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ISOPERIMETRY AND SYMMETRIZATION FOR LOGARITHMIC SOBOLEV INEQUALITIES

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Abstract. Using isoperimetry and symmetrization we provide a unified framework to study the classical and logarithmic Sobolev inequalities. In particular, we obtain new Gaussian symmetrization inequalities and connect them with logarithmic Sobolev inequalities. Our methods are very general and can be easily adapted to more general contexts.

Contents

1. Introduction

The classical L^2-Sobolev inequality states that

$$|\nabla f|\in L^2(\mathbb{R}^n)\Rightarrow f\in L^{p_n^*}(\mathbb{R}^n),\text{ where }\frac{1}{p_n^*}=\frac{1}{2}-\frac{1}{n}.$$

Consequently, $\lim_{n\to\infty}p_n^*=2$ and, therefore, the improvement on the integrability of f disappears as $n\to\infty$. On the other hand, Gross [20] showed that, if one replaces dx by the Gaussian measure $d\gamma_n(x)=(2\pi)^{-n/2}e^{-|x|^2/2}dx$, we have

$$(1.1)\qquad \int|f(x)|^2\ln|f(x)|\,d\gamma_n(x)\leq\int|\nabla f(x)|^2\,d\gamma_n(x)+\|f\|_2^2\ln\|f\|_2\,.$$

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This is Gross' celebrated logarithmic Sobolev inequality (= lS inequality), the starting point of a new field, with many important applications to PDEs, Functional Analysis, Probability, etc. (as a sample, and only a sample, we mention [2], [13], [23], [7], and the references therein). The inequality (1.1) gives a logarithmic improvement on the integrability of f, with constants independent of n, that persists as $n \mapsto \infty$, and is best possible. Moreover, rescaling (1.1) leads to L^p variants of this inequality, again with constants independent of the dimension (cf. [20]),

$$\int |f(x)|^p \ln |f(x)|\, d\gamma_n(x) \leq \frac{p}{2(p-1)} \operatorname{Re} < Nf, f_p > + \|f\|_p^p \ln \|f\|_p ,$$

where $< f, g > = \int f\bar{g} d\gamma_n$, $< Nf, f > = \int |\nabla f(x)|^2 \, d\gamma_n(x)$, $f_p = (sgn(f)) \, |f|^{p-1}$.

In a somewhat different direction, Feissner's thesis [17] under Gross, takes up the embedding implied by (1.1), namely

$$W_2^1(\mathbb{R}^n, d\gamma_n) \subset L^2(LogL)(\mathbb{R}^n, d\gamma_n),$$

where the norm of $W_2^1(\mathbb{R}^n, d\gamma_n)$ is given by

$$\|f\|_{W_2^1(\mathbb{R}^n,d\gamma_n)} = \|\nabla f\|_{L^2(\mathbb{R}^n,d\gamma_n)} + \|f\|_{L^2(\mathbb{R}^n,d\gamma_n)} ,$$

and extends it to L^p, even Orlicz spaces. A typical result[1] from [17] is given by

$$W_p^1(\mathbb{R}^n, d\gamma_n) \subset L^p(LogL)(\mathbb{R}^n, d\gamma_n), p \geq 2. \tag{1.2}$$

The connection between lS inequalities and the classical Sobolev estimates has been investigated intensively. For example, it is known that (1.1) follows from the classical Sobolev estimates with sharp constants (cf. [3], [4] and the references therein). In a direction more relevant for our development here, using the argument of Ehrhard [15], we will show, in section 5 below, that (1.1) follows from the symmetrization inequality of Pólya-Szegö for Gaussian measure (cf. [16] and Section 4)

$$\|\nabla f^{\circ}\|_{L^2(\mathbb{R},d\gamma_1)} \leq \|\nabla f\|_{L^2(\mathbb{R}^n,d\gamma_n)} ,$$

where f° is the Gaussian symmetric rearrangement of f with respect to Gaussian measure (cf. Section 2 below).

The purpose of this paper is to give a new approach to lS inequalities through the use of symmetrization methods. While symmetrization methods are a well established tool to study Sobolev inequalities, through the combination of symmetrization and isoperimetric inequalities we uncover new rearrangement inequalities and connections, that provide a context in which we can treat the classical and logarithmic Sobolev inequalities in a unified way. Moreover, with no extra effort we are able to extend the functional lS inequalities to the general setting of rearrangement invariant spaces. In particular, we highlight a new extreme embedding which clarifies the connection between lS, the concentration phenomenon and the John-Nirenberg lemma. Underlying this last connection is the apparently new observation that concentration inequalities self improve, a fact we shall treat in detail in a separate paper (cf. [30]).

The key to our method are new symmetrization inequalities that involve the isoperimetric profile and, in this fashion, are strongly associated with geometric measure theory. In previous papers (cf. [32] and the references therein) we had

[1] For the most part the classical work on functional lS inequalities has focussed on L^2, or more generally, L^p and Orlicz spaces.

obtained the corresponding inequalities in the classical case without making explicit reference to the Euclidean isoperimetric profile. Using isoperimetry we are able to connect each of the classical inequalities with their corresponding (new) Gaussian counterparts. We will show that the difference between the classical and the new Gaussian inequalities can be simply explained in terms of the difference of the corresponding isoperimetric profiles. In particular, in the Gaussian case, the isoperimetric profile is independent of the dimension, and this accounts for the fact that our rearrangement inequalities in this setting have this property. Another bonus is that our method is rather general, and amenable to considerable generalization: to Sobolev inequalities in general measure spaces, metric Sobolev spaces, even discrete Sobolev spaces. We hope to return to some of these developments elsewhere.

To describe more precisely our results let us recall that the connection between isoperimetry and Sobolev inequalities goes back to the work of Maz'ya and Federer and can be easily explained by combining the formula connecting the gradient and the perimeter (cf. [27]):

$$\|\nabla f\|_1 = \int_0^\infty Per(\{|f| > t\})dt, \tag{1.3}$$

with the classical Euclidean isoperimetric inequality:

$$Per(\{|f| > t\}) \geq n\varpi_n^{1/n} \left(|\{|f| > t\}|\right)^{\frac{n-1}{n}}, \tag{1.4}$$

where $\varpi_n =$ volume of unit ball in $\mathbb{R}^n$. Indeed, combining (1.4) and (1.3) yields the sharp form of the Gagliardo-Nirenberg inequality

$$(n-1)\varpi_n^{1/n} \|f\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^1(\mathbb{R}^n)}. \tag{1.5}$$

In [32], we modified Maz'ya's truncation method[2], to develop a sharp tool to extract symmetrization inequalities from Sobolev inequalities like (1.5). In particular, we showed that, given any rearrangement invariant norm (r.i. norm) $\|.\|$, the following optimal Sobolev inequality[3] holds (cf. [33])

$$\left\|(f^{**}(t) - f^*(t))t^{-1/n}\right\| \leq c(n,X)\|\nabla f\|, \ f \in C_0^\infty(\mathbb{R}^n). \tag{1.6}$$

An analysis of the role that the power $t^{-1/n}$ plays in this inequality led us to connect (1.6) to isoperimetric profile of $(\mathbb{R}^n, dx)$. In fact, observe that we can formulate (1.4) as

$$Per(A) \geq I_n(vol_n(A)),$$

where $I_n(t) = n\varpi_n^{1/n} t^{(n-1)/n}$ is the "isoperimetric profile" or the "isoperimetric function", and equality is achieved for balls.

The corresponding isoperimetric inequality for Gaussian measure (i.e. $\mathbb{R}^n$ equipped with Gaussian measure $d\gamma_n(x) = (2\pi)^{-n/2}e^{-|x|^2/2}dx$), and the solution to the Gaussian isoperimetric problem, was obtained by Borell [11] and Sudakov-Tsirelson [36], who showed[4] that

$$Per(A) \geq I(\gamma_n(A)),$$

[2] we termed this method "symmetrization via truncation".

[3] This inequality is optimal and includes the problematic borderline "end points" of the L^p theory.

[4] Erhard [14] provides an approach using symmetrization. Erhard also proves using this method a Gaussian version of the Brunn-Minkowski inequality but only for convex bodies. This restriction

with equality achieved for half spaces[5], and where $I = I_\gamma$ is the Gaussian profile[6] (cf. (2.2) below for the precise definition of I). To highlight a connection with the lS inequalities, we only note here that I has the following asymptotic formula near the origin (say $t \leq 1/2$, see Section 2 below),

$$I(t) \simeq t\left(\log\frac{1}{t}\right)^{1/2}. \tag{1.7}$$

As usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq c\,f$, while the symbol $f \preceq g$ means that $f \leq c\,g$.

