SELF IMPROVING SOBOLEV-POINCARÉ INEQUALITIES, TRUNCATION AND SYMMETRIZATION

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Abstract. In [12] we developed a new method to obtain symmetrization inequalities of Sobolev type for functions in $W_0^{1,1}(\Omega)$. In this paper we extend our method to Sobolev functions that do not vanish at the boundary.

1. Introduction

In our recent paper [12] we developed a new principle of "symmetrization by truncation" to obtain symmetrization inequalities of Sobolev type via truncation. In this note we consider the corresponding results for Sobolev spaces on domains, without assuming that the Sobolev functions vanish at the boundary.

The explicit connection between Sobolev-Poincaré inequalities and isoperimetric inequalities appears in the work of Maz'ya. In [13] it is shown that if $\Omega \subset \mathbb{R}^n$ is an arbitrary open set with finite volume, $1 \leq p \leq n/(n-1)$, then the Sobolev-Poincaré

$$\left(\int_\Omega |f(x) - f_\Omega|^p \, dx\right)^{1/p} \leq C \int_\Omega |\nabla f(x)| \, dx, \ \forall f \in W^{1,1}(\Omega), \tag{1.1}$$

($f_\Omega = \frac{1}{|\Omega|}\int_\Omega f$) holds if and only if the following $p-$isoperimetric inequality is satisfied: there exists a constant $M \in (0, |\Omega|)$ such that

$$U_{1/p}(M) = \sup \frac{|\mathcal{S}|^{1/p}}{s(\partial \mathcal{S})} < \infty, \tag{1.2}$$

where the sup is taken over all $\mathcal{S}$ open bounded subsets of Ω such that $\Omega \cap \partial\mathcal{S}$ is a manifold of class C^∞ and $|\mathcal{S}| \leq M$, and s denotes the $(n-1)-$dimensional area. If (1.2) is satisfied we shall say that Ω belongs to the Maz'ya class $\mathcal{J}_{1/p}$. For example, if Ω is a bounded domain, starshaped with respect to a ball, or having the cone property, or Ω is a Lipschitz domain, then Ω belongs to the class $\mathcal{J}_{1-1/n}$; if Ω is a $s-$John domain then $\Omega \in \mathcal{J}_{(n-1)s/n}$; if Ω is a domain with one $\beta-$cusp then it belong to the Mazy'a class $\mathcal{J}_{\frac{\beta(n-1)}{\beta(n-1)+1}}$ (cf. [13], [3]).

Sobolev-Poincaré inequalities are known to self improve. For example, if (1.1) holds for $p = \frac{n}{n-1}$, then (cf. [17, Theorem 2.4.1]) the inequality

$$\left(\int_\Omega |f(x) - f_\Omega|^{pn/(n-p)} \, dx\right)^{\frac{n-p}{np}} \leq C \left(\int_\Omega |\nabla f(x)|^p \, dx\right)^{1/p},$$

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holds for $1<p<n$. More generally, if $|\Omega|<\infty$, and if inequality (1.1) holds for a fixed p, $1\leq p\leq n/(n-1)$, then

$$\left(\int_{\Omega}|f(x)-f_{\Omega}|^{s}\,dx\right)^{1/s}\leq C\left(\int_{\Omega}|\nabla f(x)|^{q}\,dx\right)^{1/q}, \tag{1.3}$$

where $q<p/(p-1)$ and $s=pq/(p+q-pq)$. In particular, in some sense, "all" L^p Sobolev-Poincaré inequalities follow from the Sobolev-Poincaré inequality (1.1) or, equivalently, from a suitable version of an isoperimetric inequality.

As is well known, the sharp versions of these L^p inequalities fall outside the L^p scale and need to be formulated using $L(p,q)$ spaces. Recently (cf. [1], [14], [10]), we have shown that using a simple modification of the definition of the $L(p,q)$ spaces we also obtain the "best" results including the problematic borderline inequalities. Moreover, these sharper limiting results cannot be obtained using, for example, the usual extrapolations from the L^p inequalities but require new sharp symmetrization inequalities.

More generally, symmetrization inequalities play a fundamental role in the study of Sobolev-Poincaré inequalities in the general setting of rearrangement invariant spaces. In our program we formulate self improving properties of Sobolev-Poincaré inequalities in terms of symmetrization inequalities. In this fashion instead of showing that a particular inequality implies other inequalities one case at a time, we aim to prove a symmetrization inequality that implies "all" other Sobolev-Poincaré inequalities. One difficulty in dealing with rearrangement inequalities on domains is that the usual inequalities are only valid for certain range of the values of the variable. For example, suppose that for some $1<p\leq n/(n-1)$, the Sobolev-Poincaré inequality (1.1) holds, then (cf. [10]),

$$f^{**}(t)-f^{*}(t)\leq Ct^{1-1/p}\left|\nabla f\right|^{**}(t),\ t\in(0,\left|\Omega\right|/2),\ f\in W^{1,1}(\Omega), \tag{1.4}$$

where $f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)ds$. However in [10] we show that if we work with symmetrization inequalities of "Sobolev-Poincaré" type (i.e. inequalities where f is replaced by $f-f_{\Omega}$) then we can eliminate the restriction $t\in(0,\left|\Omega\right|/2)$ in (1.4). Indeed, under the assumption that (1.1) holds for some $1<p\leq n/(n-1)$, then in [10] we showed that, for all $f\in W^{1,1}(\Omega)$, we have

$$\inf_{c\in\mathbb{R}}\left((f-c)^{**}(t)-(f-c)^{*}(t)\right)\leq C_{\Omega}t^{1-1/p}\left|\nabla f\right|^{**}(t),\ \text{a.e. } t\in(0,|\Omega|). \tag{1.5}$$

Notice that (1.5) implies that for any r.i. space $X(0,|\Omega|)$, with upper Boyd[1] index $\beta_X<1$, we have (cf. [10])

$$\inf_{c\in\mathbb{R}}\left\|t^{1/p-1}\left(f-c\right)^{**}(t)-\left(f-c\right)^{*}(t)\right\|_{X}\leq C\left\|\nabla f\right\|_{X},$$

where $C=C(n,\left|\Omega\right|,X)$. For example, if $X=L^q$, $q>1$, $q<\frac{p}{p-1}$, and $s=pq/(p+q-pq)$, then

$$\left\|f-f_{\Omega}\right\|_{L^{s,q}(\Omega)}\leq C\left\|\nabla f\right\|_{L^{q}(\Omega)},\quad\forall f\in W^{1,q}(\Omega).$$

Since $L^{s,q}(\Omega)\subset L^{s}(\Omega)$, for $s>q$, this last inequality is the well known (optimal) improvement of (1.3). Moreover, in the limiting case $q=\frac{p}{p-1}$, then $s=\infty$ and we

[1] The restriction on the Boyd indices is only required to guarantee that the inequality $\left\|g^{**}\right\|_X\leq c_X\left\|g\right\|_X$, holds for all $g\in X$.

