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POINTWISE SYMMETRIZATION INEQUALITIES FOR SOBOLEV FUNCTIONS AND APPLICATIONS

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Abstract. We develop a technique to obtain new symmetrization inequalities that provide a unified framework to study Sobolev inequalities, concentration inequalities and sharp integrability of solutions of elliptic equations.

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1. Introduction

Symmetrization is a very useful classical tool in PDE's and the theory of Sobolev spaces. The standard symmetrization inequalities, like many other inequalities in the theory of Sobolev spaces, are often formulated as norm inequalities. One drawback is that these inequalities need to be (re)proven separately for different classes of spaces (e.g. L^p, Lorentz, Orlicz, Lorentz-Karamata, etc.). For this purpose interpolation can be a useful tool, but one may lose information in the extreme cases. Moreover, the end point Sobolev embeddings usually require a different type of spaces (often called "extrapolation spaces"). Thus, for example, the optimal embeddings of L^p based Sobolev spaces on $n-$dimensional Euclidean space are the Lorentz $L(p^*,p)$ spaces, where $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}$, $1\leq p<n$, but for the limiting case $p=n$ it is necessary to replace the Lorentz norms by suitable variants in order to accommodate exponential integrability. One way to deal with this problem is to use pointwise rearrangement inequalities; among the many contributions in this direction here we only mention just a few [54], [113], [114], [67], [9], [18], [52], [4], [35], [33], [78], [79], [106], [74], and refer the reader to the references therein. An added complication arises because different geometries produce different types of optimal spaces: a dramatic example is provided by Gaussian measure, where the optimal target spaces for the embeddings of L^p based Sobolev spaces are the $L^p(LogL)^{p/2}$ spaces (cf. [56], [51], [16], [17], and the references therein). Likewise, in the study of integrability of solutions of elliptic equations, the corresponding optimal results depend on the geometry. As a consequence, although many of the methods used in the treatment of the different cases are similar each case still requires a separate treatment.

In our recent work (cf. [87], [83], [84]) we have developed new symmetrization inequalities that address all these issues and can be applied to provide a unified treatment of sharp Sobolev-Poincaré inequalities, concentration inequalities and sharp integrability of solutions of elliptic equations. Our inequalities combine three basic features, each of which may have been considered before but, apparently, not all of them simultaneously; namely our inequalities are (i) pointwise rearrangement inequalities, (ii) incorporate in their formulation the isoperimetric profile and (iii) are formulated in terms of oscillations.

The first feature (i) allows us to treat without effort the class of all rearrangement invariant function norms. Let us illustrate this point with the classical Pólya-Szegö principle. On $\mathbb{R}^n$ this principle can be informally stated as

$$\left\|\nabla f^{\circ}\right\|_{L^p(\mathbb{R}^n)}\leq\left\|\nabla f\right\|_{L^p(\mathbb{R}^n)},\ 1\leq p\leq\infty, \tag{1.1}$$

where f° is the symmetric rearrangement of f (see (10.17) below). This inequality leaves open the question of what would be the corresponding results for other function norms, indeed, different types of norms are often treated one case at a time in the literature. The formulation of (1.1) we use takes the form

$$\left|\nabla f^{\circ}\right|^{**}(t)\leq\left|\nabla f\right|^{**}(t), \tag{1.2}$$

where $f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)ds$, and f^* is the non increasing rearrangement of f with respect to Lebesgue measure on $\mathbb{R}^n$. The point is that (1.2) readily implies

$$\left\|\nabla f^{\circ}\right\|_{X(\mathbb{R}^n)}\leq\left\|\nabla f\right\|_{X(\mathbb{R}^n)}, \tag{1.3}$$

for all rearrangement invariant spaces X on $\mathbb{R}^n$ (see Section 2.1 below).

The fact that our inequalities incorporate the isoperimetric profile [feature (ii)] allows us to treat different geometries from a unified point of view. Indeed, it is the isoperimetric profile itself that helps us determine the correct function spaces! For example, as we show below (cf. Theorem 1), the isoperimetric inequality can be reformulated on metric probability spaces (Ω, d, μ), (cf. [84], and also [15], [67], [87], [83], for Euclidean or Gaussian versions, see also [39] for a somewhat different perspective) as follows[1]

$$f_{\mu}^{**}(t) - f_{\mu}^{*}(t) \leq \frac{t}{I(t)} \left|\nabla f\right|_{\mu}^{**}(t), \tag{1.4}$$

where $f_{\mu}^{**}(t) = \frac{1}{t}\int_0^t f_{\mu}^{*}(s)ds$, and f_{μ}^{*} is the non increasing rearrangement of f with respect to the measure μ and $I(t) = I_{(\Omega,d,\mu)}(t)$ is the corresponding isoperimetric profile. If we apply a rearrangement invariant function norm X on Ω (see Section 2.1 below) to (1.4) we obtain Sobolev-Poincaré type estimates of the form[2]

$$\|f\|_{LS(X)} := \left\| \left(f_{\mu}^{**}(t) - f_{\mu}^{*}(t)\right) \frac{I(t)}{t} \right\|_{\bar{X}} \leq \left\| \left|\nabla f\right|_{\mu}^{**} \right\|_{\bar{X}}. \tag{1.5}$$

These embeddings turn out to be best possible in all the classical cases, at least for spaces that are far from L^1 (the integrated form of (1.4) can be used to cope with this problematic end point as well, see Proposition 1 below and [87] for the Euclidean case). To see how the isoperimetric profile helps to determine the correct spaces consider the following basic model cases: (a) $\mathbb{R}^n$ with Euclidean measure, if we let $X = L^p$, $1 \leq p \leq n$, and let p^* be the usual Sobolev exponent defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, then[3]

$$\left\| \left(f^{**}(t) - f^{*}(t)\right) \frac{I(t)}{t} \right\|_{L^p} \simeq \left\| \left(f^{**}(t) - f^{*}(t)\right) \right\|_{L(p^*,p)}, \tag{1.6}$$

follows from the fact that $I(t) = c_n t^{1-1/n}$, and Hardy's inequality.

