

Bifurcation of critical periods from Pleshkan’s isochrones

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Abstract. Pleshkan proved in 1969 that, up to a linear transformation and a constant rescaling of time, there are four isochrones in the family of cubic centers with homogeneous nonlinearities \mathcal{C}_3 . In this paper we prove that if we perturb any of these isochrones inside \mathcal{C}_3 , then at most two critical periods bifurcate from its period annulus. Moreover we show that, for each $k = 0, 1, 2$, there are perturbations giving rise to exactly k critical periods. As a byproduct, we obtain a partial result for the analogous problem in the family of quadratic centers \mathcal{C}_2 . Loud proved in 1964 that, up to a linear transformation and a constant rescaling of time, there are four isochrones in \mathcal{C}_2 . We prove that if we perturb three of them inside \mathcal{C}_2 , then at most one critical period bifurcates from its period annulus. In addition, for each $k = 0, 1$, we show that there are perturbations giving rise to exactly k critical periods. The quadratic isochronous center that we do not consider displays some peculiarities that are discussed at the end of the paper.

1 Introduction and statement of the results

In this paper we study the period function of the centers of planar vector fields with homogeneous nonlinearities, more concretely, differential systems of the form

$$\begin{cases} \dot{x} = -y + P_n(x, y), \\ \dot{y} = x + Q_n(x, y), \end{cases} \quad (1)$$

where $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous real polynomials of degree n . We denote by \mathcal{H}_n the family of vector fields of the above form and by \mathcal{C}_n the subset of those systems in \mathcal{H}_n with a center at the origin. Finally, \mathcal{I}_n stands for the set of nonlinear isochrones in \mathcal{H}_n . Accordingly $\mathcal{I}_n \subset \mathcal{C}_n \subset \mathcal{H}_n$. We restrict ourselves to the cases $n = 2$ and $n = 3$, that are the degrees for which the centers and the isochrones have been classified.

2000 *AMS Subject Classification*: 34C23; 37C10; 37C27.

Key words and phrases: critical period; isochronous center; period function; bifurcation; unfolding.

M. Grau is partially supported by a MCYT/FEDER grant number MTM-2008-00694 and by a CIRIT grant number 2005-SGR-550. J. Villadelprat is partially supported by the CONACIT through the grant 2009-SGR-410 and by the DGES through the grant MTM-2008-03437.

Recall that the *period annulus* of a center is the biggest punctured neighbourhood foliated by periodic orbits and in what follows we shall denote it by \mathcal{P} . Compactifying \mathbb{R}^2 to the Poincaré disc, the boundary of \mathcal{P} has two connected components: the center itself and a polycycle. We call them respectively the *inner* and *outer boundary* of the period annulus. The *period function* of the center assigns to each periodic orbit γ in \mathcal{P} its period. If all the periodic orbits in \mathcal{P} have the same period, then the center is called *isochronous*. Since the period function is defined on the set of periodic orbits in \mathcal{P} , usually the first step is to parameterize this set, let us say $\{\gamma_s\}_{s \in (0,1)}$, and then one can study the qualitative properties of the period function by means of the map $s \mapsto \text{period of } \gamma_s$, which is analytic on $(0,1)$. The *critical periods* are the critical points of this function and their number, character (maximum or minimum) and distribution do not depend on the particular parameterization of the set of periodic orbits used. We are interested in the *bifurcation* of critical periods. Roughly speaking, the disappearance or emergence of critical periods as we perturb a center. There are three different situations to study (see [19] for details):

- (a) Bifurcation of the period function from the inner boundary of \mathcal{P} (i.e., the center itself).
- (b) Bifurcation of the period function from \mathcal{P} .
- (c) Bifurcation of the period function from the outer boundary of \mathcal{P} (i.e., the polycycle).

In this paper we are only concerned with bifurcation of the period function from \mathcal{P} , i.e. case (b), and when we perturb an isochronous center. To be more precise, we study the bifurcation of critical periods from \mathcal{P} of the isochrones in \mathcal{I}_n when perturbed inside \mathcal{C}_n . Our initial motivation was to study this problem for $n = 3$ and the result we have obtained is stated in Theorem A below. As a byproduct of the tools we develop to prove it, we get a partial result of the problem for $n = 2$, see Theorem B.

As the authors explain in [7, 8] many problems dealing with the period function and critical periods are the counterpart of problems about the Poincaré return map and limit cycles. The problem studied here is the counterpart of the question of how may limit cycles bifurcate from a center of an integrable system when perturbed in a non-conservative direction. The return map of the unperturbed system is the identity and one asks for the number of fixed points that persist. In our case the map that we perturb is constant (the period function of an isochronous center) and we ask for the number of zeros of the derivative that persist. Let us comment in this setting that the bifurcation of the period function from \mathcal{P} , i.e., case (b), when the unperturbed center is not isochronous is out of reach for the moment. Its counterpart for limit cycles is the so-called *blue-sky* bifurcation, which is usually undetectable.

The classification of \mathcal{C}_3 and \mathcal{I}_3 is due to Malkin [16] and Pleshkan [22], respectively. The latter proved that, up to a linear change of coordinates and a constant rescaling of time (see [5, 21]), there are four isochronous centers in \mathcal{H}_3 :

$$\begin{aligned} (S_1^*) \quad \begin{cases} \dot{x} = -y - 3x^2y + y^3, \\ \dot{y} = x + x^3 - 3xy^2, \end{cases} & (S_2^*) \quad \begin{cases} \dot{x} = -y + x^2y, \\ \dot{y} = x + xy^2, \end{cases} \\ (S_3^*) \quad \begin{cases} \dot{x} = -y + 3x^2y, \\ \dot{y} = x - 2x^3 + 9xy^2, \end{cases} & (\bar{S}_3^*) \quad \begin{cases} \dot{x} = -y - 3x^2y, \\ \dot{y} = x + 2x^3 - 9xy^2. \end{cases} \end{aligned} \tag{2}$$

The first result that we shall prove is the following. Let us point out that in its statement \mathcal{P} refers to the period annulus of the center at the origin.

Theorem A. *If we perturb (S_1^*) , (S_2^*) , (S_3^*) or (\bar{S}_3^*) inside \mathcal{C}_3 , then at most two critical periods bifurcate from \mathcal{P} . Moreover, for each $k = 0, 1, 2$, there are perturbations that give rise to exactly k critical periods bifurcating from \mathcal{P} .*

To be more precise, if X_0 denotes the vector field associated to one of the systems in (2), then we consider the unfolding given by $X_\varepsilon = X_0 + Y_\varepsilon$ with

$$Y_\varepsilon(x, y) := (a(\varepsilon)x^2y + b(\varepsilon)y^3)\partial_x + (c(\varepsilon)x^3 + d(\varepsilon)xy^2)\partial_y \tag{3}$$

where a , b , c and d are analytic functions vanishing at $\varepsilon = 0$. Since X_ε is reversible, it is clear that it belongs to \mathcal{C}_3 for all ε . The fact that there is no other perturbation with this property follows from Malkin's result (see also [23]). Of course the problem under consideration only makes sense in case that the perturbation is not isochronous itself. This, let us say, *isochronous direction of perturbation* that we must avoid is $d + 3c = a + 3c = b - c = 0$ for (S_1^*) ; $b = c = a - d = 0$ for (S_2^*) ; and $b = 2a + 3c = 2d + 9c = 0$ for (S_3^*) and (\tilde{S}_3^*) . As we will see, the existence of such a direction prevents the parameters in the perturbation from acting independently and it explains the reason why three critical periods cannot appear even though we consider a 4-parameter unfolding.

There are some previous results related to Theorem A that should be referred. Rousseau and Toni prove in [23] that if we perturb any isochronous center in (2) inside \mathcal{C}_3 , then at most two critical periods bifurcate from the inner boundary of \mathcal{P} . They also study the perturbation of the linear (isochronous) center $X_0 = -y\partial_x + x\partial_y$. In this case there are three different unfoldings of X_0 inside \mathcal{C}_3 and the authors prove that they give rise, respectively, to at most 0, 1 and 3 critical periods bifurcating from the inner boundary of \mathcal{P} . In [4] the authors also study unfoldings of the linear center. They provide some lower bounds for the maximum number of critical periods that can bifurcate from \mathcal{P} .

