SOME REITERATION RESULTS FOR INTERPOLATION METHODS DEFINED BY MEANS OF POLYGONS

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Abstract. We continue the research on reiteration results between interpolation methods associated to polygons and the real method. Applications are given to N-tuples of function spaces, of spaces of bounded linear operators and Banach algebras.

1. Introduction

This paper deals with interpolation methods for finite families (N-tuples) of Banach spaces defined by means of a convex polygon Π in the plane $\mathbb{R}^2$ and a point (α, β) in the interior of Π. These methods were introduced by Cobos and Peetre in [13], further investigations have been done by Cobos, Kühn and Schonbek [10], Cobos, Fernández-Martínez and Schonbek [9], Cobos, Fernández-Martínez and Martínez [7], Ericsson [15], Cobos, Fernández-Martínez, Martínez and Raynaud [8], Cobos and Martín [11] and Fernández-Cabrera and Martínez [18], among other authors. Thinking of the Banach spaces as sitting on the vertices of Π they introduced K- and J-functionals with two parameters and then they define K- and J-spaces by using an (α, β)-weighted L_q-norm (the precise definitions are recalled in Section 2). For the special choice of Π as the simplex, these methods give back (the first nontrivial case of) spaces introduced by Sparr [26], and if Π is the unit square they recover spaces studied by Fernandez [16]. Other references

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on interpolation methods for N-tuples can be found in the monographs by Triebel [27] and Brudnyǐ and Krugljak [4].

It was shown in [7] and [15] that reiteration formulae between methods associated to polygons and the real method are important to describe K- and J-spaces in certain cases. In the present paper we continue their research. First we complement a result of Ericsson on interpolation using the unit square of a 4-tuple formed by spaces of class θ_j with respect to a couple $\{X, Y\}$. As we show with an example, in this result is essential that (α, β) does not lie in any diagonal of the square. The example refers to a 4-tuple of the kind $\{X, Y, Y, X\}$ with $X \hookrightarrow Y$. We also characterize the K-spaces generated by this 4-tuple and we show that they are extrapolation spaces when (α, β) is in the diagonal $\beta = 1-\alpha$. Then, assuming a mild condition on the θ_j and that q takes only the value 1 or ∞, we establish results that work for general polygons Π and for any (α, β) in its interior, even if (α, β) lies in any diagonal of Π. Applications are given to 4-tuples of Lorentz function spaces, Besov spaces, Lorentz operator spaces and N-tuples of spaces of bounded linear operators. We also establish a result on interpolation of Banach algebras.

The paper is organized as follows. In Section 2 we review some basic notions on K- and J-spaces associated to polygons. In Section 3 we show the reiteration results for the unit square and their applications to function spaces and to Lorentz operator spaces. Finally, in Section 4, we establish the results for general polygons.

2. Preliminaries

By a Banach N-tuple $\overline{A} = \{A_1, \ldots, A_N\}$ we mean N-Banach spaces $A_j, j = 1, \ldots, N$, which are continuously embedded in a common Hausdorff topological vector space. We put $\Sigma(\overline{A}) = A_1 + \cdots + A_N$ and $\Delta(\overline{A}) = A_1 \cap \cdots \cap A_N$. When $N = 2$ we simply call $\{A_1, A_2\}$ a Banach couple.

Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$. Given any N-tuple $\overline{A}$ we imagine each space A_j as sitting on the vertex P_j and we define K- and J-functionals by

$$K(t, s; a) = K(t, s; a; \overline{A}) = \inf \Big\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : \ a = \sum_{j=1}^{N} a_j, \ a_j \in A_j \Big\},$$

$$J(t, s; a) = J(t, s; a; \overline{A}) = \max \big\{ t^{x_j} s^{y_j} \|a\|_{A_j} : \ 1 \le j \le N \big\}.$$

Here t and s stand for positive numbers.

Now let (α,β) be an interior point of Π, $(\alpha,\beta)\in \operatorname{Int}\Pi$, and let $1\le q\le\infty$. The K-space $\overline{A}_{(\alpha,\beta),q;K}$ consists of all those $a\in\Sigma(\overline{A})$ for which the norm

$$\|a\|_{\overline{A}_{(\alpha,\beta),q;K}} = \left(\int_0^\infty\int_0^\infty \left(t^{-\alpha}s^{-\beta}K(t,s;a)\right)^q \frac{dt}{t}\frac{ds}{s}\right)^{\frac{1}{q}}$$

is finite (the integral should be replaced by the supremum if $q=\infty$).

The J-space $\overline{A}_{(\alpha,\beta),q;J}$ is formed by all those $a\in\Sigma(\overline{A})$ which can be represented as

$$a=\int_0^\infty\int_0^\infty u(t,s)\,\frac{dt}{t}\frac{ds}{s} \tag{2.1}$$

where $u(t,s)$ is a strongly measurable function with values in $\Delta(\overline{A})$ and satisfies

$$\left(\int_0^\infty\int_0^\infty \left(t^{-\alpha}s^{-\beta}J(t,s;u(t,s))\right)^q \frac{dt}{t}\frac{ds}{s}\right)^{\frac{1}{q}} < \infty. \tag{2.2}$$

The norm on $\overline{A}_{(\alpha,\beta),q;J}$ is

$$\|a\|_{\overline{A}_{(\alpha,\beta),q;J}} = \inf\left\{\left(\int_0^\infty\int_0^\infty \left(t^{-\alpha}s^{-\beta}J(t,s;u(t,s))\right)^q \frac{dt}{t}\frac{ds}{s}\right)^{\frac{1}{q}}\right\},$$

where the infimum is taken over all representations u satisfying (2.1) and (2.2).

These spaces were introduced by Cobos and Peetre in [13]. If we take Π equal to the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$, we recover spaces studied by Fernandez [16], [17] for 4-tuples. If Π is equal to the simplex $\{(0,0),(1,0),(0,1)\}$, then K- and J-spaces coincide with those considered by Sparr in [26] for 3-tuples.

Note the analogy of these constructions with the real interpolation space $(X,Y)_{\theta,q}$ for Banach couples $\{X,Y\}$. The space $(X,Y)_{\theta,q}$ can be described by a similar scheme, but working with $\mathbb{R}$ instead of $\mathbb{R}^2$, with the segment $[0,1]$ taking the role of the polygon Π and $0<\theta<1$ being an interior point of the segment $[0,1]$. The space X should be imagined as sitting on the point 0 and Y on the point 1. The relevant functionals are now

$$\begin{aligned}\bar{K}(t,a) &= \bar{K}(t,a;X,Y)\\ &= \inf\{\|a_0\|_X + t\|a_1\|_Y : a = a_0 + a_1, a_0 \in X, a_1 \in Y\},\ a \in X+Y.\end{aligned}$$

and

$$\bar{J}(t,a) = \bar{J}(t,a;X,Y) = \max\{\|a\|_X, t\|a\|_Y\}, \qquad a \in X \cap Y.$$

It turns out that

$$(X,Y)_{\theta,q} = \Big\{a \in X+Y : \|a\|_{(X,Y)_{\theta,q}} = \Big(\int_0^\infty \Big(t^{-\theta}\bar{K}(t,a)\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty\Big\}$$

$$= \Big\{a \in X+Y : a = \int_0^\infty u(t)\,\frac{dt}{t} \text{ with } \Big(\int_0^\infty \Big(t^{-\theta}\bar{J}(t,u(t))\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty\Big\}$$

(see [2] or [27]).

