Weak Leopoldt's conjecture for Hecke characters of imaginary quadratic fields.*

Francesc Bars †

Abstract

We give a proof of weak Leopoldt's conjecture a la Perrin-Riou, under some technical condition, for the p-adic realizations of the motive associated to Hecke characters over an imaginary quadratic field K of class number 1, where p is a prime > 3 and where the CM elliptic curve associated to the Hecke character has good reduction at the primes above p in K. This proof makes use of the 2-variable Iwasawa main conjecture proved by Rubin. Thus we prove the Jannsen conjecture for the above p-adic realizations for almost all Tate twists.

1 The formulation of weak Leopoldt's conjecture and weak Jannsen's conjecture for Hecke characters

Let us first introduce the p-adic integer realizations associated to Hecke characters. The aim of this paper is verify for these realizations the weak Leopoldt conjecture a la Perrin-Riou and Jannsen's conjecture for almost all Tate twists.

Let E be a fixed elliptic curve with CM, which is defined over an imaginary quadratic field K, with CM by $\mathcal{O}_K$, the ring of integers of K. Note that this implies $cl(K) = 1$. Associated to this elliptic curve there is a CM-character $\underline{\varphi}$ of the imaginary quadratic field K with conductor $\mathfrak{f}$, (an ideal of $\mathcal{O}_K$ that coincides with the conductor of the elliptic curve E).

Let us consider once for all p be an odd prime where E has good reduction at the primes above p in K, ordinary or supersingular. We impose $p \neq 3$ in order that any primitive p-root of unity is in K. Thus in particular $p \nmid D_K$ the discriminant of K.

Let us introduce the motives and the p-adic realizations on which we study the conjectures. Consider the category of Chow motives $\mathcal{M}_{\mathbb{Q}}(K)$ over K with morphisms induced by graded correspondences in Chow theory tensored with $\mathbb{Q}$. Then, the motive of the elliptic curve E has a canonical decomposition $h(E)_{\mathbb{Q}} = h^0(E)_{\mathbb{Q}} \oplus h^1(E)_{\mathbb{Q}} \oplus h^2(E)_{\mathbb{Q}}$. The motive $h^1(E)_{\mathbb{Q}}$ has a multiplication by K [5]. Let us consider the motive $\otimes^w_{\mathbb{Q}} h^1(E)_{\mathbb{Q}}$, for w a strictly positive integer, which has multiplication by $T_w := \otimes^w_{\mathbb{Q}} K$. Observe that T_w has a decomposition $\prod_\theta T_\theta$ as a product of fields $T_\theta \cong K$, where θ runs through the $Aut(\mathbb{C})$-orbits of $\daleth^w = Hom(T_w, \mathbb{C})$, where $\daleth = Hom(K, \mathbb{C})$. This decomposition defines some idempotents e_θ and gives also a decomposition of the motive and its realizations. Let us fix once and for all an immersion $\lambda : K \to \mathbb{C}$ as in [5, p.135].

The L-function associated to the motive $e_\theta(\otimes^w h^1(E)_{\mathbb{Q}})$ corresponds to the L-function associated to $\underline{\psi}_\theta = e_\theta(\otimes^w \underline{\varphi}) : \mathbb{A}^*_K \to T^*_\theta$ ([5, §1.3.1]) a CM-character, which, with the fixed embedding λ, corresponds to $\underline{\varphi}^a \overline{\underline{\varphi}}^b$, where $a, b \geq 0$ are integers such that $w = a + b$ and we suppose $a \neq b$ once and for all, otherwise

*Mathematics Subject Classification (2000): 11R23 (11R34, 11G40)
†Work partially supported by MTM2006-11391

the CM-character is a power of the cyclotomic character. The pair (a, b) is called the infinite type for ψ_θ. We note that there are different θ with the same infinite type. Every θ gives two elements of $\beth^w$, one given by the infinite type $\vartheta \in \theta \cap Hom_K(T_w, \mathbb{C})$ and the other comes from considering the other embedding of K as the fixed one.

There are at least three equivalent notions of a Hecke character, see [7, p.48]. One is the notion of CM-character used above, [7, p.48, definition 2]. For Hecke L-functions and the Galois group action on the p-adic realization associated to the Hecke motive, we use the notion of a character which is trivial on K^* and with image in some idèle group, $\psi_\theta : I_K/K^* \to I_{T_\theta}$ [7, p.48, definition 3]. The associated complex Hecke character, in order to define the Hecke L-function, is constructed from ψ_θ by taking the archimedian places of I_{T_θ} which correspond to the fixed immersion of K in $\mathbb{C}$ in our situation. The character constructed from ψ_θ by taking the components of the places of I_{T_θ} above p is called $\psi_{\theta,p}$ and is related with the Galois action on the p-adic realization associated to the motive. The character $\psi_{\theta,p}$ factors through $Gal(K^{ab}/K)$ and has image in $(T_\theta \otimes \mathbb{Z}_p)^*$. We will use the term Hecke character when we want to consider this notion from now on.

Take the integral Chow motive h^1E (similar to $h^1E_\mathbb{Q}$ but without tensoring by $\mathbb{Q}$ the correspondences). It has multiplication by $\mathcal{O}_K$. Consider then the integral Chow motive $\otimes^w h^1E$,, it has multiplication by $\mathcal{O}_w := \otimes^w_\mathbb{Z} \mathcal{O}_K$. Notice that e_θ is not integral in general for $w > 1$, but is contained in $\mathcal{O}_K[1/D_K]$. Let's denote by

$$M_\theta := e_\theta(\otimes^w h^1(E)_\mathbb{Q} \otimes_{\mathcal{O}_K} \mathcal{O}_K[1/D_K]),$$

considered as an Chow motive with coefficients in $\mathcal{O}_K[1/D_K]$ (i.e. we tensor by $\otimes_{\mathcal{O}_K} \mathcal{O}_K[1/D_K]$ the correspondences), and by $M_{\theta\mathbb{Q}}$ its image in $\mathcal{M}_\mathbb{Q}(K)$.

The p-adic realization of the motive $M_\theta(n)$ is given by $H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Q}_p(n))$ and denote it by $M_{\theta\mathbb{Q}_p}(n)$, and is isomorphic to

$$e_\theta(\otimes^w (T_pE \otimes \mathbb{Q}_p))(n - w),$$

which Galois action given by $\overline{\psi}_{\theta,p}\chi^{n-w}_{cycl} \otimes \mathbb{Q}_p$ (see [2, Lemma 2.2]), where χ_{cycl} denotes the cyclotomic character.

Denote by S a finite set of non-archimedian places of K which contains the primes above p and the ones where the p-adic realization is ramified, denote by K_S the maximal field extension of K which is unramified outside the primes of S, and we denote G_S by $Gal(K_S/K)$.

For $M_{\theta\mathbb{Q}_p}(n) = H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Q}_p(n))$ the set S contains the finite primes of K which divide $p\mathfrak{f}_\theta$, where $\mathfrak{f}_\theta$ is the conductor of ψ_θ. We have also a Galois invariant lattice associated to it $M_{\theta\mathbb{Z}_p}(n) := H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(n)) \otimes_{\mathcal{O}_K} \mathcal{O}_K[1/D_K]$, which is isomorphic $(p \nmid D_K)$ to

$$e_\theta(\otimes^w T_pE)(n - w)$$

where the Galois action is given by $\overline{\psi}_{\theta,p}\chi^{n-w}_{cycl}$ [2, Lemma 2.2]. For an extended explanation of these motives we refer to [2, §2].

In these realizations, if we consider that $e_\theta(\otimes^w h^1(E)_\mathbb{Q}) \subseteq h^w(E^w)_\mathbb{Q}$ the Jannsen conjecture for $h^w(E^w)$ implies the following conjecture for the Hecke characters.

Conjecture 1.1 (weak Jannsen's conjecture). *Let p be an odd prime which E has good reduction over the primes over p in K. Let S be a set of primes of K which contains the primes of K over p and the primes of $\mathfrak{f}$. Then,*

$$H^2(G_S, M_{\theta\mathbb{Q}_p}(n)) = 0,$$

for all n up to a finite set of twist.

