} & \prod_{\beta \geq \alpha} (M \otimes Q_\beta) \end{array}.$$

As $(\prod_{\beta \geq \alpha} (u_\alpha \otimes Q_\beta))\rho'(x) = 0$ and, by hypothesis, ρ is injective, we deduce that $(u_\alpha \otimes \prod_{\beta \geq \alpha} Q_\beta)(x) = 0$. Since $M \otimes \prod_{\beta \geq \alpha} Q_\beta = \varinjlim (F_\gamma \otimes \prod_{\beta \geq \alpha} Q_\beta)$, there exists $\beta_0 \geq \alpha$ such that $x \in \ker(u_{\beta_0\alpha} \otimes \prod_{\beta \geq \alpha} Q_\beta)$. The commutativity of the diagram

$$\begin{array}{ccc} F_\alpha \otimes \prod_{\beta \geq \alpha} Q_\beta & \xrightarrow{u_{\beta_0\alpha} \otimes \prod_{\beta \geq \alpha} Q_\beta} & F_{\beta_0} \otimes \prod_{\beta \geq \alpha} Q_\beta \\ \cong \downarrow & & \cong \downarrow \\ \prod_{\beta \geq \alpha} (F_\alpha \otimes Q_\beta) & \xrightarrow{\prod_{\beta \geq \alpha} (u_{\beta_0\alpha} \otimes Q_\beta)} & \prod_{\beta \geq \alpha} (F_{\beta_0} \otimes Q_\beta) \end{array}$$

implies that, for any $\beta \geq \alpha$, $a_\beta \in \ker(u_{\beta_0\alpha} \otimes Q_\beta)$. In particular, $a_{\beta_0} \in \ker(u_{\beta_0\alpha} \otimes Q_\beta)$ which is a contradiction.

(2) \Rightarrow (1). Fix a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$.

Let $\{Q_k\}_{k \in K}$ be a family of modules of \mathcal{Q} , and let $x \in \ker \rho$ where $\rho: M \otimes \prod_{k \in K} Q_k \rightarrow \prod_{k \in K} (M \otimes Q_k)$ denotes the natural map. Since $M \otimes \prod_{k \in K} Q_k = \varinjlim (F_\alpha \otimes \prod_{k \in K} Q_k)$, there exists $\alpha \in I$ and $x_\alpha = \sum_{i=1}^n x_i \otimes (q_k^i)_{k \in K} \in F_\alpha \otimes \prod_{k \in K} Q_k$ such that $x = (u_\alpha \otimes \prod_{k \in K} Q_k)(x_\alpha)$. The commutativity of the diagram

$$\begin{array}{ccc} F_\alpha \otimes \prod_{k \in K} Q_k & \xrightarrow{u_\alpha \otimes \prod_{k \in K} Q_k} & M \otimes \prod_{k \in K} Q_k \\ \rho' \downarrow & & \rho \downarrow \\ \prod_{k \in K} (F_\alpha \otimes Q_k) & \xrightarrow{\prod_{k \in K} (u_\alpha \otimes Q_k)} & \prod_{k \in K} (M \otimes Q_k) \end{array}$$

implies that, for each $k \in K$, $\sum_{i=1}^n x_i \otimes q_k^i \in \ker(u_\alpha \otimes Q_k)$.

Let $\beta \geq \alpha$ be such that $u_{\beta\alpha}$ dominates the canonical map u_α with respect to \mathcal{Q} . The commutativity of the diagram

$$\begin{array}{ccc} F_\alpha \otimes \prod_{k \in K} Q_k & \xrightarrow{u_{\beta\alpha} \otimes \prod_{k \in K} Q_k} & F_\beta \otimes \prod_{k \in K} Q_k \\ \cong \downarrow & & \cong \downarrow \\ \prod_{k \in K} (F_\alpha \otimes Q_k) & \xrightarrow{\prod_{k \in K} (u_{\beta\alpha} \otimes Q_k)} & \prod_{k \in K} (F_\beta \otimes Q_k) \end{array}$$

implies that $(u_{\beta\alpha} \otimes \prod_{k \in K} Q_k)(x_\alpha) = 0$. Hence $x = (u_\beta u_{\beta\alpha} \otimes \prod_{k \in K} Q_k)(x_\alpha) = 0$.

By Proposition 4.4, we already know that (2) and (3) are equivalent statements.

(3) \Rightarrow (4). Let $X = \{x_1, x_2, \dots\} \subseteq M$. We shall construct inductively a countable direct system $(S_n, f_n: S_n \rightarrow S_{n+1})_{n \geq 0}$ of modules in S and a sequence of maps $(v_n: S_n \rightarrow M)_{n \geq 0}$ such that $v_n = v_{n+1} f_n$ and $\{x_1, \dots, x_n\} \subseteq v_n(S_n)$.

Set $S_0 = 0$ and let v_0 be the zero map. Let $n \geq 0$ and assume as inductive hypothesis that S_m and v_m have been constructed for any $m \leq n$. Let $u: S_n \oplus R \rightarrow M$ be defined as $u(g, r) = v_n(g) + x_{n+1}r$ for any $(g, r) \in S_n \oplus R$. By (3), there exist $S_{n+1} \in S$, $v: S_n \oplus R \rightarrow S_{n+1}$, and $v_{n+1}: S_{n+1} \rightarrow M$ such that $u = v_{n+1}v$ and $\ker(v \otimes Q) = \ker(u \otimes Q)$ for all $Q \in \mathcal{Q}$. Let $\varepsilon: S_n \rightarrow S_n \oplus R$ denote the canonical inclusion and set $f_n = v \circ \varepsilon$. Then $v_n = u\varepsilon = v_{n+1}(v\varepsilon) = v_{n+1}f_n$. This completes the induction step. Note moreover that also $\ker(v_n \otimes Q) = \ker(f_n \otimes Q)$ for all $Q \in \mathcal{Q}$.

Set $N = \varinjlim S_n$ and $v = \varinjlim v_n$. Then N is countably presented. As for any $Q \in \mathcal{Q}$, $\ker(v \otimes Q) = \varinjlim \ker(v_n \otimes Q)$ and $\ker(v_n \otimes Q) = \ker(f_n \otimes Q)$, we deduce that $v \otimes Q$ is injective.

To show that N is \mathcal{Q} -Mittag-Leffler we verify that N satisfies (2), as we already know that (1) and (2) are equivalent. By Proposition 4.4 it is enough to check the condition for the direct system $(S_n, f_n)_{n \geq 0}$ and the canonical maps $u_n: S_n \rightarrow M$.

Notice that $vu_n = v_n$. Therefore, for any $Q \in \mathcal{Q}$

$$\ker(u_n \otimes Q) \subseteq \ker(v_n \otimes Q) = \ker(f_n \otimes Q)$$

from which we conclude that f_n dominates u_n with respect to \mathcal{Q} .

(5) \Rightarrow (4) is proven similarly.

(4) \Rightarrow (1). Consider a family $(Q_k)_{k \in K}$ in \mathcal{Q} , and an element x in the kernel of $\rho: M \otimes \prod_{k \in K} Q_k \rightarrow \prod_{k \in K} (M \otimes_R Q_k)$. Then there are a \mathcal{Q} -Mittag-Leffler module N and a homomorphism $v: N \rightarrow M$ such that x lies in the image of $(v \otimes \prod_{k \in K} Q_k)$, and $v \otimes_R Q$ is a monomorphism for all $Q \in \mathcal{Q}$. In the commutative diagram

$$\begin{array}{ccc} N \otimes \prod_{k \in K} Q_k & \xrightarrow{v \otimes \prod_{k \in K} Q_k} & M \otimes \prod_{k \in K} Q_k \\ \rho' \downarrow & & \rho \downarrow \\ \prod_{k \in K} (N \otimes_R Q_k) & \xrightarrow{\prod_{k \in K} (v \otimes Q_k)} & \prod_{k \in K} (M \otimes_R Q_k) \end{array}$$

we then have that ρ' is injective because N is \mathcal{Q} -Mittag-Leffler, and $\prod_{k \in K} (v \otimes Q_k)$ is injective by assumption on v . This shows that $x = 0$.

Assume now that $R \in \mathcal{Q}$. To show (3) \Rightarrow (5), we proceed as in the proof of (2) \Rightarrow (3) in [11, Theorem 6]. We take an epimorphism $p: F \rightarrow M_0$ from a finitely generated free module F , set $u = \varepsilon p$, and construct v as in condition (3). Note that $\ker u = \ker v$ since \mathcal{Q} contains R . We thus obtain $w: M_0 \rightarrow S$ and $t: S \rightarrow M$ such that $v = wp$ and $\varepsilon = tw$. To show $\ker(\varepsilon \otimes_R Q) = \ker(w \otimes_R Q)$ for all $Q \in \mathcal{Q}$ it is enough to verify the inclusion \subseteq . So, take a left R -module $Q \in \mathcal{Q}$ and $y \in \ker(\varepsilon \otimes Q)$. Note that $y = (p \otimes Q)(x)$ for some $x \in F \otimes Q$. Then $(u \otimes Q)(x) = (\varepsilon p \otimes Q)(x) = 0$, hence $(w \otimes Q)(y) = (v \otimes Q)(x) = 0$. \square

Condition (4) in Theorem 5.1 gives the following characterization of \mathcal{Q} -Mittag-Leffler modules.

Corollary 5.2. *Let \mathcal{Q} be a class of left R -modules, and let S be a class of finitely presented right R -modules. For a fixed right R -module $M \in \varinjlim S$ denote by C the class of its countably generated submodules N such that N is \mathcal{Q} -Mittag-Leffler and the inclusion $N \subseteq M$ remains injective when tensoring with any module $Q \in \mathcal{Q}$.*

Then M is \mathcal{Q} -Mittag-Leffler if and only if M is a directed union of modules in \mathcal{C} . Moreover, if $R \in \mathcal{Q}$, the modules in \mathcal{C} can be taken countably presented and in $\varinjlim S$.

Proof. For the only-if implication, we follow the notation of Theorem 5.1 (4). We only have to prove that $v(N)$ is a \mathcal{Q} -Mittag-Leffler module and that the inclusion $\varepsilon: v(N) \rightarrow M$ remains injective when tensoring with any module $Q \in \mathcal{Q}$. Let $\{Q_k\}_{k \in K}$ be a family of modules in \mathcal{Q} . Consider the commutative diagram:

$$\begin{array}{ccccc} N \otimes \prod_{k \in K} Q_k & \xrightarrow{v \otimes \prod_{k \in K} Q_k} & v(N) \otimes \prod_{k \in K} Q_k & \xrightarrow{\varepsilon \otimes \prod_{k \in K} Q_k} & M \otimes \prod_{k \in K} Q_k \\ \rho_1 \downarrow & & \rho_2 \downarrow & & \rho \downarrow \\ \prod_{k \in K} (N \otimes_R Q_k) & \xrightarrow{\prod_{k \in K} (v \otimes Q_k)} & \prod_{k \in K} (v(N) \otimes_R Q_k) & \xrightarrow{\prod_{k \in K} (\varepsilon \otimes Q_k)} & \prod_{k \in K} (M \otimes_R Q_k) \end{array}$$

Note that $v \otimes \prod_{k \in K} Q_k: N \otimes \prod_{k \in K} Q_k \rightarrow v(N) \otimes \prod_{k \in K} Q_k$ is surjective, therefore if $x \in v(N) \otimes \prod_{k \in K} Q_k$ satisfies $(\varepsilon \otimes \prod_{k \in K} Q_k)(x) = 0$, then there exists $y \in N \otimes \prod_{k \in K} Q_k$ such that $x = (v \otimes \prod_{k \in K} Q_k)(y)$ and $(\varepsilon v \otimes \prod_{k \in K} Q_k)(y) = 0$. Since $\rho(\varepsilon v \otimes \prod_{k \in K} Q_k) = (\prod_{k \in K} (v \otimes Q_k))\rho_1$ is an injective map, we infer $y = 0$, so $x = 0$. This shows that $\varepsilon \otimes \prod_{k \in K} Q_k$ is injective.

Then also $(\prod_{k \in K} (\varepsilon \otimes Q_k))\rho_2 = \rho(\varepsilon \otimes \prod_{k \in K} Q_k)$ is injective, and so is ρ_2 .

To prove the converse implication proceed as in the proof of (4) \Rightarrow (1) of Theorem 5.1.

The statement for the case when $R \in \mathcal{Q}$ is clear because then the map v in Theorem 5.1 is injective, so N is isomorphic to $v(N)$. \square

Corollary 5.3. *Let \mathcal{Q} be a class of left R -modules containing R . Then every countably generated \mathcal{Q} -Mittag-Leffler right R -module is countably presented.*

Now we can start relating \mathcal{Q} -Mittag-Leffler and \mathcal{B} -stationary modules.