A Banach space Z is said to be an intermediate space with respect to the Banach couple $\{X,Y\}$ if $X \cap Y \hookrightarrow Z \hookrightarrow X+Y$, where $\hookrightarrow$ means continuous inclusion. The intermediate space Z is said to be of class $\mathcal{C}_K(\theta;X,Y)$ if there is a constant $C>0$ such that

$$\bar{K}(t,a) \le Ct^\theta\|a\|_Z \text{ for all } a \in Z,$$

and Z is said to be of class $\mathcal{C}_J(\theta;X,Y)$ if there is a constant $C>0$ such that

$$\|a\|_Z \le Ct^{-\theta}\bar{J}(t,a) \text{ for all } a \in X \cap Y.$$

Here $0 \le \theta \le 1$. If Z is of class $\mathcal{C}_K(\theta;X,Y)$ and of class $\mathcal{C}_J(\theta;X,Y)$ then we say that Z is of class $\mathcal{C}(\theta;X,Y)$. Clearly X is of class $\mathcal{C}(0;X,Y)$ and Y is of class $\mathcal{C}(1;X,Y)$. It is also well-known that for $0<\theta<1$ the real interpolation spaces $(X,Y)_{\theta,q}$ and the complex interpolation spaces $(X,Y)_{[\theta]}$ are spaces of class $\mathcal{C}(\theta;X,Y)$ (see [2] or [27]).

Working with the methods associated to polygons, K- and J-spaces do not coincide in general, but we have that $\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K}$ (see [13], Thm. 1.3).

If $\mathcal{R}$ is any affine bijection on $\mathbb{R}^2$ then K- and J-spaces defined by means of Π and (α,β) coincide (with equivalence of norms) with those defined by means of $\mathcal{R}(\Pi) = \overline{\mathcal{R}P_1 \cdots \mathcal{R}P_N}$ and $\mathcal{R}(\alpha,\beta)$ (see [10], Remark 4.1). We call this fact the property of invariance under affine bijections.

The geometrical elements play an important role in the theory of K- and J-spaces. Indeed, let $\mathcal{P}_{\alpha,\beta}$ be the set of all triples $\{i,k,r\}$ such that (α,β)

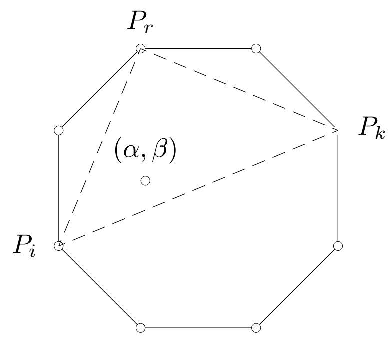

Figure 2.1

belongs to the triangle with vertices P_i, P_k, P_r (see Fig. 2.1). For each $\{i,k,r\} \in \mathcal{P}_{\alpha,\beta}$ let (c_i, c_k, c_r) be the (unique) barycentric coordinates of (α,β) with respect to P_i, P_k, P_r. It was shown in [6], Thm. 1.3, that there is a constant $C > 0$ such that for any $a \in \Delta(\overline{A})$ we have

$$\|a\|_{\overline{A}_{(\alpha,\beta),q;J}} \le C \max\{\|a\|_{A_i}^{c_i}\|a\|_{A_k}^{c_k}\|a\|_{A_r}^{c_r} : \{i,k,r\} \in \mathcal{P}_{\alpha,\beta}\}. \tag{2.3}$$

We also recall that, for any N non-negative real numbers $M_1, \dots, M_N$ we have

$$\sup_{t>0,s>0}\Big[\min_{1\le j\le N}\{t^{x_j-\alpha}s^{y_j-\beta}M_j\}\Big] = \min_{\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}}\{M_i^{c_i}M_k^{c_k}M_r^{c_r}\} \tag{2.4}$$

(see [9], Thm. 1.11).

It is possible to relate J- and K-spaces generated by an N-tuple $\overline{A}$ with those spaces generated by a subtuple $\tilde{A}$ of $\overline{A}$. Next we discuss the case when the subtuple is a 3-tuple.

Let $\{i,k,r\} \in \mathcal{P}_{\alpha,\beta}$ and suppose that (α,β) belongs to the interior of the triangle $\overline{P_iP_kP_r}$. If we put $\tilde{A} = \{A_i, A_k, A_r\}$ and we designate by $\tilde{K}, \tilde{J}$ the K- and J-functionals defined by means of the triangle, then we have

$$K(t,s;a;\overline{A}) \le \tilde{K}(t,s;a;\tilde{A}) \text{ for any } a \in A_i + A_k + A_r,$$
$$\tilde{J}(t,s;a;\tilde{A}) \le J(t,s;a;\overline{A}) \text{ for any } a \in A_i \cap A_k \cap A_r.$$

This yields the continuous embeddings

(2.5)
$$\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow (A_i,A_k,A_r)_{(\alpha,\beta),q;J} \hookrightarrow (A_i,A_k,A_r)_{(\alpha,\beta),q;K} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K}.$$

If $\{i,k,r\} \in \mathcal{P}_{\alpha,\beta}$ but (α,β) is not in the interior of the triangle, then (α,β) should be in a diagonal of Π. Say, for example, that (α,β) belongs to the diagonal joining P_i and P_k (see Fig. 2.2). The barycentric coordinates of (α,β) with respect to the points P_i, P_k, P_r are $(1-\theta_{ik}, \theta_{ik}, 0)$ for some $0 < \theta_{ik} < 1$. Then it turns out that

$$\overline{A}_{(\alpha,\beta),1;J} \hookrightarrow (A_i, A_k)_{\theta_{ik},1}\ ,\ (A_i, A_k)_{\theta_{ik},\infty} \hookrightarrow \overline{A}_{(\alpha,\beta),\infty;K} \tag{2.6}$$

(see [7], Thm. 1.5).