Remark 1.2. *Jannsen's conjecture specifies which Tate twists could be not zero, for this reason we call the above conjecture by the term weak. In our specific situation, the above cohomology group should vanish for all $n > w + 1$ and $w + 1 < 2n$ (see [3, conjecture 2.1]), moreover int the CM-situation or for realizations associated to Hecke characters this range should be smaller. For a study of the Jannsen conjecture with fixed Tate twist n and an overview of the known situations in the case of Hecke characters we refer to [3].*

Remark 1.3. *The usual formulation of the conjecture states that S contains the primes over $p\mathfrak{f}_\theta$ instead of $p\mathfrak{f}$. Using the Hochshild-Serre spectral sequence with properties for cohomology of local Galois group we obtain that the conjecture can be formulated as we do in 1.1 (see for example the proof of lemma 1.5 [3]).*

Now, we show that the above conjecture is exactly weak Leopoldt's conjecture a la Perrin-Riou for these realizations. Let us now introduce weak Leopoldt's conjecture a la Perrin-Riou for these concrete realizations and the equivalence between conjectures. We refer to Perrin-Riou [12, Appendix B] for the interested reader on weak Leopoldt's conjecture and known results (see also the paper of Kato [10] for results on modular forms).

We need to introduce some Iwasawa theory notation for imaginary quadratic fields. Let us denote $K(E[p^{i+1}]) = K_i$, $\Gamma = Gal(K(E[p^\infty]/K(\mu_{p^\infty}))$ and $\Delta = Gal(K_0/K)$. Take the following notation for some Iwasawa algebras:

$$\Lambda := \mathbb{Z}_p[[\varprojlim_n Gal(K_n/K)]]$$

which is isomorphic to $\mathbb{Z}_p[\Delta][[T_c, T_a]]$ where T_c, T_a denote two variables and Δ is isomorphic to $\mathbb{Z}/(p-1) \times \mathbb{Z}(p-1)$ in the ordinary situation (p splits in K) or $\mathbb{Z}/(p^2-1)$ in the supersingular situation (p is prime ideal of K), and

$$\Lambda(cycl) := \mathbb{Z}_p[[Gal(K(\mu_{p^\infty})/K)]],$$

where μ_{p^n} denotes the p^n roots of unity, observe $\Lambda(cycl)$ is isomorphic to $\mathbb{Z}_p[[T_c]] \times \mathbb{Z}/p-1$, $\mathbb{Z}_p[[T_c]]$ corresponds the Iwasawa algebra associated to the cyclotomic $\mathbb{Z}_p$-extension of K and $\mathbb{Z}_p[[T_a]]$ the one associated to the anticyclotomic $\mathbb{Z}_p$-extension.

Let $K_{cyc} \subseteq K(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of K and set $\Delta' = Gal(K(\mu_{p^\infty})/K_{cyc})$, we have a decomposition

$$\Lambda(cycl) = \oplus_\chi \Lambda(cycl)_\chi$$

where χ denotes the p-adic characters of Δ' and $\Lambda(cycl)_\chi = e_\chi \Lambda(cycl)$ where e_χ is the idempotent associated to χ.

Proposition 1.4. *(Jannsen, [8, Lemma 8]) The trueness of Jannsen's conjecture 1.1 is equivalent that the Galois cohomology group*

$$H^2(Gal(K_S/K(\mu_{p^\infty})), M^*_{\theta\mathbb{Q}_p}(1)/M^*_{\theta\mathbb{Z}_p}(1)) = 0,$$

where * *means* $Hom(-,\mathbb{Q}_p)$ *or* $Hom(-,\mathbb{Z}_p)$ *respectively.*

Proof. Only note that [8, Lemma 8b)] implies that the conjecture 1.1 is equivalent $H^2(Gal(K_S/K(\mu_{p^\infty})), M^*_{\theta\mathbb{Q}_p}(1)/M^*_{\theta\mathbb{Z}_p}(1))e_\chi = 0$ for all idempotents e_χ, therefore by the idempotent decomposition

$$H^2(Gal(K_S/K(\mu_{p^\infty})), M^*_{\theta\mathbb{Q}_p}(1)/M^*_{\theta\mathbb{Z}_p}(1)) = 0.$$

□

Observe that the above vanishing of the Galois cohomology group is not affected by the Tate twist, then we can formulate the next result for any $n \in \mathbb{Z}$:

Proposition 1.5. *(Perrin-Riou, [12, prop.1.3.2.]) It is equivalent:*

1. $e_\chi \varprojlim_m H^2(Gal(K_S/K(\mu_{p^m})), M_{\theta\mathbb{Z}_p}(n))$ *is a torsion* $\Lambda(cycl)_\chi$*-module for every character* χ *of* Δ'.
2. $H^2(Gal(K_S/K(\mu_{p^\infty})), M^*_{\theta\mathbb{Q}_p}(1)/M^*_{\theta\mathbb{Z}_p}(1)) = 0.$

Conjecture 1.6 (weak Leopoldt's conjecture a la Perrin-Riou). *The* $\Lambda(cycl)_\chi$*-module* $e_\chi \varprojlim_m H^2(Gal(K_S/K(\mu_{p^m})), M_{\theta\mathbb{Z}_p}(n))$ *is* $\Lambda(cycl)_\chi$*-torsion.*

The aim of this paper is to prove the conjecture 1.6, (equivalently the weak Jannsen conjecture) at ordinary and supersingular situation following the proofs given in the situation where $w = 1$, (Rubin [15] at ordinary primes, and McConnel [11] at supersingular primes). We write in detail only the supersingular situation because the ordinary situation is known to the specialists.

The results that we obtain in this paper should be a consequence of the μ vanishing in abelian extensions of K of the cyclotomic $\mathbb{Z}_p$-extension tower, see for example [3, §5]. Moreover, Erich Hecke associated to any Hecke character a modular form of some weight, then the results here are also a consequence of Kato's result [10, §15, Theorem 12.4] but our proof differs from Kato's proof.

Remark 1.7. *If we prove conjecture 1.6 with n fixed, say w, then by proposition 1.5 we prove weak Leopoldt's conjecture for any Tate twist n, and therefore weak Jannsen's conjecture. Thus in the following we can fix once and for all* $n = w$ *in order to obtain the conjectures.*

2 Generalized Selmer groups whose dual are torsion Iwasawa modules

Let us impose once and for all that the primes which divides $\mathfrak{f}$ the conductor of the elliptic curve E are in the fixed set S, and let us remember that $p > 3$ is a prime where E has good reduction at the primes above p in K. Denote by K_∞ the field extension $\cup_i K_i = K(E[p^\infty])$.

We notice here that all this section we work with the Tate twist w, but with minor changes in the proofs (mainly we work in extension fields of $K(\mu_{p^\infty})$), the results works with the Tate twist $w+i$.

The idea to prove that $e_\chi \varprojlim_m H^2(Gal(K_S/K(\mu_{p^m})), M_{\theta\mathbb{Z}_p}(w))$ is $\Lambda(cycl)_\chi$-torsion is as follows: translate the question to prove that the Pontrjagin dual of some kind of Selmer group, say e-Selmer group, is $\Lambda(cycl)$-torsion. For every field extension of K we can associate e-Selmer group which on $K(\mu_{p^\infty})$ is the one that we are interested and on K_∞ is an homomorphism module with some of the factors a Λ-torsion Iwasawa module. Then try to make Galois descent from K_∞ to $K(\mu_{p^\infty})$ in order to have some control on the characteristic polynomial in the descent. The mainly technical condition appears when we want that descent argument works, i.e. we need that some Galois cohomology group should vanish.

Let us now define the e-Selmer groups which are the useful ones for our aim (we refer to Rubin [18, Chapter I] for generalities on Selmer groups).

Let M be any field with $M \subseteq \overline{K}$. Define now the finite local conditions at places of M with $v \nmid p$ for the $Gal(\overline{M}/M)$-module $(M_{\theta\mathbb{Z}_p}(w))'$ where $' = Hom(-, \mathbb{Q}_p/\mathbb{Z}_p(1))$. Denote by $A = (M_{\theta\mathbb{Z}_p}(w))'$, $V = Hom(M_{\theta\mathbb{Z}_p}(w), \mathbb{Q}_p(1))$ and $T = Hom(M_{\theta\mathbb{Z}_p}(w), \mathbb{Z}_p(1))$, then the local finite Galois group at $v \nmid p$ with $v \in Spec(L)$ are defined by

$$H^1_f(M_v, V) := ker(H^1(M_v, V) \to H^1(I_v, V))$$

$$H^1_f(M_v, T) := i_v^{-1}(H^1_f(M_v, V))$$

$$H^1_f(M_v, A) := j_v(H^1_f(M_v, V))$$

where $i_v : H^1(M_v, T) \to H^1(M_v, V)$, $j_v : H^1(M_v, V) \to H^1(M_v, A)$ induced by $0 \to T \to V \to A \to 0$. This definition is extended to p-adic realizations V which are finite generated with a Galois invariant lattice on it T. There is an analog for $v \mid p$ of the above finite cohomology groups (see [18, chapter I] for its definition), but we do not need for our interest in this work.

