

## ON THE PERIODIC ORBITS OF HAMILTONIAN SYSTEMS

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*Dedicated to a common friend, Michał Misiurewicz, on the occasion of his 60th Birthday*

ABSTRACT. We show how to apply to Hamiltonian differential systems recent results for studying the periodic orbits of a differential system using the averaging theory. We have chosen two classical integrable Hamiltonian systems, one with the Hooke potential and the other with the Kepler potential, and we study the periodic orbits which bifurcate from the periodic orbits of these integrable systems, first perturbing the Hooke Hamiltonian with a nonautonomous potential, and second perturbing the Kepler problem with an autonomous potential.

### 1. INTRODUCTION

In this paper we study the planar motion of a particle of unitary mass under the action of a central force with Hamiltonian given by

$$H_0(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + V_0(\sqrt{x^2 + y^2}),$$

perturbed by the Hamiltonian

$$(1) \quad H(x, y, p_x, p_y, t) = H_0(x, y, p_x, p_y) + \varepsilon V(t, x, y),$$

where  $\varepsilon$  is a small parameter and  $V(t, x, y)$  is a perturbation of the potential eventually depending on the time  $t$ .

We consider a central force derived from a potential of the form

$$(2) \quad V_0(\sqrt{x^2 + y^2}) = \pm(x^2 + y^2)^{\alpha/2}.$$

The Hamilton equations associated to Hamiltonian (1) are

$$(3) \quad \begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= p_y, \\ \dot{p}_x &= -\partial(V_0(\sqrt{x^2 + y^2}) + \varepsilon V(t, x, y))/\partial x, \\ \dot{p}_y &= -\partial(V_0(\sqrt{x^2 + y^2}) + \varepsilon V(t, x, y))/\partial y, \end{aligned}$$

where the dot denotes derivative with respect to the time  $t$ . We shall apply the averaging theory for studying the periodic orbits of the Hamiltonian system (3).

The averaging method (see for instance [10]) gives a quantitative relation between the solutions of some nonautonomous differential system and the solutions of its autonomous averaged differential system, and in particular allows to study the periodic orbits of the nonautonomous differential system in function of the periodic orbits of the averaged one, see for more details [1, 3, 8, 7, 10, 11] and mainly Section 2. Our aim is to apply the *averaging theory* to two classes of Hamiltonian systems

(3) for studying their periodic solutions. But the tools that we shall use are very general and can be applied to other classes of Hamiltonian or differential systems.

Of course there is a long tradition in studying the periodic orbits of the differential systems using the averaging method, see Chapter 4 of [6], the book [10], the chapter 11 of [11], the paper [5], and many other. In the paper [5] they also use the averaging method of second order as in the present paper for studying the periodic orbits, but in that paper they studied a 2-dimensional nonautonomous Hamiltonian system using the transformation to action-angle coordinates, and here we study two 4-dimensional Hamiltonian systems, one autonomous and the other nonautonomous, without needing to pass to action-angles coordinates. Another important difference with the paper [5] is that this paper one uses the averaging method in the version given in Theorem 6, and we use Theorem 6 and also the different version given in Theorem 5.

The unique central forces coming from the central potentials of the form (2) for which all bounded orbits are periodic are the Hooke's force and the Kepler's force, that correspond to the potentials

$$(4) \quad k(x^2 + y^2), \quad \text{and} \quad -\frac{k}{\sqrt{x^2 + y^2}} \quad \text{with } k > 0,$$

respectively. This result was proved by J. Bertrand in 1873, see [4].

We will apply the averaging theory to the Hamiltonian systems (3) with the potentials  $V_0(\sqrt{x^2 + y^2})$  given by (4) for studying which periodic orbits of system (3) with  $\varepsilon = 0$  can be continued to periodic orbits of the same system with  $\varepsilon \neq 0$  sufficiently small.

Since generically the periodic orbits of Hamiltonian systems are in cylinders fulfilled of periodic orbits and every one of the periodic orbits of one of these cylinders belongs to a different level of the Hamiltonian (for more details see [2, 9]), we shall apply the averaging theory described in Section 2 to the Hamiltonian systems (3) with potential  $V_0(\sqrt{x^2 + y^2})$  given by (4) restricted to a fixed level of the Hamiltonian. To work in a fixed level of the Hamiltonian is necessary in order to apply the averaging theory for studying periodic orbits (see Theorems 5 and 6), because the periodic orbits provided by the averaging must be isolated in the set of all periodic orbits.

First we consider the planar motion of a particle of unitary mass under the action of the Hooke potential  $V_0(\sqrt{x^2 + y^2}) = (x^2 + y^2)/2$  perturbed by a non-autonomous potential  $\varepsilon V(t, x, y)$  where  $\varepsilon$  is a small parameter and  $V(t, x, y)$  is  $2\pi$ -periodic in the variable  $t$ . In cartesian coordinates the Hamiltonian governing this motion is

$$(5) \quad H(x, y, p_x, p_y, t) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \varepsilon V(t, x, y).$$

The corresponding Hamiltonian equations are

$$(6) \quad \begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= p_y, \\ \dot{p}_x &= -x - \varepsilon \partial V(t, x, y) / \partial x, \\ \dot{p}_y &= -y - \varepsilon \partial V(t, x, y) / \partial y. \end{aligned}$$

The main result on the periodic orbits of the Hamiltonian system (6) is the following one.

