A LOCAL OPTIMAL DIASTOLIC INEQUALITY ON THE TWO-SPHERE

FLORENT BALACHEFF

Abstract. We prove a local optimal inequality on the two-sphere between the area and the diastole - defined by a minimax process on the one-cycle space - in a neighborhood of the singular metric made of two equilateral triangles glued along their boundaries.

1. Introduction

Let g be a smooth Riemannian metric on the two-sphere S^2. Following Babenko [Bab97] we call *general systole*, or simply *systole* in this context, the least length of a non trivial closed geodesic and denote it by sys_0. Croke proved in [Cr88] that every Riemannian metric g on S^2 satisfies

$$\mathrm{area}(S^2, g) \geq \frac{1}{(31)^2} \mathrm{sys}_0(S^2, g)^2. \tag{1.1}$$

The constant involved in this inequality has been improved in [NR02] and [Sa04], and the best improvement is due to Rotman [Ro05] with constant $\frac{1}{32}$. A natural question is to ask what the optimal constant in inequality (1.1) is. It amounts to find the global infimum of the *systolic area* - defined as the ratio $\mathrm{area}/\mathrm{sys}_0^2$ - among smooth metrics. The round metric does not actually achieve the minimal systolic area, although it is a critical point for the systolic area (see [Bal06]). A flat metric with three conical singularities of angle $\frac{2\pi}{3}$ was conjectured by Calabi (see [Cr88]) to reach this global minimum. This metric, denoted by g_c, is made of two equilateral triangles of size 1 glued along their boundaries. Its systolic area is equal to $\frac{1}{2\sqrt{3}}$.

Conjecture 1.1 (Optimal systolic inequality for S^2)**.** For any smooth Riemannian metric g on S^2,

$$\mathrm{area}(S^2, g) \geq \frac{1}{2\sqrt{3}} \mathrm{sys}_0(S^2, g)^2.$$

Observe that the systolic area is not a continuous function of the metric, even when the space of smooth metrics is topologized by the strong topology (for a definition of strong topology, see [Hi76, p. 34]). In order to prove local results, we introduce a quantity that varies continuously with the metric g and bounds the systole from above. This quantity is the *diastole*, which has been introduced in [BS10]. The diastole of a Riemannian closed surface (M, g) is defined as the value obtained by a minimax process over the space

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of one-cycles, see section 2.1. It is denoted by $\text{dias}(M, g)$. In the case of a smooth metric, the diastole is realized as the length of an union of disjoint closed geodesics (see [Pi74, p. 468]), and so

$$\text{dias}(M, g) \geq \text{sys}_0(M, g).$$

The notion of diastole holds for any Riemannian closed surface with a finite number of conical singularities. It is proved in [BS10] that for every Riemannian closed surface (M, g) of genus k the diastole bounds from below the area:

$$\text{area}(M, g) \geq \frac{C}{k+1} \text{dias}(M, g)^2, \tag{1.2}$$

with C a positive constant. In the case of a two-sphere and since $\text{dias} \geq \text{sys}_0$ for smooth metrics, the inequality (1.2) implies the inequality (1.1) albeit with a worse constant. We call the ratio area/dias^2 the *diastolic area* and observe that it is continuous on the space of smooth metrics with a finite number of conical singularities for the uniform topology on metric spaces (see [AZ67, chapter 7] for a definition of uniform topology). The diastolic area of the singular metric g_c equals to $1/(2\sqrt{3})$ and the following conjecture appears to be rather natural:

Conjecture 1.2 (Optimal diastolic inequality for S^2)**.** For any Riemannian metric g for S^2,

$$\text{area}(S^2, g) \geq \frac{1}{2\sqrt{3}} \text{dias}(S^2, g)^2.$$

Observe that this conjecture is stronger than conjecture (1.1). By Pál's Theorem [Pá21], this conjecture holds for the special set of singular metrics on the two-sphere obtained by considering any convex disk of the plane and gluing two copies of such a disk along their boundaries.

In this paper we prove that the metric g_c is a local minimum of the diastolic area over the space of singular metrics C^1-conformal to g_c with respect to the C^1-topology. More precisely, let

$$\mathcal{M}_{g_c} := \left\{ e^{2u} \cdot g_c \mid u \in C^1(S^2, \mathbb{R}) \right\}$$

be the space of metrics C^1-conformal to g_c. The space $\mathcal{M}_{g_c}$ is naturally in bijection with $C^1(S^2, \mathbb{R}_+^*)$ and parametrizes the space of C^1-metrics on S^2 with three conical singularities of angle $2\pi/3$. We call *C^1-topology* on $\mathcal{M}_{g_c}$ the topology induced by the C^1 compact-open topology on $C^1(S^2, \mathbb{R}_+^*)$ (see [Hi76, p. 34]).