With this background one may ask: what is the Gaussian replacement of the Gagliardo-Nirenberg inequality (1.5)? The answer was provided by Ledoux who showed (cf. [24])

$$\int_0^\infty I(\lambda_f(s))ds \leq \int_{\mathbb{R}^n} |\nabla f|\, d\gamma_n(x),\ \ f \in Lip(\mathbb{R}^n). \tag{1.8}$$

In fact, following the steps of the proof we indicated for (1.5), but using the Gaussian profile instead, we readily arrive at Ledoux's inequality. This given we were therefore led to apply our method of symmetrization by truncation to the inequality (1.8). We obtained the following counterpart of (1.6)

$$(f^{**}(t) - f^*(t)) \leq \frac{t}{I(t)}\left|\nabla f\right|^{**}(t),$$

here f^* denotes the non-increasing rearrangement of f with respect to the Lebesgue measure and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$. Further analysis showed that, in agreement with the Euclidean case we had worked out in [32], all these inequalities are in fact equivalent[7] to the isoperimetric inequality[8] (cf. Section 3 below):

Theorem 1. *The following statements are equivalent (all rearrangements are with respect to Gaussian measure):*

(i) Isoperimetric inequality: For every Borel set $A \subset \mathbb{R}^n$, *with* $0 < \gamma_n(A) < 1$,

$$Per(A) \geq I(\gamma_n(A)).$$

(ii) Ledoux's inequality: for every Lipschitz function f *on* $\mathbb{R}^n$,

$$\int_0^\infty I(\lambda_f(s))ds \leq \int_{\mathbb{R}^n} |\nabla f(x)|\, d\gamma_n(x). \tag{1.9}$$

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remained an open problem until it was finally removed by Borell [12]. For a nice survey concerning these inequalities prior to 2002, see [22].

[5]In some sense one can consider half spaces as balls centered at infinity.

[6]In principle I could depend on n but by the very definition of half spaces it follows that the Gaussian isoperimetric profile is dimension free.

[7]It is somewhat paradoxical that (1.1), because of the presence of squares, needs a special treatment and is not, as fas as we know, equivalent to the isoperimetric inequality (for a partial converse in this direction cf. [23]).

[8]The equivalence between (i) and (ii) in Theorem 1 above is due to Ledoux [23], see also [9]).

(iii) Talenti's inequality[9] *(Gaussian version): For every Lipschitz function f on $\mathbb{R}^n$,*

$$(1.10)\qquad (-f^*)'(s)I(s) \leq \frac{d}{ds}\int_{\{|f|>f^*(s)\}} |\nabla f(x)|\, d\gamma_n(x).$$

(iv) Oscillation inequality (Gaussian version): For every Lipschitz function f on $\mathbb{R}^n$,

$$(1.11)\qquad (f^{**}(t) - f^*(t)) \leq \frac{t}{I(t)} |\nabla f|^{**}(t).$$

This formulation coincides with the corresponding Euclidean result we had obtained in [32], and thus, in some sense, unifies the classical and Gaussian Sobolev inequalities. More precisely, by specifying the corresponding isoperimetric profile we automatically derive the correct results in either case. Thus, for example, if in (1.9) we specify the Euclidean isoperimetric profile we get the Gagliardo-Nirenberg inequality, in (1.10) we get Talenti's original inequality [37] and in (1.11) we get the rearrangement inequality of [1].

Underlying all these inequalities is the so called Pólya-Szegö principle. The L^p Gaussian versions of this principle had been obtained earlier by Ehrhard[10] [16]. We obtain here a general version of the Pólya-Szegö principle (cf. [18] where the Euclidean case was stated without proof), what may seem surprising at first is the fact that, in our formulation, the Pólya-Szegö principle is, in fact, equivalent to the isoperimetric inequality (cf. Section 4).

Theorem 2. *The following statements are equivalent*

(i) Isoperimetric inequality: For every Borel set $A \subset \mathbb{R}^n$, with $0 < \gamma_n(A) < 1$

$$Per(A) \geq I(\gamma_n(A)).$$

(ii) Pólya-Szegö principle: For every Lipschitz function f on $\mathbb{R}^n$,

$$|\nabla f^\circ|^{**}(s) \leq |\nabla f|^{**}(s).$$

Very much like Euclidean symmetrization inequalities lead to optimal Sobolev and Poincaré inequalities and embeddings (cf. [32], [29] and the references therein), the new Gaussian counterpart (1.11) we obtain here leads to corresponding optimal Gaussian Sobolev-Poincaré inequalities as well. The corresponding analog of (1.6) is: given any rearrangement invariant space X on the interval $(0,1)$, we have the optimal inequality, valid for Lip functions (cf. Section 6 below)

$$(1.12)\qquad \|f\|_{LS(X)} := \left\| (f^{**}(t) - f^*(t)) \frac{I(t)}{t} \right\|_X \leq \|\nabla f\|_X .$$

The spaces $LS(X)$ defined in this fashion are not necessarily normed, although often they are equivalent to normed spaces[11]. As a counterpart to this defect we remark that, since the Gaussian isoperimetric profile is independent of the dimension, the inequalities (1.12) are dimension free. In particular, we note the following result here (cf. sections 6 and 6.1 below for a detailed analysis),

[9] In connection with the Euclidean version of this inequality see also [28].

[10] For comparison we mention that Ehrhard's results are formulated in terms of increasing rearrangements.

[11] For the Euclidean case a complete study of the normability of these spaces has been recently given in [35].

Theorem 3. *Let X, Y be two r.i. spaces. Then, the following statements are equivalent*

(i) For every Lipschitz function f on $\mathbb{R}^n$

$$\left\| f - \int f \right\|_Y \preceq \|\nabla f\|_X . \tag{1.13}$$

(ii) For every positive function $f \in X$ with supp$f \subset (0,1/2)$,

$$\left\| \int_t^1 f(s) \frac{ds}{I(s)} \right\|_Y \preceq \|f\|_X .$$

Part II. Let $\underline{\alpha}_X$ and $\overline{\alpha}_X$ be the lower and the upper Boyd indices of X (see Section 2 below). If $\underline{\alpha}_X > 0$, then the following statement is equivalent to (i) and (ii) above:

(iii)

$$\|f\|_Y \preceq \left\| f^*(t) \frac{I(t)}{t} \right\|_X .$$

In particular, if Y is a r.i. space such that (1.13) holds, then

$$\|f\|_Y \leq \left\| f^*(t) \frac{I(t)}{t} \right\|_X .$$

If $0 = \underline{\alpha}_X < \overline{\alpha}_X < 1$, then the following statement is equivalent to (i) and (ii) above

(iv)

$$\|f\|_Y \preceq \|f\|_{LS(X)} + \|f\|_{L^1}.$$

In particular, if Y is a r.i. space such that (1.13) holds, then

$$\|f\|_Y \leq \|f\|_{LS(X)} + \|f\|_{L^1}. .$$

To recognize the logarithmic Sobolev inequalities that are encoded in this fashion we use the asymptotic property (1.7) of the isoperimetric profile $I(s)$ and suitable Hardy type inequalities. Our result improves upon (1.2)

Corollary 1. *(see Section 6.1 below). Let $X = L^p$, $1 \leq p < \infty$. Then,*

$$\int_0^1 \left(\left(f - \int f \right)^* (s) \frac{I(s)}{s} \right)^p ds \preceq \int |\nabla f(x)|^p \, d\gamma_n(x).$$

In particular,

$$\int_0^1 f^*(s)^p (\log \frac{1}{s})^{p/2} ds \preceq \int |\nabla f(x)|^p \, d\gamma_n(x) + \int |f(x)|^p \, d\gamma_n(x).$$

In the final section of this paper we discuss briefly a connection with concentration inequalities. We refer to Ledoux [25] for a detailed account, and detailed references, on the well known connection between lS inequalities and concentration. In our setting, concentration inequalities can be derived from a limiting case of the functional lS inequalities. Namely, for $X = L^\infty$ (1.12) yields

$$\|f\|_{LS(L^\infty)} = \sup_{t<1} \left\{ (f^{**}(t) - f^*(t)) \frac{I(t)}{t} \right\} \leq \sup_t |\nabla f|^{**}(t) = \|f\|_{Lip} .$$

We denote the new space $L_{\log^{1/2}}(\infty,\infty)$ (cf. (7.2) below). Through the asymptotics of $I(s)$ we see that $L_{\log^{1/2}}(\infty,\infty)$ is a variant of the Bennett-DeVore-Sharpley [5] space[12] $L(\infty,\infty)$ = rearrangement invariant hull of BMO. As it was shown in [5], the definition of $L(\infty,\infty)$ is a reformulation of the John-Nirenberg inequality and thus yields exponential integrability. $L_{\log^{1/2}}(\infty,\infty)$ allows us to be more precise about the level of exponential integrability implied by our inequalities. In this fashion, via symmetrization and isoperimetry we have connected the John-Nirenberg inequality with the lS inequalities.