(b) $\mathbb{R}^n$ with Gaussian measure γ_n, here if we let $X = L^p$, $1 \leq p < \infty$, then (compare with [56], [51]), since $I(t) \simeq t(\log 1/t)^{1/2}$ for t near zero, we have

$$\left\| \left(f_{\gamma_n}^{**}(t) - f_{\gamma_n}^{*}(t)\right) \frac{I(t)}{t} \right\|_{L^p} \simeq \left\| \left(f_{\gamma_n}^{**}(t) - f_{\gamma_n}^{*}(t)\right) \right\|_{L^p(Log)^{p/2}}. \tag{1.7}$$

We note that feature (iii) allows us to use systematically spaces that are defined in terms of oscillations (cf. [19], [15], [97]) so that, in particular, we can treat the borderline cases in a unified fashion. For example, in the Gaussian case (1.7) we can let $p = \infty$, and we obtain the concentration result (cf. [83])

$$f \in Lip(\mathbb{R}^n) \Rightarrow \left\| \left(f_{\gamma_n}^{**}(t) - f_{\gamma_n}^{*}(t)\right) \frac{I(t)}{t} \right\|_{L^\infty} < \infty \Rightarrow f \in e^{L^2}; \tag{1.8}$$

[1] Although the Euclidean version of (1.4) is implicitly proven in [4] it is not used in this form in that paper.

[2] The spaces $\bar{X}$ are defined in Section 2.1 below.

[3] Here the symbol $f \simeq g$ indicates the existence of a universal constant $c > 0$ (independent of all parameters involved) such that $(1/c)f \leq g \leq c\, f$. Likewise the symbol $f \preceq g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq c\, g$.

while on $\mathbb{R}^n$ with Euclidean measure, $p^* = \infty$ is allowed in (1.6), indeed, when $p = n$, our condition is optimal[4] (cf. [15]) and reads[5]

$$f \in W_1^n(\mathbb{R}^n) \Rightarrow \|f^{**}(t) - f^*(t)\|_{L(\infty,n)} < \infty \Rightarrow f \in e^{L^{n'}}. \tag{1.9}$$

It also follows that if the isoperimetric profile does not depend on the dimension (e.g. this is case in the Gaussian case) then these inequalities are "dimension free".

Returning to the Pólya-Szegö inequality (1.2) note that, by construction, the inequality requires the choice of a distinguished rearrangement. A *posteriori*, one can see that the choice of the optimal symmetric rearrangement in (1.1) is ultimately connected with the solution of the isoperimetric problem on $\mathbb{R}^n$. Thus, it is not surprising that the corresponding inequality in the Gaussian case also requires a special rearrangement that is connected with the corresponding solution of the Gaussian isoperimetric problem (cf. [27], [111], [49], [34], and the references therein, and also [83] for a more recent treatment).

More generally, to obtain a general version of the Pólya-Szegö principle valid on metric spaces, we divide the problem at hand in two. First, we derive a general inequality that does not require us to make a specific choice of rearrangements but involves the isoperimetric profile, namely (cf. Theorem 1 below)

$$\int_0^t ((-f_\mu^*)'(\cdot)I(\cdot))^*(s)ds \preceq \int_0^t |\nabla f|_\mu^*(s)ds,$$

where the second rearrangement on the left hand side is with respect to the Lebesgue measure on $(0,1)$. The second step requires the construction of a suitable rearrangement. At this point we only know how to construct special rearrangements for some model cases. For more on this see the discussion in Section 4, where we consider in detail three important model examples: (a) measures on $\mathbb{R}^n$ which are products of measures of the form

$$\mu_\Phi(x) = Z_\Phi^{-1} \exp\left(-\Phi(|x|)\right) dx, \qquad x \in \mathbb{R},$$

where Φ is convex and $\sqrt{\Phi}$ is concave and where Z_Φ^{-1} is a normalization constant chosen to ensure that $\mu_\Phi(\mathbb{R}) = 1$; (b) the $n-$sphere $\mathbb{S}^n$, and (c) the model spaces of Ros [107], which we have recently studied in connection of Poincaré inequalities and concentration (cf. [86]). In each of these model cases we show that a suitable version of the Pólya-Szegö principle (1.3) holds.

In Section 5 we derive Poincaré inequalities and, using the results of Section 4, we show their sharpness in the model cases. A typical result in this section takes the form (cf. (1.5) above, and Theorem 5 below)

$$\left\| g - \int_\Omega g d\mu \right\|_{LS(X)} \preceq \|\nabla g\|_X .$$

We noticed that in the model cases these Poincaré inequalities are characterized by certain Hardy operators (="isoperimetric Hardy operators") associated with the corresponding isoperimetric profiles. This led us to single out the metric probability

[4] Thus our conditions slightly improve the exponential integrability of the borderline cases. More generally, this feature makes our inequalities and spaces relevant for the theory of concentration of inequalities (cf. [70], [83]).

[5] It is of interest to compare (1.8) and (1.9) as $n \to \infty$. Indeed, when $n \to \infty$ then $n' \to 1$ and the Euclidean result (1.9) formally converges to exponential integrability; while the Gaussian e^{L^2} integrability remains constant as $n \to \infty$.

spaces of "isoperimetric Hardy type" (cf. [86]): these are exactly the spaces where the Poincaré inequalities can be characterized in this fashion. In Section 6 we show that the remarkable equivalences between isoperimetry, Poincaré inequalities and concentration proved recently by E. Milman (cf. [93], [92], [94] and the references therein) for Riemannian manifolds satisfying suitable convexity conditions, hold for spaces of "isoperimetric Hardy type". We believe this is of interest since it offers a conceptual simplification, easier proofs, as well as an extension of the equivalences to the general metric setting, where some of the convexity concepts used by E. Milman are still to be developed.

Isoperimetric Hardy type also plays a fundamental role in Section 7, where we develop a simple transference principle that allows us to transfer Poincaré inequalities from one metric space to another, if we have a suitable majorization of the corresponding isoperimetric profiles. More precisely, we show that if for two metric probability spaces we have

$$I_{(\Omega_1,d_1,\mu_1)}(t) \geq cI_{(\Omega,d,\mu)}(t),\ t \in (0,1/2],$$

and (Ω, d, μ) is of isoperimetric Hardy type then any Poincaré inequality of the form

$$\left\| g - \int_\Omega g d\mu \right\|_{Y(\Omega)} \leq c \left\| |\nabla g| \right\|_{X(\Omega)},\ \text{for all } g \in Lip(\Omega),$$

can be transferred to a corresponding Poincaré inequality for Ω_1 (cf. Theorem 14),

$$\left\| g - \int_{\Omega_1} g d\mu_1 \right\|_{Y(\Omega_1)} \leq c \left\| |\nabla g| \right\|_{X(\Omega_1)},\ \text{for all } g \in Lip(\Omega_1).$$