The space $\mathcal{M}_{g_c}$ is adapted to the local study of the diastolic area near the metric g_c. In fact we will observe in subsection (2.2) that, thanks to the Riemann uniformization theorem, any smooth metric can be written (up to an isometry) as a metric which differs from g_c by a conformal factor which lies in $C^1(S^2, \mathbb{R}_+)$. In particular, finding the infimum of the diastolic area over $\mathcal{M}_{g_c}$ amounts to finding the infimum of the diastolic area over the space of smooth metrics, see proposition (2.2). Our main result is the following:

Theorem 1.3. *There exists an open neighborhood $\mathcal{O}$ of g_c in $\mathcal{M}_{g_c}$ with respect to the C^1-topology such that for all $g \in \mathcal{O}$,*

$$area(S^2, g) \geq \frac{1}{2\sqrt{3}}\, dias\,(S^2, g)^2$$

with equality if and only if $g = g_c$.

Our proof is based on the study of a degree 3 ramified cover of S^2 by the two-torus $\mathbb{T}^2$ and an optimal systolic inequality on the torus due to Loewner. Recently, Sabourau found an alternative proof of our theorem which does not make use of the uniformization theorem, but is based on the study of the same ramified cover [Sa09]. In section 2, we define the diastole, study the metric g_c, and then present the ramified cover over S^2 and Loewner's optimal systolic inequality. With these tools, we derive an optimal systolic inequality for singular metrics of $\mathcal{M}_{g_c}$ presented in subsection (2.4). Section 3 is devoited to the proof of our main theorem.

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2. Preliminaries and definitions

2.1. **Isodiastolic inequality for closed surfaces.** Recall from [BS10] that the *diastole* of a closed Riemannian surface (M, g) of class C^1 with a finite number of conical singularities is defined by a minimax process over the space of one-cycles and is denoted by dias (M, g).

More precisely, we denote by $\mathcal{Z}_1(M; \mathbb{Z}_*)$ the space of one-cycles of M, see [BS10] for a precise definition. Here $\mathbb{Z}_*$ denotes $\mathbb{Z}$ in the orientable case, and $\mathbb{Z}_2$ in the non orientable one. An isomorphism due to F. Almgren [Al60] between $\pi_1(\mathcal{Z}_1(M; \mathbb{Z}_*), \{0\})$ and $H_2(M; \mathbb{Z}_*) \simeq \mathbb{Z}_*$ allows us to define the diastole over the one-cycle space as

$$\text{dias}\,(M, g) := \inf_{(z_t)} \sup_{0 \leq t \leq 1} Mass(z_t)$$

where (z_t) runs over the families of one-cycles inducing a generator of $\pi_1(\mathcal{Z}_1(M; \mathbb{Z}_*), \{0\})$ and $Mass(z_t)$ represents the mass (or length) of z_t. From a result of J. Pitts [Pi74, p. 468] (see also [CC92]), this minimax principle gives rise to a union of closed geodesics (counted with multiplicity) of total length dias(M) when the metric g is smooth without conical singularities.

Recall the following result (see [BS10]).

Theorem 2.1. *There exists a constant C such that every closed surface M of genus k endowed with a Riemannian smooth metric g satisfies*

$$area(M, g) \geq \frac{C}{k+1}\, dias(M, g)^2. \tag{2.1}$$

The constant C can be taken equal to 10^{-16}.

The best constant involved in inequality (2.1) is not known for any closed surface. Nevertheless inequality (2.1) is asymptotically optimal for large genus, see [BS10]. Theorem (2.1) remains valid for Riemannian metric of class C^1 with a finite number of conical singularities, as these metrics can be obtained as uniform limit (as metric spaces) of Riemannian smooth metrics and as both the area and the diastole are continuous for this topology (see [AZ67, p.224 & 269]).