Lemma 5.4. *Let \mathcal{Q} be a class of left R -modules, and let B, M be right R -modules. Assume that M is \mathcal{B} -stationary. Write $M = \varinjlim F_\alpha$ where $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ is a direct system of finitely presented modules. If for all $\alpha, \beta \in I$ with $\beta \geq \alpha$ and all $Q \in \mathcal{Q}$*

$$\ker(u_{\beta\alpha} \otimes_R Q) = \bigcap_{h \in H_{u_{\beta\alpha}}(B)} \ker(h \otimes_R Q),$$

then M is a \mathcal{Q} -Mittag-Leffler module.

Proof. Fix $\alpha \in I$, and denote by $u_\alpha: F_\alpha \rightarrow M$ the canonical map. As M is \mathcal{B} -stationary, we infer from Proposition 4.8 that there exists $\beta \geq \alpha$ such that $u_{\beta\alpha}$

B -dominates the canonical map u_α , that is

$$\ker(u_\alpha \otimes_R Q) \subseteq \bigcap_{h \in H_{u_{\beta\alpha}}(B)} \ker(h \otimes_R Q)$$

for all left R -modules Q . Our assumption implies that

$$\ker(u_\alpha \otimes_R Q) \subseteq \ker(u_{\beta\alpha} \otimes_R Q)$$

for all $Q \in \mathcal{Q}$, so Theorem 5.1 gives the desired conclusion. \square

Before we continue our discussion of the general case, let us notice the following projectivity criteria for countably generated flat modules that improves [3, Proposition 2.5], and clarifies the proof of [16, Theorem 2.2].

Corollary 5.5. *Let M be a countably generated right flat module. Then M is projective if and only if M is R -stationary.*

Proof. To see that a countably generated projective module is R -stationary, use for example Theorem 3.11.

Assume that M is countably generated, flat and R -stationary. Then M is also $R^{(\mathbb{N})}$ -stationary by Corollary 3.9. Let $(F_\alpha, u_{\beta\alpha})_{\alpha, \beta \in I}$ be a direct system of finitely generated free modules such that $M = \varinjlim F_\alpha$. Notice that for each $\beta \in I$ we have a split monomorphism $t_\beta: F_\beta \rightarrow R^{(\mathbb{N})}$, hence $t_\beta \otimes Q$ is a split monomorphism for any left R -module Q . This implies that the criterion of Lemma 5.4 is fulfilled for any left R -module, hence M is a Mittag-Leffler module. Now we can conclude either by using [36, 2.2.2] or arguing that then M is R -Mittag-Leffler, hence countably presented by Corollary 5.5, and then use [3, Proposition 2.5]. \square

Example 5.6 ([27]). If \mathcal{Q} denotes the class of flat left R -modules, condition (5) in Theorem 5.1 is equivalent to:

- (5') For any finitely generated submodule M_0 of M there are a finitely presented module S and a homomorphism $w: M_0 \rightarrow S$ such that the embedding $\varepsilon: M_0 \rightarrow M$ factors through w .

We thus recover a characterization due to Goodearl of the modules that are Mittag-Leffler with respect to the class of flat modules [27, Theorem 1]. In particular, if R is right noetherian, then (5') is trivially satisfied, and so any right R -module is Mittag-Leffler with respect to the class of flat modules (cf. [27]). See Example 9.16 for an alternative proof and for related results.

6. RELATING \mathcal{B} -STATIONARY AND \mathcal{Q} -MITTAG-LEFFLER MODULES

Throughout this section, we fix a right R -module M together with a direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ such that $M = \varinjlim F_\alpha$.

Lemma 6.1. Let \mathcal{B} be a class of right R -modules closed under direct sums, and let \mathcal{Q} be a class of left R -modules. Assume that M is \mathcal{B} -stationary. If for any pair $\alpha, \beta \in I$ with $\beta \geq \alpha$ and for any $Q \in \mathcal{Q}$ there exists $B = B_{\beta\alpha}(Q) \in \mathcal{B}$ such that

$$\ker(u_{\beta\alpha} \otimes_R Q) = \bigcap_{h \in H_{u_{\beta\alpha}}(B)} \ker(h \otimes_R Q),$$

then M is a \mathcal{Q} -Mittag-Leffler module.

Proof. Let $\mathcal{Q}' = \{Q_k\}_{k \in K}$ be any family of modules in \mathcal{Q} . To prove the statement, we verify that \mathcal{Q}' satisfies the assumption of Lemma 5.4 for

$$B = \bigoplus_{Q \in \mathcal{Q}'} \bigoplus_{\alpha, \beta \in I, \beta \geq \alpha} B_{\beta\alpha}(Q) \in \mathcal{B}.$$

By hypothesis and by the construction of B , if we fix a pair $\alpha, \beta \in I$ with $\beta \geq \alpha \in I$, then for all $Q \in \mathcal{Q}'$

$$\ker(u_{\beta\alpha} \otimes_R Q) = \bigcap_{h \in H_{u_{\beta\alpha}}(B)} \ker(h \otimes_R Q).$$

As M is B -stationary, we conclude from Lemma 5.4 that M is \mathcal{Q}' -Mittag-Leffler. \square

Proposition 6.2. Let \mathcal{B} be a class of right R -modules closed under direct sums, and let \mathcal{Q} be a class of left R -modules. Assume that M is \mathcal{B} -stationary. If for every $Q \in \mathcal{Q}$ and every $\alpha \in I$ there exists a map $f_\alpha: F_\alpha \rightarrow B_\alpha$ such that $B_\alpha \in \mathcal{B}$ and $f_\alpha \otimes_R Q$ is a monomorphism, then M is a \mathcal{Q} -Mittag-Leffler module.

Proof. We verify the condition in Lemma 6.1. Let $\beta \geq \alpha$ in I and $Q \in \mathcal{Q}$. By hypothesis, there is $f_\beta: F_\beta \rightarrow B_\beta \in \mathcal{B}$ such that $f_\beta \otimes_R Q$ is a monomorphism. Set $h_\beta = f_\beta u_{\beta\alpha}$. Then $h_\beta \in H_{u_{\beta\alpha}}(B_\beta)$, so

$$\bigcap_{h \in H_{u_{\beta\alpha}}(B_\beta)} \ker(h \otimes_R Q) \subseteq \ker(h_\beta \otimes_R Q) = \ker(u_{\beta\alpha} \otimes_R Q)$$

and the reverse inclusion is always true. \square

We have seen several conditions implying that a \mathcal{B} -stationary module is \mathcal{Q} -Mittag-Leffler. Let us now discuss the reverse implication. We will need the following notion.

Definition 6.3. Let \mathcal{B} be a class of right R -modules, and let A be a right R -module. A morphism $f \in \text{Hom}_R(A, B)$ with $B \in \mathcal{B}$ is a \mathcal{B} -preenvelope (or a left \mathcal{B} -approximation) of A provided that the abelian group homomorphism $\text{Hom}_R(f, B'): \text{Hom}_R(B, B') \rightarrow \text{Hom}_R(A, B')$ is surjective for each $B' \in \mathcal{B}$.

Lemma 6.4. *Let \mathcal{B} be a class of right R -modules, and let $u: M \rightarrow N$ and $v: M \rightarrow M'$ be right R -module homomorphisms. Assume that M' has a \mathcal{B} -preenvelope $f: M' \rightarrow B$. Consider the following statements.*

- (1) v \mathcal{B} -dominates u .
- (2) $\ker(u \otimes_R B^*) \subseteq \ker(fv \otimes_R B^*)$.
- (3) $\ker(u \otimes_R B^*) \subseteq \ker(v \otimes_R B^*)$.

Statements (1) and (2) are equivalent, and statement (3) implies (1) and (2). Moreover, if there is a class of left R -modules \mathcal{Q} such that the character module $B^ \in \mathcal{Q}$ and $f \otimes_R Q$ is a monomorphism for all $Q \in \mathcal{Q}$, then all three statements are equivalent to*

- (4) v dominates u with respect to \mathcal{Q} .

Proof. (1) \Rightarrow (2). Since $\tilde{h} = fv \in H_v(B)$, we have $\ker(u \otimes_R Q) \subseteq \ker(\tilde{h} \otimes_R Q)$ for all left R -modules Q , so in particular for $Q = B^*$.

(2) \Rightarrow (1). By 4.5 we have $\ker(u \otimes_R Q) \subseteq \ker(\tilde{h} \otimes_R Q)$ for all left R -modules Q . Let $B' \in \mathcal{B}$. Since every $h \in H_v(B')$ factors through \tilde{h} , we further have $\ker(\tilde{h} \otimes_R Q) \subseteq \ker(h \otimes_R Q)$ for all $h \in H_v(B')$ and all Q , hence (1) holds true.

(3) \Rightarrow (2) holds true because $\ker(v \otimes_R B^*) \subseteq \ker(fv \otimes_R B^*)$.

Assume now that $B^* \in \mathcal{Q}$ and $f \otimes_R Q$ is a monomorphism for all $Q \in \mathcal{Q}$. We prove (1) \Rightarrow (4). As above we see $\ker(u \otimes_R Q) \subseteq \ker(\tilde{h} \otimes_R Q)$ for all left R -modules Q . Moreover, if $Q \in \mathcal{Q}$, then $\ker(\tilde{h} \otimes_R Q) = \ker(v \otimes_R Q)$, which yields (4).

Finally, (4) \Rightarrow (3) as $B^* \in \mathcal{Q}$. □

Proposition 6.5. *Let \mathcal{B} be a class of right R -modules. Assume that for every $\alpha \in I$ there exists a \mathcal{B} -preenvelope $f_\alpha: F_\alpha \rightarrow B_\alpha$. Assume further that M is Q -Mittag-Leffler for*

$$Q = \bigoplus_{\alpha \in I} B_\alpha^*.$$

Then M is \mathcal{B} -stationary.

Proof. For any $\alpha \in I$, denote by $u_\alpha: F_\alpha \rightarrow M$ the canonical map. By Theorem 4.8 we must show that there exists $\beta \geq \alpha$ such that $u_{\beta\alpha}$ \mathcal{B} -dominates u_α , which means $\ker(u_\alpha \otimes_R B_\beta^*) \subseteq \ker(u_{\beta\alpha} \otimes_R B_\beta^*)$ by Lemma 6.4. So, it is enough to find $\beta \geq \alpha$ such that

$$\ker(u_\alpha \otimes_R Q) \subseteq \ker(u_{\beta\alpha} \otimes_R Q).$$

To this end, we take a generating set $(x_k)_{k \in K}$ of $\ker(u_\alpha \otimes_R Q)$ and consider the diagram

$$\begin{array}{ccc} F_\alpha \otimes \prod_{k \in K} Q & \xrightarrow{u_\alpha \otimes \prod_{k \in K} Q} & M \otimes \prod_{k \in K} Q \\ \rho_\alpha \downarrow & & \downarrow \rho \\ \prod_{k \in K} (F_\alpha \otimes_R Q) & \xrightarrow{\prod_{k \in K} (u_\alpha \otimes Q)} & \prod_{k \in K} (M \otimes_R Q) \end{array}.$$

Since ρ_α is an isomorphism, there is $x \in F_\alpha \otimes \prod_{k \in K} Q$ such that $\rho_\alpha(x) = (x_k)_{k \in K} \in \prod_{k \in K} (F_\alpha \otimes_R Q)$. Then $(u_\alpha \otimes \prod_{k \in K} Q)(x) = 0$ because ρ is injective, and thus $x \in \ker(u_{\beta\alpha} \otimes_R \prod_{k \in K} Q)$ for some $\beta \geq \alpha$. From the diagram

$$\begin{array}{ccc} F_\alpha \otimes \prod_{k \in K} Q & \xrightarrow{u_{\beta\alpha} \otimes \prod_{k \in K} Q} & F_\beta \otimes \prod_{k \in K} Q \\ \rho_\alpha \downarrow & & \downarrow \rho_\beta \\ \prod_{k \in K} (F_\alpha \otimes_R Q) & \xrightarrow{\prod_{k \in K} (u_{\beta\alpha} \otimes Q)} & \prod_{k \in K} (F_\beta \otimes_R Q) \end{array}$$

we deduce that $(x_k)_{k \in K} = \rho_\alpha(x) \in \ker \prod_{k \in K} (u_{\beta\alpha} \otimes_R Q)$, that is,

$$x_k \in \ker(u_{\beta\alpha} \otimes_R Q) \quad \text{for all } k \in K,$$

and we conclude $\ker(u_\alpha \otimes_R Q) \subseteq \ker(u_{\beta\alpha} \otimes_R Q)$. \square

The previous observations are subsumed in the following result.