In a similar manner we can also treat the embedding into L^∞ using the fact that the space $L(\infty,1)=L^\infty$ (cf. [1]).

Finally, let us state that our main focus in this paper was to develop our methods and illustrate their reach, but without trying to state the results in their most general form. We refer the reader to [31] for a general theory of isoperimetry and symmetrization in the metric setting.

The section headers are self explanatory and provide the organization of the paper.

2. Gaussian Rearrangements

In this section we review well known results and establish the basic notation concerning Gaussian rearrangements that we shall use in this paper.

2.1. **Gaussian Profile.** Recall that the $n-$dimensional Gaussian measure on $\mathbb{R}^n$ is defined by

$$d\gamma_n(x)=(2\pi)^{-n/2}e^{-\frac{|x|^2}{2}}dx_1........dx_n.$$

It is also convenient to let

$$\phi_n(x)=(2\pi)^{-n/2}e^{-\frac{|x|^2}{2}}, x\in\mathbb{R}^n,$$

and therefore

$$\int_{\mathbb{R}^n}\phi_n(x)dx=\gamma_n(\mathbb{R}^n)=1. \tag{2.1}$$

Let $\Phi:\mathbb{R}\to(0,1)$ be the increasing function given by

$$\Phi(r)=\int_{-\infty}^r\phi_1(t)dt.$$

The Gaussian perimeter of a set is defined by

$$Per(\Omega)=\int_{\partial\Omega}\phi_n(x)dH_{n-1}(x),$$

where $dH_{n-1}(x)$ denotes the Hausdorff $(n-1)$ dimensional measure. The isoperimetric inequality now reads

$$Per(\Omega)\geq I(\gamma_n(\Omega)),$$

where I is the Gaussian isoperimetric function given by (cf. [23], [25])

$$I(t)=\phi_1(\Phi^{-1}(t)),\ t\in[0,1]. \tag{2.2}$$

[12] $L(\infty,\infty)(R^n,d\gamma_n)$ is defined by the condition

$$\sup_{0<t<1}(f^{**}(t)-f^*(t))=\sup_{0<t<1}\frac{1}{t}\int_0^t(f^*(s)-f^*(t))ds<\infty.$$

It was shown by Borell [11] and Sudakov-Tsirelson [36] that for the solution of the isoperimetric problem for Gaussian measures we must replace balls by half spaces. We choose to work with half spaces defined by

$$H_r = \{x = (x_1, ...x_n) : x_1 < r\},\ r \in \mathbb{R}.$$

Therefore by (2.1),

$$\gamma_n(H_r) = \int_{-\infty}^{r} \phi_1(t)dt.$$

Given a measurable set $\Omega \subset \mathbb{R}^n$, we let Ω° be the half space defined by

$$\Omega^\circ = H_r,$$

where $r \in \mathbb{R}$ is selected so that

$$\Phi(r) = \gamma_n(H_r) = \gamma_n(\Omega).$$

In other words, r is defined by

$$r = \Phi^{-1}(\gamma_n(\Omega)).$$

It follows that

$$Per(\Omega) \geq Per(\Omega^\circ) = \phi_1(\Phi^{-1}(\gamma_n(\Omega))).$$

Concerning the Gaussian profile I we note here some useful properties for our development in this paper (cf. [23] and the references therein). First, we note that, by direct computation, we have that I satisfies

$$I'' = \frac{-1}{I}, \tag{2.3}$$

and, as a consequence of (2.1), we also have the symmetry

$$I(t) = I(1-t), t \in [0,1].$$

Moreover, from (2.3) we deduce that $I(s)$ is concave has a maximum at $t = 1/2$ with $I(1/2) = (2\pi)^{-1/2}$, and since $I(0) = 0$, then $\frac{I(s)-I(0)}{s} = \frac{I(s)}{s}$ is decreasing; summarizing

$$\frac{I(s)}{s} \text{ is decreasing on } (0,1) \text{ and } \frac{s}{I(s)} \text{ is increasing on } (0,1). \tag{2.4}$$

Logarithmic Sobolev inequalities are connected with the asymptotic behavior of $I(t)$ at the origin (or at 1 by symmetry) (cf. [23])

$$\lim_{t\to 0} \frac{I(t)}{t(2\log\frac{1}{t})^{1/2}} = 1. \tag{2.5}$$

2.2. **Rearrangements.** Let $f : \mathbb{R}^n \to \mathbb{R}$. We define the non increasing, right continuous, Gaussian distribution function of f, by means of

$$\lambda_f(t) = \gamma_n(\{x \in \mathbb{R}^n : |f(x)| > t\}),\ t > 0.$$

The rearrangement of f with respect to Gaussian measure, $f^* : (0,1] \to [0,\infty)$, is then defined, as usual, by

$$f^*(s) = \inf\{t \geq 0 : \lambda_f(t) \leq s\},\ t \in (0,1].$$

In the Gaussian context we replace the classical Euclidean spherical decreasing rearrangement by a suitable Gaussian substitute, $f^\circ : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f^\circ(x) = f^*(\Phi(x_1)).$$

It is useful to remark here that, as in the Euclidean case, f° is equimeasurable with f :

$$\begin{aligned}\gamma_n(\{x : f^\circ(x) > t\}) &= \gamma_n(\{x : f^*(\Phi(x_1)) > t\})\\ &= \gamma_n(\{x : \Phi(x_1) \le \lambda_f(t)\})\\ &= \gamma_n(\{x : x_1 \le \Phi^{-1}(\lambda_f(t))\})\\ &= \gamma_1(-\infty, \Phi^{-1}(\lambda_f(t)))\\ &= \lambda_f(t).\end{aligned}$$

2.3. **Rearrangement invariant spaces.** Finally, let us recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces, and refer the reader to [6] for a complete treatment.

A Banach function space $X = X(\mathbb{R}^n)$ is called a r.i. space if $g \in X$ implies that all functions f with the same rearrangement with respect to Gaussian measure, i.e. such that $f^* = g^*$, also belong to X, and, moreover, $\|f\|_X = \|g\|_X$. The space X can then be "reduced" to a one-dimensional space (which by abuse of notation we still denote by X), $X = X(0,1)$, consisting of all $g : (0,1) \mapsto \mathbb{R}$ such that $g^*(t) = f^*(t)$ for some function $f \in X$. Typical examples are the L^p-spaces and Orlicz spaces.

We shall usually formulate conditions on r.i spaces in terms of the Hardy operators defined by

$$Pf(t) = \frac{1}{t}\int_0^t f(s)ds; \qquad Qf(t) = \int_t^1 f(s)\frac{ds}{s}.$$

It is well known (see for example [6, Chapter 3]), that if X is a r.i. space, P (resp. Q) is bounded on X if and only if the upper Boyd index $\overline{\alpha}_X < 1$ (resp. the lower Boyd index $\underline{\alpha}_X > 0$).

We notice for future use that if X is a r.i. space such that $\underline{\alpha}_X > 0$, then the operator

$$\tilde{Q}f(t) = (1 + \log 1/t)^{1/2} \int_t^1 f(s)\frac{ds}{s\,(1 + \log 1/s)^{1/2}}$$

is bounded on X. Indeed, pick $\underline{\alpha}_X > a > 0$, then since $t^a\,(1 + \log 1/t)^{1/2}$ is increasing near zero, we get

$$\tilde{Q}f(t) = \frac{t^a\,(1 + \log 1/t)^{1/2}}{t^a} \int_t^1 f(s)\frac{ds}{s\,(1 + \log 1/s)^{1/2}} \preceq \frac{1}{t^a}\int_t^1 s^a f(s)\frac{ds}{s} = Q_a f(t),$$

and Q_a is bounded on X since $\underline{\alpha}_X > a$ (see [6, Chapter 3]).

3. Proof of Theorem 1

The proof follows very closely the development in [32] with appropriate changes.