The extended Selmer group is classically defined without taking account of the local conditions at the primes dividing p and is defined by

$$Sel_{ext}(M, A) := ker(H^1(M, A) \to \prod_{v \nmid p} \frac{H^1(M_v, A)}{H^1_f(M_v, A)}).$$

Observe that the local conditions that we impose on the Selmer group for $v \nmid p$ is related to the unramified cohomology groups (see for example [18, Lemma 1.3.5]). The extended Selmer group can be used as the good e-Selmer group for our question if p is splits in K, ordinary situation.

The restricted Selmer group is defined by

$$Sel_r(M, A) := ker(H^1(M, A) \to \prod_{v \nmid p} \frac{H^1(M_v, A)}{H^1_f(M_v, A)} \prod_{v \mid p} H^1(M_v, A)).$$

Observe that the local conditions that we impose on the restricted Selmer group is related to take restrictions for cohomology classes on the Galois group of the unramified extension outside p of M, unramified cohomology group, and totally decomposed at primes above p. This is the good e-Selmer group for our question if p is inert in K, at supersingular case.

To simplify notation we denote $P_{S,M}$ for a field M such that $K \subseteq M \subseteq K_S$ by

$$\prod_{v\in S\setminus S_p} \frac{H^1(M_v,(M_{\theta\mathbb{Z}_p}(w))^{'})}{H^1_f(M_v,(M_{\theta\mathbb{Z}_p}(w))')} \prod_{v\in S_p} H^1(M_v,(M_{\theta\mathbb{Z}_p}(w))^{'}))$$

where $v \in S$ (respectively S_p or $S \setminus S_p$) denotes the set v of places of M which are above the chosen fixed finite places of K that are in S(respectively $v|p$ or $v \in S$ and $v \notin p$).

Definition 2.1. *The restricted subgroup of the Selmer group for ψ_θ for any field M such that $M \subseteq K(E[p^\infty]) \subseteq K_S$ is defined by*

$$R_M := ker(r_{S,M} : H^1(Gal(K_S/M),(M_{\theta\mathbb{Z}_p}(w))^{'}) \to P_{S,M})$$

where $r_{S,M}$ is the product of restriction maps (with the factorization through the finite cohomology groups in the places $S \setminus S_p$).

We need to introduce some notation for the Iwasawa theory of imaginary quadratic fields. Let us introduce first this Iwasawa theory notation.

Let $\mathcal{X}_\infty$ be the Galois group for the maximal abelian p-extension $M^p_{\infty,p}$ of K_∞ which is unramified outside of the primes above p. Denote by $\tilde{\mathcal{A}}_\infty$ be the Galois group for the maximal abelian p-extension $M^p_{\infty,\emptyset,p-td}$ of K_∞ which is unramified outside of the primes above p and totally decomposed at the primes above p. Denote by $\mathcal{A}_\infty$ be the Galois group for the maximal abelian p-extension $M^p_{\infty,\emptyset}$ of K_∞ which is unramified. Remember that $\mathcal{X}_\infty$ is Λ-torsion only when p is ordinary, therefore this module is not the convenient one in the supersingular situation, but $\tilde{\mathcal{A}}_\infty$ is always Λ-torsion module (we have the natural projection map $\mathcal{A}_\infty \to \tilde{\mathcal{A}}_\infty$ and $\mathcal{A}_\infty$ is Λ-torsion [16]).

Lemma 2.2. *Let p be a supersingular or ordinary prime for E then we have:*

1. $R_{K(E[p^\infty])} \cong Hom(\tilde{\mathcal{A}}_\infty,(M_{\theta\mathbb{Z}_p}(w))^{'}) \subseteq Hom(\mathcal{A}_\infty,(M_{\theta\mathbb{Z}_p}(w))^{'})$
2. *The Pontryagin dual of $R_{K(E[p^\infty])}$ denoted by $R_{K(\hat{E}[p^\infty])}$ is isomorphic to $Hom(M_{\theta\mathbb{Z}_p}(w),\tilde{\mathcal{A}}_\infty)$,*

thus $R_{K(\hat{E}[p^\infty])}$ is a torsion Λ-module.

Proof. Recall $K_\infty = K(E[p^\infty])$. Observe R_{K_∞} are the homomorphisms in

$$Hom(Gal(K_S/K_\infty),(M_{\theta\mathbb{Z}_p}(w))^{'})$$

which are non-ramified at any place of S (because over K_0 the elliptic curve has good reduction everywhere, thus $M_{\theta\mathbb{Z}_p}(w)$ is unramified everywhere over K_∞ and apply then [18, Lemma 1.3.5.iv]) and moreover are completely split at the places above p (by the local conditions at $v \in S_p$) therefore if factors through the maximal p-extension over K_∞ non-ramified in any place at which decompose totally at the places above p which is $\tilde{\mathcal{A}}_\infty$. For the last inclusion, take the functor $Hom(-,M_{\theta\mathbb{Z}_p}(w)')$ on the projection map $\mathcal{A}_\infty \to \tilde{\mathcal{A}}_\infty$.

The second statement follows by adjoint functor properties:

$$R_{K(\hat{E[p^\infty]})} = Hom_{\mathbb{Z}_p}(Hom_{\mathbb{Z}_p}(\tilde{\mathcal{A}}_\infty, (M_{\theta\mathbb{Z}_p}(w)^{'}), \mathbb{Q}_p/\mathbb{Z}_p)$$
$$\cong Hom_{\mathbb{Z}_p}(Hom_{\mathbb{Z}_p}(\tilde{\mathcal{A}}_\infty, \mathbb{Q}_p/\mathbb{Z}_p) \otimes M_{\theta\mathbb{Z}_p}(w), \mathbb{Q}_p/\mathbb{Z}_p) \cong$$
$$Hom_{\mathbb{Z}_p}(M_{\theta\mathbb{Z}_p}(w), Hom_{\mathbb{Z}_p}(Hom_{\mathbb{Z}_p}(\tilde{\mathcal{A}}_\infty, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)) \cong$$
$$\cong Hom_{\mathbb{Z}_p}(M_{\theta\mathbb{Z}_p}(w), \tilde{\mathcal{A}}_\infty).$$

□

Now, we do descent argument from K_∞ to $K(\mu_{p^\infty})$. Observe first:

Lemma 2.3. *The $\Lambda(cycl)$-modules $\varprojlim_m H^2(Gal(K_S/K(\mu_{p^m})), M_{\theta\mathbb{Z}_p}(w))$ and $R_{K(\hat{\mu_{p^\infty}})}$ are isomorphic. In particular, $e_\chi \varprojlim_m H^2(Gal(K_S/K(\mu_{p^m})), M_{\theta\mathbb{Z}_p}(w))$ is $\Lambda(cyc)_\chi$-torsion if and only if $e_\chi R_{K(\hat{\mu_{p^\infty}})}$ is $\Lambda(cyc)_\chi$-torsion.*

Proof. The result is an application of Tate-Poitou long exact sequence. Observe first that $R_{K(\mu_{p^\infty})} = \varinjlim_m R_{K(\mu_{p^m})}$. We have the exact sequence:

$$0 \to R_{K(\mu_{p^m})} \to H^1(K_S/K(\mu_{p^m}), M_{\theta\mathbb{Z}_p}(w)^{'}) \to \tag{1}$$
$$\prod_{v\in S\setminus S_p} \frac{H^1(K(\mu_{p^m})_v, (M_{\theta\mathbb{Z}_p}(w)^{'})}{H^1_f(K(\mu_{p^m})_v, (M_{\theta\mathbb{Z}_p}(w)^{'})} \prod_{v\in S_p} H^1(K(\mu_{p^m})_v, (M_{\theta\mathbb{Z}_p}(w))^{'}).$$

Denote by $L_{m,v} = H^1_f(K(\mu_{p^m})_v, M_{\theta\mathbb{Z}_p}(w))$ if $v \in S \setminus S_p$ and for $v \in S_p$ $L_{m,v} = H^1(K(\mu_{p^m})_v, M_{\theta\mathbb{Z}_p}(w))$; by local Tate duality $\frac{H^1(K(\mu_{p^m})_v,(M_{\theta\mathbb{Z}_p}(w))^{'})}{H^1_f(K(\mu_{p^m})_v,(M_{\theta\mathbb{Z}_p}(w))^{'})}$ is the Pontrjagin dual of $L_{m,v}$ for $v \in S \setminus S_p$, in particular we obtain that is isomorphic to $H^1(K(\mu_{p^m})_v, (M_{\theta\mathbb{Z}_p}(w))^{'})/L^{\perp}_{m,v}$. Taking Pontrjagin duality on the exact sequence (1), we obtain the short exact sequence:

$$\prod_{v\in S} L_{v,m} \xrightarrow{\beta_1} H^1(K_S/K(\mu_{p^m}), M_{\theta\mathbb{Z}_p}(w)^{'})^\vee \to R_{K(\hat{\mu_{p^m}})} \to 0,$$

where $\vee$ mean the Pontrjagyn dual.