obtain

$$\inf_{c\in\mathbb{R}} \|f-c\|_{L^{\infty,q}(\Omega)} \leq C\,\|\nabla f\|_{L^q(\Omega)}\,, \tag{1.6}$$

where

$$L^{\infty,q}(\Omega)=\left\{f:\|f\|^q_{L^{\infty,q}(\Omega)}=\int_0^{|\Omega|}\left(f^{**}(t)-f^*(t)\right)^q\frac{dt}{t}<\infty\right\}.$$

Once again since $L(\infty,q)(\Omega)\subset BW^q(\Omega)\subset e^{L^{q'}}(\Omega)$ (see [1]) we see that (1.6) is a sharpening of the classical limiting inequalities of Trudinger-Brezis-Wainger-Hansson. It follows that if we redefine the $L(p,q)$ spaces, $1\leq p\leq\infty$, $1\leq q\leq\infty$, using

$$\|f\|^q_{L^{p,q}(\Omega)}=\int_0^{|\Omega|}\left(f^{**}(t)-f^*(t)\right)^q t^{q/p}\frac{dt}{t},$$

then we have an attractive unified way to formulate the sharp form of the Sobolev-Poincaré inequalities, namely

$$\inf_{c\in\mathbb{R}} \|f-c\|_{L^{s,q}(\Omega)} \leq C\,\|\nabla f\|_{L^q(\Omega)}\,, 1<q\leq p/(p-1), s=pq/(q+p-pq). \tag{1.7}$$

One possible objection to (1.7) is that the important case $q=1$ is excluded. The cause for this imperfection is the presence of the "double star" operation on right hand side of (1.5). On the other hand, (1.5), for $q=1$, readily implies

$$\|f-f_\Omega\|_{L^{p,\infty}(\Omega)} \leq C\,\|\nabla f\|_{L^1(\Omega)}\,, \tag{1.8}$$

and therefore, by the truncation principle of Maz'ya (cf. [5]), we can see that (1.8) self-improves to (1.1) and even to the sharper form of the Gagliardo-Nirenberg inequality (cf. [9]),

$$\|f-f_\Omega\|_{L^{p,1}(\Omega)} \leq C\,\|\nabla f\|_{L^1(\Omega)}\,.$$

The ad-hoc argument that we needed to cope with the limiting case suggested to us that one should be able to find a sharpening of the symmetrization inequality (1.5) that implies all the Sobolev-Poincaré inequalities directly. In the case of functions vanishing at the boundary of Ω we have shown that this is indeed the case in [12]. One of the objectives of this paper is to formulate the correspoding inequalities without assuming that the Sobolev functions vanish at the boundary. Our first result is the following

Theorem 1. *Let Ω be a domain of finite measure (for simplicity we assume from now on that $|\Omega|=1$), and let $1\leq p\leq n/(n-1)$. Then the following statements are equivalent*

(i)

$$\left(\int_\Omega |f(x)-f_\Omega|^p\,dx\right)^{1/p} \preceq \int_\Omega |\nabla f(x)|\,dx,\ \forall f\in W^{1,1}(\Omega). \tag{1.9}$$

(ii) For each $f\in W^{1,1}(\Omega)$ there exists $r_f\in\mathbb{R}$ such that

$$s^{\frac{1}{p}-1}[(f-r_f)^{**}(s)-(f-r_f)^*(s)] \preceq \int_0^t |\nabla f|^*(s)ds, \tag{1.10}$$

and

$$\int_0^t s^{\frac{1}{p}-1}[(f-r_f)^{**}(s)-(f-r_f)^*(s)]ds \preceq \int_0^t |\nabla f|^*(s)ds. \tag{1.11}$$

(iii) For any r.i. space[2] $X(\Omega)$ *and for each* $f\in W_X^1(\Omega)=\{f\in X(\Omega):\nabla f\in X(\Omega)\}$, *we have*

$$\inf_{c\in\mathbb{R}}\left\|s^{\frac{1}{p}-1}[(f-c)^{**}(s)-(f-c)^{*}(s)\right\|_{\hat{X}}\preceq\|\nabla f\|_{X(\Omega)}. \tag{1.12}$$

(iv)

$$\|f-f_\Omega\|_{L^{p,1}(\Omega)}\preceq\|\nabla f\|_{L^1(\Omega)},\ \forall f\in W^{1,1}(\Omega).$$

As usual, the symbol $f\simeq g$ will indicate the existence of a universal constant $C>0$ (independent of all parameters involved) so that $(1/C)f\leq g\leq C\,f$, while the symbol $f\preceq g$ means that for a suitable constant C, $f\leq C\,g$, and likewise $f\succeq g$ means that $f\geq Cg$.

We note that Theorem 1 improves on Theorem 1 of [12] in three respects: (i) we do not assume that the Sobolev functions vanish at the boundary, (ii) in (1.12) we have eliminated the restriction on the Boyd index of X we had in [12] (this is due to our use of Lemma 2 below), and finally (iii) in [12] we only considered the limiting case $p=\frac{n}{n-1}$.

In our second main result we show that for $p=\frac{n}{n-1}$, Theorem 1 is sharp in the setting of r.i. spaces, and moreover that the verification of Sobolev-Poincaré inequalities is reduced to establish the boundedness of a certain one-dimensional Hardy type operator acting on functions defined on $(0,1)$. Interestingly this reduction is not possible for $p\neq\frac{n}{n-1}$ (see Proposition 1 below).

Theorem 2. *Let* Ω *be a domain with* $|\Omega|=1$, *and let* $X(\Omega)$, $Y(\Omega)$ *be two r.i. spaces. Assume that the following Sobolev-Poincaré inequality holds*

$$\left(\int_\Omega|f(x)-f_\Omega|^{n/(n-1)}\,dx\right)^{\frac{n-1}{n}}\preceq\int_\Omega|\nabla f(x)|\,dx,\ \forall f\in W^{1,1}(\Omega). \tag{1.13}$$

Then the following statements are equivalent

(i)

$$\|f\|_{\hat{Y}}\preceq\left\|s^{-1/n}[f^{**}(s)-f^{*}(s)]\right\|_{\hat{X}}+\|f\|_{L^1}.$$

(ii)

$$\left\|\int_t^1 s^{1/n}f(s)\frac{ds}{s}\right\|_{\hat{Y}}\preceq\|f\|_{\hat{X}},\ \forall f\in\hat{X},\ f\geq 0.$$

(iii)

$$\|f-f_\Omega\|_{Y(\Omega)}\preceq\|\nabla f\|_{X(\Omega)}.$$

Finally, we also consider suitable variants of the Polya-Szëgo symmetrization principle in a formulation that does not require the functions to vanish at the boundary

Theorem 3. *(cf. Theorem 6 below) Let* $\Omega\in\mathcal{J}_{1-1/n}$, *and let* $X(\Omega)$ *be a r.i. space. Then*

$$\inf_{c\in\mathbb{R}}\|\nabla(f-c)^{\circ}\|_{\tilde{X}(B)}\preceq\|\nabla f\|_{X(\Omega)},\ \textit{for all } f\in W^{1,1}(\Omega),$$

where f° *is the symmetric spherical decreasing rearrangement of* f *and* $\tilde{X}(B)$ *is the version of* $X(\Omega)$ *on a ball* B *centered at zero with measure 1 (see Section 4 below).*

[2] For a rearrangement invariant space (r.i. space) $X(\Omega)$ we let $\hat{X}=\hat{X}(0,1)$ be its representation as a function space on $(0,1)$ (if $X(\Omega)=L^p(\Omega)$ we shall write L^p instead of $\hat{L}^p$). We refer to [2] for further information about r.i. spaces.