(**i**) ⇒ (**ii**) By the co-area formula (cf. [27]) and the isoperimetric inequality

$$\begin{aligned}\int |\nabla f(x)|\, d\gamma_n(x) &= \int_0^\infty (\int_{\{|f|=s\}} \phi_n(x) dH_{n-1}(x)) ds \\ &= \int_0^\infty Per(\{|f|>s\}) ds \\ &\geq \int_0^\infty I(\lambda_f(s)) ds\ .\end{aligned}$$

(**ii**) ⇒ (**iii**) Let $0<t_1<t_2<\infty$. The truncations of f are defined by

$$f_{t_1}^{t_2}(x) = \begin{cases} t_2-t_1 & \text{if } |f(x)|>t_2, \\ |f(x)|-t_1 & \text{if } t_1<|f(x)|\leq t_2, \\ 0 & \text{if } |f(x)|\leq t_1. \end{cases}$$

Applying (1.9) to $f_{t_1}^{t_2}$ we obtain,

$$\int_0^\infty I(\lambda_{f_{t_1}^{t_2}}(s)) ds \leq \int_{\mathbb{R}^n} \left|\nabla f_{t_1}^{t_2}(x)\right| d\gamma_n(x).$$

We obviously have

$$\left|\nabla f_{t_1}^{t_2}\right| = |\nabla f|\, \chi_{\{t_1<|f|\leq t_2\}},$$

and, moreover,

$$\int_0^\infty I(\lambda_{f_{t_1}^{t_2}}(s)) ds = \int_0^{t_2-t_1} I(\lambda_{f_{t_1}^{t_2}}(s)) ds. \tag{3.1}$$

Observe that for $0<s<t_2-t_1$

$$\gamma_n\left(|f(x)|\geq t_2\right) \leq \lambda_{f_{t_1}^{t_2}}(s) \leq \gamma_n\left(|f(x)|\, t_1\right).$$

Consequently, we have

$$\int_0^{t_2-t_1} I(\lambda_{f_{t_1}^{t_2}}(s)) ds \geq (t_2-t_1)\min\left(I(\gamma_n\left(|f|\geq t_2\right)), I(\gamma_n\left(|f|>t_1\right))\right). \tag{3.2}$$

For $s>0$ and $h>0$, pick $t_1=f^*(s+h)$, $t_2=f^*(s)$, then

$$s \leq \gamma_n\left(|f(x)|\geq f^*(s)\right) \leq \lambda_{f_{t_1}^{t_2}}(s) \leq \gamma_n\left(|f(x)|\, f^*(s+h)\right) \leq s+h,$$

Combining (3.1) and (3.2) we have,

(3.3)

$$\begin{aligned}(f^*(s)-f^*(s+h))\min(I(s+h),I(s)) &\leq \int_{\{f^*(s+h)<|f|\leq f^*(s)\}} |\nabla f(x)|\, d\gamma_n(x) \\ &\leq \int_0^h |\nabla f|^*(t) dt,\end{aligned}$$

whence f^* is locally absolutely continuous. Thus,

$$\frac{(f^*(s)-f^*(s+h))}{h}\min(I(s+h),I(s)) \leq \frac{1}{h}\int_{\{f^*(s+h)<|f|\leq f^*(s)\}} |\nabla f(x)|\, d\gamma_n(x).$$

Letting $h\to 0$ we obtain (1.10).

(**iii**) ⇒ (**iv**) We will integrate by parts. Let us note first that using (3.3) we have that, for $0<s<t$,

$$s\left(f^*(s)-f^*(t)\right) \leq \frac{s}{\min(I(s),I(t))}\int_0^{t-s} |\nabla f|^*(s) ds. \tag{3.4}$$

Now,

$$\begin{aligned}
f^{**}(t)-f^{*}(t) &= \frac{1}{t}\int_0^t \left(f^*(s)-f^*(t)\right)ds \\
&= \frac{1}{t}\left\{\left[s\left(f^*(s)-f^*(t)\right)\right]_0^t + \int_0^t s\left(-f^*\right)'(s)ds\right\} \\
&= \frac{1}{t}\int_0^t s\left(-f^*\right)'(s)ds \\
&= A(t),
\end{aligned}$$

where the integrated term $\left[s\left(f^*(s)-f^*(t)\right)\right]_0^t$ vanishes on account of (3.4). By (2.4), $s/I(s)$ is increasing on $0<s<1$, thus

$$\begin{aligned}
A(t) &\leq \frac{1}{I(t)}\int_0^t I(s)\left(-f^*\right)'(s)ds \\
&\leq \frac{1}{I(t)}\int_0^t \left(\frac{\partial}{\partial s}\int_{\{|f|>f^*(s)\}}\left|\nabla f(x)\right|d\gamma_n(x)\right)ds \text{ (by (1.10))} \\
&\leq \frac{1}{I(t)}\int_{\{|f|>f^*(s)\}}\left|\nabla f(x)\right|d\gamma_n(x) \\
&\leq \frac{t}{I(t)}\left|\nabla f\right|^{**}(t).
\end{aligned}$$

(**iv**) $\Rightarrow$ (**i**) Let A be a Borel set with $0<\gamma_n(A)<1$. We may assume without loss that $Per(A)<\infty$. By definition we can select a sequence $\{f_n\}_{n\in N}$ of Lip functions such that $f_n \underset{L^1}{\rightarrow} \chi_A$, and

$$Per(A)=\limsup_{n\rightarrow\infty}\left\|\nabla f_n\right\|_1.$$

Therefore,

$$\begin{aligned}
\limsup_{n\rightarrow\infty} I(t)(f_n^{**}(t)-f_n^*(t)) &\leq \limsup_{n\rightarrow\infty}\int_0^t\left|\nabla f_n(s)\right|^* ds \\
&\leq \limsup_{n\rightarrow\infty}\int\left|\nabla f_n\right|d\gamma_n \\
&= Per(A).
\end{aligned} \tag{3.5}$$

As is well known $f_n \underset{L^1}{\rightarrow} \chi_A$ implies that (cf. [19, Lemma 2.1]):

$$\begin{aligned}
&f_n^{**}(t)\rightarrow\chi_A^{**}(t),\ \text{uniformly for } t\in[0,1], \text{and} \\
&f_n^{*}(t)\rightarrow\chi_A^{*}(t) \text{ at all points of continuity of } \chi_A^*.
\end{aligned}$$

Therefore, if we let $r=\gamma_n(A)$, and observe that $\chi_A^{**}(t)=\min(1,\frac{r}{t})$, we deduce that for all $t>r$, $f_n^{**}(t)\rightarrow\frac{r}{t}$, and $f_n^*(t)\rightarrow\chi_A^*(t)=\chi_{(0,r)}(t)=0$. Inserting this information back in (3.5), we get

$$\frac{r}{t}I(t)\leq Per(A),\ \forall t>r.$$

Now, since $I(t)$ is continuous, we may let $t\rightarrow r$ and we find that

$$I(\gamma_n(A))\leq Per(A),$$

as we wished to show.

4. The Pólya-Szegö principle is equivalent to the isoperimetric inequality

In this section we prove Theorem 2.

Our starting point is inequality (1.10). We claim that if A is a positive Young's function, then

$$A\left(\left(-f^{*}\right)^{'}(s)I(s)\right)\leq\frac{\partial}{\partial s}\int_{\{|f|>f^{*}(s)\}}A(|\nabla f(x)|)d\gamma_{n}(x). \tag{4.1}$$

Assuming momentarily the validity of (4.1), by integration we get

$$\int_{0}^{1}A\left(\left(-f^{*}\right)^{'}(s)I(s)\right)ds\leq\int_{\mathbb{R}^{n}}A(|\nabla f(x)|)d\gamma_{n}(x). \tag{4.2}$$

It is easy to see that the left hand side is equal to $\int_{\mathbb{R}^{n}}A(|\nabla f^{\circ}(x)|)d\gamma_{n}(x)$. Indeed, letting $s=\Phi(x_{1})$, we find

$$\begin{aligned}\int_{0}^{1}A\left(\left(-f^{*}\right)^{'}(s)I(s)\right)ds&=\int_{\mathbb{R}}A((-f^{*})^{'}(\Phi(x_{1}))I(\Phi(x_{1}))\left|\Phi'(x_{1})\right|dx\\&=\int_{\mathbb{R}^{n}}A((-f^{*})^{'}(\Phi(x_{1}))I(\Phi(x_{1}))d\gamma_{n}(x)\\&=\int_{\mathbb{R}^{n}}A(|\nabla f^{\circ}(x)|)d\gamma_{n}(x),\end{aligned}$$

where in the last step we have used the fact that

$$(-f^{*})^{'}(\Phi(x_{1}))I(\Phi(x_{1}))=(f^{*})^{'}(\Phi(x_{1}))\Phi'(x_{1})=\left|\nabla f^{\circ}(x)\right|.$$

Consequently, (4.2) states that for all Young's functions A, we have

$$\int_{\mathbb{R}^{n}}A(|\nabla f^{\circ}(x)|)d\gamma_{n}(x)\leq\int_{\mathbb{R}^{n}}A(|\nabla f(x)|)d\gamma_{n}(x),$$

which, by the Hardy-Littlewood-Pólya principle, yields

$$\int_{0}^{t}\left|\nabla f^{\circ}\right|^{*}(s)ds\leq\int_{0}^{t}\left|\nabla f\right|^{*}(s)ds,$$

as we wished to show.