This easy to formulate principle thus allows for the transference of Poincaré inequalities from all the model cases discussed above. For example, the Levi-Gromov isoperimetric inequality implies that Poincaré inequalities for the $n-$sphere can be transferred to compact connected manifolds with Ricci curvature bounded from below by $\rho > 0$ (cf. Corollary 1). Likewise, Poincaré inequalities valid for $\mathbb{R}^n$ with Gaussian measure (cf. [83]) can be transferred to Riemannian manifolds (M, d) for which (cf. Corollary 3)

$$I_{(M,d)}(t) \geq ct(\log \frac{1}{t})^{1/2},\ t \in (0,1/2].$$

In the same vein we can transfer Poincaré inequalities valid for $(\mathbb{R}^n, \mu_p^{\otimes n})$ with $\mu_p(x) = Z_p^{-1} \exp\left(-|x|^p\right) dx, 1 < p \leq 2$, this leads to simplifications to recent results of [14] (cf. Corollary 2). When the first version of our manuscript was being typed we received a query from Professor Hans Triebel concerning certain Sobolev inequalities with dimension free constants. We give a brief answer to some of Prof. Triebel's questions in Section 7.1.

In a different direction, in Section 8 we extend E. Milman's methods (based on the use of semigroup technique of Ledoux and Bakry and Ledoux (cf. [71], [72], [73], [6], and the references therein) to estimate isoperimetric profiles associated with functional inequalities involving r.i. spaces.

In Section 9, motivated by the results and methods of Gallot [52] (cf. also [112] and [9]), we extend our results and prove inequalities for the Laplacian. For example, the corresponding extension of (1.4) is given by

$$f_{\mu}^{**}(t)-f_{\mu}^{*}(t)\leq\frac{1}{t}\int_{0}^{t}\left(\frac{s}{I(s)}\right)^{2}\left|\Delta f\right|_{\mu}^{**}(s)ds. \tag{1.10}$$

When $I(t)$ is concave, a global standing assumption in this paper, then (1.10) implies the more suggestive inequality (compare with (1.4))

$$f_{\mu}^{**}(t)-f_{\mu}^{*}(t)\leq\left(\frac{t}{I(t)}\right)^{2}\frac{1}{t}\int_{0}^{t}\left|\Delta f\right|_{\mu}^{**}(s)ds.$$

As a consequence we obtain higher order Sobolev-Poincaré inequalities of the form

$$\left\|\left(f_{\mu}^{**}(t)-f_{\mu}^{*}(t)\right)\left(\frac{I(t)}{t}\right)^{2}\right\|_{\bar{X}}\preceq\left\||\Delta f|\right\|_{X}.$$

These inequalities are thus easy to iterate to produce inequalities involving higher order derivatives and lead to new sharp higher order embeddings for Sobolev spaces based on r.i. spaces. Once again the results are sharp and include sharpenings of the borderline cases. Our results in this direction extend and unify earlier Euclidean results (cf. [41], [47], [35], [97], [80] and the references therein), as well as L^{p} and Orlicz Gaussian results (cf. [51], [7], [8], [109]).

Using variants of techniques developed by Maz'ya [91], and Talenti and his school (cf. [112], [113], [114], [115], [3] and the references therein), the higher order results of Section 9 can be considerably extended in order to study the sharp integrability of *apriori* solutions of non-linear elliptic equations of the form

$$\begin{cases} -div(a(x,u,\nabla u))=fw & \text{in } \Delta,\\ u=0 & \text{on } \partial\Delta, \end{cases} \tag{1.11}$$

where Δ is an open set of $\mathbb{R}^{n}$ $(n\geq 2)$, w is a nonnegative measurable function on $\mathbb{R}^{n}$, such that the measure $\mu=w(x)dx$, is a probability measure, $a(x,\eta,\xi):\Delta\times\mathbb{R}\times\mathbb{R}^{n}\rightarrow\mathbb{R}^{n}$ is a Carathéodory function such that,

$$a(x,t,\xi).\xi\geq w(x)\left|\xi\right|^{p},\quad for\ a.e.\ \ x\in\Delta,\ \forall\eta\in\mathbb{R},\ \forall\xi\in\mathbb{R}^{n}.$$

This material is developed in Section 10 where we consider *apriori* estimates of entropy solutions of (1.11). For example, for $p=2$, we show that an entropic solution of (1.11) satisfies

$$\left\|\left(u_{\mu}^{**}(t)-u_{\mu}^{*}(t)\right)\left(\frac{I(t)}{t}\right)^{2}\right\|_{\bar{X}}\preceq\left\|f_{\mu}^{**}\right\|_{\bar{X}},$$

from where we can obtain sharp *apriori* integrability results for entropy solutions. Moreover, we also obtain estimates on the regularity of the gradient. For example, extending results in [3] we have (cf. Theorem 19 below)

$$\left|\nabla u\right|_{\mu}^{*}(t)\leq\left(\frac{2}{t}\int_{t/2}^{\mu(\Delta)}\left(\frac{I(s)}{s}f_{\mu}^{**}(s)\right)^{2}ds\right)^{1/2},$$

These estimates can be used to obtain norm estimates under suitable assumptions on $\bar{X}$ (cf. Theorem 19 below):

$$\left\|\frac{I(t)}{t}\left|\nabla u\right|_{\mu}^{*}(t)\right\|_{\bar{X}}\preceq\left\|f_{\mu}^{**}\right\|_{\bar{X}}.$$

Again we point out that the isoperimetric profile determines the nature of the correct integrability conditions.

In Section 11 we discuss the connection between Maz'ya's capacitary inequalities and the method of symmetrization by truncation. We conclude in Section 12 by recording a few (and only a few) bibliographical notes.

Finally a few words about the techniques. A common method to obtain rearrangement inequalities is via interpolation or extrapolation (cf. [32], [61]) however these methods do not necessarily produce the best possible end point results. Maz'ya [88] has shown that Sobolev inequalities self improve using his technique of smooth cut-offs. In a different direction, Maz'ya, and independently Federer and Fleming, (cf. [88], [50]), also showed the equivalence between isoperimetry and Sobolev embeddings. It is easy to see that these ideas are closely related. Indeed, consider the following three versions of the Gagliardo-Nirenberg inequality in increasing order of precision

$$\|f\|_{L(n',\infty)} \preceq \|\nabla f\|_1 \,, \text{ weak type Gagliardo-Nirenberg} \tag{1.12}$$

$$\|f\|_{L^{n'}} \preceq \|\nabla f\|_1 \,, \text{ classical Gagliardo-Nirenberg} \tag{1.13}$$

$$\|f\|_{L(n',1)} \preceq \|\nabla f\|_1 \,, \text{ sharp Gagliardo-Nirenberg,} \tag{1.14}$$

and note that for $f = \chi_A$ the left hand sides of (1.12), (1.13), (1.14) are all equal to $|A|^{1/n'}$, while the right hand sides are always a multiple of $Per(A)$, the perimeter of A. Thus, disregarding constants, the Maz'ya-Federer-Fleming equivalence theorem shows that (1.12) automatically self improves to (1.14).