The map β_1 appears in the Tate-Poitou sequence, therefore we have the following exact sequence (see for example [20, p.119]):

$$\prod_{v\in S} L_{v,m} \xrightarrow{\beta_1} H^1(K_S/K(\mu_{p^m}), M_{\theta\mathbb{Z}_p}(w)^{'})^\vee \to H^2(Gal(K_S/K(\mu_{p^m})), M_{\theta\mathbb{Z}_p}(w)) \to$$

therefore we obtain an exact sequence

$$0 \to R_{K(\hat{\mu_{p^m}})} \to H^2(Gal(K_S/K(\mu_{p^m}), M_{\theta\mathbb{Z}_p}(w)))$$
$$\to \prod_{v\in S} H^2(Gal(K_{v,S}/K(\mu_{p^m})_v), M_{\theta\mathbb{Z}_p}(w))),$$

and observe $H^2(K_{S,v}/K(\mu_{p^m})_v), M_{\theta\mathbb{Z}_p}(w)) = 0$ if $v \in S \setminus S_p$ because by [6, Proposition 1.9] is known $cd_p(Gal(K_{S,v}/K(\mu_{p^m})_v)) \le 1$ and if $v \in S_p$ because by duality this cohomology group is dual to taking the invariants which are trivial, use for example weights arguments.

□

Lemma 2.4. *As $\Lambda(cycl)$-modules we have the natural injections:*

$$R_{K(\mu_{p^\infty})} \subseteq Hom_{\mathbb{Z}_p}(\tilde{\mathcal{A}}_\infty, M_{\theta\mathbb{Z}_p}(w)')^{Gal(K(E[p^\infty])/K(\mu_{p^\infty}))}$$

$$\subseteq Hom_{\mathbb{Z}_p}(\mathcal{A}_\infty, M_{\theta\mathbb{Z}_p}(w)')^{Gal(K(E[p^\infty])/K(\mu_{p^\infty}))},$$

under the condition (for the first inclusion) that $p+1$ does not divide $b-a$ if p is inert in K (supersingular situation), and that $p-1$ does not divide $b-a$ if p splits in K (ordinary situation).

Proof. Denote by $H = Gal(K(E[p^\infty])/K(\mu_{p^\infty}))$, remember that $R_{K(E[p^\infty])} = Hom_{\mathbb{Z}_p}(\tilde{\mathcal{A}}_\infty, M_{\theta\mathbb{Z}_p}(w)')$ we have the following commutative diagram,

$$\begin{array}{ccccccc} 0 & \to & R^H_{K(E[p^\infty])} & \to & H^1(G_{K(E[p^\infty])}, (M_{\theta\mathbb{Z}_p}(w))')^H & \to & P^H_{S,K(E[p^\infty])} \\ & & \uparrow \alpha & & \uparrow \delta & & \\ 0 & \to & R_{K(\mu_{p^\infty})} & \to & H^1(G_{K(\mu_{p^\infty})}, (M_{\theta\mathbb{Z}_p}(w))') & \to & P_{S,K(\mu_{p^\infty})} \end{array}$$

therefore is enough prove that δ is injective to obtain the result. From the Hochschild-Serre spectral sequence is enough, for the injectivity of δ, to prove that

$$H^1(Gal(K(E[p^\infty])/K(\mu_{p^\infty})), (M_{\theta\mathbb{Z}_p}(w))'),$$

is zero, and because we can change the Tate twist, is enough prove that

$$H^1(Gal(K(E[p^\infty])/K(\mu_{p^\infty})), (M_{\theta\mathbb{Z}_p}(1))') \cong e_\theta(\otimes^w E[p^\infty])$$

(where the isomorphism conjugates by the complex conjugation the Galois action) is zero.

Observe that $Gal(K(E[p^\infty])/K(\mu_{p^\infty}))$ is not a p-group and denote by $\Omega_n \subset (\mathcal{O}_K/p^n)^*$ the image by the composition of the map:

$$\overline{\psi_\theta} \otimes \mathbb{Z}_p : G_K \to e_\theta(\otimes^w \mathcal{O}_K)^* \otimes \mathbb{Z}_p \cong (\mathcal{O}_K \otimes \mathbb{Z}_p)^*$$

composed by the projection to $(\mathcal{O}_K/p^n)^*$ which Ω_n acts on $\mathcal{O}_K/p^n\mathcal{O}_K$ by $z : x \mapsto \overline{z}^a z^b x$. We prove first that Ω_n is not a p-group. Let us suppose that $Gal(K_0/K(\mu_p))$ acts trivially on $e_\theta(\otimes^w T_pE)$, then $Gal(K_0/K)$ acts as a power of a cyclotomic character on $e_\theta(\otimes^w T_pE)$. Suppose first that p is inert in K. Recall that $(\mathcal{O}_K/p)^*$ has order p^2-1. Observe that φ maps bijectively $Gal(K_0/K)$ to $(\mathcal{O}_K/p)^*$ and the complex conjugation induces the unique non-trivial automorphism of K_p which is the Frobenius for $K_p/\mathbb{Q}_p$. Now if $Gal(K_0/K)$ acts as power of the cyclotomic character by $\overline{\varphi}_\theta \otimes \mathbb{Z}_p$, then we have $\overline{z}^a z^b = z^n \overline{z}^n$, we have $\overline{z}^a z^a z^{b-a} = z^n \overline{z}^n$, and then $z^{b-a} = z^{n-a} z^{p(n-a)}$ therefore $b-a \equiv (p+1)(n-a) mod(p^2-1)$, then $b-a \equiv 0 \ mod(p+1)$. Suppose now that p splits in K. Recall that $(\mathcal{O}_K/p)^*$ has order $(p-1)^2$, we study the map by $\overline{\psi_\theta} \otimes \mathbb{Z}_p$ restricted to the Galois groups $Gal(K(E[\mathfrak{p}^\infty])/K)$ and $Gal(K(E[\mathfrak{p}^{*\infty}])/K)$ where $p = \mathfrak{p}\mathfrak{p}^*$ is the decomposition of p in K. Then by [7, p.82] if is a n-power of the cyclotomic character we have $a \equiv b \ mod(p-1)$ thus $b-a \equiv 0 \ mod(p-1)$.

Then to obtain the result is enough to prove that $H^1(\Omega_n, \mathcal{O}_p/p^n)$ is zero because its inductive limit coincides with $H^1(Gal(K_\infty/K(\mu_{p^\infty})), (M_{\theta\mathbb{Z}_p}(1))')$, where $\mathcal{O}_p$ denotes once and for all $\mathcal{O}_K \otimes \mathbb{Z}_p$. Denote by Ω'_n the the prime p-part of Ω_n, therefore $H^i(\Omega'_n, \mathcal{O}_p/p^n) = 0$ for $i \geq 1$. If $H^0(\Omega'_n, \mathcal{O}_p/p^n)$ is not zero, say x a non-zero element, then $(1-z^a\overline{z}^b)x \equiv 0(mod\ p^n)$ in particular $1-z^a\overline{z}^b \equiv 0(mod\ p)$ $\forall z \in \Omega'_n$, but under $a-b \not\equiv 0(mod\ p+1)$ if p inert or $(mod\ p-1)$ if p splits we proved that is not the trivial character in $\mathcal{O}_p/p$, therefore $H^0(\Omega'_n, \mathcal{O}_p/p^n) = 0$. From the sequence:

$$0 \to H^1(\Omega_n/\Omega'_n, H^0(\Omega'_n, \mathcal{O}_p/p^n)) \to$$

$$H^1(\Omega_n, \mathcal{O}_p/p^n) \to H^1(\Omega'_n, \mathcal{O}_p/p^n),$$

we obtain that the middle term is zero, obtaining the result.