Theorem 6.6. *Let \mathcal{B} be a class of right R -modules closed under direct sums, and let \mathcal{Q} be a class of left R -modules. Assume that for every finitely presented module F there exists a \mathcal{B} -preenvelope $f: F \rightarrow B$ such that the character module $B^* \in \mathcal{Q}$ and $f \otimes_R Q$ is a monomorphism for all $Q \in \mathcal{Q}$. Then the following statements are equivalent for a right R -module M .*

- (1) M is \mathcal{B} -stationary.
- (2) M is \mathcal{Q} -Mittag-Leffler for all $Q \in \text{Add } \mathcal{Q}$.
- (3) M is \mathcal{Q} -Mittag-Leffler.

Proof. (1) implies (3) by Proposition 6.2, (2) implies (1) by Proposition 6.5, and (3) implies (2) by Theorem 1.3. \square

Example 6.7. Let B be a right R -module with the property that all finite matrix subgroups of B are finitely generated over the endomorphism ring of B , for example an endonoetherian module. Then a right R -module M is B^* -Mittag-Leffler if and only if it is B -stationary.

Proof. Set $\mathcal{B} = \text{Add } B$ and $\mathcal{Q} = \text{Add } B^*$. By Theorem 1.3 and Corollary 3.9 we know that M is B^* -Mittag-Leffler if and only if it is \mathcal{Q} -Mittag-Leffler, and M is B -stationary if and only if it is \mathcal{B} -stationary. Moreover, by [1, 3.1] every finitely presented module F has a \mathcal{B} -preenvelope $f: F \rightarrow B'$ with $B' \in \text{add } B$, hence $(B')^* \in \mathcal{Q}$. Finally, $\text{Hom}_R(f, B)$ is an epimorphism, thus applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ and using $\text{Hom} \otimes$ -adjointness, we see that $f \otimes_R Q$ is a monomorphism for all $Q \in \mathcal{Q}$. So the claim follows immediately from Theorem 6.6. \square

Definition 6.8 ([15, 2.3]). Let R be a ring. A full subcategory \mathcal{C} of $\text{Mod-}R$ is said to be *definable* if it is closed under pure submodules, direct limits and products.

To each definable category \mathcal{C} of right (left) R -modules we associate a *dual definable category* of left (right) R -modules \mathcal{C}^\vee which is determined by the property that a module M belongs to \mathcal{C} if and only if its character module $M^* \in \mathcal{C}^\vee$. This assignment defines a bijection between definable subcategories of $\text{Mod-}R$ and of $R\text{-Mod}$. For details, we refer to [31, Section 5] and [33, Section 4.2].

We now observe that when \mathcal{B} is definable, the class \mathcal{Q} in Theorem 6.6 can be taken to be \mathcal{B}^\vee . Hereby we recall that definable categories are always preenveloping by [35, Proposition 2.8, Theorem 3.3].

Corollary 6.9. *Let \mathcal{B} be a definable subcategory of $\text{Mod-}R$, and let $\mathcal{Q} = \mathcal{B}^\vee$ be the dual definable category. Let further S be a class of finitely presented right R -modules. Assume that for any $F \in S$ there exists a \mathcal{B} -preenvelope $f: F \rightarrow B$ such that $f \otimes_R Q$ is a monomorphism for all $Q \in \mathcal{Q}$. Then the following statements are equivalent for a right R -module $M \in \varinjlim S$.*

- (1) M is \mathcal{B} -stationary.
- (2) M is \mathcal{Q} -Mittag-Leffler for all $Q \in \mathcal{Q}$.
- (3) M is \mathcal{Q} -Mittag-Leffler.

Proof. Proceed as in Theorem 6.6, keeping in mind that if $B \in \mathcal{B}$, then $B^* \in \mathcal{Q}$. \square

Further applications of Theorem 6.6 are given in Section 9.

7. BAER MODULES

Throughout this section, let R be a commutative domain. A module M is said to be a *Baer module* if $\text{Ext}_R^1(M, T) = 0$ for any torsion R -module T .

Kaplansky in 1962 proposed the question whether the only Baer modules are the projective modules. He was inspired by the analogous question raised by Baer for the case of abelian groups, which was solved by Griffith in 1968.

In the general case of domains, a positive answer to Kaplansky's question was recently given by the authors in joint work with S. Bazzoni [3]. The proof uses an important result of Eklof, Fuchs and Shelah from 1990 which reduces the

problem to showing that countably presented Baer modules are projective (cf. [25, Theorem 8.22]).

Aim of this section is to give a new proof for the fact that every countably generated Baer module is projective, which uses our previous results. In fact, we are going to see that a countably generated Baer module is Mittag-Leffler. Then the result follows because countably generated flat Mittag-Leffler modules are projective (cf. [36, Corollaire 2.2.2, p. 74]).

So, let us consider a countably presented Baer module M . By Kaplansky's work, we know that M is flat, of projective dimension at most one. So M can be written as a direct system of the form

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3 \rightarrow \cdots \rightarrow F_n \xrightarrow{f_n} F_{n+1} \rightarrow \cdots$$

where, for each $n \geq 1$, F_n is a finitely generated free R -module. As the class of torsion modules is closed under direct sums, it follows from [12], see Theorem 3.11, that M is a Baer module if and only if for any torsion module T the inverse system $(\text{Hom}_R(F_n, T), \text{Hom}_R(f_n, T))$ satisfies the Mittag-Leffler condition, in other words, if and only if M is T -stationary.

Lemma 7.1. *Let R be a commutative domain.*

- (i) *Let Q be a finitely generated torsion module. For any $n \geq 0$ there exist a torsion module T and a homomorphism $h: R^n \rightarrow T$ such that $h \otimes Q$ is injective.*
- (ii) *Let Q be a finitely generated torsion-free R -module. For any $n \geq 0$ there exists a torsion module T such that*

$$\bigcap_{h \in \text{Hom}(R^n, T)} \ker(h \otimes Q) = 0.$$

Proof. (i). If $n = 0$, the claim is trivial (we assume that 0 is torsion). Fix $n \geq 1$. As Q is finitely generated and torsion, $I = \text{ann}_R(Q)$ is a nonzero ideal of R , so that $T = (R/I)^n$ is a torsion module. Note that $R/I \otimes Q \cong Q/IQ = Q$. Hence the canonical projection $R^n \rightarrow T$ satisfies the desired properties.

(ii). First we show that for any $0 \neq x \in R^n \otimes Q$ there exist a torsion module T_x and $h: R^n \rightarrow T_x$ such that $(h \otimes Q)(x) \neq 0$. Let K denote the field of quotients of R . As Q is torsion-free and finitely generated, it can be identified with a finitely generated submodule of K^m for some m . Moreover, as Q is finitely generated, multiplying by a suitable nonzero element of R , we can assume that $Q \leq R^m$.

The claim is trivial for $n = 0$. Fix $n \geq 1$. As $0 \neq x \in R^n \otimes Q = Q^n$, there is $i \in \{1, \dots, n\}$ such that the i -th component of x is nonzero. Let $\pi_i: R^n \rightarrow R$ denote the projection on the i -th component. As $(\pi_i \otimes Q)(x) \neq 0$, we only need to prove the statement for $n = 1$.

Set $x = (x_1, \dots, x_m) \in R \otimes Q = Q \subseteq R^m$. Let $j \in \{1, \dots, m\}$ be such that $x_j \neq 0$. As R is a domain, there exists $0 \neq t \in R$ such that $tR \not\subseteq x_j R$. Hence

$x \notin tQ \subseteq tR^m$. That is, if $p: R \rightarrow R/tR$ denotes the canonical projection, x is not in $tQ = \ker(Q \xrightarrow{p \otimes Q} R/tR \otimes Q)$.

To prove statement (ii) take $T = \bigoplus_{x \in R^n \otimes Q \setminus \{0\}} T_x$. \square

Theorem 7.2. *If R is a commutative domain, then any countably presented Baer module over R is Mittag-Leffler. Therefore any Baer module is projective.*

Proof. Let M_R be a countably presented Baer module. Then M_R is flat, hence a direct limit of finitely generated free modules.

Denote by \mathcal{T} and \mathcal{F} the classes of torsion and torsion-free modules, respectively. By Theorem 3.11, M_R is \mathcal{T} -stationary. Since \mathcal{T} is closed under direct sums, the previous Lemma 7.1(i), together with Proposition 6.2, implies that M is \mathcal{Q} -Mittag-Leffler where \mathcal{Q} is the class of all finitely generated modules from \mathcal{T} . By Theorem 1.3, it follows that M is \mathcal{T} -Mittag-Leffler.

Next, we show that M is also \mathcal{F} -Mittag-Leffler. Again by Theorem 1.3, it is enough to show that M is Mittag-Leffler with respect to the class of finitely generated torsion-free modules. So, let Q be a finitely generated torsion-free module, and let $u \in \text{Hom}_R(F, F')$ be a morphism between finitely generated free modules F, F' . By Lemma 7.1(ii) there exists a torsion module T such that $\bigcap_{h \in \text{Hom}(F', T)} \ker(h \otimes Q) = 0$. So, if $x \in F \otimes_R Q$ and $y = (u \otimes_R Q)(x) \neq 0$, then there must exist $h' \in \text{Hom}_R(F', T)$ such that $(h' \otimes_R Q)(y) \neq 0$, which means $(h'u \otimes_R Q)(x) \neq 0$ and shows that $x \notin \bigcap_{h \in H_u(T)} \ker(h \otimes_R Q)$. Thus we deduce that $\ker(u \otimes_R Q) = \bigcap_{h \in H_u(T)} \ker(h \otimes_R Q)$. Our claim then follows from Lemma 6.1.

Since M is flat, we now conclude from Corollary 1.5 that M is Mittag-Leffler and thus projective. \square

8. MATRIX SUBGROUPS

In [36] Raynaud and Gruson also studied modules satisfying a stronger condition, which they called *strict Mittag-Leffler modules*. In this section, we investigate the relative version of this condition and interpret it in terms of matrix subgroups. Hereby we establish a relationship with work of W. Zimmermann [40].

We start out with a stronger version of Proposition 4.4.

Proposition 8.1. *Let B be a right R -module, and let S be a class of finitely presented right R -modules. For a right R -module $M \in \varinjlim S$, the following statements are equivalent.*

- (1) *There is a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ having the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that the canonical map $u_\alpha: F_\alpha \rightarrow M$ satisfies $H_{u_\alpha}(B) = H_{u_{\beta\alpha}}(B)$.*
- (2) *Every direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ has the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that the canonical map $u_\alpha: F_\alpha \rightarrow M$ satisfies $H_{u_\alpha}(B) = H_{u_{\beta\alpha}}(B)$.*

- (3) For any finitely presented module F (belonging to S) and any homomorphism $u: F \rightarrow M$ there exist a module $S \in S$ and a homomorphism $v: F \rightarrow S$ such that u factors through v , and $H_u(B) = H_v(B)$.

Proof. Adapt the proof of Proposition 4.4 replacing Lemma 4.2 by Lemma 2.3(2) and Lemma 2.5(1). \square

In view of the characterization of B -stationary modules given in Theorem 4.8, we introduce the following terminology.

Definition 8.2. Let B be a right R -module. We say that a right R -module M is *strict B -stationary* if it satisfies the equivalent conditions of Proposition 8.1 (for some class of finitely presented modules S).

If \mathcal{B} is a class of right R -modules, then we say that M is *strict \mathcal{B} -stationary* if it is strict B -stationary for every $B \in \mathcal{B}$.

Remark 8.3.

- (1) The modules that are strict B -stationary for every right R -module B are exactly the strict Mittag-Leffler modules of [36]. In particular, every pure-projective module is strict $\text{Mod}R$ -stationary, see [36, 2.3.3].
- (2) By Proposition 4.6 and Theorem 4.8, every strict B -stationary module is B -stationary.
- (3) By Lemma 3.3 a countably presented module is strict B -stationary if and only if it is B -stationary.

The class \mathcal{B} in the definition of a strict \mathcal{B} -stationary module enjoys slightly weaker closure properties with respect to the class \mathcal{B} in the definition of \mathcal{B} -stationary modules.

Proposition 8.4. Let $\{B_j\}_{j \in J}$ be a family of right R -modules. Let $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ be a direct system of finitely presented right R -modules and $M = \varinjlim F_\alpha$. Then the following statements are equivalent.