Although in this paper we don't formally use interpolation/extrapolation theory we borrow one basic idea from this field that originates in the work of Calderón [32] (cf. also [20]), in PDE's this idea also appears in the work of Talenti ([113] and [114], see also Section 10.1 below), and was somewhat later taken up in the extrapolation theory of Jawerth-Milman [61]; namely that families of inequalities can be characterized in terms of pointwise rearrangement inequalities. Indeed, in Calderón's program [32] families of inequalities for a given operator are characterized in terms of pointwise rearrangement inequalities from which each individual functional norm inequalities follows readily. The point is that one norm inequality is not enough to effect this characterization.

Take the inequalities (1.12), (1.13), (1.14), which as we have argued above, are, in some sense, equivalent, in this case the "correct" way to express this phenomenon is via the rearrangement inequality (1.4). The technique to prove this equivalence uses systematically Maz'ya's smooth truncations method as a tool to obtain rearrangement inequalities ("symmetrization by truncation"). We notice parenthetically that truncations are also a basic tool in interpolation/extrapolation theory (for more on this see Section 3).

2. Background

We use for the most part a standard notation. For the discussion on metric spaces it will simplify the discussion somewhat to consider only probability spaces, a convention we keep for the rest of the paper.

We consider metric spaces (Ω, d, μ) equipped with a separable Borel probability measure μ. For measurable functions $u : \Omega \to \mathbb{R}$, the distribution function of u is given by

$$\mu_u(t) = \mu\{x \in \Omega : |u(x)| > t\} \qquad (t > 0).$$

The **decreasing rearrangement** u_μ^* of u is the right-continuous non-increasing function from $[0,\infty)$ into $[0,\infty]$ which is equimeasurable with u. Namely,

$$u_\mu^*(s) = \inf\{t \geq 0 : \mu_u(t) \leq s\}.$$

It is easy to see that for any measurable set $E \subset \Omega$

$$\int_E |u(x)|\, d\mu \leq \int_0^{\mu(E)} u_\mu^*(s) ds.$$

In fact, the following stronger property holds (cf. [20]),

$$\sup_{\mu(E)\leq t} \int_E |u(x)|\, d\mu = \int_0^{\mu(E)} u_\mu^*(s) ds. \tag{2.1}$$

Since u_μ^* is decreasing, the function u_μ^{**}, defined by

$$u_\mu^{**}(t) = \frac{1}{t}\int_0^t u_\mu^*(s) ds,$$

is also decreasing and, moreover,

$$u_\mu^* \leq u_\mu^{**}.$$

On occasion, when rearrangements are taken with respect to the Lebesgue measure or when the measure is clear from the context, we may omit the measure and simply write u^* and u^{**}, etc.

For a Borel set $A \subset \Omega$, the **perimeter** or **Minkowski content** of A is defined by

$$\mu^+(A) = \lim\inf_{h\to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x, A) < h\}$.

The **isoperimetric profile** $I_{(\Omega,d,\mu)}$ is defined as the pointwise maximal function $I_{(\Omega,d,\mu)} : [0,1] \to [0,\infty)$ such that

$$\mu^+(A) \geq I_{(\Omega,d,\mu)}(\mu(A)),$$

holds for all Borel sets A. A set A for which equality above is attained will be called an **isoperimetric domain.**

Condition 1. *It will facilitate our discussion to make some basic assumptions on the isoperimetric functions associated with the metric probability spaces considered in this paper. We will assume throughout the paper that the isoperimetric profile* $I_{(\Omega,d,\mu)}$ *is a concave continuous function, increasing on* $(0,1/2)$, *symmetric about the point* $1/2$ *that, moreover, vanishes at zero. We remark that these assumptions are fulfilled for a large class of metric spaces*[6].

A continuous, concave function, $I : [0,1] \to [0,\infty)$, increasing on $(0,1/2)$ and symmetric about the point $1/2$, with $I(0) = 0$, and such that

$$I_{(\Omega,d,\mu)} \geq I,$$

will be called an **isoperimetric estimator** for (Ω, d, μ).

[6]These assumptions are satisfied for the classical examples (cf. [26] [93], and the references therein)

For a Lipschitz function f on Ω (briefly $f \in Lip(\Omega)$) we define, as usual, the **modulus of the gradient** by

$$|\nabla f(x)| = \limsup_{d(x,y)\to 0} \frac{|f(x)-f(y)|}{d(x,y)}, \tag{2.2}$$

and zero at isolated points[7].

2.1. **Rearrangement invariant spaces.** We recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces and refer the reader to [20], [68], as well as [103], [104] and [105], for a complete treatment. We say that a Banach function space $X = X(\Omega)$ on (Ω, d, μ) is rearrangement-invariant (r.i.) space, if $g \in X$ implies that all $\mu-$measurable functions f with the same rearrangement function with respect to the measure μ, i.e. such that $f_\mu^* = g_\mu^*$, also belong to X, and, moreover, $\|f\|_X = \|g\|_X$.

Since $\mu(\Omega) = 1$, for any r.i. space $X(\Omega)$ we have

$$L^\infty(\Omega) \subset X(\Omega) \subset L^1(\Omega), \tag{2.3}$$

with continuous embeddings.

An r.i. space $X(\Omega)$ can be represented by a r.i. space on the interval $(0,1)$, with Lebesgue measure, $\bar{X} = \bar{X}(0,1)$, such that

$$\|f\|_X = \|f_\mu^*\|_{\bar{X}},$$

for every $f \in X$. A characterization of the norm $\|\cdot\|_{\bar{X}}$ is available (see [20, Theorem 4.10 and subsequent remarks]). Typical examples of r.i. spaces are the L^p-spaces, Lorentz spaces and Orlicz spaces.

