Isoperimetry and Symmetrization for Sobolev spaces on metric spaces

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Abstract

Using isoperimetry we obtain new symmetrization inequalities that allow us to provide a unified framework to study Sobolev inequalities in metric spaces. The applications include concentration inequalities, Poincaréinequalities, as well as metric versions of the Pólya-Szegö and Faber-Krahn principles.

Résumé

Isopérimétrie et symetrisation dans des espaces de Sobolev sur les espaces métriques. En utilisant l'isopérimétrie nous obtenons des nouvelles inégalités de symetrisation qui nous permettent de fournir un cadre unifié pour étudier des inégalités de Sobolev dans des espaces métriques. Les applications incluent des inégalités de concentration, inégalités de Poincaré, et des versions métriques des principes de Pólya-Szegö et de Faber-Krahn.

1. Introduction

This is a follow up to our recent work [10], where we obtained new symmetrization inequalities for Sobolev functions that compare the rearrangement of a function with the rearrangement of its gradient, and incorporate in their formulation the isoperimetric profile (cf. (1) below). These inequalities imply in a straightforward fashion functional inequalities for very general rearrangement invariant norms or quasi-norms (e.g. L^p, Orlicz, Lorentz, Marcinkiewicz spaces). One remarkable characteristic of these inequalities is that they preserve their form as we move from one measure space to another, the only thing that changes are the corresponding isoperimetric profiles. As a consequence we were able to provide a unified framework to study the classical Sobolev-Poincaré inequalities, logarithmic Sobolev inequalities, as well as concentration inequalities (cf. [9] and the references therein). Importantly, if the isoperimetric profile does not depend on the dimension (like in the Gaussian case) then the corresponding inequalities are dimension free.

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The purpose of this note is to outline the modifications that are necessary to extend our earlier results to the setting of metric spaces. Indeed, under relatively weak assumptions, all the tools that we need are available in the metric setting (cf. [4]), and our methods can be readily adapted to provide an almost painless extension. In particular, the results of this note, when combined with the method developed [1] in [10], produce concentration inequalities in metric spaces, as well as a Sobolev metric space version of the Pólya-Szegö principle; while our results combined with the method of [[8], Theorem 3] imply metric Faber-Krahn inequalities.

Let (Ω, d, μ) be a metric space equipped with a separable Borel probability measure μ. For measurable functions $u : \Omega \to \mathbb{R}$, the distribution function of u is given by $\lambda_u(t) = \mu(\{x \in \Omega : |f(x)| > t\}$ $(t > 0)$, the decreasing rearrangement u^* of u is defined, as usual, by $u^*(s) = \inf\{t \geq 0 : \lambda_u(t) \leq s\}$ $(t \in (0,1)])$, and we let $u^{**}(t) = \frac{1}{t}\int_0^t u^*(s)ds$. For $A \subset \Omega$, a Borel set, let $Per(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A_{\varepsilon,d}) - \mu(A)}{\varepsilon}$, where $A_{\varepsilon,d} = \{x \in \Omega : \exists y \in A \ d(x,y) < \varepsilon\}$ denotes the $\varepsilon-$extension of A with respect to the metric d. An isoperimetric inequality measures the relation between $Per(A)$ and $\mu(A)$ by means of the isoperimetric profile $I = I_{(\Omega;d;\mu)}$, which is defined as the pointwise maximal function $I : [0,1] \to [0, \infty)$, such that $Per(A) \geq I(\mu(A))$,for all Borel sets A. Finally, in this setting for a given Lipschitz function f (we shall write in what follows $f \in Lip(\Omega)$) the modulus of the gradient is defined, as usual, by $|\nabla f(x)| = \limsup_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x,y)}$.

2. Main results

Theorem 2.1 *Suppose that the isoperimetric profile I is concave, continuous, increasing on $(0, 1/2)$ and symmetric about the point $1/2$. Then the following statements hold* [2] *(and in fact are equivalent):*

$$(i): \ \forall \ f \in Lip(\Omega), \ \int_0^\infty I(\lambda_f(s))ds \leq \int_\Omega |\nabla f(x)| \, d\mu(x) \ \textit{(Ledoux)}.$$

$$(ii): \ \forall \ f \in Lip(\Omega), \ (-f^*)'(s)I(s) \leq \frac{d}{ds}\int_{\{|f| > f^*(s)\}} |\nabla f(x)| \, d\mu(x) \ \textit{(Talenti-Maz'ya)}.$$

$$(iii): \ \forall \ f \in Lip(\Omega), \ \int_0^t ((-f^*)'(.)I(.))^*(s)ds \leq \int_0^t |\nabla f|^*(s)ds \ \textit{(Pólya-Szegö)}.$$

