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THE BIANCHI VIII MODEL IS NEITHER GLOBAL ANALYTIC, NOR DARBOUX INTEGRABLE

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Abstract. We consider the Bianchi VIII model. This model has been studied during these last years but very few is known up to now on its integrability. We show that the Bianchi VIII system has neither a global analytic first integral nor a Darboux first integral which is not a function of its Hamiltonian.

1. Introduction and statement of the main results

When cosmological models are modelized by differential systems through a set of ordinary differential equations they can be studied using the rich theory of finite dynamical systems. But the differential systems coming from cosmological models present many special features which distinguish them from the typical differential system.

The Bianchi relativistic homogeneous cosmological models are presented in four dimensional manifolds, three for the space and one for the time. Here we continue the study of the integrability or non-integrability of the Bianchi VIII models which can be formulated as a Hamiltonian system with three degrees of freedom with Hamiltonian

$$H = H(P_1, P_2, P_3, Q_1, Q_2, Q_3) = -P_1^2 Q_2^2 - Q_1^2(1 + P_2^2) - 2Q_1P_1(Q_2P_2 - Q_3P_3) + Q_2Q_3(2P_2P_3 - 1) - \frac{1}{4}Q_3^2(1 + 4P_3^2),$$

see for more details Maciejewski, Strelcyn and Szydlowski [5]. In this paper the authors proved that the Bianchi VIII model as a Hamiltonian system is not completely integrable with meromorphic first integrals, for a precise statement of their results see Theorem 2 of [5].

Doing convenient non-canonical changes of variables (see for more details [5]) the Hamiltonian system of the Bianchi VIII model can be written as

$$\begin{aligned}
\dot{y}_1 &= \frac{1}{2}(y_1z_2 + y_2z_1),\\
\dot{y}_2 &= \frac{1}{2}(y_1z_1 + y_2z_2),\\
\dot{y}_3 &= y_3z_3,\\
\dot{z}_1 &= -4y_1(2y_2 + y_3),\\
\dot{z}_2 &= 2y_3(2y_2 + y_3),\\
\dot{z}_3 &= 4y_1^2 - y_3^2,
\end{aligned} \tag{1}$$

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in $\mathbb{R}^6$. It has the first integral

$$\hat{H} = -4y_1^2 - y_3(4y_2 + y_3) + z_2z_3 + \frac{1}{4}(z_2^2 - z_1^2).$$

For convenience we do the linear change of variables

$$x_1 = y_1 - y_2, \quad x_2 = y_1 + y_2, \quad x_3 = y_3,$$

and hence

$$y_1 = (x_1 + x_2)/2, \quad y_2 = (x_2 - x_1)/2, \quad y_3 = x_3.$$

Then in these new variables system (1) becomes

$$\begin{aligned} \dot{x}_1 &= \frac{1}{2}(z_2 - z_1)x_1, \\ \dot{x}_2 &= \frac{1}{2}(z_2 + z_1)x_2, \\ \dot{x}_3 &= x_3z_3, \\ \dot{z}_1 &= 2(x_1 + x_2)(x_1 - x_2 - x_3), \\ \dot{z}_2 &= 2x_3(-x_1 + x_2 + x_3), \\ \dot{z}_3 &= (x_1 + x_2)^2 - x_3^2, \end{aligned} \tag{2}$$

in $\mathbb{R}^6$. It has the first integral

$$H = -(x_1 + x_2)^2 - x_3(2(x_2 - x_1) + x_3) + z_2z_3 + \frac{1}{4}(z_2^2 - z_1^2). \tag{3}$$

The change of variables has been done in order that the hyperplanes $x_i = 0$ for $i = 1, 2, 3$ become invariant by the flow of the system.

Assume that U is an open and dense subset of $\mathbb{R}^6$, $H : U \to \mathbb{R}$ is a nonconstant function at less of class C^1, and $\mathcal{X}$ is the polynomial vector field in $\mathbb{R}^6$ associated to system (2). Then H is a *first integral* of $\mathcal{X}$ in U if H is constant on the solutions of $\mathcal{X}$ contained in U; i.e. if $\mathcal{X}H = 0$ in U. Additionally, when H is an analyic function we say that H is an *analytic first integral*. For instance if H is a polynomial, then H is an analytic first integral. These last first integrals are called *polynomial* first integrals. If H is a formal power series satisfying $\mathcal{X}H = 0$, then H is called *a formal first integral*. If now we choose U as a neighborhood of a singular point p of $\mathcal{X}$ and $H : U \to \mathbb{R}$ is an analytic first integral in U, then H is called a *local analytic* first integral of $\mathcal{X}$ at p.

The following well-known proposition (easy to prove) reduces the study of the existence of formal and analytic first integrals of the homogeneous polynomial differential system (2) of degree 2 to the study of the existence of homogeneous polynomial first integrals.

Proposition 1. *Let F be either a formal series, or an analytic function, or a polynomial function in the variables x_1, x_2, x_3, z_1, z_2 and z_3 and let $F = \sum_i F_i$ be its decomposition into homogeneous polynomials of degree i. Then F is a formal, analytic or polynomial first integral of system* (2) *if and only if each F_i is a first integral of system* (2) *for all i.*

Our main result about formal, analytic and polynomial first integrals of the homogeneous polynomial differential system (2) is the following.

Theorem 2. *The unique homogeneous polynomial first integrals of the Bianchi VIII system* (2) *are* H^m *for some positive integer number* m *where* H *is given by* (3)*.*

Another different class of functions from the analytic ones but with intersection with them is the class of the Darboux functions. The study of the Darboux first integrals is a classical problem of the integrability theory of the polynomial differential equations which goes back to Darboux [2]. Inside the class of Darboux first integrals there are the subclasses of the polynomial and rational first integrals, see for more details [3]. Since the Bianchi VIII system (2) is a polynomial differential system its Darboux first integrals can be studied.

Now we shall recall the main ingredients of the Darboux theory of integrability. A *Darboux polynomial* of system (2) is a polynomial $f \in \mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3] \backslash \mathbb{C}$ such that

(4)
$$\frac{1}{2}(z_2 - z_1)x_1\frac{\partial f}{\partial x_1} + \frac{1}{2}(z_2 + z_1)x_2\frac{\partial f}{\partial x_2} + x_3 z_3\frac{\partial f}{\partial x_3} + 2(x_1 + x_2)(x_1 - x_2 - x_3)\frac{\partial f}{\partial z_1} + 2x_3(-x_1 + x_2 + x_3)\frac{\partial f}{\partial z_2} + ((x_1 + x_2)^2 - x_3^2)\frac{\partial f}{\partial z_3} = Kf,$$

where K is a polynomial of degree at most 1, called the *cofactor* of f. Note that $f = 0$ is an *invariant algebraic hypersurface* for the flow of system (2). Again it is easy to prove the next result.