2.2. **The singular metric and the Riemann uniformization theorem.** For the following paragraph, we refer to [Tro86] (see also [Re01]). By the Riemann uniformization theorem there exists only one conformal structure on S^2 (up to diffeomorphism). So we set $S^2 = \mathbb{C} \cup \{\infty\}$. For any $\theta > 0$, the cone with angle θ defined as the set $V_\theta = \{(r,t) \mid r \geq 0\,; t \in \mathbb{R}/\theta\mathbb{Z}\}$ endowed with the metric $ds^2 = dr^2 + r^2 dt^2$ is isometric to $\mathbb{C}$ endowed with the metric $|z|^{2(\frac{\theta}{2\pi}-1)}|dz|^2$ via the application

$$(r,t) \in V_\theta \mapsto z = (\frac{\theta}{2\pi} r e^{it})^{\frac{2\pi}{\theta}} \in \mathbb{C}.$$

Furthermore for any open set U containing 0 and any harmonic function $h : U \to \mathbb{C}$, the modified metric space $(U, e^{2h}|z|^{2(\frac{\theta}{2\pi}-1)}|dz^2|)$ is still isometric to an open neighborhood of the cone of angle θ (add the Laplacian of the function h in order to get the new Gaussian curvature of the modified metric). So for any triple (a_1, a_2, a_3) of pairewise distinct points of $\mathbb{C}$, the metric

$$(|z - a_1| \cdot |z - a_2| \cdot |z - a_3|)^{-4/3} \, |dz|^2$$

induces on S^2 a flat metric with three conical points of angle $\frac{2\pi}{3}$. As there is at most one isometry class of flat metric on S^2 with prescribed conical points up to homothety, there exists some positive λ such that

$$g_c = \lambda \, (|z-1| \cdot |z| \cdot |z+1|)^{-4/3} \, |dz|^2.$$

Now let

$$\mathcal{M}_{g_c} := \left\{ e^{2u} \cdot g_c \mid u \in C^1(S^2, \mathbb{R}) \right\}$$

be the space of metrics of class C^1 with three conical singularities of angle $2\pi/3$ on S^2 conformal to g_c.

Proposition 2.2. *The infimum of the diastolic area over $\mathcal{M}_{g_c}$ equals the infimum of the diastolic area over the space of smooth metrics.*

Proof. By the Riemann uniformization theorem, every smooth Riemannian metric g is isometric to a metric of the type $e^{2v} \cdot g_0$ where g_0 denotes the round metric and $v \in C^\infty(S^2, \mathbb{R})$. As

$$g_0 = \left(\frac{2}{1 + |z|^2} \right)^2 |dz|^2,$$

we see that g is homothetic (up to diffeomorphism) to the metric

$$e^{2v} \cdot \left(\frac{2}{1+|z|^2} \right)^2 \cdot (|z+1| \cdot |z| \cdot |z-1|)^{4/3} \cdot g_c$$

which differs from g_c by a conformal factor which lies in $C^1(S^2, \mathbb{R}_+)$. So we can approximate the smooth Riemannian metric g by elements of $\mathcal{M}_{g_c}$ with respect to the C^1-topology. As the diastolic area is continuous with respect to this topology (the diastolic area is in fact continuous for the uniform topology on metric spaces), we get that the infimum of the diastolic area over $\mathcal{M}_{g_c}$ is less than the infimum of the diastolic area over the space of smooth metrics. The reverse inequality is obtained by approximating elements of $\mathcal{M}_{g_c}$ by Riemannian smooth metrics for the uniform topology on metric spaces. $\square$

2.3. **The ramified cover and the isosystolic inequality on the torus.** Let

$$f : \mathbb{T}^2 \to S^2$$

be the covering map of degree 3 ramified over the three points $\{-1, 0, 1\}$. If g is in $\mathcal{M}_{g_c}$, we denote by $\tilde{g}$ the pull-back metric of g by the map f. The metric $\tilde{g}$ is C^1 everywhere. In fact, near the ramified points, the map looks like the map $z \mapsto z^3$. So each conical singularity of angle $2\pi/3$ lift to a conical singularity of angle 2π. Observe that

$$\text{area}(\mathbb{T}^2, \tilde{g}) = 3 \cdot \text{area}(S^2, g).$$

The ramified cover can also be constructed in a more geometrical way as follows. Consider a metric g in $\mathcal{M}_{g_c}$ and cut the sphere along two minimizing geodesics going from 0 to 1 and from 0 to -1. This yields a parallelogram. Then glue three copies of this parallelogram in order to form a hexagon as in Figure 1. By identifying the opposed sides of this hexagon, we obtain a torus (see Figure 1). The covering map is then defined by sending a point of the torus to the corresponding point of the sphere.