(The second rearrangement on the left hand side is with respect to the Lebesgue measure).

$$(iv): \ \forall \ f \in Lip(\Omega), \ (f^{**}(t) - f^*(t)) \leq \frac{t}{I(t)} |\nabla f|^{**}(t). \tag{1}$$

Given any rearrangement invariant space [3] $X(\Omega)$, it follows readily from (1) that for all Lip functions, we have

$$\|f\|_{LS(X)} := \left\| (f^{**}(t) - f^*(t)) \frac{I(t)}{t} \right\|_{\bar{X}} \leq \|\nabla f\|_X. \tag{2}$$

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[1] Our method builds on a variant of Maz'ya's truncation principle, combined with the relevant isoperimetric inequalities, the co-area formula and classical arguments from real interpolation theory (cf. Calderón [7]). We call this method to obtain symmetrization inequalities "symmetrization via truncation".

[2] except where indicated all rearrangements are with respect to the measure μ.

[3] A Banach lattice of functions $X(\Omega)$ is called a rearrangement invariant (r.i.) space (cf. [2]) if $g \in X(\Omega)$ implies that all functions f with the same decreasing rearrangement, $f^* = g^*$, also belong to $X(\Omega)$, and, moreover, $\|f\|_{X(\Omega)} = \|g\|_{X(\Omega)}$. There is an essentially unique r.i. space $\bar{X}(0,1)$ of functions on the interval $(0,1)$ consisting of all $g : (0,1) \to R$ such that $g^*(t) = f^*(t)$ for some function $f \in X(\Omega)$.

Remark 1 For L^1 norms these Poincaré inequalities are a simple variant of Ledoux's inequality (i). Indeed, let m_f be a median[4] *of f, then it is easy to see that*

$$\int_\Omega |f - m_f|\, d\mu \leq \frac{1}{2I(1/2)} \int_\Omega |\nabla f(x)|\, d\mu(x). \tag{3}$$

The novelty of our inequalities, and the corresponding associated spaces $LS(X)$, is that they incorporate the isoperimetric profiles associated with the geometry in question. These spaces are not necessarily normed, although often they are equivalent to normed spaces (cf. [14]), and, in the classical cases, lead to optimal Sobolev-Poincaré inequalities (cf. [12], [10], [11] and the references therein).

We now investigate the optimality of the Poincaré type inequality (2). The following result is new in the context of r.i. spaces.

Theorem 2.2 *Let $(\Omega, \mu) = (R^n, \mu_r^{\otimes n})$, with $\mu_r(x) = \varphi(x)dx, I_{\mu_r^{\otimes n}}(t) \approx \varphi(F^{-1}(t)),\ t \in [0,1]$, where F^{-1} is the inverse of the distribution function associated to the density $\varphi(x)dx$*[5]*. Let $X(\Omega)$, $Y(\Omega)$ be r.i. spaces. Then, the following statements are equivalent*

$$(i):\ \forall\ f \in Lip(\Omega),\ \left\| f - \int f \right\|_Y \preceq \|\nabla f\|_X\,. \tag{4}$$

$$(ii):\ \left\| \int_t^1 f(s) \frac{ds}{I(s)} \right\|_{\bar{Y}} \preceq \|f\|_{\bar{X}}\,,\ \forall\ 0 \leq f \in \bar{X}, \textit{with } supp(f) \subset (0, 1/2).$$

Moreover,

(a) If the operator $Q_I\ f(t) = \frac{I(t)}{t} \int_t^1 f(s) \frac{ds}{I(s)}$ is bounded from $\bar{X}$ to $\bar{X}$, then the next inequality can be added to the list of equivalences

$$(iii):\ \|f\|_{\bar{Y}} \preceq \left\| f^*(t) \frac{I(t)}{t} \right\|_{\bar{X}}. \tag{5}$$