The classification of \mathcal{C}_2 and \mathcal{I}_2 is due to Dulac [6] and Loud [15], respectively. The latter proved that, up to a linear change of coordinates and a constant rescaling of time, there are four isochronous centers in \mathcal{H}_2 :

$$\begin{aligned} (S_1) \quad & \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x - \frac{1}{2}x^2 + \frac{1}{2}y^2, \end{cases} & (S_2) \quad & \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + y^2, \end{cases} \\ (S_3) \quad & \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + \frac{1}{4}y^2, \end{cases} & (S_4) \quad & \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x - \frac{1}{2}x^2 + 2y^2. \end{cases} \end{aligned} \tag{4}$$

Concerning these systems and the period annulus \mathcal{P} of the isochronous center at the origin, we prove the following result:

Theorem B. *If we perturb (S_2) , (S_3) or (S_4) inside \mathcal{C}_2 , then at most one critical period bifurcates from \mathcal{P} . Moreover, for each $k = 0, 1$, there are perturbations that give rise to exactly k critical periods bifurcating from \mathcal{P} .*

The perturbation of the isochrones (S_2) , (S_3) and (S_4) inside \mathcal{C}_2 can be studied with the same unfolding. Indeed, if X_0 is the associated vector field to one of these three systems, then it suffices to consider $X_\varepsilon = X_0 + Y_\varepsilon$ with

$$Y_\varepsilon(x, y) := b(\varepsilon)xy\partial_x + (d(\varepsilon)x^2 + f(\varepsilon)y^2)\partial_y \tag{5}$$

where b , d and f are analytic functions vanishing at $\varepsilon = 0$. Clearly X_ε has a center at the origin because the vector field is reversible. The fact that there is not any other perturbation of X_0 inside \mathcal{C}_2 follows from Dulac's result. On the contrary, to study the perturbation of (S_1) inside \mathcal{C}_2 it is necessary to consider two unfoldings. The first one corresponds to the perturbation inside the reversible centers and the second one to the perturbation inside the so-called generalized Lotka-Volterra centers (see [3] for details). However this is not the reason why Theorem B does not contemplate the perturbation of (S_1) . We explain what happens in this case at the end of the paper.

There are previous results related with Theorem B that must be referred. Chicone and Jacobs [3] prove that if we perturb any isochronous center in (4) inside \mathcal{C}_2 , then at most one critical period bifurcates from the inner boundary of \mathcal{P} (i.e., the center). They consider as well the perturbation of the linear isochronous center, for which the upper bound is two. The bifurcation of critical periods from the outer boundary of \mathcal{P} (i.e., the polycycle) is partially studied in [18, 19, 20]. (In [19] the authors present a conjectural bifurcation diagram of the period function of any quadratic center in the Loud form. We reproduce this diagram at the end of the paper, see Figure 1, where we shall comment how Theorem B and this conjecture fit together.) Gasull and Yu [10] prove Theorem B assuming that the perturbation (5) is *linear* in ε (or, as they say, perturbations up to first order in ε). Let us note moreover that system (S_4) has another isochronous center

apart from the one at the origin and that the authors in [10] study the simultaneous bifurcation of critical periods from both period annuli. In this paper we restrict our study to the bifurcation of critical periods from the period annulus of the center at the origin. Finally, the assertion in Theorem B corresponding to the isochronous center (S_2) is proved in [11].

In view of Theorems A and B, and all the previous results we refer above, one may pose the following question for the family \mathcal{H}_n of vector fields with homogeneous nonlinearities of degree n : Is it true that at most $n - 1$ critical periods can bifurcate from a nonlinear isochronous center when perturbed inside \mathcal{C}_n ?

The proof of Theorem A is given in Section 3 and it is based on a recent result which appears in [12], see the forthcoming Proposition 3.10. This recent result is a criterion for a collection of Abelian integrals to have a Chebyshev property and it provides, in many cases, a shortcut with respect to the classical methods for the same issue. The proof for system (S_1^*) follows the classical approach and, thus, it is much longer than the proof for the other three cases. The proof of Theorem B is given in Section 4. The following section contains the definitions and previous results needed to prove our results.

2 Definitions and notation

Let $\{X_\varepsilon\}$ be an analytic family of planar vector fields with a center at the origin for all $\varepsilon \approx 0$ and assume that the center is isochronous for $\varepsilon = 0$. Let \mathcal{P} be the *period annulus* of the center of X_0 (i.e., the biggest punctured neighbourhood of the origin foliated by periodic orbits) and let $\xi: \mathcal{I} \rightarrow \mathbb{R}^2$ be an analytic transversal section to X_0 in \mathcal{P} , with \mathcal{I} an open real interval. Let $T(s; \varepsilon)$ be the period of the periodic orbit of X_ε passing through the point $\xi(s) \in \mathcal{P}$ whenever is defined. (Here we use that the curve given by the graphic of ξ is still a transversal section to X_ε for $\varepsilon \approx 0$.) We can choose, for instance, the transversal section to be a solution of the vector field orthogonal to X_0 and in this case \mathcal{I} is its maximal interval of definition.

Definition 2.1 We say that k critical periods bifurcate from \mathcal{P} in the family $\{X_\varepsilon\}$ as $\varepsilon \rightarrow 0$ if there exist k functions $s_i(\varepsilon)$, continuous in a neighbourhood of $\varepsilon = 0$, and such that $T'(s_i(\varepsilon); \varepsilon) \equiv 0$ and $s_i(0) = s_i^* \in \mathcal{I}$ for each $i = 1, 2, \dots, k$. \square

Since $T(s; \varepsilon)$ is analytic for $\varepsilon \approx 0$, we can consider its Taylor development at $\varepsilon = 0$, say

$$T(s; \varepsilon) = \sum_{i=0}^{\infty} T_i(s) \varepsilon^i.$$

Note that in fact T_0 is constant because by assumption the center is isochronous for $\varepsilon = 0$. Then, if the center of X_ε is non-isochronous for $\varepsilon \neq 0$, there exists some $\ell \geq 1$ such that

$$T'(s; \varepsilon) = T'_\ell(s) \varepsilon^\ell + o(\varepsilon^\ell),$$

where T'_ℓ is not identically zero and the remainder is uniform in s on each compact subinterval of \mathcal{I} . In this case, from Definition 2.1 it follows that $T'_\ell(s_i^*)$ for $i = 1, 2, \dots, k$. Consequently, by applying the Weierstrass Preparation Theorem, the number of zeros of $T'_\ell(s)$ for $s \in \mathcal{I}$, counted with multiplicities, provides an upper bound for k . A lower bound for k is given by the number of simple zeros of $T'_\ell(s)$ in \mathcal{I} by using the Implicit Function Theorem.

Remark 2.2 The number of critical periods that bifurcate from \mathcal{P} as $\varepsilon \rightarrow 0$ does not depend on the particular parameterization of the set of periodic orbits in \mathcal{P} used. If two of these parameterizations yield to $T(s; \varepsilon)$ and $\widehat{T}(s; \varepsilon)$, then there exists a diffeomorphism $\zeta: \widehat{\mathcal{I}} \rightarrow \mathcal{I}$ such that $T(\zeta(s); \varepsilon) = \widehat{T}(s; \varepsilon)$. Following the obvious notation, this implies that $\widehat{T}'_\ell(s) = \zeta'(s) T'_\ell(\zeta(s))$. \square

Definition 2.3 On account of the above observation, from now on we shall work with a transversal section constructed in the following way. We take a vector field U_0 analytic on $\mathcal{P} \cup \{(0, 0)\}$, transversal to X_0 at \mathcal{P} and such that $[X_0, U_0] \equiv 0$. In the literature such a vector field is called a *commutator* of X_0 and it always exists in a neighbourhood of an isochronous center, see [24]. Here we require more; it must be analytic on $\mathcal{P} \cup \{(0, 0)\}$, but in all the examples that we shall study such a commutator exists. Then we define $T(s, \varepsilon)$ exactly as before taking a solution of U_0 as the transversal section ξ . More concretely, we choose any $q \in \mathcal{P}$ and we consider the solution $\psi(s; q)$ of U_0 with $\psi(0; q) = q$. Then we define $T(s; \varepsilon)$ as the period of the periodic orbit of X_ε passing through $\xi(s) := \psi(s; q)$. \square

Given an analytic family of functions $\{g_\varepsilon\}$ we write its Taylor series at $\varepsilon = 0$ as $g(x, y) = \sum_{i=0}^{\infty} g_i(x, y)\varepsilon^i$. We shall also use the notation $j^k(g) = \sum_{i=0}^k g_i\varepsilon^i$ for its k -jet with respect to ε . (This notation extends for vector fields in the obvious way.) Moreover, in what follows, we shall say that two vector fields X and Y are *conjugated* if there exists a change of coordinates Φ transforming X to Y , i.e., $\Phi^*X = Y$, where

$$(\Phi^*X)(p) = (D\Phi)_p^{-1}(X \circ \Phi(p)).$$

Then we say that Φ *conjugates* X and Y . A classical result shows that X has an isochronous center at p_0 if and only if there exists an analytic diffeomorphism Φ on a neighbourhood of p_0 that conjugates X with its linear part at p_0 . In this case Φ is called a *linearization*.

3 Proof of Theorem A

Let X_0 denote the vector field associated to one of the systems listed in (2) and \mathcal{P} be the period annulus of the isochronous center at the origin. Let U_0 be a commutator of X_0 which is analytic on $\mathcal{P} \cup \{(0, 0)\}$. We describe each of the considered vector fields U_0 in Lemma 3.4 and we note that they are, indeed, polynomial vector fields. Since X_0 and U_0 are transversal vector fields on \mathcal{P} , there exist two analytic functions λ and μ such that

$$Y_\varepsilon = X_\varepsilon - X_0 = \lambda X_0 + \mu U_0,$$

where Y_ε is the vector field corresponding to the perturbative part as defined in (3). In fact it is clear that

$$\lambda = \frac{Y_\varepsilon \wedge U_0}{X_0 \wedge U_0} \quad \text{and} \quad \mu = \frac{X_0 \wedge Y_\varepsilon}{X_0 \wedge U_0}.$$

Our first result, Theorem 3.2, gives an expression for the $(k+1)$ th term of the Taylor series of $T'(s; \varepsilon)$ at $\varepsilon = 0$. It depends on the $(k+1)$ th term of the development of $\lambda = \lambda(x, y; \varepsilon)$ at $\varepsilon = 0$ and to prove it we shall use the following result (see [10]). This theorem is a generalization of the result stated in the seminal paper [9], see also the references therein.