A useful property of r.i. spaces states that if

$$\int_0^r f_\mu^*(s)ds \le \int_0^r g_\mu^*(s)ds, \quad \text{holds for all } r > 0,$$

then, for any r.i. space $X = X(\Omega)$,

$$\|f\|_X \le \|g\|_X .$$

The associate space $X'(\Omega)$ of $X(\Omega)$ is the r.i. space of all measurable functions h in for which the r.i. norm given by

$$\|h\|_{X'(\Omega)} = \sup_{g\neq 0} \frac{\int_\Omega |g(x)h(x)|\, d\mu}{\|g\|_{X(\Omega)}} \tag{2.4}$$

is finite. Note that by the definition (2.4), the generalized Hölder inequality

$$\int_\Omega |g(x)h(x)|\, d\mu \le \|g\|_{X(\Omega)} \|h\|_{X'(\Omega)}$$

holds.

The **fundamental function** of X is defined by

$$\phi_X(s) = \|\chi_E\|_X ,$$

[7] In fact one can define $|\nabla f|$ for functions f that are Lipschitz on every ball in (Ω, d) (cf. [26, pp. 184, 189] for more details).

where E is any measurable subset of Ω with $\mu(E)=s$. We can assume without loss of generality that ϕ_X is concave. Moreover,

$$\phi_{X'}(s)\phi_X(s)=s. \tag{2.5}$$

For example, if X is a an Orlicz space, $X=L_N$, say (N is a Young's function), then

$$\phi_{L_N}(t)=1/N^{-1}(1/t). \tag{2.6}$$

Associated with an r.i. space X there are some useful Lorentz and Marcinkiewicz spaces, namely the Lorentz and Marcinkiewicz spaces defined by the quasi-norms

$$\|f\|_{M(X)}=\sup_t f^*(t)\phi_X(t), \quad \|f\|_{\Lambda(X)}=\int_0^1 f^*(t)d\phi_X(t).$$

Notice that

$$\phi_{M(X)}(t)=\phi_{\Lambda(X)}(t)=\phi_X(t),$$

and that

$$\Lambda(X)\subset X\subset M(X). \tag{2.7}$$

Let $p>0$ and let X be a r.i. space on Ω; the $p-$**convexification** $X^{(p)}$ of X, (cf. [76]) is defined by

$$X^{(p)}=\{x:|x|^p\in X\}, \quad \|x\|_{X^{(p)}}=\||x|^p\|_X^{1/p}.$$

We will say that X is $p-$**convex** if and only if $X^{(1/p)}$ is a Banach space.

Classically conditions on r.i. spaces are formulated in terms of the Hardy operators defined by

$$Pf(t)=\frac{1}{t}\int_0^t f(s)ds; \quad Q_af(t)=\frac{1}{t^a}\int_t^\infty s^a f(s)\frac{ds}{s}, \quad 0\le a<1,$$

(if $a=0$, we shall write Q instead of Q_0), the boundedness of these operators on r.i. spaces can be simply described in terms of the so called Boyd indices defined by

$$\bar{\alpha}_X=\inf_{s>1}\frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X=\sup_{s<1}\frac{\ln h_X(s)}{\ln s},$$

where $h_X(s)$ denotes the norm of the dilation operator on $\bar{X}$ of the dilation operator E_s, $s>0$, defined by

$$E_sf(t)=\begin{cases} f^*(\frac{t}{s}) & 0<t<s, \\ 0 & s<t<1 \end{cases}.$$

The operator E_s is bounded on $\bar{X}$ for every r.i. space $X(\Omega)$ and for every $s>0$; moreover,

$$h_X(s)\le \max(1,s). \tag{2.8}$$

For example, if $X=L^p$, then $\overline{\alpha}_{L^p}=\underline{\alpha}_{L^p}=\frac{1}{p}$. It is well known that if X is a r.i. space,

$$\begin{array}{c} P \text{ is bounded on } \bar{X} \Leftrightarrow \overline{\alpha}_X<1, \\ Q_a \text{ is bounded on } \bar{X} \Leftrightarrow \underline{\alpha}_X>a. \end{array} \tag{2.9}$$

Finally, the following result will be useful in Section 10

Lemma 1. *Let Y be a r.i, space, let $q>0$ and let $w(s)$ be a monotone function. Then*

$$\left\Vert \left(\frac{1}{t}\int_{t}^{1}\left(w(s)f^{\ast}(s)\right)^{q}ds\right)^{1/q}\right\Vert_{Y}\leq c\left\Vert wf\right\Vert_{Y}\quad \textit{if}\quad \underline{\alpha}_{X}>1/q.$$

Proof. Is an elementary adaptation of the main result in [99]. □

Remark 1. *In Section 6.1 and Section 10 we introduce new Hardy operators that are associated with isoperimetric profiles and will play an important a role in our theory.*

In [83] and [84] we introduced the "isoperimetric" spaces $LS(X)$ defined by the condition

$$\left\Vert f\right\Vert_{LS(X)}:=\left\Vert \left(f_{\mu}^{\ast\ast}(t)-f_{\mu}^{\ast}(t)\right) \frac{I(t)}{t}\right\Vert_{\bar{X}}<\infty.$$

The inequality (1.5) can be thus reformulated as

$$\left\Vert f\right\Vert_{LS(X)}\leq \left\Vert P(\left\vert \nabla f\right\vert_{\mu}^{\ast})\right\Vert_{\bar{X}}. \tag{2.10}$$

The $LS(X)$ spaces not only give sharp embedding theorems that include borderline cases but, due to the fact that their definition incorporates the isoperimetric profile, these spaces automatically "select" the correct optimal type of spaces associated with the corresponding geometry[8].

The concept of median plays a role in the study of Poincaré inequalities (cf. Section 5)

Definition 1. *Let f be a measurable function, a real number m will be called a* ***median*** *of f if*

$$\mu\left\{ f\geq m\right\} \geq 1/2\quad \textit{and}\ \mu\left\{ f\leq m\right\} \geq 1/2.$$

For most purposes to prove Poincaré inequalities (see (5.1) below) it makes no difference if we work with the median m or use the "expectation" $\int_{\Omega}fd\mu$. We record this fact in the next lemma[9]

Lemma 2. *Let X be a r.i. space on Ω. Then,*

$$\frac{1}{2}\left\Vert f-\int_{\Omega}fd\mu\right\Vert_{X}\leq \left\Vert f-m\right\Vert_{X}\leq 3\left\Vert f-\int_{\Omega}fd\mu\right\Vert_{X}.$$

Proof. By (2.3) we have

$$\left\vert \int_{\Omega}fd\mu-m\right\vert \leq \int_{\Omega}\left\vert f-m\right\vert d\mu\leq \left\Vert f-m\right\Vert_{X},$$

thus,

$$\begin{aligned}
\left\Vert f-\int_{\Omega}fd\mu\right\Vert_{X} &= \left\Vert f-m+\int_{\Omega}fd\mu+m\right\Vert_{X}\\
&\leq \left\Vert f-m\right\Vert_{X}+\left\vert \int_{\Omega}fd\mu-m\right\vert \\
&\leq 2\left\Vert f-m\right\Vert_{X}.
\end{aligned}$$

---

[8] In particular see the discussion right after (1.5) above. In the classical borderline cases these isoperimetric spaces capture exponential integrability conditions and thus seem to have a natural role in concentration inequalities (cf. example 5.2, and [70], [83]).