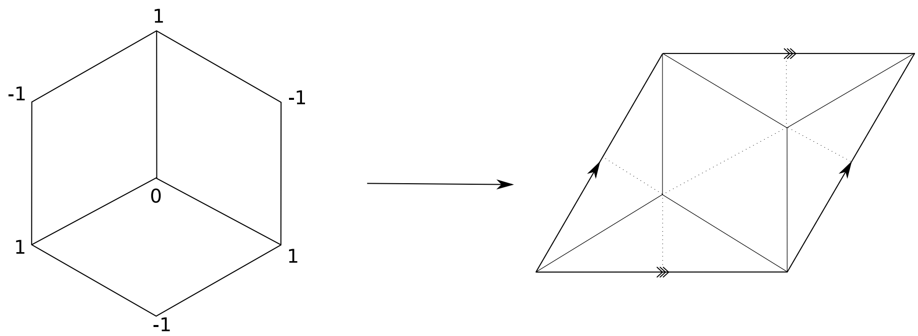

FIGURE 1. The torus above the sphere

The flat metric $\tilde{g}_c$ given by the hexagonal lattice and appearing as the pullback of the metric g_c is very special. First recall that for a closed Riemannian surface the *homotopy systole* is defined as the least length of a non-contractible closed curve and is denoted by sys_π. Loewner's theorem (unpublished, see [Pu52]) states the first known isosystolic inequality:

Theorem 2.3. *For all metric g of class C^1 on $\mathbb{T}^2$,*

$$\textit{area}(\mathbb{T}^2, g) \geq \frac{\sqrt{3}}{2} \text{sys}_\pi(\mathbb{T}^2, g)^2,$$

with equality if and only if $(\mathbb{T}^2, g)$ is homothetic to the flat torus corresponding to the hexagonal lattice $(\mathbb{T}^2, \tilde{g}_c)$.

The study of such isosystolic inequalities is well developped, see [Be00, p.104] for instance.

The systolic geometries of (S^2, g_c) and $(\mathbb{T}^2, \tilde{g}_c)$ are intimately related : to each systole of $(\mathbb{T}^2, \tilde{g}_c)$ (that is a closed geodesic realizing the homotopy systole) corresponds a geodesic loop of (S^2, g_c) whose length equals the general systole (see Figure 2). More precisely,

- Either the systole of $(\mathbb{T}^2, \tilde{g}_c)$ passes trough a point of ramification, and its image by f is a simple geodesic loop with base point a conical singularity,
- Or the systole of $(\mathbb{T}^2, \tilde{g}_c)$ never goes trough a point of ramification, and its image is a figure eight geodesic avoiding conical singularities.

The systoles of $(\mathbb{T}^2, \tilde{g}_c)$ can be classified into three one-parameter families. If we think at $(\mathbb{T}^2, \tilde{g}_c)$ as the quotient of $\mathbb{R}^2$ by the hexagonal lattice spanned by the vectors $(1,0)$ and $(1/2, \sqrt{3}/2)$, the horizontal lines, the lines parallel to the vector $(1/2, \sqrt{3}/2)$ and the lines parallel to the vector $(-1/2, \sqrt{3}/2)$ correspond to these three families of systoles. Observe now that each of these families projects onto the same one-parameter family of geodesic loops of (S^2, g_c). We denote by $\gamma_s(t) = f(t,s)$ this family where $(t,s) \in [0,1] \times [0, \sqrt{3}/2]$.

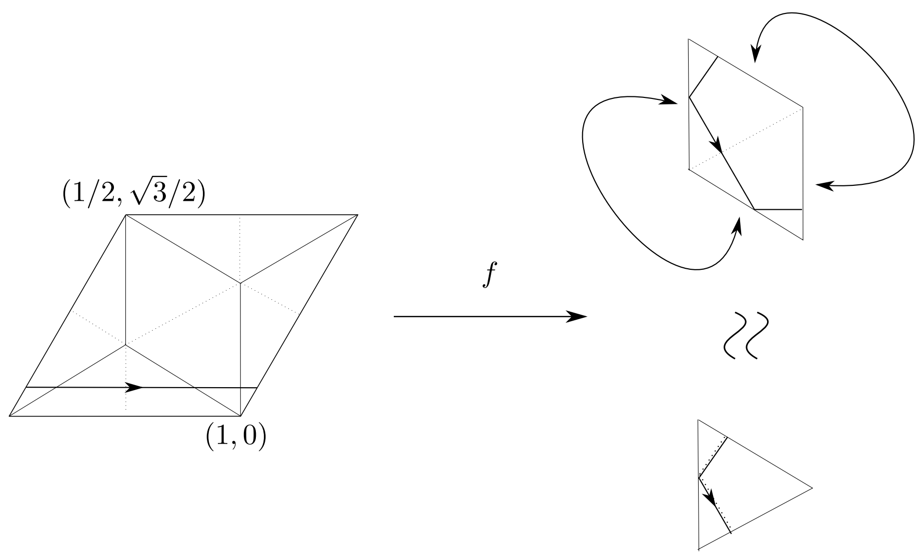

FIGURE 2. Projection of the systoles

2.4. **An optimal systolic inequality for singular spheres.** Fix a metric $g \in \mathcal{M}_{g_c}$. By Loewner's theorem (2.3), the torus $(\mathbb{T}^2, \tilde{g})$ admits a closed geodesic γ whose length satisfies

$$\frac{\sqrt{3}}{2}\, l_{\tilde{g}}(\gamma)^2 \leq \text{area}(\mathbb{T}^2, g).$$