**Theorem 1.** *For  $\varepsilon \neq 0$  sufficiently small,  $h > 0$  and every zero  $(x_0^*, y_0^*, p_{x0}^*)$  of  $f_k = f_k(x_0, y_0, p_{x0}) = 0$  for  $k = 1, 2, 3$ , where*

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} -B \sin t \\ \frac{Ap_{y_0} + B \sin t(p_{x_0} \cos t - x_0 \sin t)}{p_{y_0} \cos t - y_0 \sin t} \\ B \cos t \end{bmatrix} dt, \quad \begin{matrix} x = p_{x_0} \sin t + x_0 \cos t \\ y = p_{y_0} \sin t + y_0 \cos t \\ p_{y_0} = \sqrt{2h - p_{x_0}^2 - x_0^2 - y_0^2} \end{matrix}$$

with  $A = -V(t, x, y)/(p_{y_0} \cos t - y_0 \sin t)$ ,  $B = -\partial V(t, x, y)/\partial x$ , satisfying that

$$(7) \quad \frac{\partial(f_1, f_2, f_3)}{\partial(x_0, y_0, p_{x0})} \Big|_{x_0=x_0^*, y_0=y_0^*, p_{x0}=p_{x0}^*} \neq 0, \quad p_{y_0}^* = \sqrt{2h - p_{x_0}^{*2} - x_0^{*2} - y_0^{*2}} \neq 0,$$

there exists a  $2\pi$ -periodic solution  $\varphi(t; x_0^*, y_0^*, p_{x0}^*, \varepsilon)$  of the Hamiltonian system (6) such that  $\varphi(0; x_0^*, y_0^*, p_{x0}^*, \varepsilon) \rightarrow (x_0^*, y_0^*, p_{x0}^*, p_{y_0}^*)$  when  $\varepsilon \rightarrow 0$ .

For example an application of Theorem 1 is the next one.

**Corollary 2.** *Consider the Hamiltonian system (6) with  $V(t, x, y) = (x - x^3) \cos t$ . Then for  $\varepsilon \neq 0$  sufficiently small and  $h > 0$  the following statements hold.*

- (a) *For every  $h \notin \{2/3, 2/9\}$  system (6) has at least 14 periodic orbits bifurcating from the periodic orbits of the Hamiltonian level  $H = h$  for  $\varepsilon = 0$ .*
- (b) *If  $h = 2/3$  then system (6) has at least 8 periodic orbits bifurcating from the periodic orbits of the Hamiltonian level  $H = 2/3$  for  $\varepsilon = 0$ .*
- (c) *If  $h = 2/9$  then system (6) has at least 6 periodic orbits bifurcating from the periodic orbits of the Hamiltonian level  $H = 2/9$  for  $\varepsilon = 0$ .*

The study of the periodic orbits for the perturbed Hooke Hamiltonian systems is done in Section 3. More precisely in that section it is proved Theorem 1 and Corollary 2.

Now we consider the planar motion of a particle of unitary mass under the action of the Kepler potential  $V_0(\sqrt{x^2 + y^2}) = -1/\sqrt{x^2 + y^2}$  perturbed by a potential  $\varepsilon V(x, y)$  where  $\varepsilon$  is a small parameter. Thus its Hamiltonian in cartesian coordinates writes

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{\sqrt{x^2 + y^2}} + \varepsilon V(x, y).$$

In polar coordinates this Hamiltonian becomes

$$(8) \quad H(r, \theta, p_r, p_\theta) = \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) - \frac{1}{r} + \varepsilon V(r \cos \theta, r \sin \theta),$$

and its corresponding Hamiltonian equations are

$$(9) \quad \begin{aligned} \dot{r} &= p_r, \\ \dot{\theta} &= \frac{p_\theta}{r^2}, \\ \dot{p}_r &= \frac{p_\theta^2}{r^3} - \frac{1}{r^2} - \varepsilon \frac{\partial V}{\partial r}, \\ \dot{p}_\theta &= -\varepsilon \frac{\partial V}{\partial \theta}, \end{aligned}$$

where  $V = V(r \cos \theta, r \sin \theta)$ . The main result on the periodic orbits of the Hamiltonian system (9) is the next one.

**Theorem 3.** *For  $\varepsilon \neq 0$  sufficiently small and every zero  $(\theta_0^*, w_0^*)$  of  $f_k = f_k(\theta_0, w_0) = 0$  for  $k = 1, 2$ , where*

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} \frac{1}{e} \left( \frac{A \sin(\theta_0 - \theta) V_\theta}{B} - \frac{\cos(\theta_0 - \theta) V_u}{w_0^2} \right) \\ \frac{w_0^3 V_\theta}{B} \end{bmatrix} \Big|_{u = \frac{1}{w_0^2 + e \cos(\theta - \theta_0)}} d\theta,$$

with

$$A = \sqrt{1 + 2hw_0^2} \cos(\theta_0 - \theta) + 2, \quad B = \left( \sqrt{1 + 2hw_0^2} \cos(\theta_0 - \theta) + 1 \right)^2,$$

$$e = \sqrt{\frac{1 + 2hw_0^2}{w_0^4}}, \quad V = V\left(\frac{\cos \theta}{u}, \frac{\sin \theta}{u}\right),$$