[9] Although the result is known we include a proof for the sake of completeness.

To prove the converse we can assume that $m \geq \int_{\Omega} f d\mu$ (otherwise exchange f by $-f$). Therefore, by Chebyshev's inequality, we have

$$\begin{aligned} 1/2 &\leq \mu\{f \geq m\} \\ &\leq \mu\left\{\left|f - \int_{\Omega} f d\mu\right| \geq m - \int_{\Omega} f d\mu\right\} \\ &\leq \frac{1}{(m - \int_{\Omega} f d\mu)} \int_{\Omega} \left|f - \int_{\Omega} f d\mu\right| d\mu. \end{aligned}$$

Consequently,

$$\left(m - \int_{\Omega} f d\mu\right) \leq 2 \left\|f - \int_{\Omega} f d\mu\right\|_X ,$$

which implies

$$\left\|m - \int_{\Omega} f d\mu\right\|_X \leq 2 \left\|f - \int_{\Omega} f d\mu\right\|_X .$$

Therefore,

$$\begin{aligned} \|f - m\|_X &= \left\|f - \int_{\Omega} f d\mu - m + \int_{\Omega} f d\mu\right\|_X \\ &\leq 3 \left\|f - \int_{\Omega} f d\mu\right\|_X . \end{aligned}$$

□

3. Symmetrization using truncation and Isoperimetry

The characterization of norm inequalities in terms of pointwise rearrangement inequalities is a theme that seems to have originated in Interpolation theory. In PDE's this idea appears prominently in the work of Talenti (cf. [113] and [114]) where it appears as a comparison principle. In interpolation theory this method was developed in Calderón's masterful paper [32] (cf. also [20]), this idea is also important in the extrapolation theory developed in [61]. Interestingly, while in our work we try to characterize Sobolev norm inequalities in terms of rearrangement inequalities, we generally don't use interpolation/extrapolation. In fact, the smooth cut-off method, an idea apparently originating in the work of Maz'ya [88] (cf. also [5], [57], [116], and the references therein), shows that Sobolev inequalities have remarkable self improving properties[10]. Combining these ideas with a basic technique of interpolation/extrapolation (i.e. cutting off at levels dependent on the rearrangement of the function to which we apply the cut-off itself!) we developed the technique of "symmetrization by truncation". The main result in this section is a natural extension of similar, somewhat less general results, we obtained elsewhere (cf. [87], [83]).

Theorem 1. *Let $I : [0, 1] \to [0, \infty)$ be an isoperimetric estimator on (Ω, d, μ). The following statements hold and are in fact equivalent:*

[10]In some sense this implies that a Sobolev inequality carries the information of a family of Sobolev inequalities. If this is combined with the chain rule one can see that one Sobolev inequality also carries the "reiteration" property. Therefore, from our point of view, Sobolev inequalities need not be interpolated but can be "extrapolated".

(1) *Isoperimetric inequality:* $\forall A \subset \Omega$, *Borel set,*

$$\mu^{+}(A) \geq I(\mu(A)). \tag{3.1}$$

(2) *Ledoux's inequality:* $\forall f \in Lip(\Omega)$,

$$\int_0^\infty I(\mu_f(s))ds \leq \int_\Omega |\nabla f(x)|\, d\mu. \tag{3.2}$$

(3) *Maz'ya's inequality*[11]*:* $\forall f \in Lip(\Omega)$,

$$(-f_\mu^*)'(s)I(s) \leq \frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}} |\nabla f(x)|\, d\mu. \tag{3.3}$$

(4) *Pólya-Szegö's inequality:* $\forall f \in Lip(\Omega)$,

$$\int_0^t ((-f_\mu^*)'(.)I(.))^*(s)ds \leq \int_0^t |\nabla f|_\mu^*(s)ds. \tag{3.4}$$

(The second rearrangement on the left hand side is with respect to the Lebesgue measure).

(5) *Oscillation inequality:* $\forall f \in Lip(\Omega)$,

$$(f_\mu^{**}(t) - f_\mu^*(t)) \leq \frac{t}{I(t)} |\nabla f|_\mu^{**}(t). \tag{3.5}$$

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$. Note that $f \in Lip(\Omega)$ implies that $|f| \in Lip(\Omega)$, and, moreover, we have (cf. (2.2))

$$|\nabla f(x)| \geq |\nabla |f|(x)|.$$

By the co-area inequality applied to $|f|$ (cf. [26, Lemma 3.1]), and the isoperimetric inequality (3.1), it follows that

$$\begin{aligned}\int_\Omega |\nabla f(x)|\, d\mu &\geq \int_\Omega |\nabla |f|(x)|\, d\mu \geq \int_0^\infty \mu^{+}(\{|f|>s\})ds \\ &\geq \int_0^\infty I(\mu_f(s))ds\ .\end{aligned}$$

$(\mathbf{2}) \Rightarrow (\mathbf{3})$. Let $0 < t_1 < t_2 < \infty$. The smooth truncations of f are defined by

$$f_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1 & \text{if } |f(x)| \geq t_2, \\ |f(x)| - t_1 & \text{if } t_1 < |f(x)| < t_2, \\ 0 & \text{if } |f(x)| \leq t_1. \end{cases}$$

Applying (3.2) to $f_{t_1}^{t_2}$ we obtain,

$$\int_0^\infty I(\mu_{f_{t_1}^{t_2}}(s))ds \leq \int_\Omega \left|\nabla f_{t_1}^{t_2}(x)\right| d\mu.$$

We obviously have

$$\left|\nabla f_{t_1}^{t_2}\right| = |\nabla |f||\, \chi_{\{t_1<|f|<t_2\}},$$

and, moreover,

$$\int_0^\infty I(\mu_{f_{t_1}^{t_2}}(s))ds = \int_0^{t_2-t_1} I(\mu_{f_{t_1}^{t_2}}(s))ds. \tag{3.6}$$

Observe that, for $0 < s < t_2 - t_1$,

$$\mu\{|f| \geq t_2\} \leq \mu_{f_{t_1}^{t_2}}(s) \leq \mu\{|f| > t_1\}.$$

[11] See Mazya [91] and also Talenti [112].