It remains to prove (4.1). Here we follow Talenti's argument. Let $s>0$, then we have three different alternatives: (**a**) s belongs to some exceptional set of measure zero, (**b**) $(f^{*})^{'}(s)=0$, or (**c**) there is a neighborhood of s such that $(f^{*})'(u)$ is not zero, i.e. f^{*} is strictly decreasing. In the two first cases there is nothing to prove. In case alternative (**c**) holds then it follows immediately from the properties of the rearrangement that for a suitable small $h_{0}>0$ we can write

$$h=\gamma_{n}\left\{f^{*}(s+h)<|f|\leq f^{*}(s)\right\},\ 0<h<h_{0}.$$

Therefore, for sufficiently small h, we can apply Jensen's inequality to obtain,

$$\frac{1}{h}\int_{\{f^{*}(s+h)<|f|\leq f^{*}(s)\}}A(|\nabla f(x)|)d\gamma_{n}(x)\geq A\left(\frac{1}{h}\int_{\{f^{*}(s+h)<|f|\leq f^{*}(s)\}}\left|\nabla f(x)\right|d\gamma_{n}(x)\right).$$

Arguing like Talenti [37] we thus get

$$\frac{\partial}{\partial s}\int_{\{|f|>f^*(s)\}} A(|\nabla f(x)|)d\gamma_n(x) \geq A\left(\frac{\partial}{\partial s}\int_{\{|f|>f^*(s)\}} |\nabla f(x)|\, d\gamma_n(x)\right)$$
$$\geq A\left(\left(-f^*\right)^{'}(s)I(s)\right),$$

as we wished to show.

To prove the converse we adapt an argument in [1]. Let f be a Lipschitz function on $\mathbb{R}^n$, and let $0<t<1$. By the definition of f° we can write

$$\begin{aligned} f^*(t)-f^*(1^-) &= f^*(\Phi(\Phi^{-1}(t)))-f^*(\Phi(\infty)) \\ &= \int_{\Phi^{-1}(t)}^{\infty} |\nabla f^\circ|\,(s)ds. \end{aligned}$$

Thus,

$$f^{**}(t)-f^*(1^-)=\frac{1}{t}\int_0^t\int_{\Phi^{-1}(r)}^{\infty}|\nabla f^\circ|\,(s)dsdr.$$

Making the change of variables $s=\Phi^{-1}(z)$ in the inner integral and then changing the order of integration, we find

$$\begin{aligned} f^{**}(t)-f^*(1^-) &= \frac{1}{t}\int_0^t\int_r^1 |\nabla f^\circ|\,(\Phi^{-1}(z))\left(\Phi^{-1}(z)\right)^{'}dzdr \\ &= \int_t^1 |\nabla f^\circ|\,(\Phi^{-1}(z))\left(\Phi^{-1}(z)\right)^{'}dz+\frac{1}{t}\int_0^t z\,|\nabla f^\circ|\,(\Phi^{-1}(z))\left(\Phi^{-1}(z)\right)^{'}dz \\ &= f^*(t)-f^*(1^-)+\frac{1}{t}\int_0^t z\,|\nabla f^\circ|\,(\Phi^{-1}(z))\left(\Phi^{-1}(z)\right)^{'}dz. \end{aligned}$$

Since $\Phi'(\Phi^{-1}(z))=\phi_1(\Phi^{-1}(z)))=I(z)$, we readily deduce that $\left(\Phi^{-1}(z)\right)^{'}=\frac{1}{I(z)}$. Thus,

$$f^{**}(t)-f^*(1^-)=f^*(t)-f^*(1^-)+\frac{1}{t}\int_0^t z\,|\nabla f^\circ|\,(\Phi^{-1}(z))\frac{1}{I(z)}dz,$$

and consequently

$$\begin{aligned} f^{**}(t)-f^*(t) &= \frac{1}{t}\int_0^t z\,|\nabla f^\circ|\,(\Phi^{-1}(z))\frac{1}{I(z)}dz \\ &\leq \frac{t}{I(t)}\frac{1}{t}\int_0^t z\,|\nabla f^\circ|\,(\Phi^{-1}(z))dz \text{ (since } t/I(t) \text{ is increasing)} \\ &= \frac{1}{I(t)}\int_{-\infty}^{\Phi^{-1}(t)}|\nabla f^\circ|\,(s)\Phi'(s)ds \\ &= \frac{1}{I(t)}\int_{-\infty}^{\Phi^{-1}(t)}|\nabla f^\circ|\,(s)d\gamma_1(s) \\ &\leq \int_0^t |\nabla f^\circ|^*(s)ds \text{ (since } \gamma_1(-\infty,\Phi^{-1}(t))=t). \end{aligned}$$

Summarizing, we have shown that

$$(f^{**}(t)-f^*(t))\leq\frac{t}{I(t)}\,|\nabla f^\circ|^{**}(t),$$

which combined with our current hypothesis yields

$$(f^{**}(t)-f^{*}(t))\leq\frac{t}{I(t)}\left|\nabla f^{\circ}\right|^{**}(t)\leq\frac{t}{I(t)}\left|\nabla f\right|^{**}(t).$$

By Theorem 1 the last inequality is equivalent to the isoperimetric inequality.

Remark 1. *We note here, for future use, that the discussion in this section shows that the following equivalent form of the Pólya-Szegö principle holds*

$$\int_0^t((-f^*)'(.)I(.))^*(s)ds\leq\int_0^t\left|\nabla f\right|^*(s)ds.$$

Therefore, by the Hardy-Littlewood principle, for every r.i. space X on $(0,1)$,

$$\left\|(-f^*)'(s)I(s)\right\|_X\leq\left\|\nabla f\right\|_X. \tag{4.3}$$

5. The Pólya-Szegö principle implies Gross' inequality

We present a proof due to Ehrhard [15], showing that the Pólya-Szegö principle implies (1.1). We present full details, since Ehrhard's method is apparently not well known and some details are missing in [15].

We first prove a one dimensional inequality which, by symmetrization and tensorization, will lead to the desired result.

Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a Lip function such that f and $f'\in L^1$. By Jensen's inequality

$$\begin{aligned}\int_{-\infty}^{\infty}|f(x)|\ln|f(x)|\,dx&=\|f\|_{L^1}\int_{-\infty}^{\infty}\ln|f(x)|\frac{|f(x)|\,dx}{\|f\|_{L^1}}\\&\leq\|f\|_{L^1}\ln\Big(\int_{-\infty}^{\infty}|f(x)|\frac{|f(x)|\,dx}{\|f\|_{L^1}}\Big).\end{aligned}$$

We estimate the inner integral using the fundamental theorem of Calculus: $|f(x)|\leq\|f'\|_{L^1}$, to obtain

$$\int_{-\infty}^{\infty}|f(x)|\ln|f(x)|\,dx\leq\|f\|_{L^1}\ln\|f'\|_{L^1}.$$

Applying the preceding to f^2 we get:

$$\int_{-\infty}^{\infty}|f(x)|^2\ln|f(x)|\,dx\leq\frac{1}{2}\|f\|_{L^2}^2\ln 2\left\|ff'\right\|_{L^1}.$$

Using Hölder's inequality $\left\| ff' \right\|_{L^1} \leq \|f\|_{L^2} \left\| f' \right\|_{L^2}$, and elementary properties of the logarithm we find

(5.1)

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 \ln |f(x)| \, dx &\leq \frac{1}{2} \|f\|_{L^2}^2 \ln 2 \|f\|_{L^2} \left\| f' \right\|_{L^2} \\
&= \frac{1}{4} \|f\|_{L^2}^2 \ln 4 \|f\|_{L^2}^4 \frac{\left\| f' \right\|_{L^2}^2}{\|f\|_{L^2}^2} \\
&= \frac{1}{4} \|f\|_{L^2}^2 \ln 4 \frac{\left\| f' \right\|_{L^2}^2}{\|f\|_{L^2}^2} + \|f\|_{L^2}^2 \ln \|f\|_{L^2} \\
&\leq \left\| f' \right\|_{L^2}^2 + \|f\|_{L^2}^2 \ln \|f\|_{L^2} \text{ (in the last step we used } \ln t \leq t).
\end{aligned}$$

We apply (5.1) to $u = (2\pi e^{x^2})^{-1/4} f(x) = \phi_1(x)^{1/2} f(x)$ and compute both sides of (5.1). The left hand side becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^2 \ln |u(x)| \, dx &= \int_{-\infty}^{\infty} |f(x)|^2 \left(\ln |f(x)| + \ln(2\pi e^{x^2})^{-1/4} \right) d\gamma_1(x) \\
&= \int_{-\infty}^{\infty} |f(x)|^2 \ln |f(x)| \, d\gamma_1(x) - \frac{1}{4} \ln 2\pi \|f\|_{L^2(d\gamma_1)} \\
&- \frac{1}{4} \int_{-\infty}^{\infty} |f(x)|^2 x^2 d\gamma_1(x),
\end{aligned}$$

while the right hand side is equal to

$$\begin{aligned}
\left\| f' \right\|_{L^2}^2 &= \left\| f' \right\|_{L^2(d\gamma_1)}^2 + \frac{1}{4} \int_{-\infty}^{\infty} f(x)^2 x^2 d\gamma_1(x) - \int_{-\infty}^{\infty} f'(x) f(x) x \phi_1(x) dx \\
&= \left\| f' \right\|_{L^2(d\gamma_1)}^2 - \frac{1}{4} \int_{-\infty}^{\infty} f(x)^2 x^2 d\gamma_1(x) + \frac{1}{2} \int_{-\infty}^{\infty} f(x)^2 x^2 d\gamma_1(x) \\
&- \int_{-\infty}^{\infty} f'(x) f(x) x \phi_1(x) dx.
\end{aligned} \tag{5.2}$$