The last inclusion is the composition of the functor $Hom(-, M_{\theta\mathbb{Z}_p}(w))$ with the functor take invariants by H to the projection map $\mathcal{A}_\infty \to \tilde{\mathcal{A}}_\infty \to 0$. $\square$

Remark 2.5. *The hypothesis on the infinite type imposed in lemma 2.4 implies that the Galois action to $M_{\theta\mathbb{Z}_p}(w)$, restricted to $\Delta = Gal(K_0/K)$ is irreducible and is not any power of the cyclotomic character restricted to Δ. Therefore for any i the restriction to Δ of the Galois action on $M_{\theta\mathbb{Z}_p}(w+i)$ is irreducible and is not a power of the cyclotomic character.*

Lemma 2.6. *Under the conditions of the above lemma 2.4 we have that the following $\Lambda(cycl)$-modules $(R_{K(\hat{E}[p^\infty])})_{Gal(K(E[p^\infty])/K(\mu_{p^\infty}))}$ and $R_{K(\hat{\mu}_{p^\infty})}$ are isomorphic.*

Proof. Consider the following commutative diagram,

$$\begin{array}{ccccccc}
0 & \to & R^H_{K(E[p^\infty])} & \to & H^1(G_{K(E[p^\infty])}, (M_{\theta\mathbb{Z}_p}(w))')^H & \overset{r_{S,K_\infty}}{\longrightarrow} & P^H_{S,K(E[p^\infty])} \\
 & & \uparrow \alpha & & \uparrow \delta & & \\
0 & \to & R_{K(\mu_{p^\infty})} & \to & H^1(G_{K(\mu_{p^\infty})}, (M_{\theta\mathbb{Z}_p}(w))') & \overset{r_{S,K(\mu_{p^\infty})}}{\longrightarrow} & P_{S,K(\mu_{p^\infty})}.
\end{array} \tag{2}$$

Let us remember that $H^i(H, (M_{\theta\mathbb{Z}_p}(w))') = 0$ for $i = 1, 2$ (for $i = 1$ see the proof of lemma 2.4, and we remind $cd_pH \le 1$). Extending δ in the Hodschild-Serre spectral sequence we obtain that δ is an isomorphism.
We extend the commutative diagram (2), with a commutative diagram by the restriction maps (after checking that finite Galois cohomology groups maps to finite cohomology Galois groups):

$$\begin{array}{c}
P_{S,K(\mu_{p^\infty})} = \prod_{v\in S\setminus S_p} \frac{H^1(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Z}_p}(w))')}{H^1_f(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Z}_p}(w))')} \prod_{v\in S_p} H^1(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Z}_p}(w))') \\
\downarrow \prod \iota_v \\
P^H_{S,K_\infty} = \prod_{v\in S\setminus S_p} (\frac{H^1(K(E[p^\infty])_v, (M_{\theta\mathbb{Z}_p}(w))')}{H^1_f(K(E[p^\infty])_v, (M_{\theta\mathbb{Z}_p}(w))')})^H \prod_{v\in S_p} H^1(K(E[p^\infty])_v, (M_{\theta\mathbb{Z}_p}(w))')^H
\end{array}$$

We will prove that ι_v is a monomorphism, then α is an isomorphism, and taking Pontryagin duality we obtain the result.

We consider first the restriction map on local Galois cohomology groups. By the Hochschild-Serre spectral sequence the restriction map

$$H^1(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Z}_p}(w))') \to H^1(K(E[p^\infty])_v, (M_{\theta\mathbb{Z}_p}(w))')$$

has trivial kernel and cokernel if $H^i(H_v, (M_{\theta\mathbb{Z}_p}(w)')$ with $i = 1, 2$ are zero, where $H_v = Gal(K(E[p^\infty])_v/K(\mu_{p^\infty})_v)$. Effectively these Galois cohomology groups are zero: for $i = 2$ because $cd_pH_v \le 1$, and for $i = 1$ if H_v is a finite group, it has order prime to p, and then is clear, if not it has $\mathbb{Z}_p$ as a subgroup, denote by γ its generator, and because the kernel of $\psi_\theta \otimes \mathbb{Z}_p|H$ is not a finite extension of $K(\mu_{p^\infty})$, γ acts non-trivial on $M_{\theta\mathbb{Q}_p}(w)$ (a one dimensional $\mathcal{O}_K \otimes \mathbb{Q}_p$-module),thus $\gamma - 1$ acts exhaustively in $M_{\theta\mathbb{Q}_p}(w)/M_{\theta\mathbb{Z}_p}(w)$ and noticing $(M_{\theta\mathbb{Z}_p}(1))^{'} \cong M_{\theta\mathbb{Q}_p}(w)/M_{\theta\mathbb{Z}_p}(w)$, we obtain $H^1(H_v, (M_{\theta\mathbb{Z}_p}(w)^{'}) = 0$.

Let us now prove that ι_v is mono when $v \in S \setminus S_p$. Let us take α an element of $H^1(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Z}_p}(w)^{'})$ that by the restriction map maps to $j_v(\delta_v) \in H^1_f(K(E[p^\infty])_v, (M_{\theta\mathbb{Z}_p}(w)^{'}) = j_v(H^1_f(K(E[p^\infty])_v, (M_{\theta\mathbb{Q}_p}(w-1))^*)$ where * means $Hom(-, \mathbb{Q}_p)$. Consider the following commutative diagram:

$$\begin{array}{lcr}
H^2(H_v, (M_{\theta\mathbb{Q}_p}(w-1))^*) = 0 & & \\
\uparrow & & \\
H^1(K(E[p^\infty])_v, (M_{\theta\mathbb{Q}_p}(w-1))^*) & \stackrel{res_1}{\longrightarrow} & H^1(I_{K(E[p^\infty])_v}, (M_{\theta\mathbb{Q}_p}(w-1))^*) \\
\uparrow res & & \uparrow res \\
H^1(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Q}_p}(w-1))^*) & \stackrel{res_2}{\longrightarrow} & H^1(I_{K(\mu_{p^\infty})_v}, (M_{\theta\mathbb{Q}_p}(w-1))^*) \\
\uparrow & & \uparrow \\
H^1(H_v, (M_{\theta\mathbb{Q}_p}(w-1))^*) & & H^1(I_{K(\mu_{p^\infty})_v}/I_{K(E[p^\infty])_v}, (M_{\theta\mathbb{Q}_p}(w-1))^*) = 0 \\
 & & \uparrow \\
 & & 0
\end{array},$$

where res and res_i are restriction maps. The kernels of res_1 and res_2 are the finite Galois cohomology group for the p-adic realization $(M_{\theta\mathbb{Q}_p}(w-1))^*$, (by the ker-cokern exact sequence we cab obtain that ι_v is well defined). Observe, if $res(\alpha) \in j(H^1_f(K(E[p^\infty])_v, (M_{\theta\mathbb{Q}_p}(w-1))^*)$, $res(\alpha) = j(\delta_\alpha)$, then by the surjection between the kerns of res_i and the compatibility of j_v with restriction maps we have $res(\alpha) = j_v(\delta_\alpha) = j_v(res(\tilde{\delta}_\alpha)) = res(j_v(\tilde{\delta}_\alpha))$ with $\tilde{\delta}_\alpha$ in $H^1_f(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Q}_p}(w-1))^*)$ then by the injectivity of restriction map proved above, we have $\alpha \in H^1_f(K(\mu_{p^\infty})_v, (M_{\theta\mathbb{Q}_p}(w))^{'})$.

$\square$

Remark 2.7. *Let be F the cyclotomic $\mathbb{Z}_p$-extension of K. The above results, lemaes 2.4 and 2.6 are also true with the field F instead of $K(\mu_{p^\infty})$. The proofs needs minor changes.*

3 Weak Leopoldt's conjecture for Hecke characters

Let us take the following decomposition

$$Gal(K(E[p^\infty])/K) \cong \Gamma_{cycl} \times \Gamma_{anticyc} \times \Delta \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \Delta$$

where $\Gamma_{cycl} =< \sigma >$, σ generates the Galois group of the cyclotomic extension of K, and $\Gamma_{anticyc} =< \tau >$, where τ generates the extension $K(E[p^\infty])^\Delta / K(\mu_{p^\infty})^{\Delta'}$ where $\Delta' \cong Gal(K(\mu_p)/K)$ (we think also Δ' as a subgroup of Δ and τ as generator for the anticyclotomic $\mathbb{Z}_p$-extension of K).
Denote by $K_{cyc} = K(\mu_{p^\infty})^{\Delta'}$ the cyclotomic $\mathbb{Z}_p$-extension of K.
For Iwasawa modules notation in two variables we choose once and for all the notation $\sigma = T_c + 1$ and $\tau = T_a + 1$, therefore any $\mathbb{Z}_p[[Gal(K(E[p^\infty])^\Delta)/K]]$-module we identify with a $\mathbb{Z}_p[[T_a, T_c]]$-module.