Theorem 3.1 (Gasull-Yü). *Let X be an analytic vector field with a center at p and let \mathcal{P} be its period annulus. Let U be an analytic vector field on $\mathcal{P} \cup \{p\}$ which is transversal to X on \mathcal{P} . Then $[X, U] = \alpha X + \beta U$ for some analytic functions α and β . Fix any $q \in \mathcal{P}$ and let $\xi: \mathcal{I} \rightarrow \mathcal{P}$ be the solution of U with $\xi(0) = q$. For each $s \in \mathcal{I}$, let $\varphi(t; s)$ be the solution of X with $\varphi(0; s) = \xi(s)$ and let $T(s)$ be its period. Then*

$$T'(s) = \int_0^{T(s)} \alpha(\varphi(t; s)) \exp\left(-\int_0^t \beta(\varphi(\tau; s)) d\tau\right) dt \quad \text{for all } s \in \mathcal{I}.$$

Let us point out that in the above statement U is just a transversal vector field to X . We can now prove the following:

Theorem 3.2. Consider the parameterization of the period function $T(s; \varepsilon)$ as introduced in Definition 2.3. Let us assume that, for some $k \in \mathbb{N}$, there exists an analytic family of diffeomorphisms $\{\Phi_\varepsilon\}$, in a neighbourhood of $(0, 0)$, such that Φ_ε linearizes $j^k(X_\varepsilon)$ for each $\varepsilon \approx 0$. Then $T'_0 \equiv T'_1 \equiv \dots \equiv T'_k \equiv 0$ and

$$T'_{k+1}(s) = - \int_0^{T_0} U_0(\lambda_{k+1})|_{(x,y)=\varphi(t;s)} dt \text{ for all } s \in \mathcal{I},$$

where $\varphi(t; s)$ is the solution of X_0 with $\varphi(0; s) = \xi(s)$.

Proof. We focus in the formula for T'_{k+1} because as a byproduct it will follow that $T'_0 \equiv T'_1 \equiv \dots \equiv T'_k \equiv 0$. Note first that it suffices to show the validity of the equality for periodic orbits near the origin since the functions on the left and right of the equality depend analytically on s . To this end we use the family of diffeomorphisms $\{\Phi_\varepsilon\}$ that are defined in neighbourhood of the origin. (Certainly this family depends on k as well, but for the sake of shortness we omit this additional subscript.)

Define $\Psi_\varepsilon := (\Phi_0)^{-1} \circ \Phi_\varepsilon$. Then $\{\Psi_\varepsilon\}$ is an analytic family of diffeomorphisms with $\Psi_\varepsilon = \text{Id} + o(\varepsilon^0)$ and such that, for each $\varepsilon \approx 0$, Ψ_ε conjugates $j^k(X_\varepsilon)$ and X_0 , i.e., $(\Psi_\varepsilon)^*(j^k(X_\varepsilon)) = X_0$. Let us define in addition $U_\varepsilon := (\Psi_\varepsilon)_*(U_0)$ and note that then $U_\varepsilon = U_0 + o(\varepsilon^0)$. We claim that

$$[X_\varepsilon, U_\varepsilon] = \varepsilon^{k+1}(-U_0(\lambda_{k+1}) + o(\varepsilon^0))X_\varepsilon + \varepsilon^{k+1}(-U_0(\mu_{k+1}) + o(\varepsilon^0))U_\varepsilon, \quad (6)$$

where here (and in what follows) we use the notation $\lambda = \sum_{i=0}^\infty \lambda_i \varepsilon^i$ and $\mu = \sum_{i=0}^\infty \mu_i \varepsilon^i$. In order to prove the claim let us first recall that $X_\varepsilon = X_0 + \lambda X_0 + \mu U_0$. Thus, we can write

$$X_\varepsilon = j^k(X_\varepsilon) + \varepsilon^{k+1}((\lambda_{k+1} + o(\varepsilon^0))X_0 + (\mu_{k+1} + o(\varepsilon^0))U_0). \quad (7)$$

On the other hand,

$$(\Psi_\varepsilon)^*[j^k(X_\varepsilon), U_\varepsilon] = [(\Psi_\varepsilon)^*(j^k(X_\varepsilon)), (\Psi_\varepsilon)^*(U_\varepsilon)] = [X_0, (\Psi_\varepsilon)^*((\Psi_\varepsilon)_*(U_0))] = [X_0, U_0] = 0.$$

In the first equality above we use that if f is a local diffeomorphism, then $f^*[X, Y] = [f^*X, f^*Y]$, see for instance [14]. In the second one we use that Ψ_ε conjugates $j^k(X_\varepsilon)$ and X_0 and that, by definition, $U_\varepsilon = (\Psi_\varepsilon)_*(U_0)$. The last one follows from the fact that U_0 is a commutator of X_0 . We can thus conclude that $[j^k(X_\varepsilon), U_\varepsilon] = 0$. Accordingly, from (7) we obtain

$$[X_\varepsilon, U_\varepsilon] = \underbrace{[j^k(X_\varepsilon), U_\varepsilon]}_0 + \varepsilon^{k+1} \underbrace{[(\lambda_{k+1} + o(\varepsilon^0))X_0, U_\varepsilon]}_{\Delta_1} + \varepsilon^{k+1} \underbrace{[(\mu_{k+1} + o(\varepsilon^0))U_0, U_\varepsilon]}_{\Delta_2}.$$

Then, since $[fX, Y] = f[X, Y] - Y(f)X$ (see [14] again), it follows that

$$\begin{aligned} \Delta_1 &= (\lambda_{k+1} + o(\varepsilon^0))[X_0, U_\varepsilon] - U_\varepsilon(\lambda_{k+1} + o(\varepsilon^0))X_0 \\ &= (\lambda_{k+1} + o(\varepsilon^0))[X_0, U_0 + o(\varepsilon^0)] - U_0(\lambda_{k+1})X_\varepsilon + o(\varepsilon^0) = -U_0(\lambda_{k+1})X_\varepsilon + o(\varepsilon^0) \end{aligned}$$

and

$$\Delta_2 = (\mu_{k+1} + o(\varepsilon^0))[U_0, U_\varepsilon] - U_\varepsilon(\mu_{k+1} + o(\varepsilon^0))U_0 = -U_0(\mu_{k+1})U_\varepsilon + o(\varepsilon^0),$$

where we took $X_\varepsilon - X_0 = U_\varepsilon - U_0 = o(\varepsilon^0)$ and $[X_0, U_0] = 0$ into account. Hence

$$[X_\varepsilon, U_\varepsilon] = \varepsilon^{k+1}(-U_0(\lambda_{k+1}))X_\varepsilon + \varepsilon^{k+1}(-U_0(\mu_{k+1}))U_\varepsilon + o(\varepsilon^{k+1}),$$

and (6) follows after decomposing the remainder term above in the X_ε and U_ε components using that both vector fields are transversal for $\varepsilon \approx 0$. This shows that the claim is true.

Next we take the point $q \in \mathcal{P}$ in Definition 2.3 and consider the solution $\psi_\varepsilon(s; q)$ of X_ε with $\psi_\varepsilon(0; q) = q$. If $\widehat{T}(s; \varepsilon)$ denotes the period of the periodic orbit of X_ε passing through $\xi_\varepsilon(s) := \psi_\varepsilon(s; q)$, then from (6) and applying Theorem 3.1 it follows that

$$\widehat{T}'(s; \varepsilon) = -\varepsilon^{k+1} \int_0^{\widehat{T}(s; \varepsilon)} (U_0(\lambda_{k+1}) + o(\varepsilon^0))|_{(x, y) = \varphi_\varepsilon(t; s)} e^{-\varepsilon^{k+1} \int_0^t (U_\varepsilon(\mu_{k+1}) + o(\varepsilon^0))|_{(x, y) = \varphi_\varepsilon(\tau; s)} d\tau} dt,$$

where $\varphi_\varepsilon(t; s)$ is the solution of X_ε with $\varphi_\varepsilon(0; s) = \xi_\varepsilon(s)$. Therefore, on account of $X_\varepsilon - X_0 = U_\varepsilon - U_0 = o(\varepsilon^0)$, we obtain

$$\widehat{T}'(s; \varepsilon) = -\varepsilon^{k+1} \int_0^{T_0} U_0(\lambda_{k+1})|_{(x, y) = \varphi_0(t; s)} dt + o(\varepsilon^{k+1}).$$

Note at this point that, by construction, $T(s; \varepsilon) = \widehat{T}(f(s, \varepsilon); \varepsilon)$ for some function f with $f(s, 0) = s$. Taking this into account, one can easily show from the above equality that

$$T'(s; \varepsilon) = -\varepsilon^{k+1} \int_0^{T_0} U_0(\lambda_{k+1})|_{(x, y) = \varphi_0(t; s)} dt + o(\varepsilon^{k+1}).$$

Accordingly, $T'_0 \equiv T'_1 \equiv \dots \equiv T'_k \equiv 0$ and

$$T'_{k+1}(s) = - \int_0^{T_0} U_0(\lambda_{k+1})|_{(x, y) = \varphi_0(t; s)} dt.$$

This completes the proof of the result. ■

Remark 3.3 As we explain at the end of Section 2, it is well known that a vector field has an isochronous center if and only if it is linearizable. The authors do not know whether this characterization is true for families. This is the reason why the assumption in Theorem 3.2 requires the existence of an *analytic family* of linearizations instead of assuming that the center of $j^k(X_\varepsilon)$ is isochronous for $\varepsilon \approx 0$. The latter assumption provides a linearization Φ_ε for each $\varepsilon \approx 0$, but in general we know nothing about the regularity of the family $\{\Phi_\varepsilon\}$ with respect to ε . □

The following lemma (see [5, 21]) provides the necessary information in order to apply the above result to our problem with the Pleshkan's isochrones. It gives the commutator and first integral of each isochronous center in (2).