$$V_\theta = \frac{\partial V}{\partial \theta}, \quad V_u = \frac{\partial V}{\partial u},$$

satisfying that

$$(10) \quad \frac{\partial(f_1, f_2)}{\partial(\theta_0, w_0)} \Big|_{\theta_0 = \theta_0^*, w_0 = w_0^*} \neq 0, \quad 0 < 1 + 2hw_0^{*2} < 1,$$

there exists a periodic solution  $\varphi(t; \theta_0^*, w_0^*, \varepsilon)$  of the Hamiltonian system (9) such that  $\varphi(0; \theta_0^*, w_0^*, \varepsilon)$  tends to

$$\left( \frac{1}{r} = \frac{1}{w_0^{*2}} + \sqrt{\frac{1 + 2hw_0^{*2}}{w_0^{*4}}}, \theta = \theta_0^*, p_r = \sqrt{h + \frac{\sqrt{1 + 2hw_0^{*2}} + 1 - w_0^{*3}}{(\sqrt{1 + 2hw_0^{*2}} + 1)^2}}, p_\theta = w_0^* \right)$$

when  $\varepsilon \rightarrow 0$ .

As an example one application of Theorem 3 is the following.

**Corollary 4.** *Consider the Hamiltonian system (9) with  $V(x, y) = x^2$ . Then for  $\varepsilon \neq 0$  sufficiently small and  $h = -1/10$  this system has four periodic orbits which bifurcates from the elliptic periodic orbit of the Kepler problem (9) with  $\varepsilon = 0$  and such that  $(\theta_0^*, w_0^*)$  is equal to*

$$\begin{aligned} &(\pi/2, \quad -1.4262515827609759), \\ &(\pi/2, \quad 0.5654210557716061), \\ &(3\pi/2, \quad -1.4262515827609759), \\ &(3\pi/2, \quad 0.5654210557716061). \end{aligned}$$

In Section 4 we prove Theorem 3 and Corollary 4.

## 2. BASIC RESULTS

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper. The key tool for proving the algorithm is the averaging theory. For a general introduction to the averaging theory and related topics see the books [1, 8, 10, 11]. But the results that we shall use are presented in what follows.

We consider the problem of the bifurcation of  $T$ -periodic solutions from the differential system

$$(11) \quad \dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon),$$

with  $\varepsilon = 0$  to  $\varepsilon \neq 0$  sufficiently small. The functions  $F_0, F_1 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$  and  $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$  appearing in (11), are  $\mathcal{C}^2$  in their variables and  $T$ -periodic in the first variable, and  $\Omega$  is an open subset of  $\mathbb{R}^n$ . One of the main assumptions is that the unperturbed system

$$(12) \quad \dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}),$$

has an open subset of  $\Omega$  fulfilled of periodic solutions. A solution of this problem is given in the following using the averaging theory.

Let  $\mathbf{x}(t, \mathbf{z}, \varepsilon)$  be the solution of system (11) such that  $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$ . We write the linearization of the unperturbed system (12) along a periodic solution  $\mathbf{x}(t, \mathbf{z}, 0)$  as

$$(13) \quad \mathbf{y}' = D_{\mathbf{x}}F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}.$$

In what follows we denote by  $M_{\mathbf{z}}(t)$  some fundamental matrix of the linear differential system (13).

We assume that there exists an open set  $W$  with  $\text{Cl}(W) \subset \Omega$  such that for each  $\mathbf{z} \in \text{Cl}(W)$ ,  $\mathbf{x}(t, \mathbf{z}, 0)$  is  $T$ -periodic. The set  $\text{Cl}(W)$  is *isochronous* for the system (11); i.e. it is a set formed only by periodic orbits, all of them having the same period  $T$ . Then an answer to the problem of the bifurcation of  $T$ -periodic solutions from the periodic solutions  $\mathbf{x}(t, \mathbf{z}, 0)$  contained in  $\text{Cl}(W)$  is given in the next result.

**Theorem 5. (Perturbations of an isochronous set)** *We assume that there exists an open and bounded set  $W$  with  $\text{Cl}(W) \subset \Omega$  such that for each  $\mathbf{z} \in \text{Cl}(W)$ , the solution  $\mathbf{x}(t, \mathbf{z}, 0)$  is  $T$ -periodic, then we consider the function  $\mathcal{F} : \text{Cl}(W) \rightarrow \mathbb{R}^n$*

$$(14) \quad \mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}, 0) F_1(t, \mathbf{x}(t, \mathbf{z})) dt.$$

*If there exists  $a \in V$  with  $\mathcal{F}(a) = 0$  and  $\det((d\mathcal{F}/d\mathbf{z})(a)) \neq 0$ , then there exists a  $T$ -periodic solution  $\varphi(t, \varepsilon)$  of system (11) such that  $\varphi(0, \varepsilon) \rightarrow a$  as  $\varepsilon \rightarrow 0$ .*

For a proof of Theorem 5 see Corollary 1 of [3].

Now we consider the differential system

$$(15) \quad \dot{\mathbf{x}}(t) = \varepsilon F(t, \mathbf{x}(t)) + \varepsilon^2 R(t, \mathbf{x}(t), \varepsilon),$$

with  $\mathbf{x} \in U \subset \mathbb{R}^n$ ,  $U$  a bounded domain and  $t \geq 0$ . Moreover, we assume that  $F(t, \mathbf{x})$  and  $R(t, \mathbf{x}, \varepsilon)$  are  $T$ -periodic in  $t$ .