Using Theorem 3, and the characterization of the $X-$modulus of continuity as a $K-$functional (cf. [7]), it follows as in [11] that

Theorem 4. *Let Ω be an open domain in $\mathbb{R}^n$ with Lipschitz boundary with $|\Omega| = 1$, and let $X(\Omega)$ be a r.i. space. Then for all $f \in X(\Omega)$,*

$$\inf_{c\in\mathbb{R}} \omega_{\tilde{X}(B)}((f-c)^{\circ}, t) \preceq \omega_{X(\Omega)}(f,t),$$

where $\omega_{X(\Omega)}(f,t)$ is the X-modulus of continuity of f (see (4.1) below).

The paper is organized as follows: in Section 2 we deal with the modifications necessary to make the "symmetrization by truncation principle" method of [12] available in our setting, in particular this section contains a proof that (1.9) implies theorem 1 (ii), we then complete the proofs of Theorems 1 and 2 in Section 3 while we prove Theorems 3 and 4 in section 4.

2. Rearrangement Inequalities on Domains by Truncation

Let $\Omega \subset \mathbb{R}^n$ be a domain which for simplicity we suppose is such that $|\Omega| = 1$. In this section we prove (cf. Theorem 5 below) that (1.9) implies by symmetrization by truncation the rearrangement inequalities (1.10) and (1.11) of Theorem 1. These results are variants of symmetrization inequalities, which for functions vanishing at the boundary, have appeared in articles by Bastero-Milman-Ruiz [1], Martin-Milman [10], Mazy'a [13], Talenti [16], Martin-Milman-Pustylnik [12], etc. Our method of proof is by "symmetrization by truncation" developed recently in [12], therefore we shall only indicate briefly the necessary changes and refer the reader to [12] for complete details.

Throughout this section we shall assume that the following Sobolev-Poincaré inequality holds

$$\left(\int_{\Omega} |f(x) - f_{\Omega}|^p \, dx\right)^{1/p} \preceq \int_{\Omega} |\nabla f(x)| \, dx, \text{ for all } f \in W^{1,1}(\Omega). \tag{2.1}$$

We now formally introduce the truncations we use

Definition 1. *Let f be a positive measurable function. Let $0 < t_1 < t_2 < \infty$. The truncations $f_{t_1}^{t_2}$ of f are defined by*

$$f_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1 & \text{if } f(x) > t_2, \\ f(x) - t_1 & \text{if } t_1 < f(x) \leq t_2, \\ 0 & \text{if } f(x) \leq t_1. \end{cases}$$

The next useful result is a simple elementary fact that we state without proof.

Lemma 1. *Let (X, μ) be a finite measure space. If $w \geq 0$ is a measurable function such that $\mu(\{w = 0\}) \geq \mu(X)/2$, then for every $t > 0$*

$$\mu(\{x \in X : w(x) \geq t\}) \leq 2 \inf_{c\in\mathbb{R}} \mu(\{x \in X : |w(x) - c| \geq t/2\}).$$

We now state and prove the main result of this section (cf. also [12])

Theorem 5. *Let $f \in W^{1,1}(\Omega)$ then there exists $r_f \in \mathbb{R}$ such that*

a.

$$\int_0^t s^{1/p} \left(-(f - r_f)^*\right)'(s))ds \preceq \int_0^t |\nabla f|^*(s)ds.$$

b.
$$s^{\frac{1}{p}-1}[(f-r_f)^{**}(s)-(f-r_f)^{*}(s)] \preceq \int_0^t |\nabla f|^{*}(s)ds.$$
c.
$$\int_0^t s^{\frac{1}{p}-1}[(f-r_f)^{**}(s)-(f-r_f)^{*}(s)]\frac{ds}{s} \preceq \int_0^t |\nabla f|^{*}(s)ds.$$

Proof. Let r_f be such that
$$|\{f \geq r_f\}| \geq 1/2 \quad \text{and} \quad |\{f \leq r_f\}| \geq 1/2.$$
Let $u=(f-r_f)\chi_{\{f\geq r_f\}}$ and $v=(r_f-f)\chi_{\{f\leq r_f\}}$. Consider the truncations $u_{t_1}^{t_2}$ of u. Then,
$$\left|\left\{u_{t_1}^{t_2}=0\right\}\right| \geq 1/2.$$
Thus, by Lemma 1 and inequality (2.1), we see that for all $t>0$,
$$\begin{aligned}\left|\left\{u_{t_1}^{t_2}\geq t\right\}\right|^{1/p} t &\leq 2^{1/p+1}\inf_{c\in\mathbb{R}}\left|\left\{\left|u_{t_1}^{t_2}-c\right|\geq t/2\right\}\right|^{1/p} t/2 \\ &\preceq \inf_{c\in\mathbb{R}}\left(\int_\Omega \left|u_{t_1}^{t_2}-c\right|^p dx\right)^{1/p} \\ &\preceq \int_{\{t_1<u\leq t_2\}} |\nabla f(x)|\, dx.\end{aligned}$$
Let $t=t_2-t_1$, then
$$(t_2-t_1)\left|\left\{u_{t_1}^{t_2}\geq t_2-t_1\right\}\right|^{1/p} \preceq \int_{\{t_1<u\leq t_2\}} |\nabla f(x)|\, dx.$$
The last inequality combined with
$$|\{u\geq t_2\}| = \left|\left\{u_{t_1}^{t_2}\geq t_2-t_1\right\}\right|$$
yields
$$(t_2-t_1)\,|\{u\geq t_2\}|^{1/p} \preceq \int_{\{t_1<u\leq t_2\}} |\nabla f(x)|\, dx.$$
Similarly,
$$(t_2-t_1)\,|\{v\geq t_2\}|^{1/p} \preceq \int_{\{t_1<v\leq t_2\}} |\nabla f(x)|\, dx.$$
Note that $|f-r_f|=u+v$, then from the definition of u and v, it is plain that for $0<\alpha<\beta$,
$$\{\beta>u+v\geq\alpha\} = \{\beta>u\geq\alpha\}\cup\{\beta>v\geq\alpha\}.$$
Thus,
$$\begin{aligned}(t_2-t_1)\,|\{|f-r_f|\geq t_2\}|^{1/p} &= (t_2-t_1)\,(|\{u\geq t_2\}|+|\{v\geq t_2\}|)^{1/p} \\ &\leq (t_2-t_1)\left(|\{u\geq t_2\}|^{1/p}+|\{v\geq t_2\}|^{1/p}\right) \\ &\preceq \left(\int_{\{t_1<u\leq t_2\}}|\nabla f(x)|\,dx+\int_{\{t_1<v\leq t_2\}}|\nabla f(x)|\,dx\right) \\ &= \int_{\{t_1<|f-r_f|\leq t_2\}}|\nabla f(x)|\,dx.\end{aligned}$$