Lemma 3.4. (S_1^*) The vector field $X_0 = (-y - 3x^2y + y^3)\partial_x + (x + x^3 - 3xy^2)\partial_y$ has first integral

$$H(x, y) = \frac{x^2 + y^2}{\sqrt{(x^2 + (y-1)^2)(x^2 + (y+1)^2)}}$$

and commutator $U_0 = (x + x^3 - 3xy^2)\partial_x + (y + 3x^2y - y^3)\partial_y$.

(S_2^*) The vector field $X_0 = (-y + x^2y)\partial_x + (x + xy^2)\partial_y$ has first integral

$$H(x, y) = \frac{x^2 + y^2}{1 - x^2}$$

and commutator $U_0 = x(1 - x^2)\partial_x + y(1 - x^2)\partial_y$.

(S_3^*) The vector field $X_0 = (-y + 3x^2y)\partial_x + (x - 2x^3 + 9xy^2)\partial_y$ has first integral

$$H(x, y) = \frac{(x - 2x^3)^2 + y^2}{(1 - 3x^2)^3}$$

and commutator $U_0 = x(1 - 3x^2)(1 - 2x^2)\partial_x + y(1 - 3x^2)(1 - 6x^2)\partial_y$.

(\bar{S}_3^*) The vector field $X_0 = (-y - 3x^2y)\partial_x + (x + 2x^3 - 9xy^2)\partial_y$ has first integral

$$H(x, y) = \frac{(x + 2x^3)^2 + y^2}{(1 + 3x^2)^3}$$

and commutator $U_0 = x(1 + 3x^2)(1 + 2x^2)\partial_x + y(1 + 3x^2)(1 + 6x^2)\partial_y$.

Since the number of critical periods that bifurcate from \mathcal{P} does not depend on the particular parameterization of the set of periodic orbits in \mathcal{P} used (recall Remark 2.2), Theorem A will follow once we prove the next result.

Theorem 3.5. *Let X_0 be one of the four Pleshkan's isochrones in Lemma 3.4 and let U_0 be its corresponding commutator. Consider the unfolding of centers $X_\varepsilon := X_0 + Y_\varepsilon$ where*

$$Y_\varepsilon = (a(\varepsilon)x^2y + b(\varepsilon)y^3)\partial_x + (c(\varepsilon)x^3 + d(\varepsilon)xy^2)\partial_y$$

and where a, b, c and d are analytic functions in a neighbourhood of $\varepsilon = 0$ with $a(0) = b(0) = c(0) = d(0) = 0$. Let $\xi: \mathcal{I} \rightarrow \mathbb{R}^2$ be a transversal section to \mathcal{P} given by a solution of U_0 and let $T(s; \varepsilon)$ be the period of the periodic orbit of X_ε passing through $\xi(s) \in \mathcal{P}$. If $T'_0 \equiv T'_1 \equiv \dots \equiv T'_k \equiv 0$ and $T'_{k+1} \not\equiv 0$, then $T'_{k+1}(s)$ has at most two zeros for $s \in \mathcal{I}$ and there are perturbations Y_ε giving rise to 0, 1 and 2 zeros.

In order to prove this result some definitions and lemmas are needed.

Definition 3.6 Let f_0, f_1, \dots, f_{n-1} be analytic functions on an open interval L of \mathbb{R} .

(a) $(f_0, f_1, \dots, f_{n-1})$ is a *Chebyshev system* (in short, T-system) on L if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \dots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n - 1$ isolated zeros on L .

(b) $(f_0, f_1, \dots, f_{n-1})$ is a *complete Chebyshev system* (in short, CT-system) on L if $(f_0, f_1, \dots, f_{k-1})$ is a T-system for all $k = 1, 2, \dots, n$.

(c) $(f_0, f_1, \dots, f_{n-1})$ is an *extended complete Chebyshev system* (in short, ECT-system) on L if, for all $k = 1, 2, \dots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \dots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k - 1$ isolated zeros on L counted with multiplicities.

(Let us mention that, in these abbreviations, “T” stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev.) \square

Remark 3.7 If $(f_0, f_1, \dots, f_{n-1})$ is an ECT-system on L , then for each $k = 0, 1, \dots, n - 1$ there exists a linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \dots + \alpha_{n-1} f_{n-1}(x)$$

with *exactly* k simple zeros on L (see for instance [13, 17]). \square

Definition 3.8 Let f_0, f_1, \dots, f_{k-1} be analytic functions on an open interval L of \mathbb{R} . The *Wronskian* of $(f_0, f_1, \dots, f_{k-1})$ at $x \in L$ is

$$W[f_0, f_1, \dots, f_{k-1}](x) = \det \left(f_j^{(i)}(x) \right)_{0 \leq i, j \leq k-1} = \begin{vmatrix} f_0(x) & \cdots & f_{k-1}(x) \\ f'_0(x) & \cdots & f'_{k-1}(x) \\ \vdots & & \vdots \\ f_0^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

\square

The following result is well known (see [13, 17] for instance).

Lemma 3.9. $(f_0, f_1, \dots, f_{n-1})$ is an ECT-system on L if, and only if, for each $k = 1, 2, \dots, n$ it holds

$$W[f_0, f_1, \dots, f_{k-1}](x) \neq 0 \text{ for all } x \in L.$$

Recall that Theorem 3.5 deals with four different isochrones. For each one, the proof yields to a linear combination of some Abelian integrals and it is necessary to show that they form an ECT-system. This is in general extremely complicated to verify, but in three of the cases it is not that difficult because a criterion proved in [12] applies successfully. Proposition 3.10 is a simplified version of this criterion that it suffices for our purposes. In its statement we suppose that the projection of \mathcal{P} on the x -axis is given by $(-x_r, x_r)$ and that the energy level of H at the periodic orbits in \mathcal{P} ranges from $h = 0$ to $h = h_0$.

Proposition 3.10. *Let us consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2m-1} dx, \quad i = 0, 1, \dots, n-1,$$

where $f_i(x)$ are analytic functions in $(-x_r, x_r)$, $m \in \mathbb{Z}$ and where, for each $h \in (0, h_0)$, γ_h is the oval surrounding the origin inside the level curve $\{A(x) + B(x)y^2 = h\}$. Assume that A and B are even functions and let ℓ_i be the even part of f_i . Then $(I_0, I_1, \dots, I_{n-1})$ is an ECT-system on $(0, h_0)$ if $m \geq n-1$ and $(\ell_0, \ell_1, \dots, \ell_{n-1})$ is a CT-system on $(0, x_r)$.

The following lemma, proved in [12], establishes a formula to write the integrand of an Abelian integral so as to be suitable to apply the Proposition 3.10.

Lemma 3.11. *Let γ_h be the oval surrounding the origin inside the level curve $\{A(x) + B(x)y^2 = h\}$ and we consider a function F such that F/A' is analytic at $x = 0$. Then, for any $k \in \mathbb{N}$,*

$$\int_{\gamma_h} F(x) y^{k-2} dx = \int_{\gamma_h} G(x) y^k dx$$

where $G(x) = \frac{2}{k} \left(\frac{BF}{A'} \right)'(x) - \left(\frac{B'F}{A'} \right)(x)$.

The criterion in Proposition 3.10 does not apply for the isochronous center (S_1^*) . This fact makes the proof of Theorem 3.5 for this case much longer than the others. In particular we shall need some properties of the complete elliptic integrals (see [1] for instance).

Definition 3.12 The complete elliptic integrals of first and second kind are

$$\mathcal{E}(u) := \int_0^{\pi/2} \sqrt{1 - u^2 \sin^2 t} dt \quad \text{and} \quad \mathcal{K}(u) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - u^2 \sin^2 t}}$$

respectively, which are analytic real functions for $u \in (-1, 1)$. □

Lemma 3.13. *The functions \mathcal{K} and \mathcal{E} verify the linear differential equation*

$$\begin{pmatrix} \mathcal{K}' \\ \mathcal{E}' \end{pmatrix} = \frac{-1}{u} \begin{pmatrix} 1 & \frac{1}{u^2-1} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{K} \\ \mathcal{E} \end{pmatrix}.$$

Moreover, their Taylor series at $u = 0$ are

$$\mathcal{K}(u) = \frac{\pi}{2} \sum_{i \geq 0} \left(\frac{(2i-1)!!}{(2i)!!} \right)^2 u^{2i} \quad \text{and} \quad \mathcal{E}(u) = -\frac{\pi}{2} \sum_{i \geq 0} \left(\frac{(2i-1)!!}{(2i)!!} \right)^2 \frac{u^{2i}}{2i-1},$$

where $u \in (-1, 1)$.