Proposition 3. *Let* f *be a polynomial function in the variables* x_1, x_2, x_3, z_1, z_2 *and* z_3 *and let* $f = \sum_i f_i$ *be its decomposition into homogeneous polynomials of degree* i*. Then* f *is a Darboux polynomial of system* (2) *if and only if each* f_i *is a Darboux polynomial of system* (2) *for all* i *with a homogeneous polynomial of degree one as cofactor.*

From Proposition 3 we can restrict our attention to homogeneous Darboux polynomials $f = f(x_1, x_2, x_3, z_1, z_2, z_3)$ having cofactor

$$K = a_1x_1 + a_2x_2 + a_3x_3 + a_4z_1 + a_5z_2 + a_6z_3. \tag{5}$$

We note that for real polynomial differential systems, as system (2), when we look for their Darboux polynomials we are also looking for their complex Darboux polynomials and not only for their real ones. This is due to the fact that their complex Darboux polynomials for real polynomial differential sysetms appear in pairs (one and its conjugate). The fact that they appear in pairs forces that the Darboux first integral that we get using the Darboux theory of integrability will be real. So the complex Darboux polynomials are also necessary for obtaining the real first integrals. For more details about this see [3].

The Darboux polynomial or equivalently the invariant algebraic hypersurfaces are important because a sufficient number of them forces the existence of a first integral. This result is the basis of the Darboux theory of integrability, see [3] and [4]. As we will see in Section 2 (see Proposition 8) for studying the existence of Darboux polynomials we can reduce to study the irreducible Darboux polynomials.

Proposition 4. *The Bianchi VIII system* (2) *has no irreducible homogeneous Darboux polynomials different from* x_1*,* x_2 *and* x_3*.*

An *exponential factor* F of the polynomial differential system (2) is a function $F = \exp(f/g) \notin \mathbb{C}$ with $f, g \in \mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3]$ satisfying that

$$\frac{1}{2}(z_2 - z_1)x_1\frac{\partial F}{\partial x_1} + \frac{1}{2}(z_2 + z_1)x_2\frac{\partial F}{\partial x_2} + x_3z_3\frac{\partial F}{\partial x_3} + 2(x_1 + x_2)(x_1 - x_2 - x_3)\frac{\partial F}{\partial z_1} + 2x_3(-x_1 + x_2 + x_3)\frac{\partial F}{\partial z_2} + ((x_1 + x_2)^2 - x_3^2)\frac{\partial F}{\partial z_3} = LF,$$

for some polynomial $L \in \mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3]$ of degree at most one. For the same reason that we work with the complex Darboux polynomials we also work with the complex exponential factors.

The existence of exponential factors $\exp(f/g)$ is due to the fact that the multiplicity of the invariant algebraic surface $g = 0$ is larger than 1. If g is a non-zero constant then the existence of exponential factors is due to the multiplicity of the plane at infinity, for more details see [1] and [4]. In particular if $\exp(f/g)$ is an exponential factor, then g is a Darboux polynomial. The exponential factors contributes as the Darboux polynomials to the Darboux theory of integrability.

A first integral G of system (2) is called *Darboux*, if G is of the form

$$G = f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q},$$

where $f_1, \dots, f_p$ are Darboux polynomials and $F_1, \dots, F_q$ are exponential factors of system (2), and $\lambda_j, \mu_k \in \mathbb{C}$, for $j = 1, \dots, p$, $k = 1, \dots, q$.

Our main result on Daboux first integrals is the next result.

Theorem 5. *The Bianchi VIII system* (2) *has no Darboux first integrals except the ones which can be written as* $H^\lambda \exp(P(H)/H^m)^\mu$ *where* H *is given by* (3)*,* m *is some positive integer number and* λ *and* μ *are complex numbers.*

The proof of Theorem 2 is given in Section 3. The proof of Proposition 4 is given in Section 4, and the proof of Theorem 5 in Section 5. In Section 2 we provide some preliminary results that will be used through all the paper.

2. Preliminary results

Lemma 6. *Let* x *and* y *be one–dimensional variables. Given a polynomial* $f(x)$*, there exists a polynomial* $g(x, y)$ *such that*

$$f(x) + f(y) = f(x + y) + f(0) - xy\, g(x, y).$$

Proof. Writing $f(x+y) = \sum_{i=0}^{n} f_i(x+y)^i$ and using the binomial's formula for every $(x + y)^i$, the proof follows easily. □

Lemma 7. *Let* x_k *be one–dimensional variables for* $k = 1, \dots, n$ *with* $n > 1$*. Let* $f = f(x_1, \dots, x_n)$ *be a polynomial and let* $\overline{f} = f(x_1, \dots, x_n)|_{x_l = c_0}$*, where* c_0 *is a constant. Then there exists a polynomial* $g = g(x_1, \dots, x_n)$ *such that* $f = \overline{f} + (x_l - c_0)g$*.*

Proof. Do the expansion in Taylor series of the function $f(x_1, \dots, x_n)$ in the variable x_l in a neighborhood of $x_l = c_0$. □

Proposition 8. *Let* f *be a polynomial and* $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ *its decomposition into irreducible factors in* $\mathbb{C}[x_1, \dots, x_n]$*. Then,* f *is a Darboux polynomial if and only if* *all the* f_j *are Darboux polynomials. Moreover, if* K *and* K_j *are the cofactors of* f *and* f_j*, then* $K = \sum_{j=1}^{s} \alpha_j K_j$.