Apply the previous inequality using $t_1 = (f - r_f)^* (s + h)$ and $t_2 = (f - r_f)^* (s)$, where $s, h > 0$. Then dividing the resulting inequality by h and letting $h \to 0$ (following the corresponding argument in [16] and [12]) we arrive at

$$s^{1/p} \left(-(f - r_f)^*\right)' (s) \preceq \frac{\partial}{\partial s} \int_{\{|f - r_f| > (f - r_f)^*(s)\}} |\nabla f(x)| \, dx. \tag{2.2}$$

Therefore

$$\int_0^t s^{1/p} \left(-(f - r_f)^*\right)' (s)) ds \preceq \int_0^t |\nabla f|^* (s) ds, \tag{2.3}$$

follows. To prove (b) we use the definitions and integration by parts to get

$$\begin{aligned} (f - r_f)^{**} (t) - (f - r_f)^* (t) &= \frac{1}{t} \int_0^t \big((f - r_f)^* (s) - (f - r_f)^* (t)\big) ds \\ &= \frac{1}{t} \int_0^t s^{1-1/p} s^{1/p} \left(-(f - r_f)^*\right)' (s) ds \\ &\leq \frac{t^{1-1/p}}{t} \int_0^t s^{1/p} \left(-(f - r_f)^*\right)' (s) ds, \end{aligned}$$

and we conclude by (2.3).

For the proof of (c) we integrate

$$s^{1/p-1} [(f - r_f)^{**} (s) - (f - r_f)^* (s)] = s^{1/p-2} \int_0^s u \left(-(f - r_f)^*\right)' (u) du$$

and integrate by parts (cf. [12]). □

3. Proof of Theorems 1 and 2

In order to avoid putting conditions on the indices of the r.i. spaces we shall need the following technical result, which is implicit in [4], and whose prove we provide in an appendix.

Lemma 2. *Let g, h be two positive measurable functions on $(0, \infty)$ such that*

$$g(s) \preceq h^{**}(s), \textit{for all } s \in (0, \infty), \tag{3.1}$$

and

$$\int_0^t g(s) \preceq \int_0^t h^*(s) ds, \ \textit{for all } s \in (0, \infty). \tag{3.2}$$

Then

$$\int_0^t g^*(s) ds \preceq \int_0^t h^*(s) ds, \ \textit{for all } t \in (0, \infty),$$

and therefore for any r.i. space X

$$\|g\|_X \preceq \|h\|_X .$$

3.1. **The proof of Theorem 1.**

Proof. In Section 2 we proved that $(i) \rightarrow (ii)$.

$(ii) \rightarrow (iii)$. Applying Lemma 2 with

$$g(s) = s^{1/p-1}\left(f - r_f\right)^{**}(s) - \left(f - r_f\right)^{*}(s) \text{ and } h^*(s) = \left|\nabla f\right|^*(s)$$

we get

$$\inf_{c\in\mathbb{R}} \left\| s^{\frac{1}{p}-1}[(f-c)^{**}(s) - (f-c)^{*}(s) \right\|_{\hat{X}} \preceq \left\|\nabla f\right\|_{X(\Omega)}.$$

$(iii) \rightarrow (iv)$. Applying (iii) with $X(\Omega) = L^1(\Omega)$ we get

$$\inf_{c\in\mathbb{R}} \int_0^1 s^{1/p-1}\left[(f-c)^{**}(s) - (f-c)^{*}(s)\right] ds \preceq \left\|\nabla f\right\|_{L^1(\Omega)}.$$

We then note that

$$\int_0^1 s^{1/p-1}\left[(f-c)^{**}(s) - (f-c)^{*}(s)\right] ds \simeq \left\|f - c\right\|_{L^{p,1}(\Omega)},$$

and conclude with

$$\left\|f - f_\Omega\right\|_{L^{p,1}(\Omega)} \leq 2 \inf_{c\in\mathbb{R}} \left\|f - c\right\|_{L^{p,1}(\Omega)} \preceq \left\|\left|\nabla f\right|\right\|_{L^1(\Omega)}.$$

$(iv) \rightarrow (i)$ This implication is trivial since

$$L^{p,1}(\Omega) \subset L^p(\Omega).$$

□

3.2. **The proof of Theorem 2.**

Proof. $(i) \rightarrow (ii)$. Given $f \in \hat{X}$, let $h(t) = \int_t^1 s^{1/n} \left|f(s)\right| \frac{ds}{s}$, then $h(t) = h^*(t)$. Consequently, by Fubini,

$$\begin{aligned} h^{**}(t) - h^*(t) &= \frac{1}{t}\int_0^t h(s)ds = \frac{1}{t}\int_0^t \left(\int_x^1 s^{1/n}\left|f(s)\right|\frac{ds}{s}\right) dx - h(t) \\ &= \frac{1}{t}\int_0^t s^{1/n}\left|f(s)\right| ds. \end{aligned}$$

Also note that

$$\left\|h\right\|_{L^1} \leq \int_0^1 s^{1/n}\left|f(s)\right| ds \leq \left\|f\right\|_{L^1}.$$

Consequently by (i)

$$\begin{aligned} \left\|\int_t^1 s^{1/n}\left|f(s)\right|\frac{ds}{s}\right\|_{\hat{Y}} &= \left\|h\right\|_{\hat{Y}} \\ &\preceq \left\|t^{-1/n}\left(h^{**}(t) - h^*(t)\right)\right\|_{\hat{X}} + \left\|h\right\|_{L^1} \\ &= \left\|t^{-1/n}\frac{1}{t}\int_0^t s^{1/n}\left|f(s)\right| ds\right\|_{\hat{X}} + \left\|f\right\|_{L^1} \\ &\preceq \left\|f\right\|_{\hat{X}}, \end{aligned}$$

where in the last inequality we used the fact that $\left\|f\right\|_{L^1} \leq \left\|f\right\|_{\hat{X}}$, and that for any $\alpha > 0$, $\left\|t^{-\alpha}\frac{1}{t}\int_0^t \left|g(s)\right| ds\right\|_{\hat{X}} \leq \frac{1}{\alpha}\left\|g\right\|_{\hat{X}}$ (see [14, Lemma 2.7]).