It is straighforward to see that the curve $f(\gamma)$ is a closed geodesic of (S^2, g) except eventually at singular points where the curve $f(\gamma)$ cuts the conical singularity into two angles of exactly $\frac{\pi}{3}$. It seems natural to define for a singular sphere (S^2, g) with $g \in \mathcal{M}_{g_c}$ the systole as the least length of a closed curve which is geodesic except eventually at conical points where the curve must cut the conical point into two angles of $\frac{\pi}{3}$. We still denote it $\mathrm{sys}_0(S^2, g)$. As $\mathrm{area}(\mathbb{T}^2, \tilde{g}) = 3 \cdot \mathrm{area}(S^2, g)$, we obtain the following.

Proposition 2.4. *For any $g \in \mathcal{M}_{g_c}$,*

$$area(S^2, g) \geq \frac{1}{2\sqrt{3}} \mathrm{sys}_0(S^2, g)^2.$$

One may wonder whether Calabi's conjecture can be directly deduced from this result. Unfortunately, this is not the case as the following example shows. Consider the round metric and fix a circle C. Let x_1, x_2 and x_3 be three equidistant points on C and ϵ a positive small number. For each point x_i, we cut a small disk $D(x_i, \epsilon)$ and glue along the remaining boundary a small cone of angle $\frac{2\pi}{3}$. By smoothing the singularities obtained along each curve $\partial D(x_i, \epsilon)$, we obtain a metric g_ϵ of $\mathcal{M}_{g_c}$. So we can apply our proposition and obtain a systole γ_ϵ of (S^2, g_ϵ) satisfying the optimal systolic inequality. But the systole will have length almost $\frac{4\pi}{3}$ as the distance between two of the conical points x_1, x_2 and x_3 is almost $\frac{2\pi}{3}$. If we let ϵ goes to zero, the familly of systoles γ_ϵ will converge to a curve making a turn-and-back from one singularity to another along the circle C. Thus a systole of the round metric can not be obtained as the limit of systoles of the singular metrics (S^2, g_ϵ).

3. Proof of the local diastolic inequality

Let $g = e^{2u} \cdot g_c$ be a metric in $\mathcal{M}_{g_c}$ where $u \in C^1(S^2, \mathbb{R})$. By the Cauchy-Schwarz's inequality and as $dv_{\tilde{g}_c} = dt\, ds$,

$$\begin{aligned}
\int_0^{\sqrt{3}/2} l_g(\gamma_s(t)) ds &= \int_0^{\sqrt{3}/2} \left(\int_0^1 e^{u \circ f(t,s)} dt \right) ds \\
&= \int_{\mathbb{T}^2} e^{u \circ f} dv_{\tilde{g}_c} \\
&= 3 \cdot \int_{S^2} e^u dv_{g_c} \\
&\leq 3 \left(\int_{S^2} e^{2u} dv_{g_c} \right)^{1/2} \left(\int_{S^2} dv_{g_c} \right)^{1/2} \\
&\leq 3 \left(\mathrm{area}(S^2, g) \right)^{1/2} \left(\mathrm{area}(S^2, g_c) \right)^{1/2}.
\end{aligned}$$

So

$$\frac{\sqrt{3}}{2} \inf_{s \in [0, \sqrt{3}/2]} l_g(\gamma_s) \leq 3 \left(\mathrm{area}(S^2, g) \right)^{1/2} \left(\mathrm{area}(S^2, g_c) \right)^{1/2},$$

which gives that

$$\left(\inf_{s \in [0, \sqrt{3}/2]} l_g(\gamma_s) \right)^2 \leq 2\sqrt{3} \cdot \mathrm{area}(S^2, g)$$

as $\text{area}(S^2, g_c) = \frac{1}{2\sqrt{3}}$.