Theorem 9. *Suppose that a polynomial differential system defined in* $\mathbb{R}^n$ *of degree* m *admits* p *invariant algebraic hypersurfaces* $f_i = 0$ *with cofactors* K_i *for* $i = 1, \ldots, p$ *and* q *exponential factors* $F_j = \exp(g_j/h_j)$ *with cofactors* L_j *for* $j = 1, \ldots, q$*. Then, there exist* λ_j*,* $\mu_j \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ *if and only if the following real (multi–valued) function of Darboux type*

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\mu_1} \ldots F_q^{\mu_q},$$

substituting $f_i^{\lambda_i}$ *by* $|f_i|^{\lambda_i}$ *if* $\lambda_i \in \mathbb{R}$*, is a first integral.*

Proposition 10. *System* (2) *has exactly three irreducible Darboux polynomials of degree* 1*, namely* x_1*,* x_2 *and* x_3*.*

Proof. It follows easily by direct computation from the definition of Darboux polynomial. □

Let $\tau\colon \mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3] \to \mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3]$ be the automorphism defined by

$$\tau(x_i) = -x_i, \quad \tau(z_i) = z_i, \quad i = 1, 2, 3;$$

and let $\sigma\colon \mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3] \to \mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3]$ be the automorphism defined by

$$\sigma(x_1) = x_2,\ \sigma(x_2) = x_1,\ \sigma(x_3) = -x_3,\ \sigma(z_1) = -z_1,\ \sigma(z_i) = z_i,\ i = 2, 3.$$

Proposition 11. *If* g *is an irreducible homogeneous Darboux polynomial of degree* > 1 *for system* (2) *with cofactor* K *given by* (5)*, then* $f = (g \cdot \sigma(g)) \cdot \tau(g \cdot \sigma(g))$ *is a homogeneous Darboux polynomial invariant by* τ *and* σ *with a cofactor of the form* $4a_5 z_2 + 4a_6 z_3$*. Moreover* x_i *is not a factor* f *for* $i = 1, 2, 3$*.*

Proof. Since system (2) is invariant under τ, τg is a Darboux polynomial of system (2) with cofactor $\tau(K)$ and σg is a Darboux polynomial of system (2) with cofactor $\sigma(K)$. Therefore, by Proposition 8, the cofactor of $g \cdot \sigma(g)$ is $(a_1 + a_2)(x_1 + x_2) + 2a_5 z_2 + 2a_6 z_3$ and then the cofactor of f is $4a_5 z_2 + 4a_6 z_3$. Finally, x_i is not a factor f for $i = 1, 2, 3$, otherwise some x_i would be a factor of g, and g is irreducible of degree > 1. □

3. Homogeneous polynomial first integrals

Let $h = h(x_1, x_2, x_3, z_1, z_2, z_3)$ be a homogeneous polynomial first integral of system (2). Without loss of generality we can assume that h is invariant by τ and σ since otherwise, making $f = (h \cdot \sigma(h)) \cdot \tau(h \cdot \sigma(h))$ we get that f is a homogeneous polynomial first integral invariant by τ and σ. We first prove some auxiliary results.

Lemma 12. *Let* $u = u(x_1, x_2, z_1, z_2, z_3)$ *be a homogeneous polynomial satisfying*

$$\begin{aligned} &\frac{1}{2}(z_2 - z_1)x_1 \frac{\partial u}{\partial x_1} + \frac{1}{2}(z_2 + z_1)x_2 \frac{\partial u}{\partial x_2} + 2(x_1 + x_2)(x_1 - x_2)\frac{\partial u}{\partial z_1} \\ &+ (x_1 + x_2)^2 \frac{\partial u}{\partial z_3} = -\Big(a(z_2 - z_1) + b z_3\Big)u, \end{aligned} \tag{6}$$

with $(a,b)\in\mathbb{C}^2\setminus\{(0,0)\}$. *Then* $u=0$.

Proof. We assume $u\neq 0$ and we will reach to a contradiction. We consider two different cases.

Case 1: u is not divisible by x_1. Then we have that if we denote by u_0 the restriction of u to $x_1=0$, that is, $u_0=u(0,x_2,z_1,z_2,z_3)$ then we have that $u_0\neq 0$. In this case u_0 satisfies (6) restricted to $x_1=0$, that is,

$$\frac{1}{2}(z_2+z_1)x_2\frac{\partial u_0}{\partial x_2}-2x_2^2\frac{\partial u_0}{\partial z_1}+x_2^2\frac{\partial u_0}{\partial z_3}=-\Big(a(z_2-z_1)+bz_3\Big)u_0. \tag{7}$$

Hence from (7) we have that u_0 must be divisible by x_2. We write $u_0=x_2^m h$ where $m\geq 1$, $h\neq 0$ and h is not divisible by x_2. From (7) we obtain that h satisfies

$$\frac{1}{2}(z_2+z_1)x_2\frac{\partial h}{\partial x_2}-2x_2^2\frac{\partial h}{\partial z_1}+x_2^2\frac{\partial h}{\partial z_3}=-\Big(a(z_2-z_1)+bz_3+\frac{m}{2}(z_2+z_1)\Big)h.$$

Hence h must be divisible by x_2, a contradiction. Thus, Case 1 is not possible.

Case 2: u is divisible by x_1. In this case we write $u=x_1^m h$ where $m\geq 1$, $h\neq 0$ and h is not divisible by x_1. Clearly h satisfies

$$\begin{aligned}&\frac{1}{2}(z_2-z_1)x_1\frac{\partial h}{\partial x_1}+\frac{1}{2}(z_2+z_1)x_2\frac{\partial h}{\partial x_2}+2(x_1+x_2)(x_1-x_2)\frac{\partial h}{\partial z_1}+(x_1+x_2)^2\frac{\partial h}{\partial z_3}\\&=-\Big(\frac{2a+m}{2}(z_2-z_1)+bz_3\Big)h.\end{aligned} \tag{8}$$

We write $h_0=h(0,x_2,z_1,z_2,z_3)$, that is, h_0 satisfies (8) restricted to $x_1=0$, that is

$$\frac{1}{2}(z_2+z_1)x_2\frac{\partial h}{\partial x_2}-2x_2^2\frac{\partial h_0}{\partial z_1}+x_2^2\frac{\partial h_0}{\partial z_3}=-\Big(\frac{2a+m}{2}(z_2-z_1)+bz_3\Big)h_0. \tag{9}$$

Then h_0 must be divisible by x_2. We can write $h_0=x_2^l\bar{h}$, where $l\geq 1$, $\bar{h}\neq 0$ and $\bar{h}$ is not divisible by x_2. Clearly, $\bar{h}$ satisfies

$$\frac{1}{2}(z_2+z_1)x_2\frac{\partial \bar{h}}{\partial x_2}-2x_2^2\frac{\partial \bar{h}}{\partial z_1}+x_2^2\frac{\partial \bar{h}}{\partial z_3}=-\Big(\frac{2a+m}{2}(z_2-z_1)+bz_3+\frac{l}{2}(z_2+z_1)\Big)\bar{h}. \tag{10}$$

It follows from (10) that $\bar{h}$ must be divisible by x_2, a contradiction. Thus, Case 2 is not possible. This concludes the proof of the lemma. □

Lemma 13. *Let* $\bar{f}=f(x_1,x_2,0,z_1,z_2,z_3)$ *be a homogeneous polynomial first integral of system* (2) *restricted to* $x_3=0$ *that is invariant by* τ *and* σ. *Then* $\bar{f}=\bar{f}(z_2,\bar{H})$, *where* $\bar{H}=H|_{x_3=0}=-(x_1+x_2)^2+z_2z_3+\frac{1}{4}(z_2^2-z_1^2)$.