$(ii) \to (iii)$. Pick $r_f \in \mathbb{R}$, such that

(3.3)

$$\left\| s^{-1/n}[(f-r_f)^{**}(s) - (f-r_f)^{*}(s) \right\|_{\hat{X}} \leq 2 \inf_{c\in\mathbb{R}} \left\| s^{-1/n}[(f-c)^{**}(s) - (f-c)^{*}(s) \right\|_{\hat{X}}$$

By the fundamental theorem of calculus

$$(f-r_f)^{**}(t) = \int_t^1 [(f-r_f)^{*}(s) - (f-r_f)^{*}(s)] \frac{ds}{s} + \int_0^1 (f-r_f)^{*}(s) ds.$$

Thus,

$$\begin{aligned} \|f-r_f\|_Y &\leq \left\|(f-r_f)^{**}\right\|_{\hat{Y}} \\ &= \left\| \int_t^1 s^{1/n} \left(s^{-1/n}[(f-r_f)^{*}(s) - (f-r_f)^{*}(s)] \right) \frac{ds}{s} \right\|_{\hat{Y}} \\ &+ \|f-r_f\|_{L^1(\Omega)}. \\ &\preceq \left\| s^{-1/n}[(f-r_f)^{*}(s) - (f-r_f)^{*}(s)] \right\|_{\hat{X}} + \|f-r_f\|_{L^1(\Omega)} \\ &\preceq \|\nabla f\|_{X(\Omega)} + \|f-r_f\|_{L^1(\Omega)} \qquad \text{(by (3.3) and (1.12))}. \end{aligned}$$

Therefore

$$\inf_{c\in\mathbb{R}} \|f-c\|_{Y(\Omega)} \preceq \|\nabla f\|_{X(\Omega)} + \inf_{c\in\mathbb{R}} \|f-c\|_{L^1(\Omega)}. \tag{3.4}$$

To estimate the second term to the right we observe that

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|f-c\|_{L^1(\Omega)} &\leq \inf_{c\in\mathbb{R}} \left(\int_\Omega |f(x)-c|^{n/(n-1)} dx \right)^{(n-1)/n} \\ &\preceq \int_\Omega |\nabla f(x)| dx \qquad \text{(by (1.13))} \\ &\leq \|\nabla f\|_{X(\Omega)}. \end{aligned}$$

Inserting this estimate back into (3.4) we find that

$$\inf_{c\in\mathbb{R}} \|f-c\|_{Y(\Omega)} \preceq \|\nabla f\|_{X(\Omega)},$$

which combined with the elementary inequality

$$\|f-f_\Omega\|_{Y(\Omega)} \preceq 2 \inf_{c\in\mathbb{R}} \|f-c\|_{Y(\Omega)},$$

gives us (iii).

$(iii) \to (ii)$. We assume, without loss of generality, that $0 \in \Omega$. Let $\sigma > 0$ so that the ball centered at 0 and having measure σ is contained in Ω. Given a positive function $g \in \hat{X}$, with supp $g \subset [0, \sigma]$, define

$$u(x) = \int_{\gamma_n |x|^n}^1 g(s) s^{1/n-1} ds,$$

where $\gamma_n =$ measure of the unit ball in $\mathbb{R}^n$. Observe that for $h \in \hat{X}$ we have that

$$|\{x \in B : h(\gamma_n |x|^n) > \lambda\}| = |\{t \in (0,1) : h(t) > \lambda\}|.$$

Consequently

$$u^*(t) = \int_t^1 s^{1/n} g(s) \frac{ds}{s}.$$

Moreover, and easy computation shows that $|\nabla u|(x) = ng(\gamma_n |x|^n)$. It follows from (iii) that

$$\|u\|_{Y(\Omega)} = \|u^*\|_{\hat{Y}} = \left\| \int_t^1 g(s)s^{1/n-1}ds \right\|_{\hat{Y}} \preceq \|\nabla u\|_{X(\Omega)} + \|u_\Omega\|_{Y(\Omega)}$$
$$= \|g\|_{\hat{X}} + \|u_\Omega\|_{Y(\Omega)}$$

We conclude observing that

$$\|u_\Omega\|_{Y(\Omega)} = \left\| \int_\Omega \int_{\gamma_n |x|^n}^1 g(s)s^{1/n-1}ds \right\|_{Y(\Omega)} \leq \left\| \int_{\gamma_n |x|^n}^1 |g(s)|\, s^{1/n-1}ds \right\|_{L^1(\Omega)}$$
$$= \left\| \int_t^1 s^{1/n} |g(s)| \frac{ds}{s} \right\|_{L^1} \leq \left\| \int_t^1 |g(s)| \frac{ds}{s} \right\|_{L^1} \leq \|g\|_{L^1} \leq \|g\|_{\hat{X}}.$$

Now, let $g \geq 0$ be an arbitrary function from $\hat{X}$. Then

$$\left\| \int_t^1 g(s)s^{1/n-1}ds \right\|_{\hat{Y}} \leq \left\| \int_t^1 g(s)s^{1/n-1}\chi_{(0,\sigma)}(s)ds \right\|_{\hat{Y}}$$
$$+ \left\| \int_t^1 g(s)s^{1/n-1}\chi_{(\sigma,1)}(s)ds \right\|_{\hat{Y}}$$
$$\leq \|g\|_{\hat{X}} + \left\| \int_t^1 g(s)s^{1/n-1}\chi_{(\sigma,1)}(s)ds \right\|_{\hat{Y}}.$$

The last term on the right hand side can be readily estimated using Minkowski's inequality

$$\left\| \int_t^1 g(s)s^{1/n-1}\chi_{(\sigma,1)}(s)ds \right\|_{\hat{Y}} \leq \|g\|_{\hat{Y}} \int_\sigma^1 s^{1/n-1}ds$$
$$\preceq \|g\|_{\hat{Y}}$$

and (ii) follows.

$(ii) \rightarrow (i)$. By the fundamental theorem of calculus

$$f^{**}(t) = \int_t^1 [f^{**}(s) - f^*(s)]\frac{ds}{s} + \int_0^1 f^*(s)ds.$$

Thus

$$\|f\|_{\hat{Y}} \leq \left\| \int_t^1 s^{1/n}s^{-1/n}[f^{**}(s) - f^*(s)]\frac{ds}{s} \right\|_{\hat{Y}} + \|f\|_{L^1}$$
$$\preceq \left\| s^{-1/n}[f^{**}(s) - f^*(s)] \right\|_{\hat{X}} + \|f\|_{L^1}.$$

□

Theorem 2 raises the question of whether it is possible to prove similar results for $p \neq \frac{n}{n-1}$.