The aim of this section is to obtain that the $\mathbb{Z}_p[\Delta'][[T_c]]$-module $R_{\hat{K(\mu_{p^\infty})}}$ is torsion, therefore the weak conjectures for Hecke characters. We proved in §2 that $R_{K(\hat{E[p^\infty]})}$ is Λ-torsion and $R_{K(\hat{\mu_{p^\infty}})}$ is obtained by taking coinvariants from $R_{K(\hat{E[p^\infty]})}$ (Lemma 2.6). Therefore we need to ensure that the characteristic power of the Λ-module $R_{K(\hat{E[p^\infty]})}$ does not vanish when we take coinvariants.

Let us denote by κ_i the $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p =: \mathcal{O}_p$-character giving the action of the Galois group $Gal(K(E[p^\infty])/K)$ on $M_{\theta\mathbb{Z}_p}(w+i)$, with i an integer.

Define

$$e_i(T_a) := (\kappa_i^{-1}(\tau)(T_a+1)-1),$$

which does not depend of i, then we denote $e_i(T_a)$ by $e(T_a)$ (here $\kappa_i^{-1}(\tau) \in \mathcal{O}_p$ is given by κ_i applied to τ and taking it inverse in $\mathcal{O}_p^*$). Observe moreover that κ_i is given by the action $\varphi^a\overline{\varphi}^b\chi_{cycl}^i = \varphi^{max(0,a-b)}\overline{\varphi}^{max(0,b-a)}\chi_{cycl}^{min(b,a)+i}$ on $\mathcal{O}_K \otimes \mathbb{Z}_p \cong M_{\theta\mathbb{Z}_p}(w+i)$ (or on $e_\theta(\otimes^w T_pE)(i)$ by $\overline{\varphi^a\overline{\varphi}^b}\chi_{cycl}^i$), where χ_{cycl} denotes the cyclotomic character. Denote by $\tilde{w} := max(a-b, b-a)$. Because for defining e uses only the antyciclotomic generator, we can define e_i changing κ_i by $\tilde{\kappa}$ where $\tilde{\kappa}$ is the action given by $\varphi^{max(0,a-b)}\overline{\varphi}^{max(0,b-a)}$, which is the action of the Galois group on $M_{(\tilde{w},0)\mathbb{Z}_p}(\tilde{w})$ if $\tilde{w} = a-b$ or on $M_{(0,\tilde{w})\mathbb{Z}_p}(\tilde{w})$ if $\tilde{w} = b-a$.

Denote by χ_i the restriction of κ_i to the subgroup Δ, which is irreducible (see remark 2.5), and let us consider the idempotent:

$$e_{\chi_i} := \frac{1}{\#\Delta}\sum_{\tau\in\Delta}(\chi_i(\tau))\tau^{-1} \in (\mathcal{O}_K \otimes \mathbb{Z}_p)[\Delta].$$

For every $\mathbb{Z}_p[\Delta]$-module Z we denote by $(Z \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^{\chi_i}$ or Z^{χ_i} the module $e_{\chi_i}(Z \otimes_{\mathbb{Z}_p} \mathcal{O}_K)$, and observe that $\Lambda^{\chi_i} \cong \mathcal{O}_p[[T_c, T_a]]$.

Lemma 3.1. *$e_\chi R_{K(\hat{\mu_{p^\infty}})}$ is a $\Lambda(cycl)_\chi \cong \mathcal{O}_p[[T_c]]$-torsion module for all p-adic characters of Δ' if $e(T)$ does not divide the characteristic power series $f_i(T_a, T_c)$ of the $(\mathbb{Z}_p[[Gal(K(E[p^\infty])/K)]])^{\chi_i}$-module*

$$(\mathcal{A}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^{\chi_i},$$

for all i between 0 and $p-2 = \#\Delta' - 1$.

Proof. Let $g_i(T_c, T_a)$ denote the characteristic power series for

$$Hom_{\mathcal{O}_K\otimes\mathbb{Z}_p}(M_{\theta\mathbb{Z}_p}(w+i), (\mathcal{A}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^{\chi_i})$$

the $(\mathbb{Z}_p[[Gal(K(E[p^\infty])/K)]])^{\chi_i} \cong \mathcal{O}_p[[T_c, T_a]]$-module. The characteristic power series $f_i(T_c, T_a)$ and $g_i(T_c, T_a)$ are related by

$$f_i(T_c, T_a) = g_i(\kappa_i^{-1}(\sigma)(T_c+1)-1, \kappa_i^{-1}(\tau)(T_a+1)-1),$$

(remember that $M_{\theta\mathbb{Z}_p}(w+i)$ is a rank 1 free $\mathcal{O}_p$-module, and use the identification $Hom_{\mathbb{Z}_p}(M_{\theta\mathbb{Z}_p}(w+i), M)^\Delta = Hom_{\mathcal{O}_K\otimes\mathbb{Z}_p}(M_{\theta\mathbb{Z}_p}(w+i), (M \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^{\chi_i})$). Observes that $e(T_a)$ divides $f_i(T_c, T_a)$ if and only if T_a divides $g_i(T_c, T_a)$.

Let us now consider the $\Lambda(cycl)$-module $R_{K(\hat{\mu_{p^\infty}})}$. By the idempotent decomposition, in order to prove that $e_\chi R_{K(\hat{\mu_{p^\infty}})}$ is $\Lambda(cycl)_\chi$-torsion is enough to prove $e_{\chi_i}((R_{K(\hat{E[p^\infty]})})_{\Gamma_{anticyc}})$ is a $\mathcal{O}_p[[T_c]]$-torsion for all i between 0 and $p-2$. Rewrite now §2, with $w+i$ instead of w, denote by R^i instead of R the Selmer groups defined by $w+i$ instead of w in the definition. For $i>0$ is equivalent to prove that $e_{\chi_i}((R_{K(\hat{E[p^\infty]})})_{\Gamma_{anticyc}})$ is $\mathcal{O}_p[[T_c]]$-torsion that $e_{\chi_i}((R^i_{K(\hat{E[p^\infty]})})_{\Gamma_{anticyc}})$ is $\mathcal{O}_p[[T_c]]$-torsion. We have seen above in this proof that $e_{\chi_i}((R^i_{K(\hat{E[p^\infty]})})_{\Gamma_{anticyc}})$ is $\mathcal{O}_p[[T_c]]$-torsion if $e_i(T_a)$ does not divide the polynomial $g_i(T_c, T_a)$. □

We do an explicit argument only in the non-split situation. Similar results are in the split situation following the argument of [15, §4] with the results of de Shalit [6] with lemma 3.3. The split situation is known to the specialists on the subject (nevertheless see remark 3.7).

Let us impose that p is inert in K from now on. We study this non-divisibility inside the Coates-Wiles morphisms. We impose moreover that $a > b$ once and for all. The condition $a > b$ allow us to give precise references of the properties that we need here for obtaining the results (properties (3), (4) and (5)), and moreover we have here a very good understanding which is different from the situation $b > a$ see for example [1]. I believe that with the work of Tsuji [19] and Kato [10] and using the dual exponential map instead of Coates-Wiles morphism which is the morphism that we use here something can be done when $b > a$.

We follow the definition of elliptic units $\overline{\mathcal{C}}_\infty$ given by Rubin in [17] once and for all. U_∞ denotes the projective limit of the principal local units at p for the tower of fields $K(E[p^n])$. Consider first the $\tilde{w}$-th Coates-Wiles logarithmic derivative

$$\delta_{\tilde{w},0} : U_\infty \to \mathcal{O}_K \otimes \mathbb{Z}_p$$
$$u \mapsto (D^k log g_u(Z))|_{Z=0},$$

where D is defined for example in [17, 7.19], and g_u means the Coleman power series associated to u.

The Coates-Wiles homomorphism is compatible with the Iwasawa action,

$$\delta_{\infty,[\tilde{w}]}(Fu) = \varphi^w(F)\delta_{\tilde{w},0}(u) = F(\tilde{\kappa}(\sigma) - 1, \tilde{\kappa}(\tau) - 1)\delta_{\tilde{w},0}(u) \tag{3}$$

where F any element of $\mathcal{O}_p[[Gal(K_\infty/K_0)]]$ where $\tilde{\kappa}$ the Galois action given in $M_{(\tilde{w},0)}(\tilde{w})$.