Consequently, by the properties of I, we have

$$\int_0^{t_2-t_1} I(\mu_{f_{t_1}^{t_2}}(s))ds \geq (t_2-t_1)\min\left(I(\mu\left\{|f|\geq t_2\right\}), I(\mu\left\{|f|>t_1\right\}\right).$$

Let us see that f_μ^* is locally absolutely continuous. Indeed, for $s>0$ and $h>0$, pick $t_1=f_\mu^*(s+h)$, $t_2=f_\mu^*(s)$, then

$$s\leq\mu\left\{|f(x)|\geq f_\mu^*(s)\right\}\leq\mu_{f_{t_1}^{t_2}}(s)\leq\mu\left\{|f(x)|>f_\mu^*(s+h)\right\}\leq s+h. \tag{3.7}$$

Combining (3.6) and (3.7) we have,

$$\left(f_\mu^*(s)-f_\mu^*(s+h)\right)\min(I(s+h),I(s))\leq\int_{\left\{f_\mu^*(s+h)<|f|<f_\mu^*(s)\right\}}|\nabla|f|(x)|\,d\mu \tag{3.8}$$

which implies that f_μ^* is absolutely continuous in $[a,b]$ $(0<a<b<1)$. Indeed, for any finite family of non-overlapping intervals $\{(a_k,b_k)\}_{k=1}^m$, with $(a_k,b_k)\subset[a,b]$, and such that, moreover, $\sum_{k=1}^m(b_k-a_k)\leq\delta$, we have

$$\sum_{k=1}^m\mu\left\{\cup_{k=1}^m\left\{f_\mu^*(b_k)<|f|<f_\mu^*(a_k)\right\}\right\}\leq\sum_{k=1}^m(b_k-a_k)\leq\delta.$$

Therefore, combining this fact with (3.8), we have

$$\begin{aligned}\sum_{k=1}^m\left(f_\mu^*(a_k)-f_\mu^*(b_k)\right)\min(I(a),I(b)) &\leq\sum_{k=1}^m\left(f_\mu^*(a_k)-f_\mu^*(b_k)\right)\min(I(a_k),I(b_k))\\ &\leq\sum_{k=1}^m\int_{\left\{f_\mu^*(b_k)<|f|<f_\mu^*(a_k)\right\}}|\nabla f(x)|\,d\mu\\ &=\sum_{k=1}^m\int_{\cup_{k=1}^m\left\{f_\mu^*(b_k)<|f|<f_\mu^*(a_k)\right\}}|\nabla f(x)|\,d\mu\\ &\leq\int_0^\delta|\nabla|f||_\mu^*(t)dt\\ &\leq\int_0^\delta|\nabla f|_\mu^*(t)dt.\end{aligned}$$

The local absolute continuity follows.

Finally, using (3.8) again we get,

$$\begin{aligned}\frac{\left(f_\mu^*(s)-f_\mu^*(s+h)\right)}{h}\min(I(s+h),I(s)) &\leq\int_{\left\{f_\mu^*(s+h)<|f|<f_\mu^*(s)\right\}}|\nabla|f|(x)|\,d\mu\\ &\leq\frac{1}{h}\int_{\left\{f_\mu^*(s+h)<|f|\leq f_\mu^*(s)\right\}}|\nabla|f|(x)|\,d\mu\\ &\leq\frac{1}{h}\int_{\left\{f_\mu^*(s+h)<|f|\leq f_\mu^*(s)\right\}}|\nabla f(x)|\,d\mu.\end{aligned}$$

Letting $h\to0$ we obtain (3.3).

(2) $\Rightarrow$ **(4)**. As before, the truncation argument shows that

$$\int_0^{t_2-t_1}I(\mu_{f_{t_1}^{t_2}}(s))ds\leq\int_{\{t_1<|f|<t_2\}}|\nabla|f||\,\chi_{\{t_1<|f|<t_2\}}d\mu.$$

Observe that for $0 < s < t_2 - t_1$

$$\mu_{f_{t_1}^{t_2}}(s) = \mu\{|f| > t_1 + s\} = \mu_f(t_1 + s),$$

thus

$$\int_0^{t_2-t_1} I(\mu_{f_{t_1}^{t_2}}(s))ds = \int_{t_1}^{t_2} I(\mu_f(s))ds.$$

We have seen in part (2) $\Rightarrow$ (3) that f_μ^* is absolutely continuous, thus we get

$$\int_{t_1}^{t_2} I(\mu_f(s))ds = \int_{\mu_f(t_2)}^{\mu_f(t_1)} I(\mu_f(f_\mu^*(s))) \left(-f_\mu^*\right)'(s)ds. \tag{3.9}$$

Let m be the Lebesgue on $[0,\infty)$, then (see [40, Lemma 1, pag. 84])

$$s - m\left\{r \in (0,\infty) : f_\mu^*(r) = f_\mu^*(s)\right\} \leq m_{f_\mu^*}(f_\mu^*(s)) \leq s. \tag{3.10}$$

Recall that since f and f_μ^* are equimeasurable,

$$\mu_f(s) = m_{f_\mu^*}(s), \text{ for all } s \geq 0.$$

Inserting this in (3.10) we find

$$s - m\left\{r \in (0,\infty) : f_\mu^*(r) = f_\mu^*(s)\right\} \leq \mu_f(f_\mu^*(s)) \leq s.$$

It follows that $\mu_f(f_\mu^*(s)) = s$, unless that s belongs to an interval where f_μ^* is constant, in which case $\left(f_\mu^*\right)' = 0$. Therefore, if we set $t_1 = f_\mu^*(a)$ and $t_2 = f_\mu^*(b)$ $(a < b)$ in (3.9), we obtain

$$\begin{aligned} \int_{f_\mu^*(a)}^{f_\mu^*(b)} I(\mu_f(s))ds &= \int_{\mu_f(f_\mu^*(a))}^{\mu_f(f_\mu^*(b))} I(\mu_f(f_\mu^*(s))) \left(-f_\mu^*\right)'(s)ds \\ &= \int_a^b I(s) \left(-f_\mu^*\right)'(s)ds. \end{aligned} \tag{3.11}$$