We simplify the last expression integrating by parts the third integral to the right,

$$\begin{aligned}
\frac{1}{2} \int_{-\infty}^{\infty} f(x)^2 x^2 d\gamma_1(x) &= -\frac{1}{2} \int_{-\infty}^{\infty} f(x)^2 x d(((2\pi)^{-1/2} e^{-x^2})) \\
&= -\frac{1}{2} f(x)^2 x((2\pi)^{-1/2} e^{-x^2}) \Big|_{-\infty}^{\infty} + \\
&\frac{1}{2} \int_{-\infty}^{\infty} ((2\pi)^{-1/2} e^{-x^2})[2f(x)f'(x)x + f^2(x)] dx \\
&= \int_{-\infty}^{\infty} f(x) f'(x) x \phi_1(x) dx + \frac{1}{2} \|f\|_{L^2(d\gamma_1)}^2 .
\end{aligned}$$

We insert this back in (5.2) and then comparing results and simplifying we arrive at

$$\int_{-\infty}^{\infty} |f(x)|^2 \ln|f(x)|\, d\gamma_1(x) \leq \left\| f' \right\|_{L^2(d\gamma_1)}^2 + \|f\|_{L^2(d\gamma_1)}^2 \ln \|f\|_{L^2(d\gamma_1)}^2 + \frac{\ln(2\pi e^2)}{4} \|f\|_{L^2(d\gamma_1)}^2 . \tag{5.3}$$

Let f be a Lipchitz function on $\mathbb{R}^n$. We form the symmetric rearrangement f° considered as a one dimensional function. Then, (5.3) applied to f°, combined with the fact that f° is equimeasurable with f and the Pólya-Szegö principle, yields

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|^2 \ln|f(x)|\, d\gamma_n(x) &= \int_{\mathbb{R}} |f^\circ(x)|^2 \ln|f^\circ(x)|\, d\gamma_1(x) \\ &\leq \left\| f^{\circ\prime} \right\|_{L^2(d\gamma_1)}^2 + \|f\|_{L^2(d\gamma_1)}^2 \ln \|f\|_{L^2(d\gamma_1)}^2 \\ &+ \frac{\ln(2\pi e^2)}{4} \|f\|_{L^2(d\gamma_1)}^2 \\ &= \| |\nabla f^\circ(x)| \|_{L^2(d\gamma_n)}^2 + \|f^\circ\|_{L^2(d\gamma_n)}^2 \ln \|f^\circ\|_{L^2(d\gamma_n)}^2 \\ &+ \frac{\ln(2\pi e^2)}{4} \|f^\circ\|_{L^2(d\gamma_n)}^2 \\ &\leq \|\nabla f\|_{L^2(d\gamma_n)}^2 + \|f\|_{L^2(d\gamma_n)}^2 \ln \|f\|_{L^2(d\gamma_n)}^2 \\ &+ \frac{\ln(2\pi e^2)}{4} \|f\|_{L^2(d\gamma_n)}^2 . \end{aligned} \tag{5.4}$$

We now use tensorization to prove (1.1). Note that, by homogeneity, we may assume that f has been normalized so that $\|f\|_{L^2(d\gamma_n)} = 1$. Let $l \in N$, and let F be defined on $(\mathbb{R}^n)^l = \mathbb{R}^{nl}$ by $F(x) = \prod_{k=1}^{l} f(x_k)$, where $x_k \in \mathbb{R}^n, k = 1, ..l$. The $\mathbb{R}^{nl}$ version of (5.4) applied to F, and translated back in terms of f, yields

$$l \int_{\mathbb{R}^n} |f(x)|^2 \ln|f(x)|\, d\gamma_n(x) \leq l \|\nabla f\|_{L^2(d\gamma_n)}^2 + \frac{\ln(2\pi e^2)}{4}.$$

Therefore, upon diving by l and letting $l \to \infty$, we obtain

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln|f(x)|\, d\gamma_n(x) \leq \|\nabla f\|_{L^2(d\gamma_n)}^2 ,$$

as we wished to show.

6. Poincaré type inequalities

We consider L^1 Poincaré inequalities first. Indeed, for L^1 norms the Poincaré inequalities are a simple variant of Ledoux's inequality. Let f be a Lipschitz function on $\mathbb{R}^n$, and let m a median[13] of f. Set $f^+ = \max(f-m, 0)$ and $f^- = -\min(f-m, 0)$

[13] i.e. $\gamma_n(f \geq m) \geq 1/2$ and $\gamma_n(f \leq m) \geq 1/2$.

so that $f - m = f^{+} - f^{-}$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} |f-m|\, d\gamma_n &= \int_{\mathbb{R}^n} f^{+} d\gamma_n + \int_{\mathbb{R}^n} f^{-} d\gamma_n \\ &= \int_0^{\infty} \lambda_{f^{+}}(s) ds + \int_0^{\infty} \lambda_{f^{-}}(s) ds \\ &= (A)\end{aligned}$$

We estimate each of these integrals using the properties of the isoperimetric profile and Ledoux's inequality (1.9). First we use the fact that $\frac{I(s)}{s}$ is decreasing on $0 < s < 1/2$, combined with the definition of median, to find that

$$2\lambda_g(s) I(\frac{1}{2}) \leq I(\lambda_g(s)), \text{ where } g = f^{+} \text{ or } g = f^{-}.$$

Consequently,

$$\begin{aligned}(A) &\leq \frac{1}{2I(\frac{1}{2})} \left(\int_0^{\infty} I(\lambda_{f^{+}}(s)) ds + \int_0^{\infty} I(\lambda_{f^{-}}(s)) ds \right) \\ &\leq \frac{1}{2I(\frac{1}{2})} \left(\int_{\mathbb{R}^n} \nabla f^{+}(x) d\gamma_n(x) + \int_{\mathbb{R}^n} \nabla f^{+}(x) d\gamma_n(x) \right) \text{ (by (1.9)} \\ &= \frac{1}{2I(1/2)} \int_{\mathbb{R}^n} |\nabla f(x)|\, d\gamma_n(x).\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n} |f-m|\, d\gamma_n \leq \frac{1}{2I(1/2)} \int_{\mathbb{R}^n} |\nabla f(x)|\, d\gamma_n(x). \tag{6.1}$$

We now prove Theorem 3.

Proof. (**i**) $\rightarrow$ (**ii**). Obviously condition (1.13) is equivalent to

$$\|f-m\|_Y \preceq \|\nabla f\|_X,$$

where m is a median of f. Let f be a positive measurable function with *suppf* $\subset (0, 1/2)$. Define

$$u(x) = \int_{\Phi(x_1)}^{1} f(s) \frac{ds}{I(s)}, \quad x \in \mathbb{R}^n.$$

It is plain that u is a Lipschitz function on $\mathbb{R}^n$ such that $\gamma_n(u=0) \geq 1/2$, and therefore it has 0 median. Moreover,

$$|\nabla u(x)| = \left| \frac{\partial}{\partial x_1} u(x) \right| = \left| -f(\Phi(x_1)) \frac{\Phi'(x_1)}{I(\Phi(x_1))} \right| = f(\Phi(x_1)).$$

It follows that

$$u^*(t) = \int_t^1 f(s) \frac{ds}{I(s)}, \text{ and } |\nabla u|^*(t) = f^*(t).$$

Consequently, from

$$\|u - 0\|_Y \preceq \|\nabla u\|_X$$

we deduce that

$$\left\| \int_t^1 f(s) \frac{ds}{I(s)} \right\|_Y \preceq \|f\|_X.$$

(**ii**) → (**i**). Let f be a Lipschitz function f on $\mathbb{R}^n$. Write

$$f^*(t) = \int_t^{1/2} (-f^*)'(s)ds + f^*(1/2).$$

Thus,

$$\begin{aligned}
\|f\|_Y = \|f^*\|_Y \leq 2\left\|f^*\chi_{[0,1/2]}\right\|_Y &\preceq \left\|\int_t^{1/2} (-f^*)'(s)ds\right\|_Y + f^*(1/2)\|1\|_Y \\
&\leq \left\|\int_t^{1/2} (-f^*)'(s)I(s)\frac{ds}{I(s)}\right\|_Y + 2\|1\|_Y\|f\|_{L_1} \\
&\preceq \left\|(-f^*)'(s)I(s)\right\|_X + \|f\|_{L_1} \\
&\preceq \|\nabla f\|_X \text{ (by (6.1) and (4.3))}.
\end{aligned}$$