Recall that the double factorial of an integer n , with $n \geq -1$, is defined as

$$n!! = \begin{cases} n(n-2) \dots 5 \cdot 3 \cdot 1 & \text{if } n > 0 \text{ and } n \text{ odd,} \\ n(n-2) \dots 6 \cdot 4 \cdot 2 & \text{if } n > 0 \text{ and } n \text{ even,} \\ 1 & \text{if } n = -1, 0. \end{cases}$$

Finally, once we show the following lemma we will be in position to prove Theorem 3.5.

Lemma 3.14. *Following the notation in the statement of Theorem 3.5, let us write $X_\varepsilon - X_0 = \lambda X_0 + \mu U_0$ and, setting $\lambda = \sum_{i=1}^{\infty} \lambda_i \varepsilon^i$, define*

$$R(s) = \int_0^{T_0} U_0(\lambda_k) |_{(x,y)=\varphi(t;s)} dt,$$

where $\varphi(t; s)$ is the solution of X_0 with $\varphi(0; s) = \xi(s)$. Then $R(s) = \alpha I_0(s) + \beta I_1(s) + \gamma I_2(s)$ where (I_0, I_1, I_2) is an ECT-system on \mathcal{I} and $(\alpha, \beta, \gamma) = \phi(a_k, b_k, c_k, d_k)$ with ϕ being a surjective linear mapping such that

- (a) $\text{Ker}(\phi) = \{d_k + 3c_k = a_k + 3c_k = b_k - c_k = 0\}$ in case that X_0 is (S_1^*) ,
- (b) $\text{Ker}(\phi) = \{b_k = c_k = a_k - d_k = 0\}$ in case that X_0 is (S_2^*) ,
- (c) $\text{Ker}(\phi) = \{b_k = 2a_k + 3c_k = 2d_k + 9c_k = 0\}$ in case that X_0 is (S_3^*) or (\bar{S}_3^*) .

Proof. First we shall write $R(s)$ as an Abelian integral taking advantage of the fact that each isochronous center X_0 has a first integral H , see Lemma 3.4. Thus, if γ_s denotes the periodic orbit of X_0 inside the energy level $\{H(x, y) = H(\xi(s))\}$ and $X_0 = P_0 \partial_x + Q_0 \partial_y$, then

$$R(s) = \int_{\gamma_s} \frac{\langle \nabla \lambda_k(x, y), U_0(x, y) \rangle}{P_0(x, y)} dx,$$

where \langle, \rangle denotes the inner product. In fact, setting $\eta(s) := H(\xi(s))$, for simplicity in the exposition we shall study

$$I(h) := R(\eta^{-1}(h)) = \int_{\gamma_h} \frac{\langle \nabla \lambda_k(x, y), U_0(x, y) \rangle}{P_0(x, y)} dx,$$

where γ_h is now the oval inside the level curve $\{H(x, y) = h\}$. Note moreover that, by definition, $Y_\varepsilon = X_\varepsilon - X_0$ and $X_\varepsilon - X_0 = \lambda X_0 + \mu U_0$, so that

$$\lambda = \frac{Y_\varepsilon \wedge U_0}{X_0 \wedge U_0}.$$

We are now in position to prove the result for each different case.

The case (S_2^*) . From the above equality, taking the expression of X_0 and U_0 given by Lemma 3.4 into account, an easy computation shows that

$$\lambda(x, y; \varepsilon) = -\frac{a(\varepsilon)y^2x^2 + b(\varepsilon)y^4 - c(\varepsilon)x^4 - d(\varepsilon)x^2y^2}{x^2 + y^2},$$

and accordingly, since $\lambda = \sum_{i=1}^{\infty} \lambda_i \varepsilon^i$, it turns out that

$$\lambda_k(x, y) = -\frac{a_k y^2 x^2 + b_k y^4 - c_k x^4 - d_k x^2 y^2}{x^2 + y^2}.$$

From now on, when there is no risk of ambiguity, we shall omit the subscript k for the sake of shortness. In fact it does not play any role at all because $\lambda_k(x, y) = F(a_k, b_k, c_k, d_k, x, y)$ with F not depending on k .

Since $H(x, y) = \frac{x^2+y^2}{1-x^2}$, the projection of \mathcal{P} on the x -axis is $(-1, 1)$ and the energy level in γ_h ranges from $h = 0$ to $h_0 = +\infty$. It turns out that

$$I(h) = 2 \int_{\gamma_h} \frac{by^4 + (a-d)x^2y^2 - cx^4}{y(x^2 + y^2)} dx = \frac{2}{h} \int_{\gamma_h} \frac{by^4 + (a-d)x^2y^2 - cx^4}{y(1-x^2)} dx,$$

where in the last equality we use that $x^2 + y^2 = (1-x^2)h$. Consequently

$$I(h) = \frac{2}{h} (bI_0(h) + (a-d)I_1(h) - cI_2(h))$$

with

$$I_0(h) = \int_{\gamma_h} \frac{y^3 dx}{1-x^2}, \quad I_1(h) = \int_{\gamma_h} \frac{x^2 y dx}{1-x^2} \quad \text{and} \quad I_2(h) = \int_{\gamma_h} \frac{x^4 dx}{y(1-x^2)}.$$

Clearly, since $\eta(s) = H(\xi(s))$ is a diffeomorphism and $\eta(\mathcal{I}) = (0, +\infty)$, setting $\phi(a, b, c, d) = (b, a-d, -c)$, the proof of Lemma 3.14 will follow for this case once we show that (I_0, I_1, I_2) is an ECT-system on $(0, +\infty)$. To this end we shall apply Proposition 3.10. With this aim in view, the application of Lemma 3.11 to $I_1(h)$ with $k = 3$ gives

$$I_1(h) = \int_{\gamma_h} f_1(x) y^3 dx \quad \text{where} \quad f_1(x) = \frac{1-4x^2}{3(1-x^2)}.$$

Similarly, but now applying twice Lemma 3.11, first with $k = 1$ and then with $k = 3$, we obtain

$$I_2(h) = \int_{\gamma_h} f_2(x) y^3 dx \quad \text{where} \quad f_2(x) = \frac{8x^4 - 8x^2 + 1}{1-x^2}.$$

On the other hand, note that

$$I_0(h) = \int_{\gamma_h} f_0(x) y^3 dx \quad \text{where} \quad f_0(x) = \frac{1}{1-x^2}.$$

Our goal with these manipulations is twofold. Firstly, we want that y has the same exponent in all the Abelian integrals. Secondly, that this exponent is large enough so that, with the notation in Proposition 3.10, $m \geq n-1$. Now, by applying this result, if (f_0, f_1, f_2) is a CT-system on $(0, 1)$, then (I_0, I_1, I_2) is an ECT-system on $(0, +\infty)$, and we are done. This is clear because f_0, f_1 and f_2 share the same denominator and each numerator is an even polynomial of degree exactly $2i$ for $i = 0, 1, 2$.

The case (\mathbf{S}_3^*) . We omit many of the explanations for the sake of brevity because the proof in this case is completely analogous to the previous one. Now the Abelian integral $I(h) = R(\eta^{-1}(h))$ is given by

$$I(h) = \frac{2}{h} \int_{\gamma_h} \left(\frac{cx^4(1-2x^2)}{(1-3x^2)^3 y} + \frac{x^2(1-2x^2)(d-a+4x^2(3a-d))y}{(1-3x^2)^3} - \frac{b(1-18x^2+48x^4)y^3}{(1-3x^2)^3} \right) dx,$$

where we used the transversal commutator and the first integral given by Lemma 3.4. The projection of \mathcal{P} on the x -axis is $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $h_0 = +\infty$. By applying Lemma 3.11 exactly as in the previous case we obtain

$$I(h) = \frac{2}{3h} \left((d-a-3b+3c)I_0(h) + 6(9a+9b-9c-5d)I_1(h) - 48(6a+3b-2d)I_2(h) \right),$$

where

$$I_i(h) = \int_{\gamma_h} f_i(x) y^3 dx \quad \text{with} \quad f_i(x) = \frac{x^{2i}}{(1-3x^2)^3} \quad \text{for} \quad i = 0, 1, 2.$$

By applying Proposition 3.10, (I_0, I_1, I_2) is an ECT-system on $(0, +\infty)$ because it is clear that (f_0, f_1, f_2) is a CT-system on $(0, \frac{1}{\sqrt{3}})$. Thus, taking $\phi(a, b, c, d) = (d - a - 3b + 3c, 9a + 9b - 9c - 5d, 6a + 3b - 2d)$, the result follows in this case.