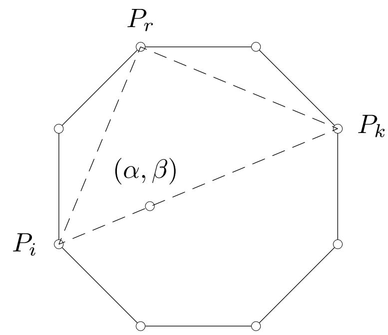

Figure 2.2

Assume that the polygon Π is placed in such a way that $y_j \geq 0$ for $j = 1,\dots,N$, and let $\overline{A}$ be an N-tuple formed by spaces A_j of class θ_j with respect to a given Banach couple $\{X,Y\}$. Suppose also that there are real numbers $\delta,\delta',\rho,\rho'$ such that $\delta\delta' > 0$, $\rho,\rho' \neq 0$, $0 < \delta\alpha+\rho\beta, \delta'\alpha+\rho'\beta < 1$ and

$$\delta x_j + \rho y_j \leq \theta_j \leq \delta' x_j + \rho' y_j \quad \text{for} \quad j = 1,\dots,N.$$

It was shown in [15], Lemma 2, that if A_j is of class $\mathcal{C}_K(\theta_j; X, Y)$ then

$$\overline{A}_{(\alpha,\beta),q;K} \hookrightarrow (X,Y)_{\delta\alpha+\rho\beta,q} + (X,Y)_{\delta'\alpha+\rho'\beta,q} \tag{2.7}$$

and if A_j is of class $\mathcal{C}_J(\theta_j; X, Y)$ then

$$(X,Y)_{\delta\alpha+\rho\beta,q} \cap (X,Y)_{\delta'\alpha+\rho'\beta,q} \hookrightarrow \overline{A}_{(\alpha,\beta),q;J}. \tag{2.8}$$

The following result is a consequence of (2.7), (2.8) and the invariance under affine bijection (see [15], Cor. 4).

Let $\Pi = \overline{P_1P_2P_3}$ be a triangle, let $(\alpha,\beta) \in \operatorname{Int}\Pi$ with barycentric coordinates (c_1,c_2,c_3) with respect to P_1,P_2,P_3 and let $1 \leq q \leq \infty$. If A_j is a space of class $\mathcal{C}(\theta_j; X, Y)$ with $0 \le \theta_j \le 1$, $j = 1, 2, 3$, and the θ_j are not all equal then we have with equivalent norms

$$(A_1, A_2, A_3)_{(\alpha,\beta),q;J} = (A_1, A_2, A_3)_{(\alpha,\beta),q;K} = (X, Y)_{\theta,q} \tag{2.9}$$

where $\theta = c_1\theta_1 + c_2\theta_2 + c_3\theta_3$.

3. Interpolation over the unit square.

In this section we take $\Pi = \overline{P_1P_2P_3P_4}$ equal to the unit square, that is to say, $P_1 = (0, 0), P_2 = (1, 0), P_3 = (0, 1), P_4 = (1, 1)$. Let $(\alpha, \beta) \in \operatorname{Int} \Pi$ such that (α, β) *does not lie on any diagonal of* Π and let $Q = (1/2, 1/2)$. The point (α, β) is in only one internal triangle $\overline{P_iP_kQ}$ and so it is in the two triangles $\overline{P_iP_kP_r}, \overline{P_iP_kP_s}$ formed by vertices of Π. Figure 3.1 illustrate the situation for $i = 1, k = 2, r = 3, s = 4$. Let (c_i, c_k, c_r) and (d_i, d_k, d_s) be the

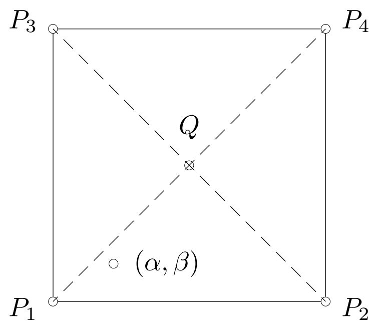

Figure 3.1

barycentric coordinates of (α, β) with respect to P_i, P_k, P_r and P_i, P_k, P_s, respectively. The following results improves [15], Thm. 6, by removing several restrictions on the class of the spaces A_j.

Theorem 3.1. *Let $\{X, Y\}$ be a Banach couple, let A_j be a space of class $\mathcal{C}(\theta_j; X, Y), 0 \le \theta_j \le 1, j = 1, 2, 3, 4$, and let $1 \le q \le \infty$. We suppose that $\theta_i \ne \theta_k$ where i, k are the indices of the vertices P_i, P_k of the (unique) internal triangle $\overline{P_iP_kQ}$ containing (α, β). Put*

$$\theta_{ikr} = c_i\theta_i + c_k\theta_k + c_r\theta_r \quad , \quad \theta_{iks} = d_i\theta_i + d_k\theta_k + d_s\theta_s. \tag{3.1}$$

Then we have, with equivalent norms,

$$(A_1, A_2, A_3, A_4)_{(\alpha,\beta),q;K} = (X, Y)_{\theta_{ikr},q} + (X, Y)_{\theta_{iks},q} \tag{3.2}$$

and

$$(A_1, A_2, A_3, A_4)_{(\alpha,\beta),q;J} = (X, Y)_{\theta_{ikr},q} \cap (X, Y)_{\theta_{iks},q}. \tag{3.3}$$

Proof. First we assume that (α,β) lies in $\overline{P_1P_2Q}$. Then $\theta_1 \neq \theta_2$ and (α,β) is in the triangles $\overline{P_1P_2P_3}$ and $\overline{P_1P_2P_4}$. Using (2.5) and (2.9) we get

$$(X,Y)_{\theta_{123},q} + (X,Y)_{\theta_{124},q} \hookrightarrow (A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;K}$$

and

$$(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;J} \hookrightarrow (X,Y)_{\theta_{123},q} \cap (X,Y)_{\theta_{124},q}\,.$$

In order to check the converse embeddings we consider the affine bijection

$$\mathcal{R}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \theta_2-\theta_1 & -\theta_1-2 \\ 0 & \theta_3+2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $\mathcal{R}P_j = P_j' = (x_j', y_j')$. Then $P_1' = (\theta_1, 0), P_2' = (\theta_2, 0), P_3' = (-2, \theta_3 + 2), P_4' = (\theta_2-\theta_1-2, \theta_3+2)$, so $y_j' \geq 0$ for $j = 1,2,3,4$. Put

$$(\alpha',\beta') = \mathcal{R}(\alpha,\beta) = (\theta_1 + \alpha(\theta_2-\theta_1) + \beta(-\theta_1-2)\,,\, \beta(\theta_3+2)).$$

Now we distinguish two cases. If

$$\theta_{123} \leq \theta_{124}\ ,\ \text{that is}\ ,\ \theta_3-\theta_1 \leq \theta_4-\theta_2$$

then we choose

$$\delta = 1\,,\ \rho = 1\,,\ \delta' = 1\,,\ \rho' = \frac{\theta_4-\theta_2+\theta_1+2}{\theta_3+2}\,.$$