Lemma 3.1. *There exists an open neighborhood $\mathcal{O}$ of g_c in $\mathcal{M}_{g_c}$ such that for all $g \in \mathcal{O}$,*

$$dias(S^2, g) \leq \inf_{s\in[0,\sqrt{3}/2]} l_g(\gamma_s).$$

Proof of the Lemma. To each geodesic loop γ_s of (S^2, g_c) we associate a one-parameter family of one-cycles $\{z_s^\alpha\}$ where $\alpha \in [0, 1]$ such that

- $z_s^{1/2} = \gamma_s$,
- $\{z_s^\alpha\}$ starts and ends at one-cycles made of one or two points,
- $\{z_s^\alpha\}$ induces a generator of $\pi_1(\mathcal{Z}_1(S^2; \mathbb{Z}), \{0\}) \simeq \mathbb{Z}$,
- each z_s^α is made of one or two closed curves and have length

  $$l_{g_c}(z_s^\alpha) = 1 - 2 \cdot |\alpha - 1/2|$$

  where $|\cdot|$ denote the absolute value.

For this we consider two cases.

First case. If $s = 0, \frac{1}{2\sqrt{3}}$ or $\frac{1}{\sqrt{3}}$, then the geodesic loop γ_s goes through a single singularity. Thus γ_s bounds two disks D_1 and D_2 each one containing in its interior a conical singularity. For $i = 1, 2$ we homotope γ_s in D_i to the singularity lying in its interior through a C^0-family $\{z_{i,s}^\beta\}_{\beta\in[0,1]}$ of geodesic loops (whose base point lies on the edge of the triangle) of length

$$l_{g_c}(z_{i,s}^\beta) = 1 - \beta.$$

Then we set

$$z_s^\alpha = \begin{cases} z_{1,s}^{1-2\alpha} & \text{if } \alpha \in [0, \frac{1}{2}] \\ z_{2,s}^{2\alpha-1} & \text{if } \alpha \in [\frac{1}{2}, 1] \end{cases},$$

see Figure 3, First case.

Second case. If $s = \frac{k}{2\sqrt{3}} + s'$ where $k = 0, 1, 2$ and $s' \in]0, \frac{1}{2\sqrt{3}}[$, then γ_s consists of a figure eight geodesic which avoids singularities. So γ_s decomposes into the concatenation of two simple closed geodesic loops $\gamma_{1,s}$ and $\gamma_{2,s}$. Let D_i be the disk bounding by $\gamma_{i,s}$ and containing a single singularity for $i = 1, 2$. Each $\gamma_{i,s}$ can be homotoped in D_i to the singularity lying in its interior through a family $\{z_{i,s}^\beta\}_{\beta\in[0,1]}$ of geodesic loops (whose base point lies on the edge of the triangle) of length

$$l_{g_c}(z_{i,s}^\beta) = (1 - \beta) l_{g_c}(\gamma_{i,s}).$$

In a similar way, the piecewise geodesic $\tilde{\gamma}_s$ obtained by the concatenation of $\gamma_{1,s}$ and $\gamma_{2,s}^{-1}$ bounds an open disk containing a single singularity in its interior, and we denote by D its closure. Again $\tilde{\gamma}_s$ can be homotoped in D to the singularity lying in the interior of D through a family $\{z_s^\beta\}_{\beta\in[0,1]}$ of piecewise geodesics of length

$$l_{g_c}(z_s^\beta) = (1 - \beta).$$

Each piecewise geodesic z_s^β consists of two geodesic arcs $z_{1,s}^\beta$ and $z_{2,s}^\beta$.

Then we set

$$z_s^\alpha = \begin{cases} z_{1,s}^{1-2\alpha} + z_{2,s}^{1-2\alpha} \text{ if } \alpha \in [0, \frac{1}{2}] \\ z_s^{2\alpha-1} \text{ if } \alpha \in [\frac{1}{2}, 1] \end{cases},$$

see Figure 3, Second case.

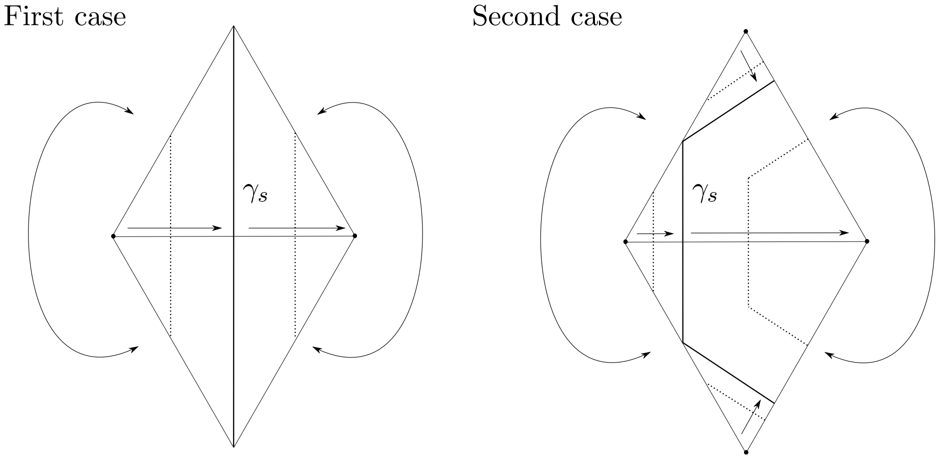

FIGURE 3. Families of one-cycles.