Proposition 1. *Suppose that the Sobolev-Poincaré inequality (1.13) holds for some* $1 \leq p \leq \frac{n}{n-1}$, *and let* $X(\Omega)$ *and* $Y(\Omega)$ *be two r.i. spaces. We have*

(i) if $X(\Omega)$ *and* $Y(\Omega)$ *are such that*

$$\left\| \int_t^1 u^{1-1/p} g(u) \frac{du}{u} \right\|_{\hat{Y}} \leq c \|g\|_{\hat{X}}, \qquad \forall g \in \hat{X}; \tag{3.5}$$

then

$$\|f - f_\Omega\|_{Y(\Omega)} \preceq \|\nabla f\|_{X(\Omega)} . \tag{3.6}$$

(ii) If $p \neq \frac{n}{n-1}$, then it is not necessarily true, in general, that (3.6) implies (3.5).

Proof. (i) The proof given in Theorem 2 for $p = \frac{n}{n-1}$ works without any changes in the general case.

(ii) Let $1 < s < \frac{n}{n-1}$, and let Ω be an $s-$John domain. Then $\Omega \in \mathcal{J}_{(n-1)s/n}$ (cf. [6]) therefore the following Sobolev-Poincaré inequality holds

$$\left(\int_\Omega |f(x) - f_\Omega|^{\frac{n}{(n-1)s}}\, dx\right)^{(n-1)s/n} \preceq \int_\Omega |\nabla f(x)|\, dx.$$

Let $t > 1$ be such that $s > \frac{t-1}{n-1}$, and let $r = \frac{nt}{(n-1)s+(1-t)}$. Note that $1 < t < r$. We will show that the validity of the Sobolev-Poincaré inequality for $s-$John domains (cf. [8])

$$\|f - f_\Omega\|_r \preceq \|\nabla f\|_t ,$$

(this corresponds to the choice $Y = L^r, X = L^t$ in Theorem 2) does not imply that the Hardy operator $Hg(t) = \int_t^1 u^{-(n-1)s/n} g(u)du$ is a bounded operator, $H : L^t \to L^r$. The boundedness of H can be reformulated as a weighted norm inequality for the operator $g \to \int_x^1 g(u)du$, namely

$$\left\|\int_x^1 g(u)du\right\|_{L^r} \leq c \left\|g(x)x^{(n-1)s/n}\right\|_{L^t} . \tag{3.7}$$

It is well known that (3.7) holds iff (cf. [13, Theorem 3 page 44])

$$\sup_{a>0} \left(\int_0^a 1\right)^{1/r} \left(\int_a^1 \left(u^{(n-1)st/n}\right)^{\frac{-1}{t-1}} du\right)^{\frac{t-1}{t}} < \infty. \tag{3.8}$$

Now, since $s < \frac{n}{n-1}$, it follows that $\frac{-(n-1)st}{n(t-1)} + 1 < 0$, and for a near zero we have

$$\begin{aligned}\left(\int_0^a 1\right)^{1/r} \left(\int_a^1 \left(s^{(n-1)st/n}\right)^{\frac{-1}{t-1}}\right)^{\frac{t-1}{t}} &\simeq a^{1/r} \left(a^{\frac{-(n-1)st+n(t-1)}{n(t-1)}} - 1\right)^{\frac{t-1}{t}} \\ &\simeq a^{1/r} a^{\frac{-(n-1)st+n(t-1)}{nt}} \\ &\simeq a^{\frac{(n-1)(1-t)(s-1)}{nt}} .\end{aligned}$$

Consequently, since $\frac{(n-1)(1-t)(s-1)}{nt} < 0$, (3.8) cannot hold. □

Remark 1. *Let h and g be continuous, positive functions on an open set $\Omega \subset \mathbb{R}^n$, and furthermore suppose that $\int_\Omega h(x)dx < \infty$ (for simplicity we assume that $\int_\Omega h(x)dx = 1$). Let $1 < p < \infty$, and assume that for every[3] $f \in C^\infty(\Omega)$, we have[4]*

$$\left(\int_\Omega |f(x) - f_{\Omega,h}|^p\, h(x)dx\right)^{\frac{1}{p}} \leq c \int_\Omega |\nabla f(x)|\, g(x)dx, \tag{3.9}$$

---

[3] If the weights are are sufficiently nice the standard proof of density applies in order to extend this inequality to Sobolev spaces.

[4] Several inequalities of the type (3.9) where Ω is a $s-$John domain ($s \geq 1$), $h(x) = \varrho(x)^a$ and $g(x) = \varrho(x)^b$ with $\varrho(x) = dist(x, \partial\Omega)$ can be found in [8] and [6].

(here $f_{\Omega,h}=\int_\Omega f(x)h(x)dx$*). Let* $d\mu(x)=h(x)dx$*, then we can rewrite (3.9) as*

$$\left(\int_\Omega |f(x)-f_{\Omega,h}|^p\,d\mu(x)\right)^{\frac{1}{p}} \le c\int_\Omega |\nabla f(x)|\,\frac{g(x)}{h(x)}d\mu(x). \tag{3.10}$$

If we denote by f_μ^* *the decreasing rearrangement of* f *with respect to the measure* μ *and* $f_\mu^{**}(t)=\frac{1}{t}\int_0^t f_\mu^*(s)ds$*, then with the same proof of Theorem 1, we see that (3.10) and the following statements are equivalent:*

(i) *There exists* $r_f\in\mathbb{R}$ *such that*

$$s^{\frac{1}{p}-1}[(f-r_f)_\mu^{**}(s)-(f-r_f)_\mu^{*}(s)]ds \preceq \int_0^t |\nabla f|_\mu^*(s)ds$$

and

$$\int_0^t s^{\frac{1}{p}-1}[(f-r_f)_\mu^{**}(s)-(f-r_f)_\mu^{*}(s)]ds \preceq \int_0^t |\nabla f|_\mu^*(s)ds.$$

(ii) *For any rearrangement invariant space* X

$$\inf_c \left\| s^{\frac{1}{p}-1}[(f-c)_\mu^{**}(s)-(f-c)_\mu^{*}(s)]\right\|_X \preceq \|\nabla f\|_X\,.$$

(iii)

$$\|f-f_{\Omega,h}\|_{L^{p,1}(\Omega,d\mu)} \preceq \|\nabla f\|_{L^1(\Omega,d\mu)}\,.$$

4. Symmetrization and Moduli of continuity

In this brief section we formulate versions of the Pólya-Szegö principle for functions on domains.