We have

$$\delta x_j' + \rho y_j' \leq \theta_j \leq \delta' x_j' + \rho' y_j'\,, \quad j = 1,2,3,4, \tag{3.4}$$

with $\delta\alpha' + \rho\beta' = \theta_{123}$ and $\delta'\alpha' + \rho'\beta' = \theta_{124}$. Therefore, by (2.7), (2.8) and the invariance under affine bijection, we derive

$$(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;K} \hookrightarrow (X,Y)_{\theta_{123},q} + (X,Y)_{\theta_{124},q} \tag{3.5}$$

and

$$(X,Y)_{\theta_{123},q} \cap (X,Y)_{\theta_{124},q} \hookrightarrow (A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;J}. \tag{3.6}$$

If

$$\theta_{124} \leq \theta_{123}\ ,\ \text{so}\ ,\ \theta_4-\theta_2 \leq \theta_3-\theta_1$$

then we choose

$$\delta = 1\,,\ \rho = \frac{\theta_4-\theta_2+\theta_1+2}{\theta_3+2}\,,\ \delta' = 1\,,\ \rho' = 1\,.$$

Again (3.4) holds. This time $\delta\alpha' + \rho\beta' = \theta_{124}$ and $\delta'\alpha' + \rho'\beta' = \theta_{123}$. Hence, (3.5) and (3.6) follows as in the previous case.

If (α,β) lies in an internal triangle different from $\overline{P_1P_2Q}$ then we use the symmetry of the unit square to lead the situation to the result that we have just established. Assume, for example, that (α,β) is in $\overline{P_2P_4Q}$ (see Fig. 3.2). The remaining cases can be treated in the same way. Then we know

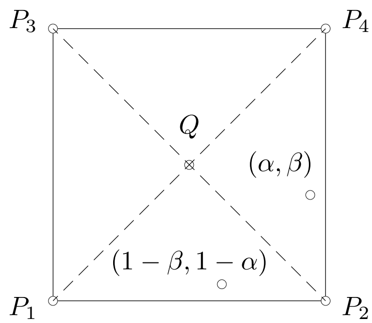

Figure 3.2

that $\theta_2 \neq \theta_4$. The relevant numbers for (3.2) and (3.3) are θ_{234}, θ_{124}. It follows directly from the definition of K-spaces over the unit square that

$$(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;K} = (A_1,A_3,A_2,A_4)_{(\beta,\alpha),q;K}$$

and

$$(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;K} = (A_4,A_3,A_2,A_1)_{(1-\alpha,1-\beta),q;K}$$

with analogous formulae for J-spaces. Hence

$$(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;K} = (A_4,A_2,A_3,A_1)_{(1-\beta,1-\alpha),q;K}$$

and the point $(1-\beta,1-\alpha)$ is in $\overline{P_1P_2Q}$. Consider the 4-tuple $B_1 = A_4, B_2 = A_2, B_3 = A_3, B_4 = A_1$, write θ_j^* for the class of B_j with respect to $\{X,Y\}$ and define θ_{ikr}^* as in (3.1) but using the barycentric coordinates of $(1-\beta,1-\alpha)$ and the θ_j^*. We have $\theta_1^* = \theta_4 \neq \theta_2 = \theta_2^*$, hence we can apply the result that we have established in the first part of the proof and derive that

$$\begin{aligned}(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;K} &= (B_1,B_2,B_3,B_4)_{(1-\beta,1-\alpha),q;K}\\ &= (X,Y)_{\theta_{123}^*,q} + (X,Y)_{\theta_{124}^*,q} = (X,Y)_{\theta_{234},q} + (X,Y)_{\theta_{124},q}.\end{aligned}$$

This proves the K-formula. The J-formula follows similarly.

The proof is complete. □

Using Theorem 3.1 we can complement [15], Example 1, by reducing the conditions on the parameters. Let us write down the outcome. Take any

σ-finite measure space (Ω,μ) and for $1<p<\infty$ and $1\le q\le\infty$, let $L_{p,q}$ be the Lorentz function space

$$L_{p,q}=\left\{f:\|f\|_{L_{p,q}}=\left(\int\limits_0^\infty\left(t^{\frac{1}{p}-1}\int\limits_0^t f^*(s)ds\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\right\}$$

where

$$f^*(s)=\inf\{\gamma>0:\mu\{x\in\Omega:|f(x)|>\gamma\}\le s\}$$

(see [2] or [27]). We have $(L_1,L_\infty)_{\theta,q}=L_{p,q}$ for $1/p=1-\theta$. As a direct consequence of Theorem 3.1 we obtain the following.

Corollary 3.2. *Let* $1<p_j<\infty, 1\le q_j,q\le\infty, j=1,2,3,4$. *Suppose that* $\alpha>\beta$ *,* $\alpha+\beta<1$ *and* $p_1\ne p_2$. *Put*

$$\frac{1}{p_{123}}=\frac{1-\alpha-\beta}{p_1}+\frac{\alpha}{p_2}+\frac{\beta}{p_3}\ ,\ \frac{1}{p_{124}}=\frac{1-\alpha}{p_1}+\frac{\alpha-\beta}{p_2}+\frac{\beta}{p_4}.$$

Then we have, with equivalence of norms,

$$(L_{p_1,q_1},L_{p_2,q_2},L_{p_3,q_3},L_{p_4,q_4})_{(\alpha,\beta),q;K}=L_{p_{123},q}+L_{p_{124},q}$$

and

$$(L_{p_1,q_1},L_{p_2,q_2},L_{p_3,q_3},L_{p_4,q_4})_{(\alpha,\beta),q;J}=L_{p_{123},q}\cap L_{p_{124},q}.$$

Corollary 3.2 refers to the case when (α,β) lies in the internal triangle $\overline{P_1P_2Q}$. Similar results holds when (α,β) is in any of the other three internal triangles.

In order to give a second application we recall the (Fourier-analytical) definition of Besov spaces. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing complex infinitely differentiable functions on $\mathbb{R}^n$ and the space of tempered distributions on $\mathbb{R}^n$, respectively. For $f\in\mathcal{S}'(\mathbb{R}^n)$, the Fourier transform and its inverse are defined in the usual way and denoted by $\hat f$ and $\check f$, respectively. Let φ be a C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp}\varphi\subset\{\xi\in\mathbb{R}^n:\|\xi\|_{\mathbb{R}^n}\le 2\}\quad\text{and}\quad\varphi(\xi)=1\quad\text{if}\quad\|\xi\|_{\mathbb{R}^n}\le 1.$$

We put $\varphi_0=\varphi$ and for $j\in\mathbb{N}$ we write $\varphi_j(\xi)=\varphi(2^{-j}\xi)-\varphi(2^{-j+1}\xi)$.