Proof. We consider system (2) restricted to $x_3=0$, that is,

$$\begin{aligned}\dot{x}_1&=\frac{1}{2}(z_2-z_1)x_1,\\ \dot{x}_2&=\frac{1}{2}(z_2+z_1)x_2,\\ \dot{z}_1&=2(x_1+x_2)(x_1-x_2),\\ \dot{z}_2&=0,\\ \dot{z}_3&=(x_1+x_2)^2.\end{aligned} \tag{11}$$

Let $\bar{f}$ be a homogeneous polynomial first integral of system (11) and let $\bar{f}_1$ be the restriction of $\bar{f}$ to $x_1 = 0$. Then $\bar{f}_1$ is a first integral of system (11) restricted to $x_1 = 0$, that is, of system

$$
\begin{aligned}
\dot{x}_2 &= \frac{1}{2}(z_2 + z_1)x_2, \\
\dot{z}_1 &= -2x_2^2, \\
\dot{z}_2 &= 0, \\
\dot{z}_3 &= x_2^2.
\end{aligned}
\tag{12}
$$

Clearly if we denote by $\bar{H}_1$ the restriction of $\bar{H}$ to $x_1 = 0$ we get that $\bar{H}_1 = -x_2^2 + z_2z_3 + (z_2^2 - z_1^2)/4$ and is a first integral of system (12). In addition we have that $g = 2z_3 + z_1$ is also a first integral of system (12). Since $z_2, \bar{H}_1$ and g are independent polynomial first integrals, we have that

$$\bar{f}_1 = \bar{f}_1(z_2, \bar{H}_1, g),$$

where $\bar{f}_1$ is a polynomial first integral. Therefore in view of Lemmas 7 and 6 (note that $\bar{H} = \bar{H}_1 - x_1(2x_2 + x_1)$) we get

$$\bar{f} = \bar{f}_1(z_2, \bar{H}_1, g) + x_1g_1 = \bar{f}_1(z_2, \bar{H}, g) + x_1g_2. \tag{13}$$

Since $\bar{f}$ must be invariant by τ and σ and $g = 2z_3 + z_1$, we get that

$$\bar{f}_1(z_2, \bar{H}, 2z_3 + z_1) + x_1g_2 = \bar{f} = \sigma(\bar{f}) = \bar{f}_1(z_2, \bar{H}, 2z_3 - z_1) + x_2\sigma(g_2),$$

which is impossible unless $\bar{f}_1 = \bar{f}_1(z_2, \bar{H})$. Hence it follows from (13) that

$$\bar{f} = \bar{f}_1(z_2, \bar{H}) + x_1g_2. \tag{14}$$

Then, imposing that $\bar{f}$, z_2 and $\bar{H}$ are first integrals of system (11), we get that g_2 satisfies (6) with $a = 1/2$ and $b = 0$. From Lemma 12 we get that $g_2 = 0$. The lemma follows now from (14). □

Lemma 14. *Let h be a homogeneous polynomial first integral of system* (2) *invariant by τ and σ. Then $h = H^m + x_3g$ for some non-negative integer m and with $g = g(x_1, x_2, x_3, z_1, z_2, z_3)$ a homogeneous polynomial of degree $2m - 1$.*

Proof. In view of Lemmas 7, 6 and 13 (note that $H = \bar{H} + x_3(2(x_1 - x_2) + x_3)$) we can write h as

$$h = \bar{h}(z_2, \bar{H}) + x_3\bar{g} = \bar{h}(z_2, H) + x_3g, \tag{15}$$

where $g = g(x_1, x_2, x_3, z_1, z_2, z_3)$ is a homogeneous polynomial in its variables. We denote by r the degree of $\bar{h}$ and we write

$$\bar{h}(z_2, H) = \sum_{l=0}^{[r/2]} \bar{h}_l z_2^{r-2l} H^l.$$

Using that h and H are first integrals of system (2), after removing the common factor x_3 we have

$$
\begin{aligned}
&2(-x_1+x_2+x_3)\sum_{l=0}^{[r/2]}\bar{h}_l(r-2l)z_2^{r-2l-1}H^l+z_3g+\frac{1}{2}(z_2-z_1)x_1\frac{\partial g}{\partial x_1}\\
&+\frac{1}{2}(z_2+z_1)x_2\frac{\partial g}{\partial x_2}+x_3z_3\frac{\partial g}{\partial x_3}+2(x_1+x_2)(x_1-x_2-x_3)\frac{\partial g}{\partial z_1}\\
&+2x_3(-x_1+x_2+x_3)\frac{\partial g}{\partial z_2}+((x_1+x_2)^2-x_3^2)\frac{\partial g}{\partial z_3}=0.
\end{aligned}
\tag{16}
$$

Evaluating (16) on $x_1=x_2=z_3=0$, and after removing the common factor x_3 we get

$$
2\sum_{l=0}^{[r/2]}\bar{h}_l(r-2l)z_2^{r-2l-1}\Big(-x_3^2+\frac{1}{4}(z_2^2-z_1^2)\Big)^l=-x_3\left(2\frac{\partial g}{\partial z_2}-\frac{\partial g}{\partial z_3}\right)\Big|_{x_1=x_2=z_3=0}.
\tag{17}
$$

Evaluating (17) on $x_3=0$ we get

$$
\sum_{l=0}^{[r/2]}\bar{h}_l(r-2l)z_2^{r-2l-1}\Big(\frac{1}{4}(z_2^2-z_1^2)\Big)^l=0.
\tag{18}
$$

If r is odd, then computing the different powers of z_1^2 in (18) we get that $\bar{h}_l=0$ for $l=0,\dots,[r/2]$ and then $\bar{h}=0$. If r is even then computing the different powers of z_1^2 in (18) we get that $\bar{h}_l=0$ for $l=0,\dots,r/2-1$ and then $\bar{h}=\bar{h}_{r/2}H^{r/2}$. Without loss of generality we can assume that $\bar{h}_{r/2}=1$. This completes the proof of the lemma. □