The averaged system associated to system (15) is defined by

$$(16) \quad \dot{\mathbf{y}}(t) = \varepsilon f(\mathbf{y}(t)),$$

where

$$(17) \quad f(\mathbf{y}) = \frac{1}{T} \int_0^T F(s, \mathbf{y}) ds.$$

The next theorem says us under which conditions the singular points of the averaged system (16) provide  $T$ -periodic orbits of system (15). For a proof see Theorem 2.6.1 of [10], Theorems 11.5 and 11.6 of [11], and Theorem 4.1.1 of [6].

**Theorem 6.** *We consider system (15) and assume that the vector functions  $F$ ,  $R$ ,  $D_{\mathbf{x}}F_1$ ,  $D_{\mathbf{x}}^2F_1$  and  $D_{\mathbf{x}}R$  are continuous and bounded by a constant  $M$  (independent of  $\varepsilon$ ) in  $[0, \infty) \times U$  with  $-\varepsilon_0 < \varepsilon < \varepsilon_0$ . Moreover, we suppose that  $F$  and  $R$  are  $T$ -periodic in  $t$ , with  $T$  independent of  $\varepsilon$ .*

- (a) *If  $a \in U$  is a singular point of the averaged system (16) such that  $\det(D_{\mathbf{x}}f(a)) \neq 0$  then, for  $|\varepsilon| > 0$  sufficiently small there exists a unique  $T$ -periodic solution  $\mathbf{x}_\varepsilon(t)$  of system (15) such that  $\mathbf{x}_\varepsilon(0) \rightarrow a$  as  $\varepsilon \rightarrow 0$ .*
- (b) *If the singular point  $a$  of the averaged system (16) is hyperbolic then, for  $|\varepsilon| > 0$  sufficiently small, the corresponding periodic solution  $\mathbf{x}_\varepsilon(t)$  of system (15) is hyperbolic and of the same stability type as  $a$ .*

### 3. PERIODIC ORBITS OF THE PERTURBED PLANAR HOOKE PROBLEM

In this section we shall prove Theorem 1 and Corollary 2.

*Proof of Theorem 1.* We want to apply Theorem 5 to the Hamiltonian system (6). Since Theorem 5 needs that the periodic orbits of system (6) are isolated in the set of all periodic orbits of the system, we must restrict our study to every fixed Hamiltonian level, otherwise the set of periodic orbits would not be isolated. Thus we consider a fixed Hamiltonian level  $h$  and we restrict the Hamiltonian system (6) to this Hamiltonian level.

We isolate  $p_y$  in the Hamiltonian level

$$\frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \varepsilon V(t, x, y) = h,$$

and expand the obtained expression of  $p_y$  in power series of  $\varepsilon$  obtaining

$$p_y = \sqrt{2h - p_x^2 - x^2 - y^2} - \varepsilon \frac{V(t, x, y)}{\sqrt{2h - p_x^2 - x^2 - y^2}} + O(\varepsilon^2).$$

Therefore system (6) restricted to the Hamiltonian level  $H(x, y, p_x, p_y, t) = h$  becomes

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= \sqrt{2h - p_x^2 - x^2 - y^2} - \varepsilon \frac{V(t, x, y)}{\sqrt{2h - p_x^2 - x^2 - y^2}} + O(\varepsilon^2), \\ \dot{p}_x &= -x - \varepsilon \frac{\partial V(t, x, y)}{\partial x}. \end{aligned} \tag{18}$$

We will apply the averaging theory described in Section 2 for studying the periodic orbits of system (18). More precisely we shall analyze which periodic orbits of system (18) with  $\varepsilon = 0$  can be continued to periodic orbits of system (18) with  $\varepsilon \neq 0$  sufficiently small.

The general solution of system (18) with  $\varepsilon = 0$  in the Hamiltonian level  $H(x, y, p_x, p_y) = h$  and with initial conditions  $x(0) = x_0$ ,  $y(0) = y_0$  and  $p_x(0) = p_{x0}$  is

$$\begin{aligned} x(t) &= p_{x0} \sin t + x_0 \cos t, \\ y(t) &= p_{y0} \sin t + y_0 \cos t, \\ p_x(t) &= p_{x0} \cos t - x_0 \sin t, \end{aligned} \tag{19}$$

where  $p_{y0} = \sqrt{2h - p_{x0}^2 - x_0^2 - y_0^2}$ . All these periodic solutions form a three dimensional open set of periodic solutions with period  $2\pi$  in the Hamiltonian level

$H(x, y, p_x, p_y) = h$ . So system (18) with  $\varepsilon$  sufficiently small satisfies the assumptions of Theorem 5.