Part II. Case $0 <\underline{\alpha}_X$:
(**ii**) → (**iii**) Let $0 < t < 1/4$, then

$$f^*(2t) \preceq \int_t^{2t} f^*(s)\frac{ds}{s} \leq \int_t^{1/2} f^*(s)\frac{I(s)}{s}\frac{ds}{I(s)},$$

therefore,

$$\begin{aligned}
\|f^*(2t)\|_Y &\preceq \left\|\int_t^{1/2} f^*(s)\frac{I(s)}{s}\frac{ds}{I(s)}\right\|_Y + f^*(1/2) \\
&\preceq \left\|f^*(t)\frac{I(t)}{t}\right\|_X + f^*(1/2) \text{ (by (ii))} \\
&\preceq \left\|f^*(t)\frac{I(t)}{t}\right\|_X + \|f\|_1 \\
&\preceq \left\|f^*(t)\frac{I(t)}{t}\right\|_X.
\end{aligned}$$

(**iii**) → (**ii**) By hypothesis

$$\left\|\int_t^{1/2} f^*(s)\frac{ds}{I(s)}\right\|_Y \preceq \left\|\left(\int_t^{1/2} f^*(s)\frac{ds}{I(s)}\right)\frac{I(t)}{t}\right\|_X.$$

Using that (see 2.5),

$$\frac{I(s)}{s} \simeq \sqrt{\log\frac{1}{s}} \simeq \sqrt{1+\log\frac{1}{s}}, \qquad 0 < s < 1/2$$

we have

$$\left(\int_t^{1/2} f(s)\frac{ds}{I(s)}\right)\frac{I(t)}{t} \preceq \sqrt{1+\log\frac{1}{t}}\int_t^1 f(s)\frac{ds}{s\sqrt{1+\log\frac{1}{s}}} = \tilde{Q}f(t).$$

Now. from $\underline{\alpha}_X > 0$ it follows that $\tilde{Q}$ is a bounded operator on X (see Section 2.3) and thus we are able to conclude.

Part II. Case $0 =\underline{\alpha}_X < \overline{\alpha}_X < 1$:

(ii) → **(iv)** By the fundamental theorem of Calculus and (ii), we have

$$\begin{aligned}\left\|f^{**}\chi_{(0,1/2)}\right\|_Y &\preceq \left\|\int_t^{1/2}\left(f^{**}(s)-f^*(s)\right)\frac{ds}{s}\right\|_Y + f^{**}(1/2)\left\|1\right\|_Y \\ &\preceq \left\|\int_t^1 \frac{I(s)}{s}\left(f^{**}(s)-f^*(s)\right)\chi_{(0,1/2)}(s)\frac{ds}{I(s)}\right\|_Y + \left\|f\right\|_1 \\ &\preceq \left\|(f^{**}(t)-f^*(t))\chi_{(0,1/2)}(t)\frac{I(t)}{t}\right\|_X + \left\|f\right\|_1 \\ &\preceq \left\|(f^{**}(t)-f^*(t))\frac{I(t)}{t}\right\|_X + \left\|f\right\|_1 .\end{aligned}$$

(iv) → **(i)** Let f be a Lipschitz function on $\mathbb{R}^n$, let m be a median of f and let $g = f - m$. By hypothesis we have

$$\left\|g\right\|_Y \preceq \left\|(g^{**}(t)-g^*(t))\frac{I(t)}{t}\right\|_X + \left\|g\right\|_1 .$$

From (see [1])

$$g^{**}(t) - g^*(t) \leq P(g^*(s/2) - g^*(s))(t) + g^*(t/2) - g^*(t),$$

and using the fact that $\frac{I(t)}{t}$ decreases,

$$P(g^*(s/2) - g^*(s))(t)\frac{I(t)}{t} \leq P(g^*(s/2) - g^*(s)\frac{I(s)}{s})(t).$$

Therefore,

$$\begin{aligned}\left\|(g^{**}(t)-g^*(t))\frac{I(t)}{t}\right\|_X &\leq \left\|P(g^*(s/2)-g^*(s)\frac{I(s)}{s})(t)\right\|_X + \left\|(g^*(t/2)-g^*(t)\frac{I(t)}{t}\right\|_X \\ &\preceq \left\|(g^*(t/2)-g^*(t)\frac{I(t)}{t}\right\|_X \quad (\text{since } \overline{\alpha}_X < 1).\end{aligned}$$

We compute the right hand side,

$$\begin{aligned}\left\|(g^*(t/2)-g^*(t)\frac{I(t)}{t}\right\|_X &= \left\|\left(\int_{t/2}^t \left(-g^*\right)'(s)ds\right)\frac{I(t)}{t}\right\|_X \\ &\leq \left\|\int_{t/2}^t \left(-g^*\right)'(s)\frac{I(s)}{s}ds\right\|_X \\ &\leq \left\|\frac{2}{t}\int_{t/2}^t \left(-g^*\right)'(s)I(s)ds\right\|_X \\ &\leq 2\left\|\frac{1}{t}\int_0^t \left(-g^*\right)'(s)I(s)ds\right\|_X \\ &\preceq \left\|\left(-g^*\right)'(t)I(t)\right\|_X \\ &\preceq \left\|\nabla f\right\|_X \quad (\text{by } (4.3)).\end{aligned}$$

Summarizing, we have obtained

$$\left\|g\right\|_Y \preceq \left\|\nabla f\right\|_X + \left\|g\right\|_1 \preceq \left\|\nabla f\right\|_X \quad (\text{by } (6.1)).$$

□

6.1. **Feissner type inequalities.** Theorem 3) readily improves upon Feissner's inequalities (1.2). Indeed, for the particular choice $X = L^p$ $(1 \leq p < \infty)$, Theorem 3 yields

$$\int_0^1 \left(\left(f - \int f \right)^* (s) \frac{I(s)}{s} \right)^p ds \preceq \int |\nabla f(x)|^p \, d\gamma_n(x).$$

In particular, using again the asymptotics of $I(s)$, $0 < s < 1/2$, we get

$$\int_0^1 f^*(s)^p (\log \frac{1}{s})^{p/2} ds \preceq \int |\nabla f(x)|^p \, d\gamma_n(x) + \int |f(x)|^p \, d\gamma_n(x).$$

Moreover, the space $L^p(LogL)^{1/2}$ is best possible among r.i. spaces Y for which the Poincaré inequality $\|f - \int f\|_Y \preceq \|\nabla f\|_{L^p}$ holds.

The case $X = L^\infty$, which is new is more interesting. Indeed, since $I(t)/t$ decreases,

$$\sup_{0<t<1} f^*(t) \frac{I(t)}{t} < \infty \;\Leftrightarrow\; f = 0.$$

But Theorem 3 ensures that

$$\left\| \left(\left(f - \int f \right)^{**} (t) - \left(f - \int f \right)^* (t) \right) \frac{I(t)}{t} \right\|_{L^\infty} \preceq \|\nabla f\|_{L^\infty} . \tag{6.2}$$

Furthermore, for every r.i space Y such that

$$\left\| f - \int f \right\|_Y \preceq \|\nabla f\|_{L^\infty} ,$$

the following embedding holds

$$\|f\|_Y \preceq \left\| (f^{**}(t) - f^*(t)) \frac{I(t)}{t} \right\|_{L^\infty} + \|f\|_1 .$$

Notice that due to the cancellation afforded by $f^{**}(t) - f^*(t)$, the corresponding space $LS(L^\infty)$ is nontrivial. The relation between concentration and $LS(L^\infty)$ will be studied in the next section.

7. On limiting embeddings and concentration

Elsewhere [14] (cf. [30]) we shall explore in detail the connection between concentration inequalities and symmetrization, including the self improving properties of concentration. In this section we merely wish to call attention to the connection between a limiting lS inequality that follows from (1.11) and concentration. We have argued that, in the Gaussian world, Ledoux's embedding corresponds to the Gagliardo-Nirenberg embedding. In the classical $n-$dimensional Euclidean case the "other" borderline case for the Sobolev embedding theorem occurs when the index of integrability of the gradients in the Sobolev space, say p, is equal to the dimension i.e. $p = n$. In this case, as is well known, from $|\nabla f| \in L^n(\mathbb{R}^n)$ we can deduce the exponential integrability of $|f|^{n'}$ (cf. [38]). A refinement of this result, which follows from the Euclidean version of (1.11), is given by the following inequality from [1]

$$\left\{ \int_0^\infty (f^{**}(s) - f^*(s))^n \frac{ds}{s} \right\}^{1/n} \preceq \left\{ \int_0^\infty |\nabla f(x)|^n \, dx \right\}^{1/n} .$$

[14] In particular the method of symmetrization by truncation can be extended to this setting.