Let $\Omega\in\mathcal{J}_{1-1/n}$ be a domain of finite measure (for simplicity we assume that $|\Omega|=1$), and let $X(\Omega)$ be a r.i. space. Given $f\in X(\Omega)$ the symmetric spherical decreasing rearrangement f° of f is defined by

$$f^\circ(x)=f^*\left(\gamma_n |x|^n\right),\quad x\in B,$$

where $\gamma_n=$ measure of the unit ball in $\mathbb{R}^n$ and B is the ball centered at the origin with $|B|=1$. Since f° is equimeasurable with f, $(f^\circ)^*=f^*$, $X(\Omega)$ has also a representation as a function space on $\tilde{X}(B)$ such that

$$\|f\|_{X(\Omega)}=\|f^\circ\|_{\tilde{X}(B)}\,.$$

Let us also recall that[5] (see [11]).

$$\|f^\circ-g^\circ\|_{\tilde{X}(B)}\le \|f-g\|_X \quad f,g\in X.$$

Our first result of this section is an extension of the classical Pólya-Szegö inequality for domains of class $\mathcal{J}_{1-1/n}$.

Theorem 6. *Let* $\Omega\in\mathcal{J}_{1-1/n}$ *and* $X(\Omega)$ *a r.i. space. Then for any* $f\in W^{1,1}(\Omega)$ *we get that*

$$\inf_{c\in\mathbb{R}}\|\nabla(f-c)^\circ\|_{\tilde{X}(B)}\preceq\|\nabla f\|_{X(\Omega)}\,.$$

[5]We refer the reader to [15] for further information about symmetric spherical rearrangement.

Proof. Let $f \in W^{1,1}(\Omega)$, arguing as in [12] a slight modification to the proof of (2.2) above yields that there is $r_f \in \mathbb{R}$ such that for any Young function Φ

$$\int_0^1 \Phi\left(s^{1/p}\left(-(f-r_f)^*\right)'(s)\right)ds \preceq \int_\Omega \Phi(|\nabla f(x)|)dx$$

Since

$$\int_0^1 \Phi\left(s^{1-1/n}\left(-(f-r_f)^*\right)'(s)\right)ds \simeq \int_B \Phi(|\nabla (f-r_f)^\circ(x)|)dx$$

we obtain,

$$\int_0^t (|\nabla (f-r_f)^\circ|)^*(s)ds \preceq \int_0^t |\nabla f|^*(s)ds.$$

Summarizing we get

$$\inf_{c\in\mathbb{R}} \|\nabla (f-c)^\circ\|_{\tilde{X}(B)} \leq \|\nabla (f-r_f)^\circ\|_{\tilde{X}(B)} \preceq \|\nabla f\|_{X(\Omega)}.$$

□

Given $f \in X(\Omega)$ the $X(\Omega)-$modulus of continuity of f is defined by

$$\omega_X(f,t) = \sup_{0<|h|\leq t} \left\|(f(\cdot+h)-f(\cdot))\chi_{\Omega(h)}\right\|_{X(\Omega)}, \tag{4.1}$$

with $\Omega(h) = \{x \in \Omega : x+\rho h \in \Omega,\ 0 \leq \rho \leq 1\}$, $h \in \mathbb{R}^n$.

Let $W_X^1 = W_X^1(\Omega) = \{f \in X(\Omega) : \nabla f \in X(\Omega)\}$. Then using the previous result, the fact that (cf. [7])

$$\inf_{g\in W_X^1}\{\|f-g\|_{X(\Omega)} + t\|\nabla g\|_{X(\Omega)}\} \simeq \omega_X(t,h),$$

and the proof of Theorem 1 in [11] we obtain

Theorem 7. *Let Ω be an open domain in $\mathbb{R}^n$ with Lipschitz boundary and such that $|\Omega| = 1$. Let $X(\Omega)$ a r.i. space, and let $f \in X(\Omega)$. Then*

$$\inf_{c\in\mathbb{R}} \omega_{\tilde{X}(B)}((f-c)^\circ, t) \preceq \omega_{X(\Omega)}(f,t) \qquad 0<t<1.$$

Corollary 1. *Let Ω be an open domain in $\mathbb{R}^n$ with Lipschitz boundary and such that $|\Omega| = 1$. Let $X(\Omega)$ a r.i. space, and $f \in X(\Omega)$. Then*

$$\inf_{c\in\mathbb{R}} \left((f-c)^{**}(t) - (f-c)^*(t)\right) \preceq \frac{\omega_{X(\Omega)}\left(t^{1/n}, f\right)}{\phi_X(t)}, \qquad 0<t<1/2,$$

where $\phi_X(s)$ is the fundamental function of $X(\Omega)$: $\phi_X(s) = \|\chi_E\|_X$, with E any measurable subset of Ω with $|E| = s$.

Proof. By the previous theorem and since $\left((f-c)^\circ\right)^* = (f-c)^*$ it is enough to check that for any $c \in \mathbb{R}$

$$(f-c)^{**}(t) - (f-c)^*(t) \preceq \frac{\omega_{\tilde{X}(B)}((f-c)^\circ, t^{1/n})}{\phi_X(t)} \qquad 0<t<1/2,$$

and this follows easily from Theorem 2 of [11]. □

5. Appendix

In this section for completeness sake we provide a proof of Lemma 2. In fact the proof given below is implicitly contained in the proof of Theorem 1.2 of [4].

Proof. The main step is to show that for every finite family of intervals (a_i, b_i), $i = 1, \ldots, m$, with $0 < a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m < \infty$, there is a constant c such that

$$\sum_{i=1}^{n} \int_{a_i}^{b_i} g(s)ds \leq c \int_0^{\sum_{i=1}^{n}(b_i - a_i)} h^*(s)ds, \tag{5.1}$$

If (5.1) holds then by a routine limiting process we can show that for any measurable set $E \subset (0, \infty)$, we have

$$\int_E g(s) \leq c \int_0^{|E|} h^*(s)ds,$$

and the desired inequality follows

$$\int_0^t g^*(s) \leq c \int_0^t h^*(s)ds, t > 0.$$

It remains to prove (5.1). Fix $j \in \{1, \ldots, m\}$, then by (3.1)

$$\begin{aligned} \sum_{i \geq j} \int_{a_i}^{b_i} g(s) &\leq c \int_0^\infty \chi_{\cup_{i\geq j}(a_i,b_i)}(r) \left(\frac{1}{r} \int_0^r h^*(s)ds \right) dr \\ &= c \int_0^\infty h^*(s) \left(\int_s^\infty \chi_{\cup_{i\geq j}(a_i,b_i)}(r) \frac{dr}{r} \right) ds. \end{aligned} \tag{5.2}$$