- (1) M is strict $\prod_{j \in J} B_j$ -stationary.
- (2) For any $\alpha \in I$ there exists $\beta \geq \alpha$ such that the canonical map $u_\alpha: F_\alpha \rightarrow M$ satisfies $H_{u_\alpha}(B_j) = H_{u_{\beta\alpha}}(B_j)$ for any $j \in J$.
- (3) M is strict $\bigoplus_{j \in J} B_j$ -stationary.

Proof. Proceed as in Proposition 3.8, cf. Remark 2.2. \square

Corollary 8.5. Let \mathcal{B} be a class of right R -modules. Let M be a strict \mathcal{B} -stationary right R -module. Then the following statements hold true.

- (i) M is strict \mathcal{B}' -stationary where \mathcal{B}' denotes the class of all modules isomorphic either to a locally split submodule or to a pure quotient of a module in \mathcal{B} .
- (ii) M is strict $\text{Add } \mathcal{B}$ -stationary if and only if it is strict $\text{Prod } \mathcal{B}$ -stationary if and only if there exists a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $\varinjlim F_\alpha \cong M$ having the property that for any $\alpha \in I$ there exists $\beta \geq \alpha$ such that the canonical map $u_\alpha: F_\alpha \rightarrow M$ satisfies $H_{u_\alpha}(B) = H_{u_{\beta\alpha}}(B)$ for any $B \in \mathcal{B}$.

(iii) M is strict Add B - and strict Prod B -stationary for every $B \in \mathcal{B}$.

Proof. Adapt the proof of Corollary 3.9 replacing Lemma 2.5(2) by Lemma 2.5(4), and Proposition 3.8 by Proposition 8.4. \square

We now recall some notions from [40].

Definition 8.6. Given two right R -modules A, B , an integer $n \in \mathbb{N}$, and an element $\underline{a} = (a_1, \dots, a_n) \in A^n$, we consider the End B -linear map

$$\varepsilon_{\underline{a}} : \text{Hom}_R(A, B) \rightarrow B^n, \quad f \mapsto f(\underline{a}) = (f(a_1), \dots, f(a_n))$$

and define

$$H_{A, \underline{a}}(B) = \text{Im } \varepsilon_{\underline{a}} = \{f(\underline{a}) \mid f \in \text{Hom}_R(A, B)\}.$$

If $n = 1$, then $H_{A, \underline{a}}(B) = H_{A, a}(B)$ is called a *matrix subgroup* of B , and it is called a *finite matrix subgroup* if the module A is finitely presented.

The subgroups $H_{A, \underline{a}}(B)$ are related to H -subgroups as follows.

Lemma 8.7. Let A, B, M be right R -modules, $n \in \mathbb{N}$, and $\underline{a} = (a_1, \dots, a_n) \in A^n$.

- (1) If a_1, \dots, a_n is a generating set of A , then $\varepsilon_{\underline{a}}$ is a monomorphism.
- (2) If $v \in \text{Hom}_R(A, M)$, then $\varepsilon_{\underline{a}}(H_v(B)) = H_{M, \underline{m}}(B)$ where $\underline{m} = v(\underline{a}) \in M^n$.
- (3) If $\underline{m} = (m_1, \dots, m_n) \in M^n$, then $H_{M, \underline{m}}(B) = \varepsilon_{\underline{e}}(H_u(B))$ where $\underline{e} = (e_1, \dots, e_n)$ is given by the canonical basis e_1, \dots, e_n of the free module R^n , and $u : R^n \rightarrow M$ is defined by $u(r_1, \dots, r_n) = \sum_{i=1}^n m_i r_i$.

Proof. The proof is left to the reader. \square

We will need some further terminology from [40].

Definition 8.8. Let $n \in \mathbb{N}$. A pair (A, \underline{a}) consisting of a right R -module A and an element $\underline{a} = (a_1, \dots, a_n) \in A^n$ will be called an *n -pointed module*. A morphism of *n -pointed modules* $h : (A, \underline{a}) \rightarrow (M, \underline{m})$ is an R -module homomorphism $h : A \rightarrow M$ such that $h(\underline{a}) = \underline{m}$.

Consider now a direct system of right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with direct limit $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ and canonical maps $u_\alpha : F_\alpha \rightarrow M$. If for every $\alpha \in I$ the elements $\underline{x}_\alpha \in F_\alpha^n$ are chosen in such a way that the $u_{\beta\alpha} : (F_\alpha, \underline{x}_\alpha) \rightarrow (F_\beta, \underline{x}_\beta)$ are morphisms of n -pointed modules, then $((F_\alpha, \underline{x}_\alpha), u_{\beta\alpha})_{\beta, \alpha \in I}$ is called a *direct system of n -pointed modules*. Setting $\underline{m} = u_\alpha(\underline{x}_\alpha)$ for some $\alpha \in I$, we have that also the $u_\alpha : (F_\alpha, \underline{x}_\alpha) \rightarrow (M, \underline{m})$ are morphisms of n -pointed modules. We then write $(M, \underline{m}) = \varinjlim (F_\alpha, \underline{x}_\alpha)$.

We now show that the strict B -stationary modules are precisely the modules studied by Zimmermann in [40, 3.2]. Like the Mittag-Leffler modules, they can be characterized in terms of the injectivity of a natural transformation.

Let ${}_S B_R$ be an S - R -bimodule, and let ${}_S V$ be a left S -module. For any right R -module M_R there is a natural transformation

$$\nu = \nu(M, B, V) : M \otimes_R \operatorname{Hom}_S(B, V) \rightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), V)$$

defined by

$$\nu(m \otimes \varphi) : f \mapsto \varphi(f(m)).$$

Notice that when $B = V$ and ${}_S B_R$ is faithfully balanced, then ν is induced by the evaluation map of M inside its bidual.

If M_R is finitely presented and ${}_S V$ is injective, then ν is an isomorphism (cf. [20, Theorem 3.2.11]). The case that ν is a monomorphism for all injective modules ${}_S V$ was studied by Zimmermann in [40, 3.2]. We are going to see below that this happens precisely when M is strict B -stationary. So, like Q -Mittag-Leffler modules, strict B -stationary modules can be characterized in terms of the injectivity of a natural transformation, in this case the injectivity of ν . Let us first discuss how the injectivity of ν behaves under direct sums.

If $M = \bigoplus_{i \in I} M_i$ then, for each $i \in I$, the canonical inclusion $M_i \rightarrow M$ induces an inclusion

$$\operatorname{Hom}_S(\operatorname{Hom}_R(M_i, B), V) \rightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), V).$$

This family of inclusions induces an injective map

$$\Phi : \bigoplus_{i \in I} \operatorname{Hom}_S(\operatorname{Hom}_R(M_i, B), V) \rightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), V)$$

given by the rule

$$\Phi((g_i)_{i \in I_0}) : f \mapsto \sum_{i \in I_0} g_i(f|_{M_i}).$$

This allows us to deduce the following result.

Lemma 8.9. *The map $\nu(\bigoplus_{i \in I} M_i, B, V)$ is injective if and only if, for any $i \in I$, $\nu(M_i, B, V)$ is injective.*

We observe the following relationship between the injectivity of the natural transformations ν and ρ .

Lemma 8.10. *Let M be a right R -module, let ${}_S B_R$ be a bimodule, and let $\{V_i\}_{i \in I}$ be a family of left S -modules. Then the map $\nu = \nu(M, B, \prod_{i \in I} V_i)$ is injective if and only if so are all maps $\nu_i = \nu(M, B, V_i)$, $i \in I$, and the map*

$$\rho : M \otimes_R \prod_{i \in I} \operatorname{Hom}_S(B, V_i) \rightarrow \prod_{i \in I} M \otimes_R \operatorname{Hom}_S(B, V_i).$$

Proof. The claim follows from the commutativity of the diagram

$$\begin{array}{ccc}
 M \otimes_R \operatorname{Hom}_S \left(B, \prod_{i \in I} V_i \right) & \xrightarrow{\quad \nu \quad} & \operatorname{Hom}_S \left(\operatorname{Hom}_R(M, B), \prod_{i \in I} V_i \right) \\
 \downarrow \cong & & \uparrow \cong \\
 & & \prod_{i \in I} \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), V_i) \\
 & & \uparrow (v_i)_{i \in I} \\
 M \otimes_R \prod_{i \in I} \operatorname{Hom}_S(B, V_i) & \xrightarrow{\quad \rho \quad} & \prod_{i \in I} M \otimes_R \operatorname{Hom}_S(B, V_i)
 \end{array}$$

using that the v_i are injective if so is ν because V_i is a direct summand of $\prod_{i \in I} V_i$. \square

Theorem 8.11. *Let B and M be right R -modules. The following statements are equivalent.*

- (1) M is strict B -stationary.
- (2) For every $n \in \mathbb{N}$, every element $\underline{m} \in M^n$ and every direct system of n -pointed modules $((F_\alpha, \underline{x}_\alpha), (u_{\beta\alpha})_{\beta, \alpha \in I})$ with all F_α being finitely presented and $(M, \underline{m}) = \varinjlim (F_\alpha, \underline{x}_\alpha)$ there is $\beta \in I$ such that $H_{M, \underline{m}}(B) = H_{F_\beta, \underline{x}_\beta}(B)$.
- (3) For every $n \in \mathbb{N}$ and every element $\underline{m} \in M^n$ there are a finitely presented right R -module A , an element $\underline{a} \in A^n$ and a morphism of n -pointed modules $h : (A, \underline{a}) \rightarrow (M, \underline{m})$ such that $H_{M, \underline{m}}(B) = H_{A, \underline{a}}(B)$.

Let S be a ring such that ${}_S B_R$ is a bimodule, and let ${}_S U$ be an injective cogenerator of $S\text{-Mod}$. Then the following statements are further equivalent.

- (4) For every injective left S -module V , the canonical map

$$\nu : M \otimes_R \operatorname{Hom}_S(B, V) \rightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), V)$$

defined by $\nu(m \otimes \varphi) : f \mapsto \varphi(f(m))$ is a monomorphism.

- (5) For any set I , the canonical map

$$\nu : M \otimes_R \operatorname{Hom}_S(B, U^I) \rightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), U^I)$$

is a monomorphism.

- (6) The canonical map

$$\nu : M \otimes_R \operatorname{Hom}_S(B, U) \rightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, B), U)$$

is a monomorphism and M is $\operatorname{Hom}_S(B, U)$ -Mittag-Leffler.

Proof. The equivalence of (2), (3) and (4) is due to Zimmermann, see [40, 3.2].

(1) \Rightarrow (3). As in Lemma 8.7(3), we write $H_{M,\underline{m}}(B) = \varepsilon_{\underline{e}}(H_u(B))$ where $\underline{e} = (e_1, \dots, e_n)$ is given by the canonical basis e_1, \dots, e_n of the free module R^n , and $u : R^n \rightarrow M$, $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n m_i r_i$. By assumption there exist a finitely presented module A , a homomorphism $v : R^n \rightarrow A$ and a homomorphism $h : A \rightarrow M$ such that $u = hv$, and $H_u(B) = H_v(B)$. Set $\underline{a} = v(\underline{e}) \in A^n$. Then $h(\underline{a}) = u(\underline{e}) = \underline{m}$. So, we obtain a morphism of n -pointed modules $h : (A, \underline{a}) \rightarrow (M, \underline{m})$. Moreover, we see as in Lemma 8.7(2) that $H_{M,\underline{m}}(B) = \varepsilon_{\underline{e}}(H_v(B)) = H_{A,\underline{a}}(B)$.

(2) \Rightarrow (1). Take a direct system of finitely presented right R -modules $(F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$ with $M = \varinjlim (F_\alpha, u_{\beta\alpha})_{\beta, \alpha \in I}$, and fix $\alpha \in I$. Moreover, choose a generating set a_1, \dots, a_n of F_α , and set $\underline{x}_\alpha = (a_1, \dots, a_n)$. Set further $\underline{x}_\beta = u_{\beta\alpha}(\underline{x}_\alpha)$ for all $\beta \geq \alpha$, and $\underline{m} = u_\alpha(\underline{x}_\alpha)$. Then $(F_\beta, \underline{x}_\beta)_{\beta \in I, \beta \geq \alpha}$ is a direct system of n -pointed modules with direct limit (M, \underline{m}) . So, by assumption there is $\beta \in I$ such that $H_{M,\underline{m}}(B) = H_{F_\beta, \underline{x}_\beta}(B)$. By Lemma 8.7(2) this means $\varepsilon_{\underline{x}_\alpha}(H_{u_\alpha}(B)) = \varepsilon_{\underline{x}_\alpha}(H_{u_{\beta\alpha}}(B))$. Since $\varepsilon_{\underline{x}_\alpha}$ is a monomorphism, we conclude $H_{u_\alpha}(B) = H_{u_{\beta\alpha}}(B)$.