Denote by $\|\cdot\|_\infty$ the uniform norm on $C^1(S^2, \mathbb{R})$. For each $\alpha \in [0, 1/2[\cup]1/2, 1]$, the family of 1-cycles

$$\{z_s^{t/2+(1-t)\alpha} \mid t \in [0, 1]\}$$

covers a domain Ω_s^α whose boundary is the reunion of γ_s and z_s^α. We can lift this domain to a domain $\tilde{\Omega}_s^\alpha$ of $\mathbb{T}^2$ as in Figure 4 which decomposes into two domains $\tilde{\Omega}_{1,s}^\alpha$ and $\tilde{\Omega}_{2,s}^\alpha$. One of this domain is reduced to a point if $s = 0, 1/(2\sqrt{3})$ or $1/\sqrt{3}$. Each domain $\tilde{\Omega}_{i,s}^\alpha$ is bounded by three curves: $\tilde{\gamma}_{i,s}$, $\tilde{z}_{i,s}^\alpha$ and $\tilde{c}_{i,s}$. Note that $\tilde{c}_{i,s}$ is composed of two connected components. The curve $\tilde{c}_{i,s}$ is oriented such that the concatenation of the first connected component of $\tilde{c}_{i,s}$, $\tilde{z}_{i,s}^\alpha$ and the second component of $\tilde{c}_{i,s}$ makes sense and is an arc homotopic to $\tilde{\gamma}_{i,s}$. We endow $\tilde{\Omega}_{i,s}^\alpha$ with the orientation such that its boundary is the concatenation of $\tilde{\gamma}_{i,s}$, the first connected component of $-\tilde{c}_{i,s}$, $-\tilde{z}_{i,s}^\alpha$ and the second component of $-\tilde{c}_{i,s}$. We can easily compute that

$$\text{area}(\tilde{\Omega}_{i,s}^\alpha, \tilde{g}_c) = \frac{1}{4} \tan\frac{\pi}{6} \left(l_{g_c}(\gamma_{i,s})^2 - l_{g_c}(z_{i,s}^\alpha)^2\right).$$

Observe that

$$\text{area}(\tilde{\Omega}_{i,s}^\alpha, \tilde{g}_c) \leq \frac{1}{2} \tan\frac{\pi}{6} \left(l_{g_c}(\gamma_{i,s}) - l_{g_c}(z_{i,s}^\alpha)\right)$$

as $l_{g_c}(\gamma_{i,s}) + l_{g_c}(z_{i,s}^\alpha) \leq 2$.

For all $g = e^{2u} \cdot g_c \in \mathcal{M}_{g_c}$ and by Stokes' Theorem,

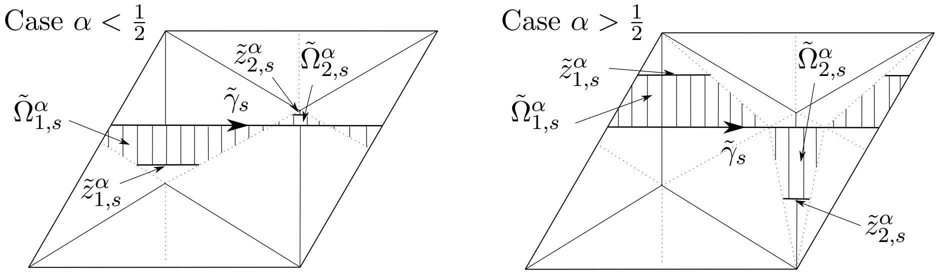

FIGURE 4. Domains.