Since for $R \geq 0$ we have (see [4, formula (3.37) pag. 63])

$$\int_0^R \left(\int_s^\infty \chi_{\cup_{i\geq j}(a_i,b_i)}(r) \frac{dr}{r} \right) ds \leq \left(1 + \sum_{i \geq j} \log\left(\frac{b_i}{a_i}\right) \right) \int_0^R \chi_{\left[0, \sum_{i\geq j}(b_i - a_i)\right]}(s)ds$$

by Hardy's Lemma (see [2, Proposition 3.6 pag 63]) it follows that

$$\begin{aligned} &\int_0^\infty h^*(s) \left(\int_s^\infty \chi_{\cup_{i\geq j}(a_i,b_i)}(r) \frac{dr}{r} \right) ds \\ &\leq \left(1 + \sum_{i \geq j} \log\left(\frac{b_i}{a_i}\right) \right) \int_0^\infty h^*(s) \chi_{\left[0, \sum_{i\geq j}(b_i - a_i)\right]}(s)ds, \end{aligned}$$

which combined with (5.2) gives

$$\sum_{i \geq j} \int_{a_i}^{b_i} g(s)ds \leq c \left(1 + \sum_{i \geq j} \log\left(\frac{b_i}{a_i}\right) \right) \int_0^{\sum_{i\geq j}(b_i - a_i)} h^*(s)ds. \tag{5.3}$$

If $\sum_{i\geq j} \log\left(\frac{b_i}{a_i}\right) \leq 1$, then (5.1) follows simply by choosing $j = 1$. If $\sum_{i=1}^{m} \log\left(\frac{b_i}{a_i}\right) > 1$, it is easily seen that there exist and index j_0 and a positive number c_{j_0} such that $a_{j_0} \leq c_{j_0} \leq b_{j_0}$ and

$$1 < \log\left(\frac{b_{j_0}}{c_{j_0}}\right) + \sum_{i > j_0} \log\left(\frac{b_i}{a_i}\right) \leq 2. \tag{5.4}$$

Applying (5.3) with $j = j_0$ replacing a_{j_0} by c_{j_0} , we get

$$\begin{aligned}
&\int_{c_{j_0}}^{b_{j_0}} g(s)ds + \sum_{i>j_0} \int_{a_i}^{b_i} g(s)ds \\
&\leq c \left(1 + \log\left(\frac{b_{j_0}}{c_{j_0}}\right) + \sum_{i>j_0} \log\left(\frac{b_i}{a_i}\right)\right) \int_0^{(b_{j_0} - c_{j_0}) + \sum_{i>j_0}(b_i - a_i)} h^*(s)ds \\
&\leq 3c \int_0^{(b_{j_0} - c_{j_0}) + \sum_{i>j_0}(b_i - a_i)} h^*(s)ds \quad \text{(by (5.4))}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\log\left(\frac{b_{j_0}}{c_{j_0}}\right) + \sum_{i>j_0} \log\left(\frac{b_i}{a_i}\right) &= \int_{c_{j_0}}^{\infty} \chi_{\left((c_{j_0}, b_{j_0}) \cup \left(\cup_{i>j_0}(a_i, b_i)\right)\right)}(s) \frac{ds}{s} \\
&\leq \frac{1}{c_{j_0}} \left((b_{j_0} - c_{j_0}) + \sum_{i>j_0}(b_i - a_i)\right),
\end{aligned}$$

and since by (5.4)

$$c_{j_0} < \left((b_{j_0} - c_{j_0}) + \sum_{i>j_0}(b_i - a_i)\right), \tag{5.5}$$

it follows that

$$\begin{aligned}
&\sum_{1 \leq j < j_0} \int_{a_i}^{b_i} g(s)ds + \int_{a_{j_0}}^{c_{j_0}} g(s)ds \\
&\leq \int_0^{c_{j_0}} g(s)ds \leq c \int_0^{c_{j_0}} h^*(s)ds \quad \text{(by (3.2))} \\
&\leq c \int_0^{(b_{j_0} - c_{j_0}) + \sum_{i>j_0}(b_i - a_i)} h^*(s)ds \quad \text{(by (5.5))}.
\end{aligned}$$

Summarizing

$$\begin{aligned}
\sum_{i=1}^{n} \int_{a_i}^{b_i} g(s)ds &\leq \sum_{1 \leq j < j_0} \int_{a_i}^{b_i} g(s)ds + \int_{a_{j_0}}^{c_{j_0}} g(s)ds + \int_{c_{j_0}}^{b_{j_0}} g(s)ds + \sum_{i>j_0} \int_{a_i}^{b_i} g(s)ds \\
&\leq 4c \int_0^{(b_{j_0} - c_{j_0}) + \sum_{i>j_0}(b_i - a_i)} h^*(s)ds \\
&\leq 4c \int_0^{\sum_{i=1}^{n}(b_i - a_i)} h^*(s)ds.
\end{aligned}$$

□

References

[1] J. Bastero, M. Milman and F. Ruiz, *A note on $L(\infty, q)$ spaces and Sobolev embeddings*, Indiana Univ. Math. J. **52** (2003), 1215-1230.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988

[3] S. Buckley and P. Koskela, *Sobolev-Poincaré implies John,* Math. Res. Lett. **2** (1995), 577–593.

[4] A. Cianchi, *Symmetrization and second-order Sobolev inequalities,* Annali di Matematica **183** (2004), 45-77.

[5] P. Hajlasz, *Sobolev inequalities, truncation method, and John domains*, Papers in Analysis, Rep. Univ. Jyväskylä Dep. Math. Stat. 83, Univ. Jyväskylä, Jyväskylä, 2001, pp 109-126.
[6] P. Hajlasz and P. Koskela, *Isoperimetric inequalities and Imbedding theorems in irregular domains,* J. London Math. Soc. **58** (1998), 425–450.
[7] H. Johnen and K. Scherer, *On the equivalence of the $K-$functional and moduli of continuity and some applications*, in Constructive theory of functions of several variables, pp. 119-140, Lecture Notes in Math. 571, Springer, Berlin, 1977
[8] T. Kilpeläinen and J. Maly, *J. Sobolev inequalities on sets with irregular boundaries,* Z. Anal. Anwendungen **19** (2000), 369–380.
[9] P. Koskela and P. Onninen, *Shap inequalities via truncation,* J. Math. Anal. Appl. **278** (2003), 324–334.
[10] J. Martín and M. Milman, *Higher order symmetrization inequalities and applications*, J. Math. Anal. Appl. **330** (2007), 91-113
[11] J. Martín and M. Milman, *Symmetrization inequalites and Sobolev embeddings*, Proc. Amer. Math. Soc. **134** (2006), 2335-2347.
[12] J. Martín, M. Milman and E. Pustylnik, *Self Improving Sobolev Inequalities, Truncation and Symmetrization,* to appear in J. Funct. Anal.
[13] V. G. Maz'ya, *Sobolev Spaces,* Springer-Verlag, New York, 1985.
[14] M. Milman and E. Pustylnik, *On sharp higher order Sobolev embeddings*, Comm. Contemp. Math. **6** (2004), 495-511.
[15] Mossino, J, *Inégalités isopérimertriques et applications on physique,* Trabaux en Cours, Hermann, Paris, 1984.
[16] G. Talenti, *Inequalities in rearrangement-invariant function spaces*, Nonlinear Analysis, Function Spaces and Applications, Prometheus, Prague vol. 5, 1995, pp. 177-230.
[17] W. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics **120** (Springer, 1989).

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