Proof of Theorem 2. Let h be a homogeneous polynomial first integral of system (2). It follows from Lemma 14 that

$$
h=H^m+x_3g,\quad \text{with}\quad g=g(x_1,x_2,x_3,z_1,z_2,z_3)
$$

a homogeneous polynomial of degree $2m-1$. Then since h and H are first integrals of system (2) we have that g satisfies

$$
\begin{aligned}
&\frac{1}{2}(z_2-z_1)x_1\frac{\partial g}{\partial x_1}+\frac{1}{2}(z_2+z_1)x_2\frac{\partial g}{\partial x_2}+x_3z_3\frac{\partial g}{\partial x_3}+2(x_1+x_2)(x_1-x_2-x_3)\frac{\partial g}{\partial z_1}\\
&+2x_3(-x_1+x_2+x_3)\frac{\partial g}{\partial z_2}+((x_1+x_2)^2-x_3^2)\frac{\partial g}{\partial z_3}=-z_3g.
\end{aligned}
\tag{19}
$$

We claim that $g=0$. We note that if we prove the claim the theorem is proved. In short we are left with the proof that $g=0$. We proceed by contradiction. Assume that $g\neq 0$. We consider two different cases.

Case 1: g is not divisible by x_3. Then we have that if we denote by $g_0=g(x_1,x_2,0,z_1,z_2,z_3)$ the restriction of g to $x_3=0$, then $g_0\neq 0$ and it satisfies (6) with $a=0$ and $b=1$. Then using Lemma 12 with u replaced by g_0 we get that $g_0=0$ a contradiction. Hence this case is not possible.

Case 2: g is divisible by x_3. We write $g = x_3^j h$ where $j \geq 1$, $h \neq 0$ and h is not divisible by x_3. Furthermore h satisfies

(20)
$$\frac{1}{2}(z_2 - z_1)x_1\frac{\partial h}{\partial x_1} + \frac{1}{2}(z_2 + z_1)x_2\frac{\partial h}{\partial x_2} + x_3z_3\frac{\partial h}{\partial x_3} + 2(x_1 + x_2)(x_1 - x_2 - x_3)\frac{\partial h}{\partial z_1} + 2x_3(-x_1 + x_2 + x_3)\frac{\partial h}{\partial z_2} + ((x_1 + x_2)^2 - x_3^2)\frac{\partial h}{\partial z_3} = -(1 + j)z_3h.$$

Then if we denote by $h_0 = h(x_1, x_2, 0, z_1, z_2, z_3)$ we get that $h_0 \neq 0$ and satisfies (20) restricted to $x_3 = 0$. That is h_0 satisfies (6) with $a = 0$ and $b = 1 + j$. Therefore using Lemma 12 with u replaced by h_0 we get that $h_0 = 0$ a contradiction. Hence this case is not possible. This completes the proof of the theorem. □

4. Homogeneous Darboux polynomials with non–zero cofactor

The objective of this section is to study the homogeneous Darboux polynomials of system (2) with non–zero cofactor.

Proposition 15. *System* (2) *has no homogeneous Darboux polynomials invariant by* τ *and* σ *with cofactor* $K = a_5z_2 + a_6z_3$ *with* $(a_5, a_6) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *that are coprime with* x_1, x_2 *and* x_3.

Proof. Let f be a homogeneous Darboux polynomial invariant by τ and σ with cofactor $K = a_5z_2 + a_6z_3$, $(a_5, a_6) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ that is coprime with x_1, x_2 and x_3. By Proposition 10, the degree n of f satisfies that $n > 1$, and x_i is not a factor f for $i = 1, 2, 3$. Now the proof of the proposition will be completed if we reach a contradiction. We consider different cases.

Case 1: Either $a_6 \neq 0$, or $a_6 = 0$ and $a_5 \neq j/2$ for some $j \in \mathbb{N}$. In this case we write $f_0 = f_0(x_1, x_2, z_1, z_2, z_3) = f(x_1, x_2, 0, z_1, z_2, z_3)$. Since x_3 is not a factor of f, then $f_0 \neq 0$. We divide the proof in two different cases.

Case 1.1: f_0 is not divisible by x_1. In this case we write $f_{0,0} = f_0(0, x_2, z_1, z_2, z_3)$. Then the polynomial $f_{0,0}$ is a Darboux polynomial of system (2) restricted to $x_1 = x_3 = 0$, and thus it satisfies the equation

$$\frac{1}{2}(z_2 + z_1)x_2\frac{\partial f_{0,0}}{\partial x_2} - 2x_2^2\frac{\partial f_{0,0}}{\partial z_1} + x_2^2\frac{\partial f_{0,0}}{\partial z_3} = (a_5z_2 + a_6z_3)f_{0,0}. \qquad (21)$$

Hence $f_{0,0}$ is divisible by x_2. We write $f_{0,0} = x_2^m g$ where $1 \leq m \leq n$, $\deg(g) = n - m$, $g \neq 0$ and g is not divisible by x_2. We have that $m < n$; otherwise, $f_{0,0} = bx_2^n$ with $b \in \mathbb{C} \setminus \{0\}$, and from (21) we get a contradiction. From (21) and Proposition 8, we obtain that g is a Darboux polynomial of system (2) restricted to $x_1 = x_3 = 0$ satisfying

$$\frac{1}{2}(z_2 + z_1)x_2\frac{\partial g}{\partial x_2} - 2x_2^2\frac{\partial g}{\partial z_1} + x_2^2\frac{\partial g}{\partial z_3} = \Big(a_5z_2 + a_6z_3 - \frac{m}{2}(z_2 + z_1)\Big)g.$$

Therefore, since $m < n$ and either $a_6 \neq 0$, or $a_6 = 0$ and $a_5 \neq j/2$ for some $j \in \mathbb{N}$, then g is divisible by x_2, a contradiction. Thus, Case 1.1 is not possible.

Case 1.2: f_0 is divisible by x_1. In this case we write $f_0 = x_1^m g$ where $1 \leq m \leq n$, $\deg(g) = n - m$, $g \neq 0$ and g is not divisible by x_1. We have that $m < n$; otherwise, $f_0 = bx_1^m$ with $b \in \mathbb{C} \setminus \{0\}$ and from (4) with $x_3 = 0$ we obtain

$$\frac{m}{2}(z_2 - z_1)bx_1^m = (a_5z_2 + a_6z_3)bx_1^m,$$

a contradiction with the hypotheses of Case 1.