Let $1<p<\infty, 1\le q\le\infty$ and $s\in\mathbb{R}$. The space $B^s_{p,q}$ is the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}}=(\sum_{j=0}^\infty 2^{jsq}\|(\varphi_j\hat f)\check{}\,\|^q_{L_p(\mathbb{R}^n)})^{1/q}<\infty$$

(with the usual modification if $q = \infty$). We refer to the monographs by Triebel [28], [29], [30] for details on Besov spaces. It is clear from the definition that $B^s_{p,q} \hookrightarrow B^u_{p,q}$ if $u < s$. The following interpolation formula holds

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} = B^s_{p,q}.$$

Here $-\infty < s_0 \neq s_1 < \infty, 1 < p < \infty, 1 \leq q_0, q_1, q \leq \infty, 0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$. Consequently, according to Theorem 3.1, we obtain the following.

Corollary 3.3. *For $j = 1,2,3,4$, let $-\infty < s_j < \infty, 1 \leq q_j, q \leq \infty$ and let $1 < p < \infty$. Suppose that $\alpha > \beta$, $\alpha + \beta < 1$ and $s_1 \neq s_2$. Put*

$$s_{123} = (1-\alpha-\beta)s_1 + \alpha s_2 + \beta s_3 \ , \ s_{124} = (1-\alpha)s_1 + (\alpha-\beta)s_2 + \beta s_4,$$

$$\bar{s} = \min\{s_{123}, s_{124}\} \quad , \quad \breve{s} = \max\{s_{123}, s_{124}\}.$$

Then we have, with equivalence of norms,

$$(B^{s_1}_{p,q_1}, B^{s_2}_{p,q_2}, B^{s_3}_{p,q_3}, B^{s_4}_{p,q_4})_{(\alpha,\beta),q;K} = B^{\bar{s}}_{p,q}$$

and

$$(B^{s_1}_{p,q_1}, B^{s_2}_{p,q_2}, B^{s_3}_{p,q_3}, B^{s_4}_{p,q_4})_{(\alpha,\beta),q;J} = B^{\breve{s}}_{p,q}.$$

Similar results hold if (α,β) lies in an internal triangle different from $\overline{P_1P_2Q}$.

If (α,β) lies in any diagonal of the square then it should be in two internal triangles at least (see Fig. 3.3). In this case, Theorem 3.1 is not valid in

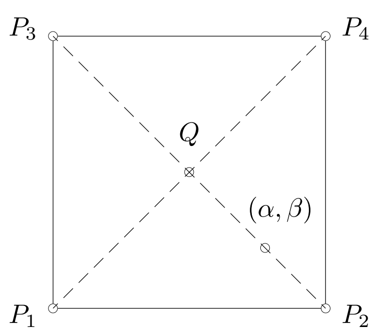

Figure 3.3

general even if we assume that $\theta_i \neq \theta_j$ for any triangle $\overline{P_iP_kQ}$ containing (α,β). We show it with an example.

Counterexample 3.4. Let $\ell_1(w_n)$ be the weighted ℓ_1-space with weights w_n. Put

$$X = \ell_1(n^{-1/2})\,,\; Y = \ell_1(n^{-1})\,,\; A_1 = A_4 = X \text{ and } A_2 = A_3 = Y.$$

Then $X \hookrightarrow Y$ so spaces $(X,Y)_{\theta,1}$ increase with the parameter θ (see [27], Thm. 1.3.3). The spaces A_j are of class $\mathcal{C}(\theta_j; X, Y)$ with $\theta_1 = \theta_4 = 0$ and $\theta_2 = \theta_3 = 1$. Take $(\alpha,\beta) = Q = (1/2,1/2)$. This point is in any of the four internal triangles and $\theta_i \neq \theta_k$ for each $\overline{P_iP_kQ}$ of them. However $\overline{A}_{(1/2,1/2),1;K}$ does not coincide with $(X,Y)_{\theta,1}$ for any value of θ because (see [9], Example 2.8)

$$\Big(\ell_1(n^{-1/2}), \ell_1(n^{-1}), \ell_1(n^{-1}), \ell_1(n^{-1/2})\Big)_{(1/2,1/2),1;K} = \ell_1(\frac{1+\log n}{n}).$$

The next result characterizes the K-interpolation spaces for a 4-tuple (X,Y,Y,X) with $X \hookrightarrow Y$ when (α,β) lies in any diagonal. Recall that for $1 \le q \le \infty$ the extrapolation space $(X,Y)_{1,q}$ is defined to be the collection of all $a \in Y$ which have a finite norm

$$\|a\|_{(X,Y)_{1,q}} = \left(\int_1^\infty (t^{-1}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} \qquad (\text{if } \; 1 \le q < \infty),$$

$$\|a\|_{(X,Y)_{1,\infty}} = \sup_{t \ge 1}\{t^{-1}\bar{K}(t,a)\}$$

(see [20] and [23]). Here $\bar{K}$ is the K-functional with respect to the couple $\{X,Y\}$. Note that, since $t^{-1}\bar{K}(t,a)$ is a decreasing function of t, we have $\|a\|_{(X,Y)_{1,\infty}} = \bar{K}(1,a) = \|a\|_{X+Y}$. Therefore, $(X,Y)_{1,\infty} = X + Y$.

Theorem 3.5. *Let $\{X,Y\}$ be a Banach couple with $X \hookrightarrow Y$, let $0 < \alpha < 1$ and $1 \le q \le \infty$. Then we have, with equivalent norms,*

$$(X,Y,Y,X)_{(\alpha,\alpha),q;K} = \begin{cases} (X,Y)_{2\alpha,q} & \text{if } \;\; 0 < \alpha < 1/2, \\ (X,Y)_{2(1-\alpha),q} & \text{if } \;\; 1/2 < \alpha < 1, \\ (X,Y)_{1,q} & \text{if } \;\; \alpha = 1/2, \end{cases}$$

and

$$(X,Y,Y,X)_{(\alpha,1-\alpha),q;K} = (X,Y)_{1,q} \quad \text{for any} \quad 0 < \alpha < 1.$$

Proof. Take any $a \in Y$ and suppose that $1 \le q < \infty$. It is easy to check that

$$K(t,s;a) = \min\{1,ts\}\bar{K}\left(\frac{\min\{t,s\}}{\min\{1,ts\}}, a\right).$$