We write system (18) into the form (11)

$$\begin{aligned}\dot{x} &= F_{0,1}(x, y, p_x) + \varepsilon F_{1,1}(x, y, p_x) + O(\varepsilon^2), \\ \dot{y} &= F_{0,2}(x, y, p_x) + \varepsilon F_{1,2}(x, y, p_x) + O(\varepsilon^2), \\ \dot{p}_x &= F_{0,3}(x, y, p_x) + \varepsilon F_{1,3}(x, y, p_x) + O(\varepsilon^2),\end{aligned}$$

where  $F_0 = (F_{0,1}, F_{0,2}, F_{0,3})$  is

$$F_{0,1} = p_x, \quad F_{0,2} = \sqrt{2h - p_x^2 - x^2 - y^2}, \quad F_{0,3} = -x,$$

and  $F_1 = (F_{1,1}, F_{1,2}, F_{1,3})$  is

$$(20) \quad F_{1,1} = 0, \quad F_{1,2} = -\frac{V(t, x, y)}{\sqrt{2h - p_x^2 - x^2 - y^2}}, \quad F_{1,3} = -\frac{\partial V(t, x, y)}{\partial x}.$$

The periodic solution  $\mathbf{x}(t, \mathbf{z}, 0)$  of system (11) with  $\varepsilon = 0$  now is the periodic solution  $(x(t), y(t), p_x(t))$  given by (19) of system (18) with initial conditions  $\mathbf{z} = (x_0, y_0, p_{x0})$ . After an easy but tedious computation the fundamental matrix  $M_{\mathbf{z}}(t)$  of the differential system (13) such that  $M_{\mathbf{z}}(0)$  is the identity matrix of  $\mathbb{R}^3$  is

$$(21) \quad M_{\mathbf{z}}(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ -\frac{x_0 \sin t}{p_{y0}} & \frac{p_{y0} \cos t - y_0 \sin t}{p_{y0}} & -\frac{p_{x0} \sin t}{p_{y0}} \\ -\sin t & 0 & \cos t \end{pmatrix}.$$

We note that  $p_{y0} \neq 0$  in order that the matrix  $M_{\mathbf{z}}(t)$  be well defined. This is the reason that in the statement of Theorem 5 we have the assumption  $p_{y0}^* \neq 0$ . In fact this is a technical restriction that would be able to be avoided working with another fundamental matrix, but here we do not take care of this.

By Theorem 5 we must study the zeros  $(x_0, y_0, p_{x0})$  in the set

$$W = \{(x_0, y_0, p_{x0}) \in \mathbb{R}^3 : 0 < x_0^2 + y_0^2 + p_{x0}^2 < R\},$$

where  $R > 0$  is an arbitrary constant, of the system  $\mathcal{F}(x_0, y_0, p_{x0}) = (f_1(x_0, y_0, p_{x0}), f_2(x_0, y_0, p_{x0}), f_3(x_0, y_0, p_{x0})) = 0$ , where according with (14) if we denote  $f_k = f_k(x_0, y_0, p_{x0})$  we have

$$(22) \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \int_0^{2\pi} M_{\mathbf{z}}^{-1}(t) \begin{bmatrix} F_{1,1}(x, y, p_x) \\ F_{1,2}(x, y, p_x) \\ F_{1,3}(x, y, p_x) \end{bmatrix} \begin{matrix} x = p_{x0} \sin t + x_0 \cos t \\ y = p_{y0} \sin t + y_0 \cos t \\ p_x = p_{x0} \cos t - x_0 \sin t \\ p_{y0} = \sqrt{2h - p_{x0}^2 - x_0^2 - y_0^2} \end{matrix} dt.$$

where  $F_{k,1}$  for  $k = 1, 2, 3$  are as in (20).

Note that in (22) only  $F_{1,2}(x, y, p_x)$  depends on the variable  $p_x$  through the expression  $\sqrt{2h - p_x^2 - x^2 - y^2}$ . But when we restrict the integrand function of (22) on the periodic solution of system (18) with  $\varepsilon = 0$ , i.e. on  $x = p_{x0} \sin t + x_0 \cos t$ ,  $y = p_{y0} \sin t + y_0 \cos t$  and  $p_x = p_{x0} \cos t - x_0 \sin t$ , we get that  $\sqrt{2h - p_x^2 - x^2 - y^2} = p_{y0} \cos t - y_0 \sin t$ . Now doing an easy computation, using (21) and (22), we get the expression of  $(f_1, f_2, f_3)$  given in the statement of Theorem 1. Hence Theorem 5 completes the proof of Theorem 1.  $\square$

*Proof of Corollary 2.* We apply Theorem 1 to our Hamiltonian system (6) with  $V(t, x, y) = (x - x^3) \cos t$ . Since

$$A = \frac{(x^3 - x) \cos t}{p_{y_0} \cos t - y_0 \sin t}, \quad B = (3x^2 - 1) \cos t,$$

we obtain after some tedious computations that the functions  $f_k = f_k(x_0, y_0, p_{x_0})$  are