(4) \Leftrightarrow (5). Since products of injective modules are injective, statement (4) implies (5). As ${}_S U$ is a cogenerator, any injective left S -module V is a direct summand of U^I for a suitable set I . Since $v(M, B, U^I)$ is injective, so is $v(M, B, V)$, and we conclude that (5) implies (4).

(5) \Rightarrow (6). Follows from Lemma 8.10. □

Here are some consequences of the previous theorem.

Corollary 8.12. *Let B be a right R -module.*

- (1) [40, 3.6] *The class of strict B -stationary modules is closed under pure submodules and pure extensions.*
- (2) *A direct sum of modules is strict B -stationary if and only if so are all direct summands.*
- (3) [40, 3.8] *The module B is Σ -pure-injective if and only if every right R -module is strict B -stationary.*

The characterization of strict B -stationary modules in terms of the injectivity of v allows us to provide a wide class of examples of such modules. It is the analog of the class discussed in Proposition 1.9.

Proposition 8.13. *Let S be a class of right R modules that is strict \mathcal{B} -stationary with respect to a class $\mathcal{B} \subseteq S^\perp$. Then any module isomorphic to a direct summand of an $S \cup \text{Add } R$ -filtered module is strict \mathcal{B} -stationary.*

Proof. As projective modules are strict $\text{Mod } R$ -stationary and $S^\perp = (S \cup \text{Add } R)^\perp$, so we can assume that S contains $\text{Add } R$. By Corollary 8.12, the class of strict \mathcal{B} -stationary modules is closed by direct summands. So, we only need to prove the statement for S -filtered modules. Also by Corollary 8.12, we know that arbitrary direct sums of modules in S are strict \mathcal{B} -stationary.

Let M be an S -filtered right R -module. Let τ be an ordinal such that there exists an S -filtration $(M_\alpha)_{\alpha \leq \tau}$ of M . Observe that for any $\beta \leq \alpha \leq \tau$, M_α and M_α/M_β are S -filtered modules, so they belong to ${}^\perp\mathcal{B}$ by [26, 3.1.2]. For the rest of the proof we fix $B \in \mathcal{B}$, a ring S such that ${}_S B_R$ is a bimodule, and an injective left S -module V . We shall show that M is strict \mathcal{B} -stationary proving by induction that, for any $\alpha \leq \tau$, the canonical map

$$\nu_\alpha: M_\alpha \otimes_R \text{Hom}_S(B, V) \rightarrow \text{Hom}_S(\text{Hom}_R(M_\alpha, B), V)$$

is injective.

As $M_0 = 0$, the claim is true for $\alpha = 0$. If $\alpha < \tau$, then the exact sequence

$$0 \rightarrow M_\alpha \rightarrow M_{\alpha+1} \rightarrow M_{\alpha+1}/M_\alpha \rightarrow 0$$

and the fact that $\mathcal{B} \subseteq S^\perp$ yield a commutative diagram with exact rows

$$\begin{array}{ccccccc} M_\alpha \otimes_R \text{Hom}_S(B, V) & \longrightarrow & M_{\alpha+1} \otimes_R \text{Hom}_S(B, V) & \longrightarrow & & & \\ \nu_\alpha \downarrow & & \nu_{\alpha+1} \downarrow & & & & \\ 0 \longrightarrow \text{Hom}_S(\text{Hom}_R(M_\alpha, B), V) & \longrightarrow & \text{Hom}_S(\text{Hom}_R(M_{\alpha+1}, B), V) & \longrightarrow & & & \\ & & & & \longrightarrow & (M_{\alpha+1}/M_\alpha) \otimes_R \text{Hom}_S(B, V) & \longrightarrow 0 \\ & & & & & \downarrow \nu & \\ & & & & & \longrightarrow \text{Hom}_S(\text{Hom}_R(M_{\alpha+1}/M_\alpha, B), V) & \longrightarrow 0 \end{array}$$

The natural map ν is injective because $M_{\alpha+1}/M_\alpha$ is strict \mathcal{B} -stationary. So, if ν_α is injective, then $\nu_{\alpha+1}$ is also injective.

Let $\alpha \leq \tau$ be a limit ordinal, and assume that ν_β is injective for any $\beta < \alpha$. We shall prove that ν_α is injective. Let $x \in \text{Ker } \nu_\alpha$. There exists $\beta < \alpha$ and $y \in M_\beta \otimes_R \text{Hom}_S(B, V)$ such that $x = (\varepsilon_\beta \otimes_R \text{Hom}_S(B, V))(y)$, where $\varepsilon_\beta: M_\beta \rightarrow M_\alpha$ denotes the canonical inclusion. As $M_\alpha/M_\beta \in {}^\perp\mathcal{B}$, ε_β induces an injective map $\varepsilon: \text{Hom}_S(\text{Hom}_R(M_\beta, B), V) \rightarrow \text{Hom}_S(\text{Hom}_R(M_\alpha, B), V)$. Considering the commutative diagram

$$\begin{array}{ccc} M_\beta \otimes_R \text{Hom}_S(B, V) & \xrightarrow{\varepsilon_\beta \otimes_R \text{Hom}_S(B, V)} & M_\alpha \otimes_R \text{Hom}_S(B, V) \\ \nu_\beta \downarrow & & \downarrow \nu_\alpha \\ \text{Hom}_S(\text{Hom}_R(M_\beta, B), V) & \xrightarrow{\varepsilon} & \text{Hom}_S(\text{Hom}_R(M_\alpha, B), V) \end{array}$$

we see that $0 = \varepsilon v_\beta(\gamma)$. As v_β and ε are injective, $\gamma = 0$. Therefore $x = 0$, and v_α is injective. \square

Next, we further investigate the relationship between relative Mittag-Leffler modules and strict stationary modules. It was shown by Azumaya [10, Proposition 8] that a module M is strict Mittag-Leffler if and only if every pure-epimorphism $X \rightarrow M$, $X \in \text{Mod-}R$, is locally split. We will now see that also the dual property plays an important role in this context. According to [42], we will say that a right R -module B is *locally pure-injective* if every pure-monomorphism $B \rightarrow X$, $X \in \text{Mod-}R$, is locally split.

Moreover, in the following, for a right R -module B , we will indicate by B^\bullet a left R -module which is obtained from B by some duality, that is, by taking a ring S such that ${}_S B_R$ is a bimodule together with an injective cogenerator ${}_S V$ of $S\text{-Mod}$, and setting ${}_R B^\bullet = \text{Hom}_S(B, V)$. For example, B^\bullet can be the character module B^* of B . But it can also be the *local dual* B^+ of B , which is obtained as above by choosing $S = \text{End}_R B$. For a left R -module C , the notation C^\bullet is used correspondingly.

Proposition 8.14. *Let M and B be right R -modules, and let C be a left R -module. The following statements hold true.*

- (1) [40, 3.3(1)] *M is a C -Mittag-Leffler module if and only if M is strict C^\bullet -stationary.*
- (2) *If M is strict B -stationary, then M is B^\bullet -Mittag-Leffler. The converse holds true if B is locally pure-injective.*

Proof. (2) The first part of the statement is shown by Zimmermann [40, 3.3(2)(a)] and it follows also from Theorem 8.11. For the converse, assume that B is locally pure-injective. Let ${}_R B^\bullet = \text{Hom}_S(B, V)$ where S is a ring such that ${}_S B_R$ is a bimodule and ${}_S V$ is an injective cogenerator of $S\text{-Mod}$. Consider a ring T such that ${}_S V_T$ is a bimodule, let U_T be an injective cogenerator of $\text{Mod-}T$, and assume without loss of generality that $V_T \subseteq U_T$. Then ${}_R B_T^\bullet$ is also a bimodule, and we can consider $B^{\bullet\bullet} = \text{Hom}_T(B^\bullet, U)$. By (1), M is strict $B^{\bullet\bullet}$ -stationary. Furthermore, the evaluation map $B \rightarrow B^{\bullet\bullet}$ is a pure monomorphism (see e.g. [43, 1.2(4)]), hence locally split. By Corollary 8.5(i) it follows that M is strict B -stationary. \square

Example 9.11 shows that the converse of Proposition 8.14 (2) is not true in general.

Example 8.15. Let B be a locally pure-injective right R -module with the property that all finite matrix subgroups of B are finitely generated over the endomorphism ring of B . Then a right R -module M is strict B -stationary if and only if it is B -stationary.

In particular, this applies to the case when B is a pure-projective right R -module over a left pure-semisimple ring R .

Proof. The statement follows by combining Example 6.7 with Proposition 8.14(2).

When R is a left pure-semisimple ring, all finitely presented right R -modules are endofinite [30]. Hence every pure-projective right R -module B is locally pure-injective by [42, 2.4], and endonoetherian by [44]. So, the assumptions are satisfied in this case. \square

Restricting to local duals, we can employ recent work of Dung and Garcia [18] to obtain a criterion for endofiniteness of finitely presented modules.

Proposition 8.16. *Let ${}_R C$ be a finitely presented left R -module. The following statements are equivalent.*

- (1) C is endofinite.
- (2) C is Σ -pure-injective, and C^+ is C -Mittag-Leffler.
- (3) C is Σ -pure-injective, and all cyclic $\text{End } C^+$ -submodules of C^+ are finite matrix subgroups.

Proof. (1) \Rightarrow (2). C is endofinite if and only if it satisfies the descending and the ascending chain condition on finite matrix subgroups. Hence C is Σ -pure-injective, and every right R -module is C -Mittag-Leffler, see Example 1.6(3).

(2) \Rightarrow (3). By Proposition 8.14 the module C^+ is strict C^+ -stationary. By condition (3) in Theorem 8.11 for $n = 1$, it follows that all matrix subgroups of C^+ of the form $H_{C^+, m}(C^+)$ with $m \in C^+$ are finite matrix subgroups, and of course, the matrix subgroups of such form are precisely the cyclic $\text{End } C^+$ -submodules of C^+ .

(3) \Rightarrow (1). Since C is Σ -pure-injective, the module C^+ satisfies the ascending chain condition on finite matrix subgroups, see [44, Proposition 3]. Furthermore, every finitely generated $\text{End } C^+$ -submodule of C^+ is a finite sum of cyclic submodules, hence a finite matrix subgroup, because the class of finite matrix subgroups is closed under finite sums, see e.g. [43, 2.5]. So, we conclude that C^+ satisfies the ascending chain condition on finitely generated $\text{End } C^+$ -submodules, in other words, C^+ is endonoetherian, see also [41]. Now the claim follows from [18, 4.2], where it is shown that a finitely presented module is endofinite provided its local dual is endonoetherian. \square

Using [17, 4.1], we obtain the following observation.

Corollary 8.17. *A left pure-semisimple ring has finite representation type if and only if the local dual C^+ of any finitely presented left R -module C is C -Mittag-Leffler.*

Before turning in more detail to pure-semisimple rings, let us apply our results to the following setting.

9. COTORSION PAIRS

In this section, we shall see that the theory of relative Mittag-Leffler modules and (strict) stationary modules fits very well into the theory of cotorsion pairs.

Definition 9.1.

- (1) Let $\mathcal{M}, \mathcal{L} \subseteq \text{Mod-}R$ be classes of modules. The pair $(\mathcal{M}, \mathcal{L})$ is said to be a *cotorsion pair* provided $\mathcal{M} = {}^\perp \mathcal{L}$ and $\mathcal{L} = \mathcal{M}^\perp$.
The cotorsion pair $(\mathcal{M}, \mathcal{L})$ is said to be *complete* if for every module X there are short exact sequences $0 \rightarrow X \rightarrow L \rightarrow M \rightarrow 0$ and $0 \rightarrow L' \rightarrow M' \rightarrow X \rightarrow 0$ where $L, L' \in \mathcal{L}$ and $M, M' \in \mathcal{M}$.
- (2) If S is a set of right R -modules, we obtain a cotorsion pair $(\mathcal{M}, \mathcal{L})$ by setting $\mathcal{L} = S^\perp$ and $\mathcal{M} = {}^\perp(S^\perp)$. It is called the cotorsion pair *generated*¹ by S , and it is a *complete* cotorsion pair (cf. [26, Theorem 3.2.1]).
- (3) We will say that a cotorsion pair $(\mathcal{M}, \mathcal{L})$ is of *finite type* provided it is generated by a set of modules $S \subseteq \text{mod-}R$. Note that we can always assume $S = \mathcal{M} \cap \text{mod-}R$.
- (4) Dually, if S is a set of right R -modules, we obtain a cotorsion pair $(\mathcal{M}, \mathcal{L})$ by setting $\mathcal{M} = {}^\perp S$ and $\mathcal{L} = ({}^\perp S)^\perp$. It is called the cotorsion pair *cogenerated* by S .