Splitting the double integral of the norm of the K-space in the sets

$$\Omega_1 = \{(t,s) \in \mathbb{R}^2 : 0 < t \le 1\,,\, 0 < s \le t\},$$
$$\Omega_2 = \{(t,s) \in \mathbb{R}^2 : 0 < t \le 1\,,\, t < s \le 1/t\},$$
$$\Omega_3 = \{(t,s) \in \mathbb{R}^2 : 0 < t \le 1\,,\, 1/t < s < \infty\},$$
$$\Omega_4 = \{(t,s) \in \mathbb{R}^2 : 1 < t < \infty\,,\, 0 < s \le 1/t\},$$
$$\Omega_5 = \{(t,s) \in \mathbb{R}^2 : 1 < t < \infty\,,\, 1/t < s \le t\},$$
$$\Omega_6 = \{(t,s) \in \mathbb{R}^2 : 1 < t < \infty\,,\, t < s < \infty\},$$

it follows that

$$\text{(3.7)}\quad \|a\|_{(X,Y,Y,X)_{(\alpha,\alpha),q;K}} \sim \left(\int_1^\infty (t^{-2(1-\alpha)}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \bar{K}(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty (t^{-2\alpha}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q}.$$

Here $\sim$ means equivalence of norms. For the other diagonal we have

$$\text{(3.8)}\quad \|a\|_{(X,Y,Y,X)_{(\alpha,1-\alpha),q;K}} \sim \left(\int_1^\infty (t^{-1}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 (t^{-(2\alpha-1)}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 (t^{-(1-2\alpha)}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q}.$$

We may assume, without loss of generality, that the norm of the embedding $X \hookrightarrow Y$ is 1. Then

$$\bar{K}(t,a) = t\|a\|_Y \quad \text{for all} \quad 0 < t \le 1.$$

This yields that for any $\delta > -1$, $0 < \gamma \le 1$ and $0 < \theta < 1$

$$\text{(3.9)}\quad \left(\int_0^1 (t^{\delta}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} = ((1+\delta)q)^{-1/q}\,\|a\|_Y \le \left((1+\delta)\gamma q^2\right)^{-1/q} \left(\int_1^\infty (t^{-\gamma}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q}$$

and

$$\text{(3.10)}\quad \|a\|_{(X,Y)_{\theta,q}} \sim \left(\int_1^\infty (t^{-\theta}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q}.$$

Let $\alpha < 1/2$, so $-2(1-\alpha) < -2\alpha$. By (3.7), (3.9) and (3.10), we obtain

$$\|a\|_{(X,Y,Y,X)_{(\alpha,\alpha),q;K}} \sim \left(\int_1^\infty (t^{-2\alpha}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} \sim \|a\|_{(X,Y)_{2\alpha,q}}.$$

If $1/2 < \alpha$ then $-2\alpha < -2(1-\alpha)$ and we get

$$\|a\|_{(X,Y,Y,X)_{(\alpha,\alpha),q;K}} \sim \left(\int_1^\infty (t^{-2(1-\alpha)}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} \sim \|a\|_{(X,Y)_{2(1-\alpha),q}}.$$

For the other diagonal, from (3.8) and (3.9) we derive

$$\|a\|_{(X,Y,Y,X)_{(\alpha,1-\alpha),q;K}} \sim \left(\int_1^\infty (t^{-1}\bar{K}(t,a))^q \frac{dt}{t}\right)^{1/q} = \|a\|_{(X,Y)_{1,q}}.$$

The case $q = \infty$ can be treated analogously. □

Next we specialize Theorem 3.5 to two concrete cases. Let (Ω, μ) be a finite measure space with $\mu(\Omega) = 1$. Recall that the Zygmund spaces $L \log L$ is formed by all μ-measurable functions f on Ω for which

$$\|f\|_{L\log L} = \int_0^1 (\frac{1}{t}\int_0^t f^*(s)ds)dt < \infty$$

(see [1]). As it was pointed out in [20], Example 2.6, the space $L \log L$ can be obtained by extrapolation from the couple $\{L_\infty, L_1\}$. Indeed, by [2], Thm. 5.2.1, the K-functional for $\{L_1, L_\infty\}$ is given by

$$\bar{K}(t, f; L_1, L_\infty) = \int_0^t f^*(s)ds \quad \text{so} \quad (L_\infty, L_1)_{1,1} = L \log L.$$

As a direct consequence of Theorem 3.5 we get the following result.

Corollary 3.6. *Let $0 < \alpha < 1$. Then*

$$(L_\infty, L_1, L_1, L_\infty)_{(\alpha,1-\alpha),1;K} = L \log L.$$

Now take a Hilbert space H and let $\mathcal{L}(H)$ be the space of all bounded linear operators acting from H into H. The singular numbers of $T \in \mathcal{L}(H)$ are

$$s_n(T) = \inf\{\|T - R\| : \operatorname{rank} R < n\} \quad , \quad n \in \mathbb{N}.$$

For $1 \le p \le \infty$, the Schatten p-class $\mathcal{L}_p(H)$ consists of all $T \in \mathcal{L}(H)$ having a finite norm

$$\|T\|_{\mathcal{L}_p(H)} = \left(\sum_{n=1}^\infty s_n(T)^p\right)^{1/p} .$$

We refer to [19] for details on these spaces. In a more general way, given $1 < p < \infty$ and $1 \le q \le \infty$ the Lorentz operator space $\mathcal{L}_{p,q}(H)$ is defined as the collection of all $T \in \mathcal{L}(H)$ for which

$$\|T\|_{\mathcal{L}_{p,q}(H)} = \left(\sum_{n=1}^{\infty} (n^{(1/p)-1} \sum_{j=1}^{n} s_j(T))^q n^{-1} \right)^{1/q} < \infty$$

(see [27]). Spaces $\mathcal{L}_{p,q}(H)$ are the analogues for operators to the Lorentz function spaces $L_{p,q}$. From the point of view of interpolation theory, both families of spaces behave in a similar way. Namely,

$$(\mathcal{L}_1(H), \mathcal{L}(H))_{\theta,q} = \mathcal{L}_{p,q}(H)\ ,\ \frac{1}{p} = 1 - \theta\ ,\ 0 < \theta < 1\ ,\ 1 \leq q \leq \infty.$$

Hence, writing down Theorem 3.1 for these spaces we obtain a similar result to Corollary 3.2 but replacing $L_{p,q}$ by $\mathcal{L}_{p,q}(H)$.

In order to specialize Theorem 3.5, we recall that for $1 \leq q < \infty$ the space $\mathcal{L}_{\infty,q}(H)$ is formed by all $T \in \mathcal{L}(H)$ for which

$$\|T\|_{\mathcal{L}_{\infty,q}(H)} = \left(\sum_{n=1}^{\infty} s_n(T)^q n^{-1} \right)^{1/q} < \infty$$

(see [22] and [14]). These spaces correspond to the limit case $p = \infty$ in the Lorentz scale but the general theory of Lorentz operator spaces does not cover the case of $\mathcal{L}_{\infty,q}$-spaces (see [14], p. 325). It is shown in [12], Cor. 4.3, that

$$(\mathcal{L}_1(H), \mathcal{L}(H))_{1,q} = \mathcal{L}_{\infty,q}(H).$$

Consequently, Theorem 3.5 gives the following formulae.