The case $(\bar{\mathbf{S}}_3^*)$. Exactly in the same way as in the previous cases, the Abelian integral $I(h) = R(\eta^{-1}(h))$ is given by

$$I(h) = \frac{2}{h} \int_{\gamma_h} \left(\frac{cx^4(1+2x^2)}{(1+3x^2)^3 y} + \frac{x^2(1+2x^2)(d-a+4x^2(d-3a))y}{(1+3x^2)^3} - \frac{b(1+18x^2+48x^4)y^3}{(1+3x^2)^3} \right) dx.$$

The projection of the period annulus \mathcal{P} on the x -axis is $(-\infty, +\infty)$ and $h_0 = \frac{4}{27}$. By applying Lemma 3.11,

$$I(h) = \frac{2}{3h} \left((d-a-3b+3c)I_0(h) - 6(9a+9b-9c-5d)I_1(h) - 48(6a+3b-2d)I_2(h) \right),$$

where

$$I_i(h) = \int_{\gamma_h} f_i(x)y^3 dx \text{ with } f_i(x) = \frac{x^{2i}}{(1+3x^2)^3} \text{ for } i = 0, 1, 2.$$

We are under the hypothesis of Proposition 3.10 and, since it is obvious that (f_0, f_1, f_2) is a CT-system on $(0, +\infty)$, we conclude that (I_0, I_1, I_2) is an ECT-system on $(0, \frac{4}{27})$. Consequently the result follows taking, as before, $\phi(a, b, c, d) = (d - a - 3b + 3c, 9a + 9b - 9c - 5d, 6a + 3b - 2d)$.

The case (\mathbf{S}_1^*) . The proof for the perturbation of this isochronous center is longer and more complicated than the others because the condition for the first integral to take the form $A(x) + B(x)y^2$, as established in Proposition 3.10, is not verified. Then we must apply Lemma 3.9, that characterizes the Chebyshev property in terms of Wronskians. For this reason, instead of introducing Abelian integrals, we keep the expression of $R(s)$ in terms of the solution of the unperturbed system and, for the sake of convenience, we use complex notation. Recall that we now study the unfolding

$$\begin{cases} \dot{x} = -y + (-3 + a(\varepsilon))x^2y + (1 + b(\varepsilon))y^3, \\ \dot{y} = x + (1 + c(\varepsilon))x^3 + (-3 + d(\varepsilon))xy^2. \end{cases}$$

Taking the commutator given by Lemma 3.4, an easy computation shows that

$$\lambda(x, y; \varepsilon) = \frac{x^2(1+x^2-3y^2)(c(\varepsilon)x^2+d(\varepsilon)y^2) - y^2(1+3x^2-y^2)(a(\varepsilon)x^2+b(\varepsilon)y^2)}{(x^2+y^2)(1+2(x^2-y^2)+(x^2+y^2)^2)}.$$

The coordinate transformation $z = x + iy$ brings the unfolding to $\dot{z} = f(z) + ip(z, \bar{z})$ with

$$f(z) = iz(1+z^2) \quad \text{and} \quad p(z, \bar{z}) = (\alpha z^3 + \beta z^2 \bar{z} + \gamma z \bar{z}^2 + \delta \bar{z}^3)/8,$$

where $\alpha = b + c - d - a$, $\beta = 3c + d - a - 3b$, $\gamma = a + 3b + 3c + d$ and $\delta = a - b + c - d$. An easy computation shows that the solution of the unperturbed system, i.e., $\dot{z} = f(z)$, with initial condition $z = h \in (0, +\infty)$ at $t = 0$ is given by

$$\varphi(t; h) = \frac{he^{it}}{\sqrt{1+h^2-e^{2it}h^2}}.$$

Accordingly, following the usual notation $\alpha(\varepsilon) = \sum_{i=1}^{\infty} \alpha_i \varepsilon^i$, we split the function under consideration as

$$R(h) = \int_0^{2\pi} \langle \nabla \lambda_k(x, y), U_0(x, y) \rangle|_{x+iy=\varphi(t; h)} dt = \alpha_k I_3(h) + \beta_k I_2(h) + \gamma_k I_1(h) + \delta_k I_0(h).$$

Long but straightforward manipulations show that $I_3 \equiv 0$ and that

$$R(h) = \frac{-4h^4}{(1+h^2)^2} \left(\beta_k \bar{I}_2(h) + \gamma_k \pi + \delta_k \bar{I}_0(h) \right)$$

where, setting $\mu(h) = \frac{2h\sqrt{1+h^2}}{1+2h^2}$,

$$\bar{I}_2(h) := \frac{2h^2}{1+2h^2} \mathcal{K}(\mu(h)) - \frac{2+4h^2}{h^2} \mathcal{E}(\mu(h))$$

and

$$\bar{I}_0(h) := \frac{(1+2h^2)(1+2h^2+2h^4)}{h^6} \mathcal{E}(\mu(h)) - \frac{(1+h^2+h^4)(1+3h^2+3h^4)}{h^6(1+2h^2)} \mathcal{K}(\mu(h)).$$

The fact that $I_3 \equiv 0$ is not unexpected at all. If $\beta = \gamma = \delta = 0$, then the perturbed system is holomorphic, it does not depend on \bar{z} , and consequently the center is isochronous for all ε , so that $T'(s; \varepsilon) \equiv 0$, see [25].

At this point, taking $\phi(a, b, c, d) = (3c + d - a - 3b, a + 3b + 3c + d, a - b + c - d)$, the proof of Lemma 3.14 for the case (S_1^*) reduces to the verification of the fact that $(\pi, \bar{I}_2, \bar{I}_0)$ is an ECT-system on $(0, +\infty)$. Since the first function is a non-zero constant, we compute the two-dimensional Wronskian by using Lemma 3.13,

$$W[\pi, \bar{I}_2](h) = \bar{I}_2'(h) = \frac{2+2h^2}{h^3} \left(\mathcal{E}(\mu(h)) + \frac{\mathcal{K}(\mu(h))}{1+2h^2} \right).$$

Clearly it is different from zero for all $h \in (0, +\infty)$ because \mathcal{E} and \mathcal{K} are strictly positive functions. Next, taking Lemma 3.13 into account again, we compute the three-dimensional Wronskian. The key point is that it factorizes as

$$W[\pi, \bar{I}_2, \bar{I}_0](h) = \begin{vmatrix} \bar{I}_2' & \bar{I}_0' \\ \bar{I}_2'' & \bar{I}_0'' \end{vmatrix} = \frac{72(1+h^2)^3}{h^{11}} \left(\mathcal{E}(\mu(h)) + L_+(h) \mathcal{K}(\mu(h)) \right) \left(\mathcal{E}(\mu(h)) + L_-(h) \mathcal{K}(\mu(h)) \right),$$

where

$$L_{\pm}(h) := \frac{-1 - 2h^2 - 2h^4 \pm 2(1 + 4h^2 + 5h^4 + 2h^6 + h^8)^{1/2}}{3(1 + 2h^2)^2}.$$

Note that $L_-(h) < 0$ for all $h \in (0, +\infty)$. It is also easy to show that $L_+(h) > 0$ for all $h \in (0, +\infty)$. Therefore, we will see that this Wronskian does not vanish once we prove that

$$\mathcal{R}(h) := \mathcal{E}(\mu(h)) + L_-(h) \mathcal{K}(\mu(h)) \neq 0 \text{ for all } h \in (0, +\infty).$$

From now on we will use the variable $u = \mu(h)$ because then the expressions that we obtain are shorter. Thus, due to $\mu(0, +\infty) = (0, 1)$, we must show that

$$\mathcal{L}(u) := \mathcal{R}(\mu^{-1}(u)) = \mathcal{E}(u) + \frac{1}{6} \left(u^2 - 2 - \sqrt{16 - 16u^2 + u^4} \right) \mathcal{K}(u)$$

does not vanish for all $u \in (0, 1)$. To prove this claim we first note that by applying Lemma 3.13 one can check that $\mathcal{L}(u)$ verifies the differential equation

$$\mathcal{L}''(u) = g_1(u) \mathcal{L}'(u) + g_0(u) \mathcal{L}(u), \tag{8}$$

where

$$g_1(u) = \frac{48 - 64u^2 + 17u^4}{u(1 - u^2)(16 - 16u^2 + u^4)} + \frac{\sqrt{16 - 16u^2 + u^4}}{u(1 - u^2)}$$

and

$$g_0(u) = \frac{16 - 12u^2 + (8 - u^2)\sqrt{16 - 16u^2 + u^4}}{(1 - u^2)(16 - 16u^2 + u^4)}.$$

On the other hand, from Lemma 3.13 once again, we get that $\mathcal{L}(u) = -\frac{3\pi}{4096}u^8 + o(u^9)$. We are now in position to prove the claim. By contradiction, assume that there exists some $u_1 \in (0, 1)$ such that $\mathcal{L}(u_1) = 0$. Then, since $\mathcal{L}(u) < 0$ for $u \approx 0$, this forces the existence of a local minimum of \mathcal{L} , say u_0 , with $\mathcal{L}(u_0) < 0$. The evaluation of the differential equation in (8) at $u = u_0$ shows that $\mathcal{L}''(u_0) = g_0(u_0)\mathcal{L}(u_0)$. Since $g_0(u) > 0$ for all $u \in (0, 1)$, this implies that $\mathcal{L}''(u_0) < 0$, which contradicts the fact that $u = u_0$ is a local minimum. Hence the claim is true and, therefore, the three-dimensional Wronskian is different from zero. In short, $(\pi, \bar{I}_2, \bar{I}_0)$ is an ECT-system on $(0, +\infty)$ and this completes the proof of the result for the case (S_1^*) . ■

Proof of Theorem 3.5 We prove the result for the perturbation of the isochronous center (S_1^*) only because the other cases follow exactly in the same way. Thus, consider the vector field X_0 with the isochronous center (S_1^*) at the origin and, see Lemma 3.4, let U_0 be its commutator. In addition, let Φ be a linearization of X_0 . As usual we consider the family of centers $X_\varepsilon = X_0 + Y_\varepsilon$ and we denote by $T(s; \varepsilon)$ the period function of X_ε using a solution of U_0 as transversal section.