For more information on cotorsion pairs, we refer to [26].

In view of Theorem 6.6, complete cotorsion pairs provide a good setting for relative stationarity and Mittag-Leffler properties. As a first approach we give the following result.

Proposition 9.2. *Let $(\mathcal{M}, \mathcal{L})$ be a cotorsion pair in $\text{Mod-}R$. Set $C = \mathcal{M}^\perp$. Then the following hold true.*

- (1) *If $(\mathcal{M}, \mathcal{L})$ is complete and \mathcal{L} is closed by direct sums, then any \mathcal{L} -stationary right R -module is C -Mittag-Leffler.*
- (2) *If $(\mathcal{M}, \mathcal{L})$ is generated by (a set of) countably presented modules and \mathcal{L} is closed by direct sums, then any module in \mathcal{M} is strict \mathcal{L} -stationary (and thus C -Mittag-Leffler).*
- (3) *Assume that $(\mathcal{M}, \mathcal{L})$ is generated by a class S of finitely presented modules with the property that the first syzygy of any module in S is also finitely presented. Then a countably generated module M belongs to \mathcal{M} if and only if M belongs to $\varinjlim S$ and is (strict) \mathcal{L} -stationary.*

Proof. (1) The hypotheses in (1) imply that any right R -module X fits into an exact sequence $0 \rightarrow X \rightarrow L \rightarrow M \rightarrow 0$ where $L \in \mathcal{L}$ and $M \in \mathcal{M}$. Hence the statement follows from Proposition 6.2.

- (2) To prove (2) observe first that a countably presented module in \mathcal{M} is strict \mathcal{L} -stationary by Example 3.13(1) and Remark 8.3(3).

Now if $M \in \mathcal{M}$ then, by [26, Corollary 3.2.4], M is a direct summand of a module N filtered by countably presented modules in \mathcal{M} . By the observation above, N is filtered by strict \mathcal{L} -stationary modules. Then M is strict \mathcal{L} -stationary by Proposition 8.13. As the cotorsion pair is complete by Definition 9.1(2), we infer from (1) that M is also C -Mittag-Leffler.

¹This terminology differs from previous use, cf. [26].

- (3) By [8, 2.3] \mathcal{M} is included in $\varinjlim S$, and, by [20, Lemma 10.2.4], \mathcal{L} is closed under direct sums. So, the only-if part follows from (2). For the converse implication, let M be a countably generated module in $\varinjlim S$ which is \mathcal{L} -stationary. Statement (1) implies that M is C -Mittag-Leffler. As $R \in C$, we deduce from Corollary 5.3 that M is countably presented. Therefore, and because \mathcal{L} is closed under direct sums, we can apply Theorem 3.11 to conclude that $\text{Ext}_R^1(M, L) = 0$ for any $L \in \mathcal{L}$. Thus $M \in \mathcal{M}$. \square

From [39, 1.9] we immediately obtain the following consequence.

Example 9.3. Let R be an \aleph_0 -noetherian hereditary ring. If $(\mathcal{M}, \mathcal{L})$ is a cotorsion pair in $\text{Mod-}R$ such that \mathcal{L} is closed by direct sums, then any module in \mathcal{M} is strict \mathcal{L} -stationary (and thus C -Mittag-Leffler).

In the following results, we use again the notation B^\bullet to indicate a module obtained from B by some duality, like the character module, or the local dual of B . For a class of modules S , we write S^\bullet in order to indicate a class consisting of modules that are obtained by some duality from the modules of S . Note that we are not assuming a functorial relationship between S and S^\bullet .

Lemma 9.4. Let $(\mathcal{M}, \mathcal{L})$ be a cotorsion pair of finite type in $\text{Mod-}R$. Set $S = \mathcal{M} \cap \text{mod-}R$, and $C = \mathcal{M}^\top$. Then the following hold true.

- (1) $C = S^\top$, and ${}^\top C = \varinjlim S = \varinjlim \mathcal{M}$.
- (2) If $\mathcal{D} = C^\perp$, then $(\overline{C}, \overline{\mathcal{D}})$ is the cotorsion pair cogenerated by S^\bullet .
- (3) A right R -module B belongs to \mathcal{L} if and only if B^\bullet belongs to C .
- (4) A left R -module C belongs to C if and only if C^\bullet belongs to \mathcal{L} .
- (5) A right R -module B belongs to $\varinjlim S$ if and only if B^\bullet belongs to \mathcal{D} .
- (6) Assume that the cotorsion pair $(\overline{C}, \overline{\mathcal{D}})$ is of finite type. Then a left R -module X belongs to \mathcal{D} if and only if X^\bullet belongs to $\varinjlim S$.
- (7) If $\mathcal{E} = (\varinjlim S)^\perp$, then $(\varinjlim S, \mathcal{E})$ is the cotorsion pair cogenerated by the pure-injective modules from \mathcal{L} .
- (8) Let $f : N \rightarrow M$ be a monomorphism with $M \in \varinjlim S$. Then $f \otimes C$ is a monomorphism for all $C \in C$ if and only if $\text{coker } f \in \varinjlim S$.

Proof. (1) By [8, 2.3] we have $S \subseteq \mathcal{M} \subseteq \varinjlim S = {}^\top(S^\top)$, hence $C = \mathcal{M}^\top = S^\top$, and ${}^\top C = \varinjlim S$. By the well-known Ext-Tor relations we further obtain $C = {}^\perp(S^\bullet)$, hence (2), and also statements (3)–(5).

For statement (6), we assume that the cotorsion pair $(\overline{C}, \overline{\mathcal{D}})$ is of finite type. Then $X \in \mathcal{D}$ if and only if $\text{Ext}(C, X) = 0$ for all $C \in \overline{C} \cap \text{mod } R$, which is equivalent to $\text{Tor}(X^\bullet, C) = 0$ for all $C \in \overline{C} \cap \text{mod } R$. But since $C \subseteq \varinjlim (C \cap \text{mod } R)$ by [8, 2.3], and Tor commutes with direct limits, the latter means that $X^\bullet \in {}^\top C = \varinjlim S$.

Statement (7) is [8, 2.4].

(8) If $0 \rightarrow N \xrightarrow{f} M \rightarrow Z \rightarrow 0$ is exact, then $\text{Tor}(M, C) = 0$ for all $C \in \mathcal{C}$ by (1). Hence $f \otimes C$ is a monomorphism for all $C \in \mathcal{C}$ if and only if $Z \in {}^\perp C = \varinjlim S$. \square

As an application of our previous results, we obtain the following result.

Theorem 9.5. *Let $(\mathcal{M}, \mathcal{L})$ be a cotorsion pair of finite type in $\text{Mod-}R$. Set $S = \mathcal{M} \cap \text{mod-}R$ and $C = \mathcal{M}^\perp$, and denote by \mathcal{L}' the class of all locally pure-injective modules from \mathcal{L} . Then the following statements are equivalent for a right R -module M .*

- (1) M is \mathcal{L} -stationary.
- (2) M is C -Mittag-Leffler for all $C \in \mathcal{C}$.
- (3) M is C -Mittag-Leffler.
- (4) M is strict C^\bullet -stationary for all $C \in \mathcal{C}$.
- (5) M is strict \mathcal{L}' -stationary.

Moreover, every $M \in \mathcal{M}$ is strict \mathcal{L} -stationary (and thus C -Mittag-Leffler). If M is countably generated, then $M \in \mathcal{M}$ if and only if M belongs to $\varinjlim S$ and is (strict) \mathcal{L} -stationary.

Proof. First of all, as $\mathcal{L} = S^\perp$, the class \mathcal{L} is closed under direct sums [20, Lemma 10.2.4]. Moreover, the cotorsion pair is complete, see Definition 9.1(2). Therefore for every right R -module F there is a short exact sequence

$$0 \rightarrow F \xrightarrow{f} B \rightarrow B/F \rightarrow 0,$$

where $B \in \mathcal{L}$ and $B/F \in \mathcal{M}$. Of course, f is an \mathcal{L} -preenvelope. Further, by Lemma 9.4, $B^\bullet \in \mathcal{C}$, and $B/F \in {}^\perp C$, hence the map $f \otimes C$ is a monomorphism for all $C \in \mathcal{C}$. The equivalence of the first three statements then follows directly from Theorem 6.6. Furthermore, (2) is equivalent to (4) by Proposition 8.14.

(2) \Rightarrow (5). Let $B \in \mathcal{L}$ be locally pure-injective. Then $B^\bullet \in \mathcal{C}$ by Lemma 9.4, so M is B^\bullet -Mittag-Leffler, hence strict B -stationary by Lemma 8.14(2).

(5) \Rightarrow (4) is clear since C^\bullet is a (locally) pure-injective module in \mathcal{L} by Lemma 9.4 and [43, 1.6(2)].

The rest of the theorem follows from Proposition 9.2. \square

Corollary 9.6. *Let $(\mathcal{M}, \mathcal{L})$ be a cotorsion pair of finite type in $\text{Mod-}R$. Set $S = \mathcal{M} \cap \text{mod-}R$, and $C = \mathcal{M}^\perp$. Then a module $M \in \varinjlim S$ is C -Mittag-Leffler if and only if it is a directed union of countably presented submodules N that belong to \mathcal{M} and satisfy $M/N \in \varinjlim S$.*

Proof. By Corollary 5.2, the module M is C -Mittag-Leffler if and only if it is a directed union of countably presented submodules $N \in \varinjlim S$ such that N is C -Mittag-Leffler and the inclusion $N \subseteq M$ remains injective when tensoring with

any module $C \in \mathcal{C}$. The second condition means that $M/N \in \varinjlim S$ by Lemma 9.4(8). The statement thus follows immediately from Theorem 9.5. \square

An important example of cotorsion pairs of finite type is provided by tilting theory.

Definition 9.7. Let $n \in \mathbb{N}$. A right R -module T is *n-tilting* provided

- (T1) T has projective dimension at most n ,
- (T2) $\text{Ext}_R^i(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I , and
- (T3) there exist $r \geq 0$ and a long exact sequence $0 \rightarrow R \rightarrow T_0 \rightarrow \cdots \rightarrow T_r \rightarrow 0$ such that $T_i \in \text{Add } T$ for each $0 \leq i \leq r$.

If T is a tilting module, the class $\mathcal{L} = \{X \in \text{Mod-}R \mid \text{Ext}_R^i(T, X) = 0 \text{ for all } i \geq 1\}$ gives rise to a cotorsion pair $(\mathcal{M}, \mathcal{L})$, called the *tilting cotorsion pair* induced by T . Note that $(\mathcal{M}, \mathcal{L})$ is generated by the set $\{\Omega^i(T) \mid i \geq 0\}$ of all syzygies of T . Moreover, $\mathcal{M} \cap \mathcal{L} = \text{Add } T$.

Dually, one defines *cotilting* modules, and *cotilting cotorsion pairs*. If $(\mathcal{C}, \mathcal{D})$ is a cotilting cotorsion pair induced by a cotilting module C , then $\mathcal{C} \cap \mathcal{D} = \text{Prod } C$.

Note that tilting cotorsion pairs are always of finite type, see [12] and [14]. Moreover, if $(\mathcal{M}, \mathcal{L})$ is an n -tilting cotorsion pair induced by the tilting module T , then the cotorsion pair $(\mathcal{C}, \mathcal{D})$ constructed as in Lemma 9.4 is an n -cotilting cotorsion pair induced by the cotilting module T^* , see [4].

Corollary 9.8. Let $(\mathcal{M}, \mathcal{L})$ be a tilting cotorsion pair in $\text{Mod-}R$. Set $S = \mathcal{M} \cap \text{mod-}R$, and $\mathcal{C} = \mathcal{M}^\perp$. Let moreover T be a tilting right R -module inducing $(\mathcal{M}, \mathcal{L})$, and C a cotilting left R -module inducing the cotorsion pair $(\mathcal{C}, \mathcal{D})$. Then the following statements are equivalent for a right R -module $M \in \varinjlim S$.

- (1) M is \mathcal{L} -stationary.
- (2) M is T -stationary.
- (3) M is C -Mittag-Leffler.
- (4) M is C -Mittag-Leffler.

Moreover, every $M \in \mathcal{M}$ is strict \mathcal{L} -stationary (and thus C -Mittag-Leffler). If M is countably generated, then $M \in \mathcal{M}$ if and only if M belongs to $\varinjlim S$ and is (strict) \mathcal{L} -stationary.