Corollary 3.7. *Let $0 < \alpha < 1$ and $1 \leq q < \infty$. Then*

$$(\mathcal{L}_1(H), \mathcal{L}(H), \mathcal{L}(H), \mathcal{L}_1(H))_{(\alpha,\alpha),q;K} = \begin{cases} \mathcal{L}_{\frac{1}{1-2\alpha},q}(H) & \text{if} \quad 0 < \alpha < 1/2, \\ \mathcal{L}_{\frac{1}{2\alpha-1},q}(H) & \text{if} \quad 1/2 < \alpha < 1, \\ \mathcal{L}_{\infty,q}(H) & \text{if} \quad \alpha = 1/2, \end{cases}$$

and

$$(\mathcal{L}_1(H), \mathcal{L}(H), \mathcal{L}(H), \mathcal{L}_1(H))_{(\alpha,1-\alpha),q;K} = \mathcal{L}_{\infty,q}(H) \quad \text{for any} \quad 0 < \alpha < 1.$$

4. Interpolation over general polygons.

In this section we deal with general polygons $\Pi = \overline{P_1 \cdots P_N}$. Assuming a mild condition on the θ_j and that q takes only the values 1 or ∞, we shall establish results that work even if (α, β) lies in any diagonal.

Recall that $\mathcal{P}_{\alpha,\beta}$ is formed by all triples $\{i,k,r\}$ such that $(\alpha,\beta)\in\overline{P_iP_kP_r}$. We denote by (c_i,c_k,c_r) the barycentric coordinates of (α,β) with respect to P_i,P_k,P_r.

Theorem 4.1. *Let $\Pi=\overline{P_1\cdots P_N}$ be a convex polygon with $P_j=(x_j,y_j)$ and let $(\alpha,\beta)\in Int\,\Pi$. Assume that $\{X,Y\}$ is a Banach couple and that $\overline{A}=\{A_1,\dots,A_N\}$ is a Banach N-tuple formed by spaces A_j of class $\mathcal{C}(\theta_j;X,Y)$ with $0\le\theta_j\le 1, j=1,\dots,N$. For each $\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}$ with $(\alpha,\beta)\in Int\,\overline{P_iP_kP_r}$ we suppose that the numbers $\theta_i,\theta_k,\theta_r$ are not all equal. If $(\alpha,\beta)\in\overline{P_iP_kP_r}$ but (α,β) is not in $Int\,\overline{P_iP_kP_r}$, say because $c_i=0$, then we suppose that $\theta_k\neq\theta_r$.*

For $\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}$ we put $\theta_{ikr}=c_i\theta_i+c_k\theta_k+c_r\theta_r$ and we let

$$\bar{\theta}=\min\{\theta_{ikr}:\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}\}\quad,\quad\breve{\theta}=\max\{\theta_{ikr}:\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}\}.$$

Then we have, with equivalent norms,

(i) $\overline{A}_{(\alpha,\beta),\infty;K}=(X,Y)_{\bar{\theta},\infty}+(X,Y)_{\breve{\theta},\infty}$,
(ii) $\overline{A}_{(\alpha,\beta),1;J}=(X,Y)_{\bar{\theta},1}\cap(X,Y)_{\breve{\theta},1}$.

Proof. Let $\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}$. If $(\alpha,\beta)\in Int\,\overline{P_iP_kP_r}$, it follows from (2.5) and (2.9) that

$$(X,Y)_{\theta_{ikr},\infty}\hookrightarrow\overline{A}_{(\alpha,\beta),\infty;K}. \tag{4.1}$$

If $(\alpha,\beta)\in\overline{P_iP_kP_r}$ but it is not in its interior, we still obtain (4.1) but using now (2.6). Hence

$$(X,Y)_{\bar{\theta},\infty}+(X,Y)_{\breve{\theta},\infty}\hookrightarrow\overline{A}_{(\alpha,\beta),\infty;K}.$$

To establish the converse embedding, take any $a\in\overline{A}_{(\alpha,\beta),\infty;K}$ and let $a=\sum_{j=1}^N a_j$ be any representation of a with $a_j\in A_j$. For any $t,s,\lambda>0$, if $\bar{K}$ is the K-functional with respect to $\{X,Y\}$, we have

$$\min_{1\le j\le N}\{t^{x_j}s^{y_j}\lambda^{-\theta_j}\}\bar{K}(\lambda,a)\le\sum_{j=1}^N t^{x_j}s^{y_j}\lambda^{-\theta_j}\bar{K}(\lambda,a_j).$$

Since A_j is of class $\mathcal{C}(\theta_j;X,Y)$ there is a constant $C>0$ such that

$$\sum_{j=1}^N t^{x_j}s^{y_j}\lambda^{-\theta_j}\bar{K}(\lambda,a_j)\le C\sum_{j=1}^N t^{x_j}s^{y_j}\|a_j\|_{A_j}.$$

Hence

$$\min_{1\le j\le N}\{t^{x_j}s^{y_j}\lambda^{-\theta_j}\}\bar{K}(\lambda,a)\le CK(t,s;a).$$

This yields that

$$\sup_{t,s>0}[\min_{1\le j\le N}\{t^{x_j-\alpha}s^{y_j-\beta}\lambda^{-\theta_j}\}]\bar{K}(\lambda,a)\le C\|a\|_{\overline{A}_{(\alpha,\beta),\infty;K}}.$$

By (2.4) we know that

$$\sup_{t,s>0}[\min_{1\le j\le N}\{t^{x_j-\alpha}s^{y_j-\beta}\lambda^{-\theta_j}\}]=\min\{\lambda^{-\theta_{ikr}}:\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}\}$$

so

$$\sup_{0<\lambda\le 1}\{\lambda^{-\bar{\theta}}\bar{K}(\lambda,a)\}+\sup_{1\le\lambda<\infty}\{\lambda^{-\breve{\theta}}\bar{K}(\lambda,a)\}\le 2C\|a\|_{\overline{A}_{(\alpha,\beta),\infty;K}}.$$

Now using Holmstedt's formula (see [2], Thm. 3.6.1) we derive

$$\begin{aligned}\|a\|_{(X,Y)_{\bar{\theta},\infty}+(X,Y)_{\breve{\theta},\infty}}&=\bar{K}(1,a;\overline{X}_{\bar{\theta},\infty},\overline{X}_{\breve{\theta},\infty})\\&\le C_1[\sup_{0<\lambda\le 1}\{\lambda^{-\bar{\theta}}\bar{K}(\lambda,a)+\sup_{\lambda\ge 1}\{\lambda^{-\breve{\theta}}\bar{K}(\lambda,a)\}]\le C_2\|a\|_{\overline{A}_{(\alpha,\beta),\infty;K}}.\end{aligned}$$

This proves (i).