We claim that if $T'_0 \equiv T'_1 \equiv \dots \equiv T'_k \equiv 0$, then there exists an analytic family of diffeomorphisms $\{\Phi_\varepsilon^k\}$, in a neighbourhood of the origin, such that Φ_ε^k linearizes $j^k(X_\varepsilon)$ for all $\varepsilon \approx 0$. The proof follows by induction on k . The case $k = 0$ is trivial because $j^0(X_\varepsilon) = X_0$ and so we can take $\Phi_\varepsilon^0 = \Phi$. Assume that the claim is true for $k = n$ and let us show its validity for $k = n + 1$. So suppose that $T'_0 \equiv T'_1 \equiv \dots \equiv T'_n \equiv T'_{n+1} \equiv 0$ and, by the induction hypothesis, that there exists a linearization Φ_ε^n of $j^n(X_\varepsilon)$ for $k = 0, 1, \dots, n$. Then, by applying $n + 1$ times Theorem 3.2,

$$T'_k(s) = - \int_0^{T_0} U_0(\lambda_k)|_{(x,y)=\varphi(t;s)} dt \text{ for } k = 1, 2, \dots, n + 1,$$

and consequently, from (a) in Lemma 3.14,

$$d_k + 3c_k = a_k + 3c_k = b_k - c_k = 0 \text{ for } k = 1, 2, \dots, n + 1.$$

(This follows from the fact that each T'_k is a linear combination of three functions forming an ECT-system with the coefficients vanishing simultaneously only in case that the above relations hold.) Therefore it turns out that we can write

$$j^{n+1}(X_\varepsilon) = (-y - 3\kappa(\varepsilon)x^2y + \kappa(\varepsilon)y^3)\partial_x + (x + \kappa(\varepsilon)x^3 - 3\kappa(\varepsilon)xy^2)\partial_y$$

with $\kappa(\varepsilon) := 1 + \sum_{k=1}^{n+1} c_k \varepsilon^k$. Now, if we define $\Psi_\varepsilon(x, y) = (\kappa(\varepsilon)x, \kappa(\varepsilon)y)$, then one can easily verify that it holds $(\Psi_\varepsilon)^*(j^{n+1}(X_\varepsilon)) = X_0$. Accordingly $\Phi_\varepsilon^{n+1} := \Phi \circ \Psi_\varepsilon$ provides a linearization of the $(n + 1)$ -jet of X_ε . Thus the claim is true.

It is clear that the result under consideration only makes sense in case that the perturbation is not isochronous, so there exists some $\ell \geq 0$ such that $T'_0 \equiv T'_1 \equiv \dots \equiv T'_\ell \equiv 0$ and $T'_{\ell+1} \neq 0$. Then, on account of the claim, there exists an analytic family of linearizations $\{\Phi_\varepsilon^\ell\}$ of $j^\ell(X_\varepsilon)$ and so, by applying Theorem 3.2,

$$T'_{\ell+1}(s) = - \int_0^{T_0} U_0(\lambda_{\ell+1})|_{(x,y)=\varphi(t;s)} dt \text{ for all } s \in \mathcal{I},$$

Hence, from Lemma 3.14, $T'_{\ell+1}(s) = \alpha I_0(s) + \beta I_1(s) + \gamma I_2(s)$ where (I_0, I_1, I_2) is an ECT-system on \mathcal{I} and $(\alpha, \beta, \gamma) = \phi(a_{\ell+1}, b_{\ell+1}, c_{\ell+1}, d_{\ell+1})$ for some surjective linear mapping. Consequently, since $T'_{\ell+1}$ is not identically zero, it can have at most two zeros on \mathcal{I} . Moreover, since ϕ is exhaustive, one can choose $a_{\ell+1}, b_{\ell+1}, c_{\ell+1}$ and $d_{\ell+1}$ such that $T'_{\ell+1}(s) = 0$ has exactly 0, 1 or 2 roots for $s \in \mathcal{I}$ (recall Remark 3.7). This completes the proof of the result. ■

4 Proof of Theorem B

Recall that we consider the unfolding $X_\varepsilon = X_0 + Y_\varepsilon$, where X_0 is the vector field associated to each one of the isochrones in (4) and Y_ε is the perturbation in (5). Note first that, by means of the transformation

$(x, y, t) \mapsto (\eta x, \eta y, \eta t)$ with $\eta = 1 + b(\varepsilon)$, there is no loss of generality in assuming $b \equiv 0$, i.e., that the unfolding is given by

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + (d_0 + d(\varepsilon))x^2 + (f_0 + f(\varepsilon))y^2, \end{cases} \quad (9)$$

where d and f are analytic functions vanishing at $\varepsilon = 0$, and the value of (d_0, f_0) is $(-\frac{1}{2}, \frac{1}{2})$, $(0, 1)$, $(1, \frac{1}{4})$ and $(-\frac{1}{2}, 2)$ in case that we perturb the isochronous center (S_1) , (S_2) , (S_3) and (S_4) , respectively.

In order to show Theorem B we must first take, for each isochronous center, a commutator U_0 of X_0 defined in the whole period annulus. For instance we can use the ones in [5, 21] but, as it will be clear in a moment, we do not need their concrete expression. Then, as we did in the previous section for the Pleshkan's isochrones, we decompose the perturbation as $X_\varepsilon - X_0 = \lambda X_0 + \mu U_0$. The next result is the counterpart of Lemma 3.14 and it follows after translating the results in [10] to the language that we use here.

Lemma 4.1 (Gasull-Yu). *Setting $\lambda = \sum_{i=1}^{\infty} \lambda_i \varepsilon^i$, define*

$$R(s) = \int_0^{T_0} U_0(\lambda_k)|_{(x,y)=\varphi(t;s)} dt,$$

where $\varphi(t; s)$ is the solution of X_0 such that $\varphi(0; s) = \xi(s)$ and $\xi: \mathcal{I} \rightarrow \mathbb{R}^2$ is a fixed solution of U_0 . Then

$$R(s) = d_k I_0(s) + f_k I_1(s) \text{ for all } s \in \mathcal{I},$$

where:

- (a) (I_0, I_1) is an ECT-system on \mathcal{I} in case that $(d_0, f_0) \in \{(0, 1), (0, \frac{1}{4}), (-\frac{1}{2}, 2)\}$.
- (b) $I_0(s) = I_1(s) \neq 0$ for all $s \in \mathcal{I}$ in case that $(d_0, f_0) = (-\frac{1}{2}, \frac{1}{2})$.

This result follows from the proof of Theorem 3 in [10], but it is worth making some comments. Recall that to parameterize the periodic orbits we take a commutator U_0 of X_0 , i.e., such that $[X_0, U_0] = 0$, analytic in $\mathcal{P} \cup \{(0, 0)\}$. This provides a parameterization of the period function of X_ε , say $T(s; \varepsilon) = 2\pi + T_1(s)\varepsilon + o(\varepsilon)$, for which

$$T'_1(s) = \int_0^{T_0} U_0(\lambda_1)|_{(x,y)=\varphi(t;s)} dt.$$

(This expression is well known but it can be viewed as the case $k = 0$ of Theorem 3.2.) Instead of a commutator, the authors in [10] use a vector field \widehat{U}_0 such that $[X_0, \widehat{U}_0] = \beta \widehat{U}_0$ for some function β . This yields of course to another parameterization of the period function of X_ε , say $\widehat{T}(s; \varepsilon) = 2\pi + \widehat{T}_1(s)\varepsilon + o(\varepsilon)$. The expression of $\widehat{T}'_1(s)$ is slightly different from the previous one, but we still can take advantage of their result. Indeed, it is clear that there exists a diffeomorphism ζ verifying $T(s; \varepsilon) = \widehat{T}(\zeta(s); \varepsilon)$, so that $T'_1(s) = \zeta'(s)\widehat{T}'_1(\zeta(s))$, see Remark 2.2. The proof of Theorem 3 in [10] shows that $\widehat{T}'_1(s) = d_1 \widehat{I}_0(s) + f_1 \widehat{I}_1(s)$, where \widehat{I}_0 and \widehat{I}_1 are functions verifying, for each case, statements (a) and (b) in Lemma 4.1. This is the key point because then

$$\int_0^{T_0} U_0(\lambda_1)|_{(x,y)=\varphi(t;s)} dt = d_1 I_0(s) + f_1 I_1(s), \text{ with } I_i(s) := \zeta'(s)\widehat{I}_i(\zeta(s)),$$

and the result follows due to $\zeta'(s) \neq 0$ for all s . (To be more precise, this proves the case $k = 1$, however the subindex does not play any role at all because $\lambda_k(x, y) = F(d_k, f_k, x, y)$ with F not depending on k .)