$$(23) \quad \begin{aligned} f_1 &= -\frac{3\pi}{2} p_{x_0} x_0, \\ f_2 &= \frac{\pi}{4(p_{y_0}^2 + y_0^2)^3} \left( x_0^3 p_{y_0}^5 + 9p_{x_0}^2 x_0 p_{y_0}^5 - 4x_0 p_{y_0}^5 - 15p_{x_0}^3 y_0 p_{y_0}^4 + \right. \\ &\quad \left. 9p_{x_0} x_0^2 y_0 p_{y_0}^4 + 12p_{x_0} y_0 p_{y_0}^4 + 6x_0^3 y_0^2 p_{y_0}^3 - 18p_{x_0}^2 x_0 y_0^2 p_{y_0}^3 - \right. \\ &\quad \left. 10p_{x_0}^3 y_0^3 p_{y_0}^2 - 18p_{x_0} x_0^2 y_0^3 p_{y_0}^2 + 16p_{x_0} y_0^3 p_{y_0}^2 - 3x_0^3 y_0^4 p_{y_0} - \right. \\ &\quad \left. 3p_{x_0}^2 x_0 y_0^4 p_{y_0} + 4x_0 y_0^4 p_{y_0} - 3p_{x_0}^3 y_0^5 - 3p_{x_0} x_0^2 y_0^5 + 4p_{x_0} y_0^5 \right), \\ f_3 &= -\frac{\pi}{4} (4 - 3(p_{x_0}^2 + 3x_0^2)), \end{aligned}$$

where  $p_{y_0} = \sqrt{2h - p_{x_0}^2 - x_0^2 - y_0^2}$ . We compute the solutions  $(x_0^*, y_0^*, p_{x_0}^*)$  of  $f_k(x_0, y_0, p_{x_0}) = 0$  for  $k = 1, 2, 3$ . After another tedious computation we obtain the following 22 solutions

$$\begin{aligned} s_{1,2} &= \left( 0, 0, \mp \frac{2}{\sqrt{3}} \right), \\ s_{3,4} &= \left( 0, -\sqrt{\frac{3h-2}{2}}, \mp \frac{2}{\sqrt{3}} \right), \\ s_{5,6} &= \left( 0, \sqrt{\frac{3h-2}{2}}, \mp \frac{2}{\sqrt{3}} \right), \\ s_{7,8} &= \left( 0, \sqrt{2h - \frac{4}{3}}, \mp \frac{2}{\sqrt{3}} \right), \\ s_{9,10} &= \left( 0, -\sqrt{2h - \frac{4}{3}}, \mp \frac{2}{\sqrt{3}} \right), \\ s_{11,12} &= \left( \mp \frac{2}{3}, \frac{\sqrt{18h-4}}{3}, 0 \right), \\ s_{13,14} &= \left( \mp \frac{2}{3}, -\frac{\sqrt{18h-4}}{3}, 0 \right), \\ s_{15,16} &= \left( \mp \frac{2}{3}, -\frac{1}{6} \sqrt{99h - 22 + \sqrt{57(2-9h)^2}}, 0 \right), \\ s_{17,18} &= \left( \mp \frac{2}{3}, \frac{1}{6} \sqrt{99h - 22 + \sqrt{57(2-9h)^2}}, 0 \right), \\ s_{19,20} &= \left( \mp \frac{2}{3}, -\frac{1}{6} \sqrt{99h - 22 - \sqrt{57(2-9h)^2}}, 0 \right), \\ s_{21,22} &= \left( \mp \frac{2}{3}, \frac{1}{6} \sqrt{99h - 22 - \sqrt{57(2-9h)^2}}, 0 \right). \end{aligned}$$

The determinat (7) now becomes

$$\det(x_0, y_0, p_{x_0}) = \frac{9\pi^3 (p_{x_0}^2 - 3x_0^2) \left( \alpha p_{x_0} \sqrt{2h - p_{x_0}^2 - x_0^2 - y_0^2} - \beta x_0 y_0 \right)}{16(2h - p_{x_0}^2 - x_0^2)^3 \sqrt{2h - p_{x_0}^2 - x_0^2 - y_0^2}},$$



where

$$\begin{aligned}\alpha &= 15p_{x_0}^6 + 21x_0^2p_{x_0}^4 + 60y_0^2p_{x_0}^4 - 60hp_{x_0}^4 - 12p_{x_0}^4 - 3x_0^4p_{x_0}^2 + 40y_0^4p_{x_0}^2 + \\ & 60h^2p_{x_0}^2 - 24hx_0^2p_{x_0}^2 - 24x_0^2p_{x_0}^2 - 48x_0^2y_0^2p_{x_0}^2 - 120hy_0^2p_{x_0}^2 - 24y_0^2p_{x_0}^2 + \\ & 48hp_{x_0}^2 - 9x_0^6 + 36hx_0^4 - 12x_0^4 - 120x_0^2y_0^4 - 48h^2 - 36h^2x_0^2 + 48hx_0^2 - \\ & 108x_0^4y_0^2 + 216hx_0^2y_0^2 - 24x_0^2y_0^2 + 48hy_0^2, \\ \beta &= -81p_{x_0}^6 - 155x_0^2p_{x_0}^4 - 204y_0^2p_{x_0}^4 + 324hp_{x_0}^4 + 20p_{x_0}^4 - 67x_0^4p_{x_0}^2 - \\ & 120y_0^4p_{x_0}^2 - 324h^2p_{x_0}^2 + 296hx_0^2p_{x_0}^2 + 40x_0^2p_{x_0}^2 - 160x_0^2y_0^2p_{x_0}^2 + \\ & 408hy_0^2p_{x_0}^2 + 24y_0^2p_{x_0}^2 - 80hp_{x_0}^2 + 7x_0^6 - 28hx_0^4 + 20x_0^4 + 40x_0^2y_0^4 + \\ & 80h^2 + 28h^2x_0^2 - 80hx_0^2 + 44x_0^4y_0^2 - 88hx_0^2y_0^2 + 24x_0^2y_0^2 - 48hy_0^2.\end{aligned}$$

We note that  $p_{y_0}^* = \sqrt{2h - p_{x_0}^{*2} - x_0^{*2} - y_0^{*2}}$  is equal to zero for the solutions  $s_k$  with  $k = 7, \dots, 14$ , so we cannot apply Theorem 1 to these eight solutions.