Proof. Write M as direct limit of a direct system $(F_\alpha, u_{\beta\alpha})$ of modules in S . Then for all $\alpha \in I$ there is a short exact sequence

$$0 \rightarrow F_\alpha \xrightarrow{f_\alpha} B_\alpha \rightarrow B_\alpha/F_\alpha \rightarrow 0,$$

where $B_\alpha \in \mathcal{L}$ and $B_\alpha/F_\alpha \in \mathcal{M}$. Since $F_\alpha \in S \subseteq \mathcal{M}$, it follows that $B_\alpha \in \mathcal{L} \cap \mathcal{M} = \text{Add } T$. In particular, f_α is an $\text{Add } T$ -preenvelope. Further, $B_\alpha \in \mathcal{L} \cap \varinjlim S$, so

$B_\alpha^\bullet \in C \cap \mathcal{D} = \text{Prod} C$ by 9.4. Finally, since $B_\alpha/F_\alpha \in {}^\top C$, the map $f \otimes C$ is a monomorphism for all $C \in C$.

(1) and (4) are equivalent statements by Theorem 9.5. Moreover, the implication (1) \Rightarrow (2) and (4) \Rightarrow (3) are trivial.

(2) \Rightarrow (4). Note that M is Add T -stationary by Corollary 3.9. Now this implies (4) by Proposition 6.2.

(3) \Rightarrow (2). By Theorem 1.3, M is Q -Mittag-Leffler for all modules Q that are pure submodules of products of copies of C . In particular, we can take $Q = \bigoplus_{\alpha \in I} B_\alpha^\bullet$. The claim then follows from Proposition 6.5. \square

Corollary 9.9. *Let T be a tilting module, $S = \text{End}_R T$. Then the following hold true.*

- (1) *Every finitely generated S -submodule of T is a finite matrix subgroup.*
- (2) *${}_S T$ is noetherian if and only if T^\bullet is Σ -pure-injective.*

Proof.

- (1) Let $(\mathcal{M}, \mathcal{L})$ be the tilting cotorsion pair generated by T . We know that T belongs to the kernel $\mathcal{M} \cap \mathcal{L}$. Thus T is strict T -stationary by Corollary 9.8. By Theorem 8.11, all matrix subgroups of T of the form $H_{T,x}(T)$ with $x \in T$ are finite matrix subgroups, and of course, the matrix subgroups of such form are precisely the cyclic S -submodules of T . Observe that the class of finite matrix subgroups is closed under finite sums. So, we infer that every finitely generated S -submodule of T , being a finite sum of cyclic submodules, is a finite matrix subgroup.
- (2) We know from [44, Proposition 3] that T^\bullet is Σ -pure-injective if and only if T satisfies the ascending chain condition on finite matrix subgroups. By (1), the latter means that ${}_S T$ is noetherian. \square

If $(\mathcal{M}, \mathcal{L})$ is a cotorsion pair of finite type, then it follows from Theorem 9.5 that \mathcal{M} is contained in the class of strict \mathcal{L} -stationary modules. If $S = \mathcal{M} \cap \text{mod-}R$, then the countably generated modules in $\varinjlim S$ that are strict \mathcal{L} -stationary are precisely the countably generated modules in $\overline{\mathcal{M}}$, and they also coincide with the countably generated modules in $\varinjlim S$ that are \mathcal{L} -stationary. Raynaud and Gruson in [36, p. 76] provide examples showing that, in general, \mathcal{M} is properly contained in the class of modules in $\varinjlim S$ that are strict \mathcal{L} -stationary, and the latter class is also properly contained in the class of \mathcal{L} -stationary modules. We explain these examples for completeness' sake. First we prove the following result.

Lemma 9.10. *Let R be a ring, and let F_1 and F_2 be flat right R -modules such that there exists an exact sequence*

$$0 \rightarrow R \xrightarrow{u} F_1 \rightarrow F_2 \rightarrow 0.$$

- (i) Let \mathcal{Q} be a class of left R -modules. If F_2 is \mathcal{Q} -Mittag-Leffler, then so is F_1 .
- (ii) If F_1 is strict R -stationary, then u splits.

Proof.

- (i) The statement follows from Examples 1.6(4).
- (ii) Assume that F_1 is strict R -stationary. Let S be the class of finitely generated free modules. By Proposition 8.1 and since $F_1 \in \varinjlim S$, there exist $n > 0$ and a homomorphism $v: R \rightarrow R^n$, such that $u = \overline{t}v$ for some $t: R^n \rightarrow F_1$ and $H_u(R) = H_v(R)$. Since u is a pure monomorphism so is v , but being a pure monomorphism between finitely generated projective modules, v splits. Therefore the identity map belongs to $H_v(R) = H_u(R)$, thus u also splits. \square

The easiest instance of tilting cotorsion pair is the one generated by the class S of finitely generated free modules. Then $S^\perp = \text{Mod-}R$, and ${}^\perp(S^\perp) = \mathcal{P}$ is the class of all projective modules. The ring R is a tilting module that generates the tilting cotorsion pair $(\mathcal{P}, \text{Mod-}R)$. Note that $\varinjlim S = \mathcal{F}$ is the class of flat modules. The relative Mittag-Leffler, strict stationary and stationary modules associated to this cotorsion pair in Corollary 9.8 are the Mittag-Leffler, strict Mittag-Leffler and $(\text{Mod-}R)$ -stationary modules, respectively. If F_1 is a flat Mittag-Leffler module and $\text{Ext}_R^1(F_1, R) \neq 0$, then there is a non split exact sequence

$$0 \rightarrow R \rightarrow F_2 \rightarrow F_1 \rightarrow 0.$$

In view of Lemma 9.10, F_2 is a Mittag-Leffler module that is not strict Mittag-Leffler. An example for this situation is the following:

Example 9.11. For any set I , the abelian group \mathbb{Z}^I is a flat strict Mittag-Leffler abelian group. If I is infinite, there exist nonsplit extensions of \mathbb{Z}^I by \mathbb{Z} , hence there are flat Mittag-Leffler abelian groups that are not strict Mittag-Leffler.

Proof. Of course, \mathbb{Z}^I is flat. Since any finitely generated submodule of \mathbb{Z}^I is contained in a finitely generated direct summand of \mathbb{Z}^I [23, Proof of Theorem 19.2], we deduce from Proposition 8.1 that \mathbb{Z}^I is strict Mittag-Leffler.

When I is infinite, \mathbb{Z}^I is not a Whitehead group, that is, $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}^I, \mathbb{Z}) \neq 0$ [24, Proposition 99.2]. Then the claim follows by the remarks above. \square

We remark that our results can also be interpreted in the framework of definable categories. In fact, we are going to see that cotorsion pairs of finite type satisfy the hypotheses of Corollary 6.9, as the class \mathcal{L} is definable and C coincides with the dual definable category \mathcal{L}^\vee , cf. Definition 6.8.

Proposition 9.12. Let S be a class of right R -modules. Set $C = S^\top$.

- (1) C is definable if and only if the first syzygy $\Omega^1(M)$ of any module $M \in S$ is a C -Mittag-Leffler module.

- (2) Let $(\mathcal{M}, \mathcal{L})$ be the cotorsion pair in $\text{Mod-}R$ generated by S . If \mathcal{L} is definable, then $C = \mathcal{L}^\vee$ is the dual definable category of \mathcal{L} .

Proof. (1) The class C is always closed by direct limits. We claim that it is also closed by pure submodules. Indeed, let $C \in C$, and let C' be a pure submodule of C . Denote by $\varepsilon: C' \hookrightarrow C$ the inclusion. Given $S \in S$, fix a projective presentation of S

$$0 \rightarrow \Omega^1(S) \rightarrow P \rightarrow S \rightarrow 0,$$

and consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Tor}_1^R(S, C') & \longrightarrow & \Omega^1(S) \otimes_R C' & \longrightarrow & P \otimes C' \\ & & \text{Tor}_1^R(S, \varepsilon) \downarrow & & \Omega^1(S) \otimes \varepsilon \downarrow & & P \otimes \varepsilon \downarrow \\ 0 & \longrightarrow & 0 = \text{Tor}_1^R(S, C) & \longrightarrow & \Omega^1(S) \otimes_R C & \longrightarrow & P \otimes_R C \end{array}.$$

As ε is a pure embedding, $\Omega^1(S) \otimes \varepsilon$, and thus also $\text{Tor}_1^R(S, \varepsilon)$, are injective. Therefore $\text{Tor}_1^R(S, C') = 0$ as wanted.

By Proposition 1.10, C is closed by products if and only if $\Omega^1(M)$ is C -Mittag-Leffler for all $M \in S$.

(2) By the Ext-Tor relations a module C belongs to C if and only if $C^* \in \mathcal{L}$. So, if $C \in C$, then $L = C^* \in \mathcal{L}$ and $L^* = C^{**} \in \mathcal{L}^\vee$. Since C is a pure submodule of C^{**} and \mathcal{L}^\vee is closed under pure submodules, we infer that $C \in \mathcal{L}^\vee$.

Conversely, if $X \in \mathcal{L}^\vee$, then $L = X^* \in \mathcal{L}^{\vee\vee} = \mathcal{L}$, and similarly $L^{**} \in \mathcal{L}$. But then $L^* \in C$. Since X is a pure submodule of $X^{**} = L^*$ and C is closed under pure submodules, we conclude that $X \in C$. \square

Corollary 9.13. *Let S be a class of finitely presented right R -modules. Set $C = S^\tau$, and let $(\mathcal{M}, \mathcal{L})$ be the cotorsion pair in $\text{Mod-}R$ generated by S . The following statements are equivalent.*

- (1) \mathcal{L} is definable.
- (2) C is definable.
- (3) For any set I , $R^I \in C$.
- (4) The first syzygy of any module in S is finitely presented.
- (5) The first syzygy of any finitely generated module in \mathcal{M} is finitely presented.

Proof. We shall repeatedly use the following observation: a module M is finitely presented if and only if it is finitely generated and R -Mittag-Leffler. This follows from the fact that M is finitely generated if and only if the natural transformation $\rho_I: M \otimes R^I \rightarrow M^I$ is surjective for any set I , and M is finitely presented if and only if ρ_I is bijective for any set I .

Let us now start the proof. By Proposition 9.12(2), statement (1) implies (2). Since $R \in C$, (2) implies (3).

(3) \Rightarrow (4). By Proposition 1.10, $R^I \in C$ for any set I if and only if the first syzygy $\Omega^1(S)$ of any $S \in S$ is R -Mittag-Leffler. Since $\Omega^1(S)$ is also finitely generated, we infer that $\Omega^1(S)$ is finitely presented.

(4) \Rightarrow (5). We proceed as in [13, Section 3]. First of all, we infer from Proposition 9.2(2) that every module in \mathcal{M} is C -Mittag-Leffler, and in particular R -Mittag-Leffler. Thus all finitely generated modules in \mathcal{M} are finitely presented.

Furthermore, by Proposition 9.12(1), condition (4) also implies that C is definable. Next, we note that $C = S^\tau = \mathcal{M}^\tau$. Indeed, it is clear that $S^\tau \supseteq \mathcal{M}^\tau$, and the reverse inclusion follows from the fact that every module in \mathcal{M} is a direct summand of an S -filtered module, see [26, Corollary 3.2.4]. Now, applying again Proposition 9.12(1), we deduce that the first syzygy $\Omega^1(M)$ of any module $M \in \mathcal{M}$ is C -Mittag-Leffler, and in particular, R -Mittag-Leffler. So, if M is finitely generated, we conclude that $\Omega^1(M)$ is finitely presented.

Finally, assume (5). As the cotorsion pair is generated by a class of finitely presented modules with all first syzygies being finitely presented, (1) holds true. \square

Applying Corollary 9.13 to the cotorsion pair generated by a single finitely presented module, we obtain the following result on coherent functors. Recall that an additive functor from $\text{Mod-}R$ to the category of abelian groups is said to be *coherent* if it commutes with direct limits and products. It is well known that a subcategory of $\text{Mod-}R$ is definable if and only if it is determined by the vanishing of some set of coherent functors [15, 2.3].

Corollary 9.14. *The following statements are equivalent for a finitely presented right R -module M .*

- (1) *The functor $\text{Ext}_R^1(M, -)$ is coherent.*
- (2) *The first syzygy of M is finitely presented.*
- (3) *The functor $\text{Tor}_1^R(M, -)$ is coherent.*

Proof. (1) implies that M^\perp is definable, thus (2) follows by Corollary 9.13. Moreover, (2) implies (1) and (3). To finish the proof note, for example, that (3) implies condition (3) in Corollary 9.13. \square

Example 9.15. Let \mathcal{P}_1 denote the class of all modules of projective dimension at most one. If S is a class of modules in \mathcal{P}_1 , then $C = S^\tau$ is definable by Proposition 9.12(1). In particular, \mathcal{P}_1^τ is definable. Note, however, that in general \mathcal{P}_1^\perp is not definable. We refer to [12] for a detailed study of the definability of \mathcal{P}_1^\perp .