We have the following result [17, Theorem 7.22]:

• Consider the elliptic unit $\eta_{\mathfrak{a}}$ where $\mathfrak{a}$ is any ideal prime to $6p\mathfrak{f}$ defined in [17]. Then,

$$\delta_{\infty,[\tilde{w}]}(\eta_{\mathfrak{a}}\mathcal{O}_p) = (\tilde{w} - 1)!(N_{K/\mathbb{Q}}(\mathfrak{a}) - \tilde{\kappa}(\mathfrak{a}))\Omega^{-\tilde{w}} L(\overline{\tilde{\kappa}}, \tilde{w})\mathcal{O}_p, \tag{4}$$

where Ω is the period of the elliptic curve in the situation $(\tilde{w}, 0)$.

For our application with $\tilde{w} = 1$ we need a twisted version of the above $1th$-Coates-Wiles morphism, this is mainly done in [14, §2], we follow now [14] (now $\tilde{w} = 1$). The map defined by δ_n in [14, §2] is now,

$$\delta_{n,[\tilde{w}]} : U_\infty \to K_{\mathfrak{p}}(E_{p^{n+1}})$$

where $K_{\mathfrak{p}}$ is the completion at p of K.

Let us consider ρ a finite character of $Gal(K(E[p^\infty])/K(E[p]))$ which conductor divides p^{n+1} which is a finite character ρ of $Gal(K(E[p^\infty])/K(E[p]))$. The following map does not depends of n under the divisibility condition such that ρ factors through $Gal(K(E[p^{n+1}])/K)$ ([14, §2]):

$$\delta_{\rho,[\tilde{w}]} : U_\infty \to K_{\mathfrak{p}}(E[p^\infty])$$

defined by

$$\delta_{\rho,[\tilde{w}]}(u) := \frac{1}{p^{n+1}} \sum_{\gamma \in Gal(K(E[p^{n+1}])/K)} \rho(\gamma)\delta_{n+1,[\tilde{w}]}(u)^\gamma.$$

This map has the following two properties:

•[14, Lemma 2.1.(iii)]

$$\delta_{\rho,[\tilde{w}]}(hu) = h(\tilde{\kappa}\rho^{-1}(\sigma) - 1, \tilde{\kappa}\rho^{-1}(\tau) - 1)\delta_{\rho,[\tilde{w}]}(u)$$

for any $u \in U_\infty$ and $h \in \mathcal{O}_K \otimes \mathbb{Z}_p[[Gal(K(E[p^\infty])/K(E[p]))]]$. Here $\tilde{\kappa}$ denotes the action given over $M_{\tilde{w},0}(\tilde{w})$.

• (Coates-Wiles) Consider $\mathfrak{a}$ an ideal prime to $6p\mathfrak{f}$ satisfying $N_{K/\mathbb{Q}}(\mathfrak{a}) - \tilde{\kappa}(\mathfrak{a}) \not\equiv 0$ in $\mathcal{O}_p/p$. Exist an elliptic unit $\eta_{\mathfrak{a}}$ such that

$$\delta_{\rho,[\tilde{w}]}(\eta_{\mathfrak{a}}\mathcal{O}_p) = (\tilde{w}-1)!\Omega^{-\tilde{w}}(1 - \frac{\overline{\tilde{\kappa}}(\mathfrak{p})}{(p^2)^{\tilde{w}}})L(\rho\overline{\tilde{\kappa}}, \tilde{w})\mathcal{O}_p, \qquad (5)$$

where Ω is the period of the elliptic curve E (see [11, Theorem 11]) if $a > b$.

We know that the χ_i are non-trivial character and are not any power of the cyclotomic character under the divisibility conditions of lemma 2.4 (i.e. $p+1 \nmid \tilde{w} = |b-a|$), and moreover we impose here, in order to use Rubin's result on the main Iwasawa conjecture [16], that χ_i are nontrivial on the decomposition group of $\mathfrak{p}$ in Δ where $\mathfrak{p}$ is the prime of K above p. Under these conditions, we choose some concrete ideal $\mathfrak{a}_i$ of $\mathcal{O}_K$, which satisfies $\kappa_i(\mathfrak{a}_\mathfrak{i}) \not\equiv N_{K/\mathbb{Q}}(\mathfrak{a}_\mathfrak{i})$ mod$p$ (see for the existence of $\mathfrak{a}_\mathfrak{i}$ with this property [2, Remark 5.8]), then is known that $\eta_{\mathfrak{a}_i}$ generates the rank 1 Iwasawa module $\overline{\mathcal{C}}_\infty^{\chi_i}$ of elliptic units and we can choose such that $(N_{K/\mathbb{Q}}(\mathfrak{a}_i) - \tilde{\kappa}(\mathfrak{a}_i)) \neq 0$, thus we will apply the equality (4) with $\mathfrak{a} = \mathfrak{a}_i$. In the situation with $\tilde{w} = 1$, one can choose an $\mathfrak{a}$ as in equation (5) such that $N_{K/\mathbb{Q}}(\mathfrak{a}) - \kappa_i(\mathfrak{a}) \not\equiv 0$ in $\mathcal{O}_p/p$ because the χ_i are not any power of the cyclotomic character for any i between 0 and $p-1$.

Proposition 3.2. *Suppose that χ_i is nontrivial on the decomposition group of $\mathfrak{p}$ in Δ where $\mathfrak{p}$ is the prime of K above p, and that $p+1 \nmid \tilde{w}$ with $a > b$. Then, $e(T_a)$ does not divide $f_i(T_c, T_a)$ for all $i = 0, \ldots, p-2$.*

Proof. By Rubin's theorem on Iwasawa main conjecture [16] (from which we impose the assumption in the theorem), and McConnell result [11, prop.13,remark p.171] we have that $\overline{\mathcal{C}}_\infty^{\chi_i} \cong f_i(T_c, T_a)\overline{\mathcal{E}}_\infty^{\chi_i}$, where $\overline{\mathcal{C}}_\infty$ denotes the elliptic units for the imaginary quadratic field K given in ([17]) and $\overline{\mathcal{E}}_\infty$ is the clousure of the global units of K_∞ see for example [15].

By construction $e(\tilde{\kappa}(\tau) - 1) = 0$, and suppose that $e(T_a)$ divide $f_i(T_c, T_a)$. Consider first $\tilde{w} > 1$. The map $\delta_{\infty,[\tilde{w}]}$ satisfies:

$$\delta_{\infty,[\tilde{w}]}(f_i u) = f_i(\tilde{\kappa}(\sigma) - 1, \tilde{\kappa}(\tau) - 1)\delta_{\infty,[\tilde{w}]}(u) = 0,$$

with $u \in \overline{\mathcal{E}}_\infty^{\chi_i}$ therefore using the property (4) we have $L(\overline{\tilde{\kappa}}, \tilde{w}) = 0$ which is impossible by lemma 3.3.

Let us consider now $\tilde{w} = 1$. We follow the argument used for $\tilde{w} > 1$ but we need to consider the characters ρ in order to apply Rohrlich's result [13, Theorem 1] (lemma 3.3). Let us consider Dirichlet characters ρ such that factorizes in the cyclotomic $\mathbb{Z}_p$-extension we have,

$$\delta_{\rho,[\tilde{w}]}(f_i u) = f_i(\rho^{-1}\tilde{\kappa}(\sigma) - 1, \tilde{\kappa}(\tau) - 1)\delta_{\rho,[\tilde{w}]}(u) = 0,$$

for all these characters ρ, therefore by (5) one obtains $L(\rho\overline{\tilde{\kappa}}_w, \tilde{w}) = 0$ with contradiction with Rohrlich result [13, Theorem 1] (or lemma 3.3). □

Lemma 3.3. *Let be f the cusp form which it corresponds to $\overline{\varphi}^{\tilde{w}}$ or $\varphi^{\tilde{w}}$ which has weight $\tilde{w}+1$ (see [13, §1] and references there to the associated cusp form). Then for all but finitely many Dirichlet characters ρ unramified at $\mathfrak{p}$ the primes of K above p and also unramified at infinity, we have*

$$L(i, f, \rho) \neq 0,$$

for any fixed integer i between $[\tilde{w}/2]+1$ and $\tilde{w}$ (where $[r]$ denotes the integer part of r). Moreover, with the exception for $\tilde{w}$ odd and applied at integer $[\tilde{w}/2]+1$, the above result is true for any Dirichlet character ρ.