We turn to the proof of (ii). Using (2.5), (2.6) and (2.9), we obtain that $\bar{A}_{(\alpha,\beta),1;J}\hookrightarrow(X,Y)_{\theta_{ikr},1}$ for any $\{i,k,r\}\in\mathcal{P}_{\alpha,\beta}$. Hence

$$\bar{A}_{(\alpha,\beta),1;J}\hookrightarrow(X,Y)_{\bar{\theta},1}\cap(X,Y)_{\breve{\theta},1}.$$

To establish the converse inclusion we recall the discrete version of $(X,Y)_{\theta,1}$, $0<\theta<1$, realized as a K-space (see [2] or [27]). The space $(X,Y)_{\theta,1}$ consists of all $a\in X+Y$ such that

$$\|a\|^*_{(X,Y)_{\theta,1}}=\sum_{\nu=-\infty}^{\infty}2^{-\theta\nu}\bar{K}(2^\nu,a)<\infty.$$

Moreover, $\|\cdot\|_{(X,Y)_{\theta,1}}$ and $\|\cdot\|^*_{(X,Y)_{\theta,1}}$ are equivalent norms.

Given any $a\in(X,Y)_{\bar{\theta},1}\cap(X,Y)_{\breve{\theta},1}$, by the fundamental lemma (see [2], Lemma 3.3.2) there is a representation $a=\sum_{\nu=-\infty}^{\infty}u_\nu$ with $\{u_\nu\}\subset X\cap Y$ and $\bar{J}(2^\nu,u_\nu)\le 4\bar{K}(2^\nu,a)$ for each $\nu\in\mathbb{Z}$. We claim that the series $\sum_{\nu=-\infty}^{\infty}u_\nu$ is absolutely convergent in $\bar{A}_{(\alpha,\beta),1;J}$. Indeed, using (2.3) and that A_j is of the class $\mathcal{C}(\theta_j; X, Y)$, we have

$$\begin{aligned}
\sum_{\nu=-\infty}^{\infty} \|u_\nu\|_{(\alpha,\beta),1;J} &\leq C \sum_{\nu=-\infty}^{\infty} \max\{\|u_\nu\|_{A_i}^{c_i} \|u_\nu\|_{A_k}^{c_k} \|u_\nu\|_{A_r}^{c_r} : \{i,k,r\} \in \mathcal{P}_{\alpha,\beta}\} \\
&\leq C_1 \sum_{\nu=-\infty}^{\infty} \max\{2^{-\theta_i c_i} \bar{J}(2^\nu, u_\nu)^{c_i} 2^{-\theta_k c_k} \bar{J}(2^\nu, u_\nu)^{c_k} 2^{-\theta_r c_r} \bar{J}(2^\nu, u_\nu)^{c_r} \\
&\qquad : \{i,k,r\} \in \mathcal{P}_{\alpha,\beta}\} \\
&\leq C_1 \sum_{\nu=-\infty}^{\infty} \max\{2^{-\theta_{ikr}\nu} \bar{J}(2^\nu, u_\nu) : \{i,k,r\} \in \mathcal{P}_{\alpha,\beta}\} \\
&\leq C_1 \left(\sum_{\nu=-\infty}^{\infty} 2^{-\bar{\theta}\nu} \bar{J}(2^\nu, u_\nu) + \sum_{\nu=-\infty}^{\infty} 2^{-\breve{\theta}\nu} \bar{J}(2^\nu, u_\nu) \right) \\
&\leq 4C_1 \left(\sum_{\nu=-\infty}^{\infty} 2^{-\bar{\theta}\nu} \bar{K}(2^\nu, a) + \sum_{\nu=-\infty}^{\infty} 2^{-\breve{\theta}\nu} \bar{K}(2^\nu, a) \right).
\end{aligned}$$

Consequently, a belongs to $\overline{A}_{(\alpha,\beta),1;J}$ and

$$\|a\|_{(\alpha,\beta),1;J} \leq 4C_1 \left(\|a\|^*_{(X,Y)_{\bar{\theta},1}} + \|a\|^*_{(X,Y)_{\breve{\theta},1}} \right) \leq 8C_1 \|a\|_{(X,Y)_{\bar{\theta},1} \cap (X,Y)_{\breve{\theta},1}}.$$

This gives (ii) and completes the proof. $\square$

Next we show some direct applications of Theorem 4.1. Assume first that the Banach couple $\{X,Y\}$ is formed by Banach algebras such that multiplications in X and Y coincide on $X \cap Y$. It was shown by Bishop [3] (see also [21] and [5]) that for $0 < \theta < 1$ the space $(X,Y)_{\theta,1}$ is a Banach algebra. Multiplication in $(X,Y)_{\theta,1}$ being the same as in X and Y on $X \cap Y$. So Theorem 4.1 yields the following result.

Corollary 4.2. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon and let $(\alpha,\beta) \in Int\,\Pi$. Let $\{X,Y\}$ be a couple of Banach algebras and let $\overline{A} = \{A_1, \ldots, A_N\}$ be an N-tuple formed by spaces A_j of class $\mathcal{C}(\theta_j; X, Y)$ where the θ_j satisfy the same assumptions as in Theorem 4.1. Then $\overline{A}_{(\alpha,\beta),1;J}$ is a Banach algebra.*

Finally we consider a Banach couple $\{H_0, H_1\}$ formed by Hilbert spaces such that $H_0 \cap H_1$ is dense in H_0 and H_1. It was shown by Ovchinnikov [24], [25] that

$$(\mathcal{L}(H_0,H_0), \mathcal{L}(H_1,H_1))_{\theta,\infty} = \mathcal{L}((H_0,H_1)_{\theta,1}, (H_0,H_1)_{\theta,\infty}).$$

Hence, using Theorem 4.1, we derive the following.

Corollary 4.3. *Let* $\Pi = \overline{P_1 \cdots P_N}$ *be a convex polygon and let* $(\alpha, \beta) \in Int\,\Pi$. *Take* $X = \mathcal{L}(H_0, H_0), Y = \mathcal{L}(H_1, H_1)$ *and let* $\overline{A} = \{A_1, \ldots, A_N\}$ *be an* N*-tuple formed by spaces* A_j *of class* $\mathcal{C}(\theta_j; X, Y)$ *where the* θ_j *satisfy the same assumptions as in Theorem 4.1. Then we have*

$$\overline{A}_{(\alpha,\beta),\infty;K} = \mathcal{L}((H_0, H_1)_{\bar{\theta},1}, (H_0, H_1)_{\bar{\theta},\infty}) + \mathcal{L}((H_0, H_1)_{\breve{\theta},1}, (H_0, H_1)_{\breve{\theta},\infty}).$$

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