Since $m < n$, and f_0 and x_1 are Darboux polynomials of system (2) restricted to $x_3 = 0$, from Proposition 8, g is also a Darboux polynomial of system (2) restricted to $x_3 = 0$ satisfying

$$\frac{1}{2}(z_2 - z_1)x_1\frac{\partial g}{\partial x_1} + \frac{1}{2}(z_2 + z_1)x_2\frac{\partial g}{\partial x_2} + 2(x_1 + x_2)(x_1 - x_2)\frac{\partial g}{\partial z_1} + (x_1 + x_2)^2\frac{\partial g}{\partial z_3}$$
$$= \Big(a_5 z_2 + a_6 z_3 - \frac{m}{2}(z_2 - z_1)\Big)g.$$

We write $g_0 = g(0, x_2, z_1, z_2, z_3)$. Since g is not divisible by x_1 we have that $g_0 \neq 0$. Clearly g_0 is a Darboux polynomial of system (2) restricted to $x_1 = x_3 = 0$, and g_0 satisfies

$$\frac{1}{2}(z_2 + z_1)x_2\frac{\partial g_0}{\partial x_2} - 2x_2^2\frac{\partial g_0}{\partial z_1} + x_2^2\frac{\partial g_0}{\partial z_3} = \Big(a_5 z_2 + a_6 z_3 - \frac{m}{2}(z_2 - z_1)\Big)g_0. \tag{22}$$

Hence, g_0 is divisible by x_2 and we can write $g_0 = x_2^l h$, where $1 \leq l \leq n - m$, $h \neq 0$ and h is not divisible by x_2. We have that $l < n - m$; otherwise, $g_0 = bx_2^{n-m}$ with $b \in \mathbb{C} \setminus \{0\}$, and using (22) we get a contradiction with the assumptions of Case 1. Again from (22) we have that

$$\begin{aligned}&\frac{1}{2}(z_2 + z_1)x_2\frac{\partial h}{\partial x_2} - 2x_2^2\frac{\partial h}{\partial z_1} + x_2^2\frac{\partial h}{\partial z_3}\\ &= \Big(a_5 z_2 + a_6 z_3 - \frac{m}{2}(z_2 - z_1) - \frac{l}{2}(z_2 + z_1)\Big)h.\end{aligned} \tag{23}$$

Note that since either $a_6 \neq 0$, or $a_6 = 0$ and $a_5 \neq j/2$ for some $j \in \mathbb{N}$, we get from (23) that h must be divisible by x_2, a contradiction. Thus, Case 1.2 is not possible.

Case 2: $a_6 = 0$ and $a_5 = j/2$ for some $j \in \mathbb{N} \setminus \{0\}$. In this case we write $f_0 = f_0(x_1, x_3, z_1, z_2, z_3) = f(x_1, 0, x_3, z_1, z_2, z_3)$. Since x_2 is not a factor of f, then $f_0 \neq 0$. We divide the proof in two different cases.

Case 2.1: f_0 is not divisible by x_1. In this case we write $f_{0,0} = f_0(0, x_3, z_1, z_2, z_3)$. Then the polynomial $f_{0,0}$ is a Darboux polynomial of system (2) restricted to $x_1 = x_2 = 0$, and thus it satisfies the equation

$$x_3 z_3\frac{\partial f_{0,0}}{\partial x_3} + 2x_3^2\frac{\partial f_{0,0}}{\partial z_2} - x_3^2\frac{\partial f_{0,0}}{\partial z_3} = \frac{j}{2}z_2 f_{0,0}. \tag{24}$$

Hence $f_{0,0}$ is divisible by x_3. We write $f_{0,0} = x_3^m g$ where $1 \leq m \leq n$, $\deg(g) = n - m$, $g \neq 0$ and g is not divisible by x_3. We have that $m < n$; otherwise, $f_{0,0} = bx_3^n$ with $b \in \mathbb{C} \setminus \{0\}$, and from (24) we get a contradiction. From (24) we obtain that g is a Darboux polynomial of system (2) restricted to $x_1 = x_2 = 0$ satisfying

$$x_3 z_3\frac{\partial g}{\partial x_3} + 2x_3^2\frac{\partial g}{\partial z_2} - x_3^2\frac{\partial g}{\partial z_3} = \Big(\frac{j}{2}z_2 - m z_3\Big)g.$$

Therefore g must be divisible by x_3, a contradiction. Thus Case 2.1 is not possible.

Case 2.2: f_0 is divisible by x_1. In this case we write $f_0 = x_1^m g$ where $1 \leq m \leq n$, $\deg(g) = n - m$, $g \neq 0$ and g is not divisible by x_1. We have that $m < n$; otherwise, $f_0 = bx_1^m$ with $b \in \mathbb{C} \setminus \{0\}$ and

$$\frac{m}{2}(z_2 - z_1)bx_1^m = \frac{j}{2}z_2 bx_1^m,$$

a contradiction. Since $f_0 = x_1^m g$ with $m < n$ we have that g is also a Darboux polynomial of system (2) restricted to $x_2 = 0$ from (4) we get

$$\frac{1}{2}(z_2 - z_1)x_1\frac{\partial g}{\partial x_1} + x_3z_3\frac{\partial g}{\partial x_3} + 2x_1(x_1 - x_3)\frac{\partial g}{\partial z_1} + 2x_3(-x_1 + x_3)\frac{\partial g}{\partial z_2} + (x_1^2 - x_3^2)\frac{\partial g}{\partial z_3}$$
$$= \Big(\frac{j}{2}z_2 - \frac{m}{2}(z_2 - z_1)\Big)g.$$

We write $g_0 = g(0, x_3, z_1, z_2, z_3)$. Since g is not divisible by x_1 we have that $g_0 \neq 0$. Since g_0 is a Darboux polynomial of system (2) restricted to $x_1 = x_2 = 0$, and g_0 satisfies

$$x_3z_3\frac{\partial g_0}{\partial x_3} + 2x_3^2\frac{\partial g_0}{\partial z_2} - x_3^2\frac{\partial g_0}{\partial z_3} = \Big(\frac{j}{2}z_2 - \frac{m}{2}(z_2 - z_1)\Big)g_0. \tag{25}$$

Hence, g_0 is divisible by x_3 and we can write $g_0 = x_3^l h$, where $1 \leq l \leq n - m$, $h \neq 0$ and h is not divisible by x_3. We have that $l < n - m$; otherwise, $g_0 = bx_3^{n-m}$ with $b \in \mathbb{C} \setminus \{0\}$, and we get a contradiction using (25). Again from (25) we have that

$$x_3z_3\frac{\partial h}{\partial x_3} + 2x_3^2\frac{\partial h}{\partial z_2} - x_3^2\frac{\partial h}{\partial z_3} = \Big(\frac{j}{2}z_2 - \frac{m}{2}(z_2 - z_1) - lz_3\Big)h.$$