Proof. The main part is the deep theorem of Rohrlich [13, Theorem 1]. He proves the result at the integer value $[\tilde{w}/2]+1$ for $\tilde{w}$ odd, which is true for almost all Dirichlet characters with the imposed unramified condition. The proof in the other situations are classically known for any Dirichlet character ρ. □

Theorem 3.4 (Weak Jannsen's conjecture). *Let K be an imaginary quadratic field of class number 1. Let p be a prime which E has supersingular reduction over the primes over p in K (p is inert). Let S be a set of primes of K which contains the primes of K over p and the primes of $\mathfrak{f}$. Let us impose that θ is of infinite type (a, b) and satisfy firstly $p+1 \nmid \tilde{w} = |b-a| = a-b$ and secondly that χ_i is nontrivial on the decomposition group of $\mathfrak{p}$ in Δ where $\mathfrak{p}$ is the prime of K above p.*

Then

$$H^2(G_S, M_{\theta\mathbb{Q}_p}(n)) = 0$$

for all n up to a finite number of Tate twists n.

Proof. From proposition 3.2 and lemma 3.1 we have that the Pontryagin dual of R_{cycl} is $\Lambda(cycl)$-torsion. Use now lemma 2.3 and propositions 1.4, 1.5 to conclude. □

Theorem 3.5 (weak Leopoldt's conjecture). *With the same hypotheses of theorem 3.4, we have that $e_\chi \varprojlim_m H^2(K_S/K(\mu_{p^m}), M_{\theta\mathbb{Z}_p})$ is a torsion $\Lambda(cycl)_\chi$-module.*

Proof. Straightforward by proposition 1.5. □

In the literature appears also the so called Mazur, Schneider and Wingberg conjecture which in our situation asserts that the cohomology group

$$H^2(Gal(K_S/K_{cyc}), (M_{\theta\mathbb{Z}_p}(1))')$$

should vanish (observe that $(M_{\theta\mathbb{Z}_p}(1))' \cong e_\theta(\otimes^w E[p^\infty])$).

Theorem 3.6 (Mazur/Schneider/Wingberg conjecture). *Let K be an imaginary quadratic field of class number 1. Let p be a prime which E has good supersingular reduction at the prime above p in K. Let S be a set of primes of K which contains the primes of K over p and the primes of $\mathfrak{f}$. Let us impose that the infinite type (a, b) satisfies $p+1 \nmid \tilde{w} = |b-a| = a-b$. Then,*

$$H^2(Gal(K_S/K_{cyc}, (M_{\theta\mathbb{Z}_p}(1))') = 0.$$

Proof. Observe first $(M_{\theta\mathbb{Z}_p}(1))' \cong (M_{\theta\mathbb{Q}_p}(1))^*(1)/(M_{\theta\mathbb{Z}_p})(1))^*(1)$ (we follow the notation introduced in §1). From [11, Theorem 2] the vanishing of the Galois group is equivalent that $H^2(Gal(K_S/K_{cycl}), M_{\theta\mathbb{Z}_p}(1))$ is a $\mathcal{O}_p[[T_c]]$-torsion module. That this module is torsion, is equivalent to $R^{\hat{1-w}}_{K_{cyc}}$ is $\mathcal{O}_p[[T_c]]$-torsion (proof of lemma 2.3 with remark 2.7). Apply lemma 3.1 and proposition 3.2 to conclude. □

Remark 3.7 (The ordinary situation). *Suppose that p splits in K. Let us sketch here some traces of the proof. We impose now that $p-1 \nmid \tilde{w} = |b-a|$. We follow arguments closely to Rubin's arguments in [15, §3,§4] (which prove the result when θ has infinite type $(1,0)=(a,b)$). Define the Selmer group R in definition 2.1 without taking any condition at places over p (i.e. the restricted Selmer group instead of the extended one). Then, the arguments through §2 are unchanged with $\mathcal{X}_\infty$ instead of $\mathcal{A}_\infty$ or $\tilde{\mathcal{A}}_\infty$ (in the ordinary situation $\mathcal{X}_\infty$ is a Λ-torsion module). The annihilator for $\mathcal{X}_\infty$ by Rubin's proof of the Iwasawa main conjecture is the annihilator for global units modulo $\mathcal{C}_\infty$. Therefore with [6, Theorem 4.14] we can obtain that the L function associated to the Hecke character annihilates the Selmer group following [15, Lemma 4.1,Theorem 4.2](this is the translation in the ordinary situation of our results on non-divisibility in §3). Now use proposition 3.3 in order to obtain that the dual of the Selmer group at $K(\mu_{p^\infty})$ and at K_{cycl} is torsion (following [15, Theorem 4.4]).*

Acknowledgments

It is a big pleasure to thank Matthias Flach. After a very preliminary version of the paper [3], i sent it to him with the natural question: is already done the result for almost all twists? He very kindly answer me the main ideas for what should be the proof in the ordinary case and his convince that in the supersingular situation something could be done. This is the work from his very kind e-mail. I am also happy to thank Luis Dieulefait and Xavier Xarles for their encouragement.

References

[1] *A.Agboola*: On Rubin's variant of the p-adic Birch and Swinnerton-Dyer conjecture. Preprint available at www.math.ucsb.edu/∼agboola/

[2] *F.Bars*: The Tamagawa number conjecture for Hecke characters. Preprint Universitat Autònoma de Barcelona 2002. Submitted. http://mat.uab.es/∼francesc/TamagHecke.pdf

[3] *F. Bars*: On Jannsen's conjecture for Hecke characters of imaginary quadratic fields. Preprint Math.Depart. UAB, July 2005. Submitted.

[4] *Coates, J. and Wiles, A.*: On the conjecture of Birch and Swinnerton-Dyer. Invent.Math. 39, 223-251 (1977).

[5] *C. Deninger*: Higher regulators and Hecke L-series of imaginary quadratic field II. Ann. of Math.(2) 132, 131-158 (1990).

[6] *E. de Shalit*: Iwasawa Theory of Elliptic Curves with Complex Multiplication. Persp. in Mathematics 3, Acad.Press (1987).

[7] *T. Geisser*: A p-adic analogue of Beilinson's conjectures for Hecke characters of imaginary quadratic fields. Schriftenreihe des Math.Inst. der Universität Münster, 3 Serie, Helft 14(1995) Thesis.

[8] *U. Jannsen*: On the l-adic cohomology of varieties over number fields and its Galois cohomology. In: Ihara et al. (eds.): Galois groups over $\mathbb{Q}$, MSRI Publication (1989).

[9] *K. Kato*:Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. In Arithmetic Algebraic Geometry (Trento, 1991), LNM 1553, 50-163 (1993).

[10] *K. Kato*: p-adic Hodge theory and values of zeta functions of modular forms. Cohomologies p-adiques et applications arithmétiques. III. Astérisque No. 295, ix, 117–290, (2004).

[11] *G. McConnell*: On the Iwasawa theory of CM elliptic curves at supersingular primes. Compositio Math. 101, 1-19 (1996).

[12] *B. Perrin-Riou*: Fonctions L p-adiques des représentation p-adiques. Astérisque 229, 198pp. (1995).

[13] *D. E. Rohrlich*: L-functions and division towers. Math.Ann. 281, 611-632 (1988).

[14] *K. Rubin*: Local units, elliptic units, Heegner points and elliptic curves. Invent.Math. 88, 405-422 (1987).

[15] *K. Rubin*: On the main conjecture of Iwasawa theory for imaginary quadratic fields. Invent. Math. 93, 701-713 (1988).

[16] *K. Rubin*: The "main conjectures" of Iwasawa theory for imaginary quadratic fields. Invent. math. 103, 25-68 (1991).

[17] *K. Rubin*: Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer, in: Arithmetic Theory of Elliptic Curves. Lecture Notes in Mathematics 1716, Springer (1999).

[18] *K. Rubin*: Euler systems. Annals of Mathematics Studies, 147.Hermann Weyl Lectures. Princeton Univ. Press, Princeton, NJ 2000.

[19] *T.Tsuji*: Explicit reciprocity law and formal moduli for Lubin-Tate formal groups. J. reine angew. Math. 569, 103-173 (2004).

[20] *L.C.Washington*: Galois cohomology. In Modular Forms and Fermat's Last Theorem (Boston, MA, 1995), 101–120, Springer, New York, 1997.

Francesc Bars Cortina
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
Catalonia
Spain
e-mail: francesc@mat.uab.cat