(b) On the other hand if Q_I is not bounded from $\bar{X}$ to $\bar{X}$, but $\|f\|_X \simeq \|f^{**}\|_{\bar{X}}$, then the next inequality can be added to the list of equivalences

$$\|f\|_{\bar{Y}} \preceq \|f\|_{LS(X)} + \|f\|_{L^1}. \tag{6}$$

As a concrete illustration[6] consider the family of probability measures on the real line given by $d\mu_r(t) = \alpha_r^{-1} e^{-|t|^r} dt = \varphi_r(t)dt$, $1 < r \leq 2$, where α_r^{-1} is chosen to ensure that $\mu_r(\mathbb{R})$ =1. These probabilities form a scale between exponential and Gaussian measure. The associated isoperimetric profile is given by $I_{\mu_r}(t) = \varphi_r(F_r^{-1}(t))$, where F_r^{-1} is the inverse of the distribution function associated to the density $\varphi_r(t)$ (cf. [5]). The isoperimetric profiles $I_{\mu_r^{\otimes n}}$, associated to the product probability measures $\mu_r^{\otimes n}$, is dimension free (see [1]): there is a universal constant $c(r)$ such that $I_{\mu_r}(t) \geq \inf_{n\geq 1} I_{\mu_r^{\otimes n}}(t) \geq c(r) I_{\mu_r}(t)$. As an application let $n \geq 2$, and apply Theorem 2.2 to $X = L^p(\mathbb{R}^n, d\mu_r^{\otimes n})$, $1 \leq p < \infty$, then (cf. also [11, Theorem 3]),

$$\int_0^1 \left(\left(f - \int f \right)^* (s) \frac{I_{\mu_r}(s)}{s} \right)^p ds \preceq \int_{\mathbb{R}^n} |\nabla f(x)|^p\, d\mu_r^{\otimes n}(x),$$

with dimension free constants. In particular, since (see [5, Lemma 16.1]) $\lim_{t\to 0^+} \frac{I_{\mu_r}(t)}{t\left(\log \frac{1}{t}\right)^{1/q}} = r$, $1/r + 1/q = 1$, it follows easily that

$$\int_0^1 f^*(s)^p (\log \frac{1}{s})^{p/q} ds \preceq \int_{\mathbb{R}^n} |\nabla f(x)|^p\, d\mu_r^{\otimes n}(x) + \int_{\mathbb{R}^n} |f(x)|^p\, d\mu_r^{\otimes n}(x).$$

[4] i.e. $\mu\,(f \geq m) \geq 1/2$ and $\mu\,(f \leq m) \geq 1/2$.
[5] This choice of I is motivated by the results in [6], [3] and [1].
[6] For further examples we refer to [4], [13], and the references therein.

Moreover, for this class of measures, $L^p(LogL)^{p/q}$ is the best possible choice among all r.i. spaces Y for which the inequality $\|f-\int f\|_Y \preceq \|\nabla f\|_{L^p}$ holds. If $p=\infty$, we have

$$\|f-\int f\|_{LS(L^\infty)} = \left\| \left(\left(f-\int f\right)^{**}(t) - \left(f-\int f\right)^{*}(t) \right) \frac{I_{\mu_r}(t)}{t} \right\|_{L^\infty} \preceq \|\nabla f\|_{L^\infty}. \tag{7}$$

The relation to concentration inequalities follows directly from our main inequality. Indeed, we have

$$\sup_{t<1}\left\{ (f^{**}(t)-f^{*}(t)) \frac{I_{\mu_r}(t)}{t} \right\} \le \sup_t |\nabla f|^{**}(t) = \|f\|_{Lip},$$

which, by the asymptotic properties of I_{μ_r}, implies that $f^{**}(t)-f^{*}(t) \preceq \frac{\|f\|_{Lip}}{\left(\log\frac{1}{t}\right)^{1/q}}$ $(0<t<1/2)$. We may now proceed as in [[10], Section 7].

Let us finally consider Sobolev embeddings into L^∞. Notice that from inequality (1) we get

$$\|f\|_\infty - 2\int_0^{1/2} f^*(t) = \int_0^{1/2} (f^{**}(t)-f^*(t))\frac{dt}{t} \le \int_0^{1/2}\left(\frac{1}{t}\int_0^t |\nabla f|^*(s)ds\right)\frac{dt}{I_{\mu_r}(t)} = \int_0^{1/2} |\nabla f|^*(s)\int_s^{1/2}\frac{ds}{I_{\mu_r}(s)s}.$$

Using the asymptotics of $I_{\mu_r}(s)$ combined with the Poincaré inequality 3 yields

$$\|f-m_f\|_\infty \preceq \int_0^{1/2} |\nabla f|^*(s) \frac{ds}{s\left(\log\frac{1}{s}\right)^{1/q}}.$$

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