Therefore h must be divisible by x_3, a contradiction. Thus Case 2.2 is not possible. This completes the proof of the proposition. □

Proof of Proposition 4. By Proposition 10, if g is an irreducible homogeneous Darboux polynomial of degree 1, it must be x_1, x_2 or x_3. Now we assume that g is an irreducible homogeneous Darboux polynomial of degree $n > 1$ for system (2) with non–zero cofactor K of the form (5). Then, from Proposition 11, we can assume that $f = (g \cdot \sigma g) \cdot \tau(g \cdot \sigma g)$ is a homogeneous Darboux polynomial invariant by τ and σ, with degree $4n$ and non–zero cofactor of the form $4(a_5z_2 + a_6z_3)$ and such that x_i is not a factor f for $i = 1, 2, 3$. From Proposition 15, we get that $a_5 = a_6 = 0$, otherwise we have a contradiction. Hence f is a homogeneous polynomial first integral of system (2). By Theorem 2, $f = aH^{2n}$ with n even and $a \in \mathbb{C} \setminus \{0\}$. Therefore, from the definition of f and since H and g are irreducible and invariant for σ and τ, it follows that $g = bH^{n/2}$, in contradiction with the fact that the cofactor of g is not zero. □

5. Darboux first integrals

The equation defining an exponential factor $F = \exp(h/g)$ with cofactor L is

$$\dot{x}_1\frac{d}{dx_1}\frac{h}{g} + \dot{x}_2\frac{d}{dx_2}\frac{h}{g} + \dot{x}_3\frac{d}{dx_3}\frac{h}{g} + \dot{z}_1\frac{d}{dz_1}\frac{h}{g} + \dot{z}_2\frac{d}{dz_2}\frac{h}{g} + \dot{z}_3\frac{d}{dz_3}\frac{h}{g} = L, \tag{26}$$

where we have simplified the common factor F, and

$$L = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4z_1 + b_5z_2 + b_6z_3. \tag{27}$$

According to Theorem 2 and Propositions 4 and 8, if system (2) has exponential factors, they must be of the form $\exp(h/(x_1^{n_1}x_2^{n_2}x_3^{n_3}))$, $\exp(h/(x_1^{n_1}x_2^{n_2}x_3^{n_3}H^{n_4}))$, where h is a polynomial in $\mathbb{C}[x_1, x_2, x_3, z_1, z_2, z_3]$, and n_1, n_2, n_3, n_4 are non–negative integers.

We shall need the following auxiliary result.

Proposition 16. *The unique irreducible homogeneous Darboux polynomials with non–zero cofactor of system* (2) *restricted to* $x_1 = 0$ *are* x_2 *and* x_3. *Moreover, the unique irreducible homogeneous Darboux polynomials with non–zero cofactor of system* (2) *restricted to* $x_2 = 0$ *are* x_1 *and* x_3 *and the unique irreducible homogeneous Darboux polynomials with non–zero cofactor of system* (2) *restricted to* $x_3 = 0$ *are* x_1 *and* x_2.

Proof. The proof is analogous to the proof of Proposition 15. □

Proposition 17. *The unique exponential factors of system* (2) *are of the form* $\exp(P(H)/H^m)$, *where* P *is a polynomial in* $\mathbb{C}[H]$ *and* m *is a non-negative integer. Moreover such exponential factors have cofactor zero.*

Proof. We start showing that if system (2) has an exponential factor of the form $\exp(h/H^m)$, with m a non-negative integer, then h is a polynomial in H. Using that H is a polynomial first integral, applying (26) with $h/g = h/H^m$ we get

$$\dot{x}_1 \frac{dh}{dx_1} + \dot{x}_2 \frac{dh}{dx_2} + \dot{x}_3 \frac{dh}{dx_3} + \dot{z}_1 \frac{dh}{dz_1} + \dot{z}_2 \frac{dh}{dz_2} + \dot{z}_3 \frac{dh}{dz_3} = LH^m, \tag{28}$$

with L given by (27). Taking $x_1 = x_2 = x_3 = 0$ in (28), since $H_{|x_1=x_2=x_3=0} \neq 0$, it holds that $b_0 = b_4 = b_5 = b_6 = 0$. Restricting (28) to $x_1 = x_2 = z_3 = 0$, we get

$$x_3^2 \left(2\frac{\partial h}{\partial z_2} - \frac{\partial h}{\partial z_3} \right)\Big|_{x_1=x_2=z_3=0} = b_3 x_3 \overline{H}^m, \tag{29}$$

where $\overline{H} = -x_3^2 + (z_2^2 - z_1^2)/4$. From (29) we obtain that either $b_3 = 0$, or $\overline{H}^m$ must be divisible by x_3. This last case is impossible and then $b_3 = 0$. In a similar way restricting (28) to $x_1 = x_3 = 0$, $z_2 = -z_1$ we get $b_2 = 0$, and restricting (28) to $x_2 = x_3 = 0$ and $z_2 = z_1$ we get $b_1 = 0$. Thus we have $L = 0$. Therefore from (28) h is a polynomial first integral. From Theorem 2, h is a polynomial function of H.

Suppose now that $\exp(h/(x_1^{n_1} x_2^{n_2} x_3^{n_3} H^{n_4}))$ is an exponential factor of system (2), where n_1, n_2, n_3, n_4 are non–negative integers with at least one of them positive, and h is coprime with x_1, x_2, x_3 and H. Then h satisfies

$$\begin{aligned} &\dot{x}_1 \frac{\partial h}{dx_1} + \dot{x}_2 \frac{\partial h}{dx_2} + \dot{x}_3 \frac{\partial h}{dx_3} - \left(\frac{\dot{x}_1}{x_1} n_1 + \frac{\dot{x}_2}{x_2} n_2 + \frac{\dot{x}_3}{x_3} n_3 \right) h + \\ &\dot{z}_1 \frac{\partial h}{dz_1} + \dot{z}_2 \frac{\partial h}{dz_2} + \dot{z}_3 \frac{\partial h}{dz_3} = L x_1^{n_1} x_2^{n_2} x_3^{n_3} H^{n_4} \end{aligned} \tag{30}$$

where we have simplified by the common factor $\exp(h/(x_1^{n_1} x_2^{n_2} x_3^{n_3} H^{n_4}))$ and multiplied by $x_1^{n_1} x_2^{n_2} x_3^{n_3} H^{n_4}$. We consider three different situations.