Let us note that analogous computations to the ones carried out in the previous section and the application of the criterion given in [12] also yield to the proof of Lemma 4.1. For the sake of shortness we prefer to take advantage of the results in [10] instead.

Proof of Theorem B Let us fix any $(d_0, f_0) \in \{(0, 1), (0, \frac{1}{4}), (-\frac{1}{2}, 2)\}$ and consider the unfolding X_ε in (9) of the isochronous center X_0 . Let us also take the corresponding commutator U_0 of X_0 and, as usual, denote the period function of X_ε using a solution of U_0 as transversal section by $T(s; \varepsilon)$. Finally, let Φ be a linearization of X_0 .

We claim that if $T'_0 \equiv T'_1 \equiv \dots \equiv T'_k \equiv 0$, then $j^k(X_\varepsilon) = X_0$. The proof follows by induction on k . The case $k = 0$ is trivial because $j^0(X_\varepsilon) = X_0$. Assume that the claim is true for $k = n$ and let us show its validity for $k = n + 1$. So suppose that $T'_0 \equiv T'_1 \equiv \dots \equiv T'_{n+1} \equiv 0$ and, by the induction hypothesis, that $j^n(X_\varepsilon) = X_0$. Then Φ is a linearization of $j^k(X_\varepsilon) = X_0$ for $k = 0, 1, \dots, n$ and, by applying $n + 1$ times Theorem 3.2, we have that

$$T'_k(s) = - \int_0^{T_0} U_0(\lambda_k)|_{(x,y)=\varphi(t;s)} dt \text{ for } k = 1, 2, \dots, n + 1.$$

Hence, thanks to (a) in Lemma 4.1, $T'_1 \equiv T'_2 \equiv \dots \equiv T'_{n+1} \equiv 0$ implies $d_k = f_k = 0$ for $k = 1, 2, \dots, n + 1$, so that $j^{n+1}(X_\varepsilon) = X_0$. This proves the claim.

It is clear that the result under consideration only makes sense in case that the perturbation is not isochronous, so there exists some $\ell \geq 0$ such that $T'_0 \equiv T'_1 \equiv \dots \equiv T'_\ell \equiv 0$ and $T'_{\ell+1} \not\equiv 0$. Then, on account of the claim, Φ is a linearization of $j^\ell(X_\varepsilon) = X_0$ and, by applying Theorem 3.2,

$$T'_{\ell+1}(s) = - \int_0^{T_0} U_0(\lambda_{\ell+1})|_{(x,y)=\varphi(t;s)} dt \text{ for all } s \in \mathcal{I}.$$

Thus, from Lemma 3.14, $T'_{\ell+1}(s) = d_{\ell+1}I_0(s) + f_{\ell+1}I_1(s)$ where (I_0, I_1) is an ECT-system on \mathcal{I} . Consequently, since $T'_{\ell+1}$ is not identically zero, it can have at most one zero for $s \in \mathcal{I}$. Moreover, on account of Remark 3.7, one can choose $d_{\ell+1}$ and $f_{\ell+1}$ such that $T'_{\ell+1}(s) = 0$ has exactly 0 or 1 root for $s \in \mathcal{I}$. This completes the proof of the result. \blacksquare

We conclude the paper with some final remarks about how Theorem B and Figure 1 fit together. This figure gathers some of the results and conjectures that appear in [19] on the bifurcation diagram of the period function of the so-called dehomogenized Loud's systems

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases}$$

It may seem surprising that, thanks to Theorem 3.2, it is possible to obtain a result for arbitrary perturbations by taking advantage of a result, namely the one in [10], that holds only for perturbations that are linear in ε . Roughly speaking, this shows that the linear perturbations already give all the critical periods that you can get with an arbitrary perturbation. Certainly this is not a general property but it explains why we could not obtain the result for the perturbation of the isochronous center $(d_0, f_0) = (-\frac{1}{2}, \frac{1}{2})$, not even for the linear ones. Indeed, a linear perturbation corresponds in the parameter plane to move slightly travelling on a straight line that passes through the isochronous center. In addition if you can get $k \geq 0$ critical periods moving in one direction, then you also get k in the opposite one (changing the signum of ε). Clearly, see Figure 1, this is what it happens with the isochronous centers $(d_0, f_0) \in \{(0, 1), (-\frac{1}{2}, 2), (0, \frac{1}{4})\}$ but not with $(d_0, f_0) = (-\frac{1}{2}, \frac{1}{2})$. Let us note moreover that if the conjectural diagram in Figure 1 is true, then the bifurcation of $(d_0, f_0) = (0, \frac{1}{4})$ giving rise to a critical period arising from \mathcal{P} also causes the emergence of another one from $\partial\mathcal{P}$ at the same time. The bifurcation of the latter critical period would be the counterpart, for the period function, of the bifurcation of an *alien limit cycle* as defined in [2]. (In that paper, the authors present a way to study the perturbations from a Hamiltonian 2-saddle cycle which can produce limit cycles that cannot be detected using zeroes of the Abelian integral, even when it is generic.) On account of Theorem B, this, let us say, *alien critical period* does not come from a zero of the Abelian integral related with the derivative of the period function.

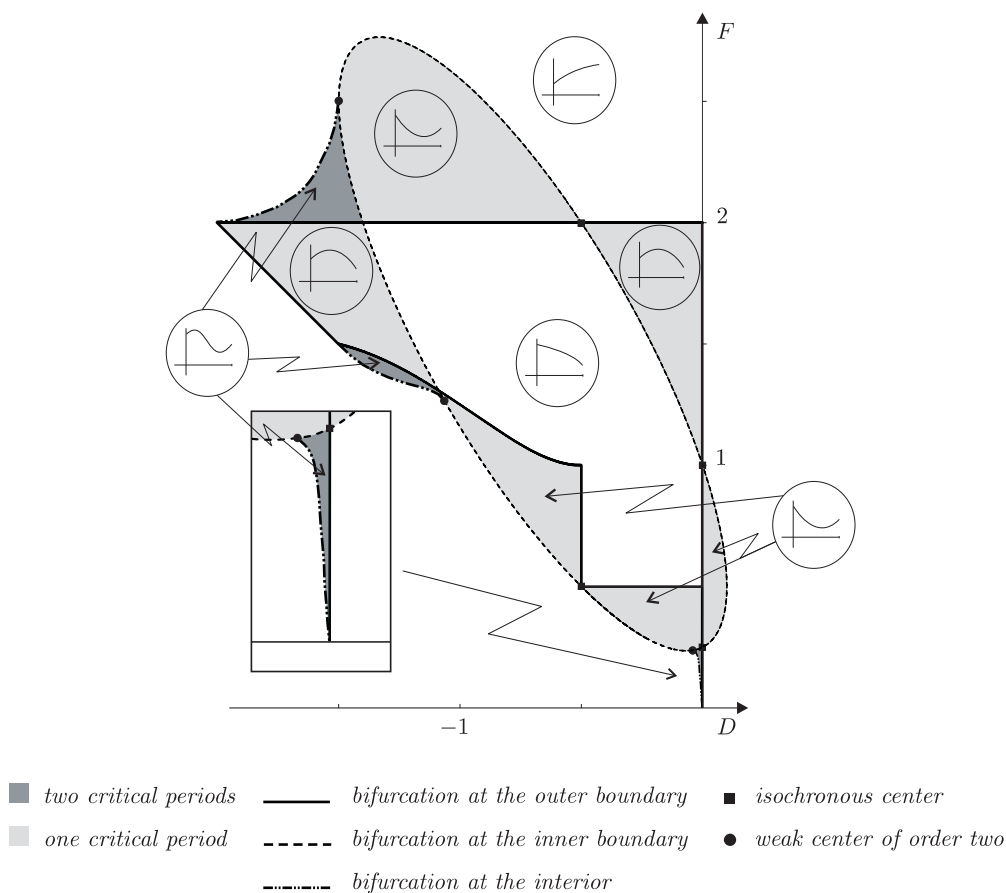


Figure 1: Conjectural bifurcation diagram of the period function of the dehomogenized Loud's systems.

On the other hand, Chicone and Jacobs prove in [3] that there are perturbations of each one of the four isochrones giving rise to exactly one critical period bifurcating from the inner boundary of \mathcal{P} . Theorem B shows that there are perturbations of the isochronous centers $(d_0, f_0) \in \{(0, 1), (-\frac{1}{2}, 2), (0, \frac{1}{4})\}$ giving rise to exactly one critical period bifurcating from \mathcal{P} . These two facts do not contradict the conjecture in Figure 1 near $(d_0, f_0) = (0, 1)$ and $(d_0, f_0) = (-\frac{1}{2}, 2)$, it simply shows that both perturbations are different.

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