Cotorsion pairs $(\mathcal{M}, \mathcal{L})$ where \mathcal{L} is definable will be studied in more detail in a forthcoming paper. We now give an example of a cotorsion pair where \mathcal{L} is not definable, but which still satisfies some of the statements of Theorem 9.5.

Example 9.16. Let $(\mathcal{M}, \mathcal{L})$ be the cotorsion pair generated by the class of all finitely presented right R -modules, and $C = \mathcal{M}^\perp$. In other words, \mathcal{L} is the class of all $f\mathcal{P}$ -injective right R -modules, and C is the class of all flat left R -modules.

We claim that the modules in \mathcal{M} are strict \mathcal{L} -stationary and C -Mittag-Leffler. Indeed, by [26, Corollary 3.2.4], every $M \in \mathcal{M}$ is a direct summand of a module N filtered by finitely presented modules. Hence, N is filtered by strict \mathcal{L} -stationary modules, and M is strict \mathcal{L} -stationary by Proposition 8.13. Similarly, as N is filtered by Mittag-Leffler modules, we deduce from Proposition 1.9 that N is Mittag-Leffler with respect to the class of flat modules.

Assume now that R is right coherent. Then a right R -module is \mathcal{L} -stationary if and only if it is C -Mittag-Leffler. This is an application of Theorem 9.5.

In particular, if R is right noetherian, then the $f\mathcal{P}$ -injective modules coincide with the injectives. So, any right R -module is strict stationary with respect to the class of injective right R -modules, and it is Mittag-Leffler with respect to the class of flat left R -modules, cf. Example 5.6.

10. THE PURE-SEMISIMPLICITY CONJECTURE

Throughout this section, we assume that R is a two-sided artinian, hereditary, indecomposable, left pure semisimple ring. It is well known that then every indecomposable finitely generated non-projective left R -module is end-term of an almost split sequence in $R\text{-Mod}$ consisting of finitely presented modules, and every indecomposable finitely generated non-injective right R -module is the first term of an almost split sequence in $\text{Mod-}R$ consisting of finitely presented modules.

We adopt the notation $A = \tau C$ and $C = \tau^{-1}A$ if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an almost split sequence, and define inductively τ^n respectively τ^{-n} . We know from [9] that there is a *preprojective component* \mathbf{p} in $\text{Mod-}R$, that is, a class of finitely generated indecomposable right R -modules satisfying the following conditions.

- (1) For any $X \in \mathbf{p}$ there are a left almost split morphism $X \rightarrow Z$ and a right almost split morphism $Y \rightarrow X$ in $\text{Mod-}R$ with Z, Y being finitely generated.
- (2) If $X \rightarrow Y$ is an irreducible map with one of the modules lying in \mathbf{p} , then both modules are in \mathbf{p} .
- (3) The Auslander-Reiten-quiver of \mathbf{p} is connected and has no oriented cycles.
- (4) For every $Z \in \mathbf{p}$ there is $m \geq 0$ such that $\tau^m Z$ is projective.

Similarly, there is a *preinjective component* in $R\text{-Mod}$, i.e., a class of finitely generated indecomposable left R -modules with the dual properties. Moreover, the two components are related by the local duality, that is, there is a bijection $\mathbf{q} \rightarrow \mathbf{p}$, ${}_R A \mapsto A_R^+$. The modules in $\text{add } \mathbf{p}$ are called *preprojective*, the modules in $\text{add } \mathbf{q}$ are called *preinjective*.

In [2], the cotorsion pair $(\mathcal{M}, \mathcal{L})$ in $\text{Mod-}R$ generated by \mathbf{p} and the cotorsion pair (C, \mathcal{D}) in $R\text{-Mod}$ cogenerated by \mathbf{q} are investigated. In particular, it is shown that there is a finitely generated product-complete tilting and cotilting left R -module W such that $C = \text{Cogen } W = {}^\perp W$ and $\mathcal{D} = \text{Gen } W = W^\perp$. Note that

$C = \mathcal{M}^\tau = \mathbf{p}^\tau$ by Lemma 9.4(2). Moreover, $(\mathcal{M}, \mathcal{L})$ is a tilting cotorsion pair in $\text{Mod-}R$ with corresponding cotilting cotorsion pair (C, \mathcal{D}) in $R\text{-Mod}$. But (C, \mathcal{D}) is also a tilting cotorsion pair in $R\text{-Mod}$, and the corresponding cotilting cotorsion pair in $\text{Mod-}R$ is $(\varinjlim \text{ add } \mathbf{p}, \mathcal{E})$, see [2, 5.2 and 5.4].

We now apply our previous results to this setup, specializing to the case where B^\bullet denotes the local dual of a module B . Let us fix a tilting module T such that $T^\perp = \mathcal{L}$.

Proposition 10.1. *The following statements hold true.*

- (1) T is noetherian over its endomorphism ring.
- (2) All right R -modules in \mathcal{M} are strict T -stationary (and hence W -Mittag-Leffler).

Proof.

- (1) Any left R -module is pure-injective, thus Σ -pure-injective. In particular, T^+ is Σ -pure-injective. By Corollary 9.9 we conclude that T is noetherian over its endomorphism ring.
- (2) holds true by Corollary 9.8. □

We remark that all left R -modules are Mittag-Leffler modules by [11], hence strict Mittag-Leffler modules, see 8.14(2) and 8.3.

As shown in [2], the validity of the Pure-Semisimplicity Conjecture is related to the question whether W is endofinite. We obtain the following criteria for endofiniteness of W .

Proposition 10.2. *The following statements are equivalent.*

- (1) W is endofinite.
- (2) W^+ is W -Mittag-Leffler.
- (3) Every (countable) direct limit of preprojective right R -modules is W -Mittag-Leffler.
- (4) Every (countable) direct system of preprojective right R -modules is T -stationary.
- (5) Every (countable) direct system of preprojective right R -modules has limit in \mathcal{M} .
- (6) If A is direct limit of a (countable) direct system of preprojective right R -modules, and L is a locally pure-injective module from \mathcal{L} , then $\text{Ext}_R^1(A, L) = 0$.

Proof. (1) \Rightarrow (5) follows from [2, 5.6], which asserts that W is endofinite if and only if the class \mathcal{M} is closed under direct limits.

(5) \Leftrightarrow (4) \Leftrightarrow (3) follows from Corollaries 3.12 and 9.8.

(3) \Rightarrow (2). $W \in C \cap \mathcal{D}$, hence $W^+ \in \varinjlim \text{ add } \mathbf{p}$ by Lemma 9.4(6).

(2) \Rightarrow (1) holds by Proposition 8.16.

(5) \Leftrightarrow (6). Of course, (5) implies (6). For the converse implication, observe first that \mathcal{L} is a definable class, hence it is closed under pure-injective envelopes. So, every module $L \in \mathcal{L}$ is isomorphic to a pure submodule of a

(locally) pure-injective module in \mathcal{L} . Note further that the class \mathcal{L}' of all locally pure-injective modules from \mathcal{L} is closed under direct sums, because this is true for the tilting class \mathcal{L} and for the class of locally pure-injective modules, see [42, 2.4]. Consider now a module A which is direct limit of a countable direct system of preprojective right R -modules. Then A is countably presented, and it follows from (6) and [12, 2.5] that A^\perp contains all pure submodules of modules in \mathcal{L}' . We conclude that $\text{Ext}_R^1(A, L) = 0$ for all $L \in \mathcal{L}$, which proves $A \in \mathcal{M}$. \square

Remark 10.3. It is well known that every cotilting class is a torsion-free class. So, let us consider the torsion pair defined by the cotilting class $\varinjlim \text{add } \mathbf{p}$, denoting by t the corresponding torsion radical. Assume the following condition holds true (cf. [34, 4.1]):

If N is a finitely generated submodule of W^+ , then $t(W^+/N)$ is finitely generated.

Then it follows that W is endofinite. In fact, since $W^+ \in \varinjlim \text{add } \mathbf{p}$, all its finitely generated submodules are in \mathcal{M} . Moreover, for every finitely generated submodule N of W^+ there is a finitely generated submodule $N' \subseteq W^+$ for which $N \subseteq N'$ and $W^+/N' \in \varinjlim \text{add } \mathbf{p}$. To see this, choose N' such that $N'/N = t(W^+/N)$ and use that N'/N is finitely generated. So, we conclude that W^+ is a directed union of finitely generated submodules N' that belong to \mathcal{M} and satisfy $W^+/N' \in \varinjlim \text{add } \mathbf{p}$. But then W^+ is W -Mittag-Leffler by Corollary 9.6, which means that W is endofinite by Proposition 10.2.

We close with a criterion for R being of finite representation type.

Proposition 10.4. *Let P be the direct sum of a set of representatives of the isomorphism classes of the modules in \mathbf{p} . Then the following statements are equivalent.*

- (1) *R has finite representation type.*
- (2) *Every countable direct system of preprojective modules is (strict) P -stationary.*

Proof. It is clear that (1) implies (2). Indeed, R is of finite representation type if and only if all right (and left) R -modules are Mittag-Leffler, hence stationary with respect to any module [11].

Assume (2) holds. To prove that R has finite representation type, it suffices to show that \mathbf{p} is finite, see [2, 3.5]. Assume on the contrary that \mathbf{p} is infinite. By applying [44, Theorem 9] to \mathbf{p} , we know that there are an infinite family $(P_i)_{i \in \mathbb{N}}$ of pairwise non-isomorphic modules in \mathbf{p} and a sequence of homomorphisms $(f_i: P_i \rightarrow P_{i+1})_{i \in \mathbb{N}}$ such that $f_i \cdots f_0 \neq 0$ for any $i \in \mathbb{N}$.

Since all modules in \mathbf{p} are endofinite by [9, 6.2], the direct system

$$P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \rightarrow \cdots \rightarrow P_n \xrightarrow{f_n} P_{n+1} \rightarrow \cdots$$

is (strict) \mathbf{p} -stationary, see Corollary 8.12(3). But by our assumption it is even P -stationary, and since $\text{Add } P = \text{Add } \mathbf{p}$, we infer from Corollary 3.9(3) that it is

Add \mathbf{p} -stationary. By Corollary 3.9(2) it follows that for any $n \in \mathbb{N}$ there exists $m \geq n$ such that

$$H_{f_m \cdots f_n}(P_i) = \bigcap_{l \geq n} H_{f_l \cdots f_n}(P_i)$$

for all $i \in \mathbb{N}$. On the other hand, as our modules are preprojective, for any $i \in \mathbb{N}$ there exists l_i such that $\text{Hom}_R(P_l, P_i) = 0$ for any $l > l_i$, hence $H_{f_l \cdots f_n}(P_i) = 0$ for any $l \geq l_i$. So, we deduce that $H_{f_m \cdots f_n}(P_i) = 0$ for all $i \in \mathbb{N}$. In particular, we have

$$f_m \cdots f_n = \text{Id}_{P_{m+1}} f_m \cdots f_n \in H_{f_m \cdots f_n}(P_{m+1}) = 0,$$

which contradicts the choice of the sequence $(f_i)_{i \in \mathbb{N}}$. Thus we conclude that \mathbf{p} is finite. \square

Acknowledgements. We would like to thank Javier Sánchez Serdà for valuable discussions on the paper [36], Silvana Bazzoni for many comments on preliminary versions of the paper, and Ivo Herzog for interesting conversations. Finally, we thank the referee for drawing our attention towards definable categories.

The research of this paper was initiated while the second author was spending a sabbatical year at the Università di Padova (Italy). She thanks her host for its kind hospitality. She also thanks the Università dell’Insubria, Varese (Italy) for its hospitality in several visits while the paper was being written. The research was continued during the visit of the authors to CRM Barcelona in September 2006 supported by the Research Programme on Discrete and Continuous Methods of Ring Theory. First author partially supported by Università di Padova, Progetto di Ateneo CDPA048343, and by PRIN 2005 “Prospettive in teoria degli anelli, algebre di Hopf e categorie di moduli”. Both authors partially supported by the DGI and the European Regional Development Fund, jointly, through Project MTM2005–00934, and by the Comissionat per Universitats i Recerca of the Generalitat de Catalunya, Project 2005SGR00206.

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KEY WORDS AND PHRASES: Mittag-Leffler inverse system; cotorsion pairs; tilting modules; Baer modules; pure semisimple rings.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 16D70, 16E30, 18E15.

Received: June 11th, 2007; revised: May 16th, 2008.

Article electronically published on August 11th, 2008.