Case 1: $n_1 > 0$. Taking $x_1 = 0$ in (30) and denoting by $\overline{h}$ the restriction of h to $x_1 = 0$, we conclude that $\overline{h}$ satisfies

$$\begin{aligned} &\frac{1}{2}(z_2 + z_1) x_2 \frac{\partial \overline{h}}{dx_2} + x_3 z_3 \frac{\partial \overline{h}}{dx_3} - 2x_2(x_2 + x_3) \frac{\partial \overline{h}}{dz_1} + 2x_3(x_2 + x_3) \frac{\partial \overline{h}}{dz_2} \\ &+ (x_2^2 - x_3^2) \frac{\partial \overline{h}}{dz_3} = \left(\frac{n_1}{2}(z_2 - z_1) + \frac{n_2}{2}(z_2 + z_1) + n_3 z_3 \right) \overline{h}. \end{aligned} \tag{31}$$

Since h is coprime with x_1, we have that $\overline{h} \neq 0$. Therefore, $\overline{h}$ is a Darboux polynomial of system (2) restricted to $x_1 = 0$. By Proposition 16 we get that $\overline{h} = c x_2^{m_2} x_3^{m_3}$ for some $c \in \mathbb{C}\setminus\{0\}$ where m_2 and m_3 are non-negative integers such that $m_2+m_3 \neq 0$. Then, from (31) we get

$$c\Big(\frac{m_2}{2}(z_2+z_1)+m_3z_3\Big)x_2^{m_2}x_3^{m_3} = c\Big(\frac{n_1}{2}(z_2-z_1)+\frac{n_2}{2}(z_2+z_1)+n_3z_3\Big)x_2^{m_2}x_3^{m_3}$$

and consequently

$$m_3 = n_3, \quad m_2 = n_1+n_2 \quad m_2 = n_2-n_1,$$

which implies $n_1 = 0$, a contradiction.

Case 2: $n_2 > 0$. Taking $x_2 = 0$ in (30) and denoting by $\overline{h}$ the restriction of h to $x_2 = 0$, we conclude that $\overline{h}$ satisfies

$$\begin{aligned}\frac{1}{2}(z_2-z_1)x_1\frac{\partial\overline{h}}{dx_1}+x_3z_3\frac{\partial\overline{h}}{dx_3}-2x_1(x_1-x_3)\frac{\partial\overline{h}}{dz_1}+2x_3(-x_1+x_3)\frac{\partial\overline{h}}{dz_2}\\ +(x_1^2-x_3^2)\frac{\partial\overline{h}}{dz_3} = \Big(\frac{n_1}{2}(z_2-z_1)+\frac{n_2}{2}(z_2+z_1)+n_3z_3\Big)\overline{h}.\end{aligned} \tag{32}$$

Since by hypothesis h is coprime with x_2, we have that $\overline{h} \neq 0$. Therefore $\overline{h}$ is a Darboux polynomial of system (2) restricted to $x_2 = 0$. By Proposition 16 we get that $\overline{h} = cx_1^{m_2}x_3^{m_3}$ for some $c \in \mathbb{C}\setminus\{0\}$ and m_2 and m_3 are non-negative integers such that $m_2+m_3 \neq 0$. Then from (32) we get

$$c\Big(\frac{m_2}{2}(z_2-z_1)+m_3z_3\Big)x_1^{m_2}x_3^{m_3} = c\Big(\frac{n_1}{2}(z_2-z_1)+\frac{n_2}{2}(z_2+z_1)+n_3z_3\Big)x_2^{m_2}x_3^{m_3}$$

and we get that

$$m_3 = n_3, \quad m_2 = n_1+n_2 \quad m_2 = n_1-n_2$$

which yields $n_2 = 0$ a contradiction.

Case 3: $n_3 > 0$. Taking $x_3 = 0$ in (30) and denoting by $\overline{h}$ the restriction of h to $x_3 = 0$, we conclude that $\overline{h}$ satisfies

$$\begin{aligned}\frac{1}{2}(z_2-z_1)x_1\frac{\partial\overline{h}}{dx_1}+\frac{1}{2}(z_2+z_1)x_2\frac{\partial\overline{h}}{dx_2}-2(x_1^2-x_2^2)\frac{\partial\overline{h}}{dz_1}+2x_3(-x_1+x_2)\frac{\partial\overline{h}}{dz_2}\\ +(x_1+x_2)^2\frac{\partial\overline{h}}{dz_3} = \Big(\frac{n_1}{2}(z_2-z_1)+\frac{n_2}{2}(z_2+z_1)+n_3z_3\Big)\overline{h}.\end{aligned} \tag{33}$$

Since by hypothesis h is coprime with x_3, we have that $\overline{h} \neq 0$. Therefore $\overline{h}$ is a Darboux polynomial of system (2) restricted to $x_3 = 0$. By Proposition 16 we get that $\overline{h} = cx_1^{m_2}x_2^{m_3}$ for some $c \in \mathbb{C}\setminus\{0\}$ and m_2 and m_3 are non-negative integers such that $m_2+m_3 \neq 0$. Then from (33) we get

$$c\Big(\frac{m_2}{2}(z_2-z_1)+\frac{m_3}{2}(z_2+z_1)\Big)x_1^{m_2}x_3^{m_3} = c\Big(\frac{n_1}{2}(z_2-z_1)+\frac{n_2}{2}(z_2+z_1)+n_3z_3\Big)x_2^{m_2}x_3^{m_3}$$

and in a similar way to the previous cases we obtain that $n_3 = 0$, a contradiction. This completes the proof of the proposition. □

Proof of Theorem 5. From Theorems 9 and 2, and Propositions 4 and 17, if system (2) has a Darboux first integral G, then

$$G = x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}H^{\lambda_4}\exp(P(H)/H^m), \quad \text{with} \quad \lambda_1,\lambda_2,\lambda_3,\lambda_4 \in \mathbb{C},$$

where P is a polynomial in the variable H, and m is a non-negative integer. Since G and H are first integrals and the cofactors of H and $\exp(P(H)/H^m)$ are zero, using Theorem 9, it must hold

$$\frac{\lambda_1}{2}(z_2 - z_1) + \frac{\lambda_2}{2}(z_2 + z_1) + \lambda_3 z_3 = 0.$$

This implies $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and completes the proof of the theorem. □

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