$$\begin{aligned}
l_g(\gamma_{i,s}) - l_g(z^{\alpha}_{i,s}) &= \int_{\tilde{\gamma}_{i,s}} e^{u\circ f}\, dt - \int_{\tilde{z}^{\alpha}_{i,s}} e^{u\circ f}\, dt \\
&= \int_{\tilde{c}_{i,s}} e^{u\circ f}\, dt + \int_{\tilde{\Omega}^{\alpha}_{i,s}} de^{u\circ f} \wedge dt \\
&= \int_{\tilde{c}_{i,s}} dt + \int_{\tilde{c}_{i,s}} (e^{u\circ f} - 1)\, dt + \int_{\tilde{\Omega}^{\alpha}_{i,s}} \frac{\partial}{\partial s} e^{u\circ f}\, ds \wedge dt \\
&\geq \int_{\tilde{c}_{i,s}} dt - \int_{\tilde{c}_{i,s}} \|e^u - 1\|_\infty\, dt - \sup_{\tilde{\Omega}^{\alpha}_{i,s}} \left| \frac{\partial}{\partial s} e^{u\circ f} \right| \cdot \operatorname{area}(\tilde{\Omega}^{\alpha}_{i,s}, \tilde{g}_c) \\
&\geq (1 - \|e^u - 1\|_\infty - \frac{1}{2} \cdot \tan\frac{\pi}{6} \cdot \sup_{\tilde{\Omega}^{\alpha}_{i,s}} \left| \frac{\partial}{\partial s} e^{u\circ f} \right|)\, (l_{g_c}(\gamma_{i,s}) - l_{g_c}(z^{\alpha}_{i,s}))
\end{aligned}$$

Observe that there exists a positive constant B such that

$$\sup_{\tilde{\Omega}^{\alpha}_{i,s}} \left| \frac{\partial}{\partial s} e^{u\circ f} \right| \leq B \cdot \|de^u\|_0$$

where $\|de^u\|_0 = \sup\{|de^u{}_x(v)| \mid (x,v) \in TS^2 \text{ with } g_0(v,v) = 1\}$ is the uniform norm on closed one-forms on S^2 associated to the round metric g_0. In fact,

$$\left| \frac{\partial}{\partial s} e^{u\circ f} \right| (x) = \left| de^u{}_{f(x)} \circ df_x(\frac{\partial}{\partial s}) \right|$$

for all $x \in \mathbb{T}^2$. But $g_0(df_x(\frac{\partial}{\partial s}), df_x(\frac{\partial}{\partial s}))$ goes to 0 when x goes to a point of ramification. This is clear from the fact that $g_c(df_x(\frac{\partial}{\partial s}), df_x(\frac{\partial}{\partial s})) = 1$ outside singularities and from the following expression of g_0:

$$g_0 = \left(\frac{2}{1 + |z|^2} \right)^2 \cdot (|z+1| \cdot |z| \cdot |z-1|)^{4/3} \cdot g_c$$

where $z \in S^2 = \mathbb{C} \cup \{\infty\}$. So we can set $B = \sup_{x\in\mathbb{T}^2} \sqrt{g_0(df_x(\frac{\partial}{\partial s}), df_x(\frac{\partial}{\partial s}))}$.

Now define the set $\mathcal{O}$ as the reunion of the open sets

$$\{e^{2u} \cdot g_c \in \mathcal{M}_{g_c} \text{ such that } \|e^u - 1\|_\infty < t \text{ and } \frac{1}{2} \tan\frac{\pi}{6}\, B\, \|de^u\|_0 < 1 - t\}$$

for $t \in]0,1[$. The set $\mathcal{O}$ is an open neighborhood of g_c in the C^1-topology and we get

$$l_g(\gamma_s) \geq l_g(z_s^\alpha)$$

for all $g \in \mathcal{O}$, and for all (s, α).

Fix $s' \in [0, \frac{\sqrt{3}}{2}]$ such that $l_g(\gamma_{s'}) = \inf_{s\in[0,\sqrt{3}/2]} l_g(\gamma_s)$. For all $g \in \mathcal{O}$, we obtain

$$\begin{aligned} \text{dias}(S^2, g) &\leq \sup_{\alpha\in[0,1]} l_g(z_{s'}^\alpha) \\ &\leq l_g(\gamma_{s'}) \\ &\leq \inf_{s\in[0,\sqrt{3}/2]} l_g(\gamma_s). \end{aligned}$$

□

Let us finish the proof of the Theorem 1.3. For all $g \in \mathcal{O}$, we have

$$(\text{dias}(S^2, g))^2 \leq 2\sqrt{3} \cdot \text{area}(S^2, g).$$

Now observe that the equality case imposes to the conformal factor e^{2u} to be constant. So g is homothetic to g_c.

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Florent Balacheff - Laboratoire Paul Painlevé, Bt. M2, Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Ascq, France

E-mail address: Florent.Balacheff@math.univ-lille1.fr