We have

$$p_{y_0}^*(s_k) = \text{constant}\sqrt{3h-2}, \quad \det(s_k) = \pm \frac{6\sqrt{3}\pi^3}{3h-2},$$

if  $k = 1, \dots, 6$  and

$$\begin{aligned}p_{y_0}^*(s_k) &= \frac{1}{6}\sqrt{6-27h \pm \sqrt{57(9h-2)^2}}, \\ \det(s_k) &= \pm \frac{3\sqrt{99h-22 \pm \sqrt{57(9h-2)^2}} \left(38-171h \pm \sqrt{57(9h-2)^2}\right) \pi^3}{2(2-9h)^2\sqrt{6-27h \pm \sqrt{57(9h-2)^2}}},\end{aligned}$$

if  $k = 15, \dots, 22$ . Therefore the solutions  $s_k$  for  $k = 1, \dots, 6$  do not satisfy the conditions (7) if  $h = 2/3$ , and the solutions  $s_k$  for  $k = 15, \dots, 22$  do not satisfy the conditions (7) if  $h = 2/9$ . Hence applying Theorem 1 the proof of the corollary is completed.  $\square$

#### 4. PERIODIC ORBITS OF THE PERTURBED PLANAR KEPLER PROBLEM

In this section we shall prove Theorem 3 and Corollary 4.

*Proof of Theorem 3.* Dividing the first and the last two equations of (9) by the second one we get

$$\begin{aligned}(24) \quad r' &= \frac{r^2 p_r}{p_\theta}, \\ p_r' &= \frac{p_\theta}{r} - \frac{1}{p_\theta} - \varepsilon \frac{r^2}{p_\theta} \frac{\partial V}{\partial r}, \\ p_\theta' &= -\varepsilon \frac{r^2}{p_\theta} \frac{\partial V}{\partial \theta},\end{aligned}$$

where the prime denotes derivative with respect to  $\theta$ . We do the change of coordinates

$$u = \frac{1}{r}, \quad v = p_r, \quad w = p_\theta.$$

In these new coordinates system (24) becomes

$$(25) \quad \begin{aligned} u' &= -\frac{v}{w}, \\ v' &= w \left( u - \frac{1}{w^2} \right) + \varepsilon \frac{1}{w} \frac{\partial V}{\partial u}, \\ w' &= -\varepsilon \frac{1}{u^2 w} \frac{\partial V}{\partial \theta}, \end{aligned}$$

where now  $V = V(\cos \theta/u, \sin \theta/u)$ .

The Hamiltonian (8) in the coordinates  $u, v, w, \theta$  becomes

$$(26) \quad H(u, v, w, \theta) = \frac{1}{2}(v^2 + w^2 u^2) - u + \varepsilon V \left( \frac{\cos \theta}{u}, \frac{\sin \theta}{u} \right).$$

The general solution of system (25) for  $\varepsilon = 0$  and  $w_0 \neq 0$  (i.e. of the planar and non-rectilinear Kepler problem) is

$$(27) \quad \begin{aligned} u(\theta) &= \frac{1}{w_0^2} + e \cos(\theta - \theta_0), \\ v(\theta) &= e w_0 \sin(\theta - \theta_0), \\ w(\theta) &= w_0, \end{aligned}$$

with  $e \geq 0$  and  $w_0, \theta_0 \in \mathbb{R}$ . When  $w_0 = 0$  the solutions of the Kepler problem (9) with  $\varepsilon = 0$  move on a straight line, and here we do not consider these particular solutions. Substituting solution (27) in the Hamiltonian (26) with  $\varepsilon = 0$  we obtain  $h = (e^2 w_0^4 - 1)/(2w_0^2)$ . It is well known that the solutions of the Kepler problem in polar coordinates  $(r, \theta)$  when  $w_0 \neq 0$  are

$$\begin{aligned} &\text{ellipses if } h < 0, \text{ or equivalently } 0 \leq e w_0^2 < 1, \\ &\text{parabolas if } h = 0, \text{ or equivalently } e w_0^2 = 1, \\ &\text{hyperbolas if } h > 0, \text{ or equivalently } e w_0^2 > 1. \end{aligned}$$

Here  $e w_0^2$  denotes the eccentricity.