In this fashion one could consider the corresponding borderline Gaussian embedding that results from (1.11) when $n = p = \infty$. The result now reads

$$\sup_{t<1} \left\{ (f^{**}(t) - f^{*}(t)) \frac{I(t)}{t} \right\} \leq \sup_{t} |\nabla f|^{**}(t) = \|f\|_{Lip}. \tag{7.1}$$

We now show how (7.1) is connected with the concentration phenomenon (cf. [25] and the references therein).

For the corresponding analysis we start by combining (7.1) with (2.5)

$$I(t) \geq ct \left(\log \frac{1}{t} \right)^{1/2}, \; t \in (0, \frac{1}{2}],$$

to obtain

$$f^{**}(t) - f^{*}(t) \preceq \frac{\|f\|_{Lip}}{\left(\log \frac{1}{t}\right)^{1/2}}, \; t \in (0, \frac{1}{2}].$$

Therefore, for $t \in (0, \frac{1}{2}]$, we have

$$\begin{aligned} f^{**}(t) - f^{**}(1/2) &= \int_t^{1/2} (f^{**}(s) - f^{*}(s)) \frac{ds}{s} \\ &\preceq \||\nabla f|\|_{\infty} \int_t^{1/2} \frac{1}{\left(\log \frac{1}{s}\right)^{1/2}} \frac{ds}{s} \\ &\leq 2 \||\nabla f|\|_{\infty} \left(\log \frac{1}{t} \right)^{1/2}. \end{aligned}$$

Thus, if $\lambda \||\nabla f|\|_{\infty}^2 \prec 1$,

$$\begin{aligned} \int_0^{1/2} e^{\lambda (f^{**}(t) - f^{**}(1/2))^2} dt &\preceq \int_0^{1/2} e^{\left(\log \frac{1}{t^{\lambda \||\nabla f|\|_{\infty}^2}} \right)} dt \\ &= \int_0^{1/2} \frac{1}{t^{\lambda \||\nabla f|\|_{\infty}^2}} dt < \infty. \end{aligned}$$

Moreover, since f^{**} is decreasing we have

$$\begin{aligned} \int_{1/2}^{1} e^{\lambda (f^{**}(t) - f^{**}(1/2))^2} dt &\leq \int_{1/2}^{1} e^{\lambda (f^{**}(1-t) - f^{**}(1/2))^2} dt \\ &= \int_0^{1/2} e^{\lambda (f^{**}(t) - f^{**}(1/2))^2} dt. \end{aligned}$$

This readily implies the exponential integrability of $(f(t) - f^{**}(1/2))$:

$$\int_{\mathbb{R}^n} e^{\lambda (f(x) - f^{**}(1/2))^2} d\gamma_n(x) < \infty,$$

and, in fact, we can readily compute the corresponding Orlicz norm.

In this fashion we are led to define a new space $L_{\log^{1/2}}(\infty,\infty)(\mathbb{R}^n, d\gamma_n)$ by the condition[15]

$$\|f\|_{L_{\log^{1/2}}(\infty,\infty)(\mathbb{R}^n,d\gamma_n)} = \sup_{0<t<1} \left(f^{**}(t) - f^{*}(t)\right)\left(\log\frac{1}{t}\right)^{1/2} < \infty. \tag{7.2}$$

Summarizing our discussion, we have

$$\|f\|_{L_{\log^{1/2}}(\infty,\infty)(\mathbb{R}^n,d\gamma_n)} \preceq \|\nabla f\|_{L^\infty(\mathbb{R}^n,d\gamma_n)}$$

and

$$L_{\log^{1/2}}(\infty,\infty)(\mathbb{R}^n, d\gamma_n) \subset e^{L^2(\mathbb{R}^n,d\gamma_n)}.$$

The scale of spaces $\{L_{\log^\alpha}(\infty,\infty)\}_{\alpha\in R_+}$ is thus suitable to measure exponential integrability. When $\alpha = 0$ we get the celebrated $L(\infty,\infty)$ spaces introduced in [5], which characterize the rearrangement invariant hull of BMO. The corresponding underlying rearrangement inequality in the Euclidean case is the following version of the John-Nirenberg lemma

$$f^{**}(t) - f^{*}(t) \preceq \left(f^{\#}\right)^{*}(t)$$

where $f^{\#}$ is the sharp maximal operator used in the definition of BMO (cf. [5] and [21]).

In fact, in our context the $L(\infty,\infty)$ space is connected to the exponential inequalities by Bobkov-Götze [8]. Proceeding as before we see that (compare with [8])

$$\left(f^{**}(t) - f^{*}(t)\right) \preceq |\nabla f|^{**}(t)\left(\log\frac{1}{t}\right)^{-1/2}, \; 0 < t < \frac{1}{2},$$

from where if follows readily that $|\nabla f| \in e^{L^2} \Longrightarrow f \in L(\infty,\infty)$, and therefore if, moreover $\int f = 0$, we can also conclude that $f \in e^{L}$.

8. Symmetrization by truncation of entropy inequalities

In this brief section we wish to indicate, somewhat informally, how our methods can be extended to far more general setting. Let (Ω,μ) be a probability measure space. As in the literature, we consider the entropy functional defined, on positive measurable functions, by

$$Ent(g) = \int g\log g d\mu - \int g d\mu \log \int g d\mu.$$

Suppose for example that Ent satisfies a lS inequality of order 1 on a suitable class of functions,

$$Ent(g) \leq c\int \Gamma(g)d\mu. \tag{8.1}$$

Here Γ is to be thought as an abstract gradient. We will make an assumption that is not made in the literature but is crucial for our method to work: we will assume

[15] More generally, the relevant spaces to measure exponential integrability to the power p are defined by

$$\sup(f^{**}(t) - f^{*}(t))(\log\frac{1}{t})^{1/p'} < \infty.$$

that Γ is *truncation friendly*, in the sense that for any truncation of f (see section 3) we have

$$\left|\Gamma(f_{h_1}^{h_2})\right| = |\Gamma(f)|\, \chi_{\{h_1<|f|\leq h_2\}}. \tag{8.2}$$

While this is a non standard assumption, as we know, the usual gradients are indeed *truncation friendly*. In order to continue we need the following elementary result that comes from [10] (Lemma 2.2)

$$Ent(g) \geq -\log \|g\|_0 \int g d\mu \tag{8.3}$$

here $\|g\|_0 = \mu\{g \neq 0\}$. Combining (8.1), (8.2), (8.3) it follows that

$$-\log \left\|f_{h_1}^{h_2}\right\|_0 \int f_{h_1}^{h_2} d\mu \leq c \int |\Gamma(f)|\, \chi_{\{h_1<|f|\leq h_2\}} d\mu$$

$$-\log \lambda_f(h_1)\mu\{h_1 < |f(x)| \leq h_2\} \leq c \int |\Gamma(f)|\, \chi_{\{h_1<|f|\leq h_2\}} d\mu$$

$$(-\log \lambda_f(h_2))\,(h_2 - h_1)\lambda_f(h_2) \leq c \int_{\{h_1<|f|\leq h_2\}} |\Gamma(f)|\, d\mu$$

Pick $h_1 = f^*(s+h)$, $h_2 = f^*(s)$, then

$$s(\log \frac{1}{s})\,(f^*(s) - f^*(s+h)) \leq c \int_{\{f^*(s+h)<|f|\leq f^*(s)\}} |\Gamma(f)|\, d\mu.$$

Thus,

$$s(\log \frac{1}{s}) \frac{(f^*(s) - f^*(s+h))}{h} \leq \frac{c}{h} \int_{\{f^*(s+h)<|f|\leq f^*(s)\}} |\Gamma(f)|\, d\mu.$$

Therefore, following the analysis of Section 4, we find that, for any Young's function A, we have

$$A\left(s(\log \frac{1}{s})(-f^*)'(s)\right) \leq \frac{d}{ds}\left(\int_{\{|f|>f^*(s)\}} A(|\Gamma(f)|)d\mu\right).$$

Integrating, and using the Hardy-Littlewood-Pólya principle exactly as in section 4, we obtain the following abstract version of the Pólya-Szegö principle

$$\int_0^t \left(s(\log \frac{1}{s})(-f^*)'(s)\right)^* (r)dr \leq \int_0^t |\Gamma(f)|^*(r)dr.$$

This analysis establishes a connection between entropy inequalities and logarithmic Sobolev inequalities via symmetrization. In particular, our inequalities extend the classical results to the setting of rearrangement invariant spaces. For more details see [30].

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