Now we shall do the change of variables  $(u, v, w) \rightarrow (e, \theta_0, w_0)$  defined by (27). Writing system (25) in the new variables  $(e, \theta_0, w_0)$  we obtain

$$(28) \quad \begin{aligned} e' &= \varepsilon \frac{(e w_0^2 \sin^2(\theta_0 - \theta) - 2 \cos(\theta_0 - \theta)) V_\theta + (e \cos(\theta_0 - \theta) w_0^2 + 1)^2 \sin(\theta_0 - \theta) V_u}{(e \cos(\theta_0 - \theta) w_0^2 + 1)^2}, \\ \theta_0' &= \frac{\varepsilon}{e} \left( \frac{(e \cos(\theta_0 - \theta) w_0^2 + 2) \sin(\theta_0 - \theta) V_\theta}{(e \cos(\theta_0 - \theta) w_0^2 + 1)^2} - \frac{\cos(\theta_0 - \theta) V_u}{w_0^2} \right), \\ w_0' &= -\varepsilon \frac{w_0^3 V_\theta}{(e \cos(\theta_0 - \theta) w_0^2 + 1)^2}, \end{aligned}$$

where

$$V_\theta = \frac{\partial}{\partial \theta} V \left( \frac{\cos \theta}{u}, \frac{\sin \theta}{u} \right), \quad V_u = \frac{\partial}{\partial u} V \left( \frac{\cos \theta}{u}, \frac{\sin \theta}{u} \right).$$

Now we shall fix our attention on the Hamiltonian level

$$(29) \quad H(e, \theta_0, w_0, \theta) = h,$$

of the perturbed system (25). Working in this Hamiltonian level we isolated  $e$  in function of the other two variables  $(\theta_0, w_0)$  and  $\theta$  which appears through the potential  $V$ . Thus we obtain

$$e = \sqrt{\frac{1 + 2hw_0^2}{w_0^4}} - \varepsilon \frac{V}{\sqrt{1 + 2hw_0^2}} + O(\varepsilon^2).$$

Then system (28) becomes

$$(30) \quad \begin{aligned} \theta'_0 &= \varepsilon \sqrt{\frac{w_0^4}{1 + 2hw_0^2}} \left( \frac{\left( \sqrt{1 + 2hw_0^2} \cos(\theta_0 - \theta) + 2 \right) \sin(\theta_0 - \theta) V_\theta}{\left( \sqrt{1 + 2hw_0^2} \cos(\theta_0 - \theta) + 1 \right)^2} - \right. \\ &\quad \left. \frac{\cos(\theta_0 - \theta) V_u}{w_0^2} \right) + O(\varepsilon^2), \\ w'_0 &= -\varepsilon \frac{w_0^3 V_\theta}{\left( \sqrt{1 + 2hw_0^2} \cos(\theta_0 - \theta) + 1 \right)^2} + O(\varepsilon^2). \end{aligned}$$

Now applying Theorem 6 of the averaging theory described in Section 2 to system (30), it follows immediately Theorem 3.  $\square$

*Proof of Corollary 4.* We apply Theorem 3 to our Hamiltonian system (9) with  $V = V(x, y) = x^2$ . After some tedious computations the function  $f_2 = f_2(\theta_0, w_0)$  is

$$(31) \quad f_2 = \frac{5(1 + 2hw_0^2) \cos \theta_0 \sin \theta_0}{8\sqrt{2}(-h)^{7/2}w_0^3}.$$

Since  $0 < 1 + 2hw_0^2 < 1$ , it follows that  $f_2 = 0$  if and only if  $\theta_0 \in \{0, \pi/2, \pi, 3\pi/2\}$ . Now we compute  $f_1 = f_1(\theta_0, w_0)$  for these four values of  $\theta_0$ , but since for the values  $\theta_0 \in \{0, \pi\}$  the function  $f_1 = 0$  has no real solutions for  $w_0$  we only provide in what follows the expression of  $f_1$  for  $\theta_0 \in \{\pi/2, 3\pi/2\}$ . Since  $f_1 = f_1(\pi/2, w_0) = f_1(3\pi/2, w_0)$  we only provide one expression for  $f_1$ :

$$\begin{aligned} & -\frac{1}{K} \left( (8h^3(6w_0^5 - 5)w_0^6 + 10w_0^5 + h(46w_0^7 - 24w_0^2) + h^2(68w_0^9 - 40w_0^4) - 5)L^6 + \right. \\ & 2hw_0^2(w_0^5 - 4h^2(2w_0^5 - 3)w_0^4 + 2h(w_0^7 + w_0^2) + \\ & \left. M(64(-h)^{5/2}(w_0^5 - 1)w_0^5 + 3\sqrt{2}(1 + 2hw_0^2)^2(2w_0^5 - 1)) \right) L^5 + \\ & (1 + 2hw_0^2)^3(4hw_0^2 + 5)(2w_0^5 - 1)L^4 - \\ & 6h(w_0 - 1)w_0^2(1 + 2hw_0^2)^2(2hw_0^2 + L + 1)(w_0^4 + w_0^3 + w_0^2 + w_0 + 1) \\ & \left. (16h^2w_0^4 + 6hw_0^2 + 1) \right), \end{aligned}$$

where

$$\begin{aligned} K &= 64L^9M(-h)^{7/2}w_0^9, \\ L &= \sqrt{1 + 2hw_0^2}, \\ M &= \sqrt{1 + hw_0^2 + \sqrt{1 + 2hw_0^2}}. \end{aligned}$$

We compute the solutions  $w_0^*$  of  $f_1 = 0$  for  $h = -1/10$ , and we obtain only the two real solutions:

$$w_0^* = -1.4262515827609759, \quad w_0^* = 0.5654210557716061.$$

So we get the four solutions given in the statement of Corollary 4. In these four solutions the determinant (10) takes the values  $-45970.28294002966$ ,  $7552084.398956253$ ,  $-45970.28294002966$ ,  $7552084.398956253$ , respectively. Therefore applying Theorem 6 the corollary is proved.  $\square$

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