Consider now a finite family of intervals (a_i, b_i), $i = 1, \ldots, m$, with $0 < a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m < 1$, then

$$\begin{aligned} \int_{\cup_{1\leq i\leq m}(a_i,b_i)} \left(-f_\mu^*\right)'(s)I(s)ds &= \sum_{i=1}^m \int_{f_\mu^*(a_i)}^{f_\mu^*(b_i)} I(\mu_f(s))ds \quad \text{(by (3.11))} \\ &\leq \sum_{i=1}^m \int_{\left\{f_\mu^*(b_i)<|f|<f_\mu^*(a_i)\right\}} |\nabla f(x)|\, d\mu \\ &= \int_{\cup_{1\leq i\leq m}\left\{f_\mu^*(b_i)<|f|<f_\mu^*(a_i)\right\}} |\nabla f(x)|\, d\mu \\ &\leq \int_0^{\sum_{i=1}^m (b_i-a_i)} |\nabla f|_\mu^*(s)ds. \end{aligned}$$

Now, by a routine limiting process we can show that, for any measurable set $E \subset (0,1)$, we have

$$\int_E (-f_\mu^*)'(s)I(s)ds \leq \int_0^{m(E)} |\nabla f|_\mu^*(s)ds.$$

Therefore,

$$\sup_{m(E)\leq t}\int_E(-f_\mu^*)'(s)I(s)ds \leq \sup_{m(E)\leq t}\int_0^{m(E)}\left|\nabla f\right|_\mu^*(s)ds$$
$$= \int_0^t \left|\nabla f\right|_\mu^*(s)ds,$$

and consequently by (2.1) we get

$$\int_0^t((-f_\mu^*)'(\cdot)I(\cdot))^*(s)ds \leq \int_0^t \left|\nabla f\right|_\mu^*(s)ds.$$

(**3**) $\Rightarrow$ (**5**). We will integrate by parts. Let us note first that using (3.8) we have that, for $0<s<t$,

$$s\left(f_\mu^{**}(s)-f_\mu^*(t)\right) \leq \frac{s}{\min(I(s),I(t))}\int_0^{t-s}\left|\nabla f\right|_\mu^*(s)ds. \tag{3.12}$$

Now,

$$\begin{aligned}
f_\mu^{**}(t)-f_\mu^*(t) &= \frac{1}{t}\int_0^t\left(f_\mu^{**}(s)-f_\mu^*(t)\right)ds\\
&= \frac{1}{t}\left\{\left[s\left(f_\mu^{**}(s)-f_\mu^*(t)\right)\right]_0^t + \int_0^t s\left(-f_\mu^*\right)'(s)ds\right\}\\
&= \frac{1}{t}\int_0^t s\left(-f_\mu^*\right)'(s)ds\\
&= A(t),
\end{aligned}$$

where the integrated term $\left[s\left(f_\mu^{**}(s)-f_\mu^*(t)\right)\right]_0^t$ vanishes on account of (3.12). Since $s/I(s)$ is increasing on $0<s<1$, we get

$$\begin{aligned}
A(t) &\leq \frac{1}{I(t)}\int_0^t I(s)\left(-f_\mu^*\right)'(s)ds\\
&\leq \frac{1}{I(t)}\int_0^t\left(\frac{\partial}{\partial s}\int_{\left\{|f|>f_\mu^*(s)\right\}}\left|\nabla f(x)\right|d\mu\right)ds \text{ (by (3.3))}\\
&\leq \frac{1}{I(t)}\int_{\left\{|f|>f_\mu^*(s)\right\}}\left|\nabla f(x)\right|d\mu\\
&\leq \frac{t}{I(t)}\left|\nabla f\right|_\mu^{**}(t).
\end{aligned}$$

(**4**) $\Rightarrow$ (**5**). Again we are going to use integration by parts. To this end, notice that for $0<s<t$,

$$\begin{aligned}
\left(f_\mu^*(s)-f_\mu^*(t)\right)\min(I(t),I(s)) &\leq \int_s^t(-f_\mu^*)'(z)I(z)dz\\
&\leq \int_0^{t-s}((-f_\mu^*)'(.)I(.))^*(z)dz\\
&\leq \int_0^{t-s}\left|\nabla f\right|_\mu^*(z)dz.
\end{aligned}$$

Thus,

$$s\left(f_\mu^*(s)-f_\mu^*(t)\right)\leq\frac{s}{\min(I(t),I(s))}\int_0^{t-s}\left|\nabla f\right|_\mu^*(z)dz. \tag{3.13}$$

Now,

$$\begin{aligned}
f_\mu^{**}(t)-f_\mu^*(t)&=\frac{1}{t}\int_0^t\left(f_\mu^*(s)-f_\mu^*(t)\right)ds\\
&=\frac{1}{t}\left\{\left[s\left(f_\mu^*(s)-f_\mu^*(t)\right)\right]_0^t+\int_0^t s\left(-f_\mu^*\right)'(s)ds\right\}\\
&\leq\frac{1}{t}\int_0^t s\left(-f_\mu^*\right)'(s)ds\\
&=B(t),
\end{aligned}$$

where the integrated term $\left[s\left(f_\mu^*(s)-f_\mu^*(t)\right)\right]_0^t$ vanishes on account of (3.13). Using the fact that $s/I(s)$ is increasing on $0<s<1$, we deduce that

$$\begin{aligned}
B(t)&\leq\frac{1}{I(t)}\int_0^t I(s)\left(-f_\mu^*\right)'(s)ds\\
&\leq\frac{1}{I(t)}\int_0^t((-f_\mu^*)'(.)I(.))^*(s)ds\\
&\leq\frac{t}{I(t)}\left|\nabla f\right|_\mu^{**}(t).
\end{aligned}$$

(**5**) $\Rightarrow$ (**1**). Let A be a Borel set with $0<\mu(A)<1$. We may assume, without loss, that $\mu^+(A)<\infty$. By definition we can select a sequence $\{f_n\}_{n\in N}$ of Lip functions such that $f_n \underset{L^1}{\rightarrow} \chi_A$, and

$$\mu^+(A)=\limsup_{n\to\infty}\left\|\nabla f_n\right\|_1.$$

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} I(t)((f_n)_\mu^{**}(t)-(f_n)_\mu^*(t))&\leq\limsup_{n\to\infty}\int_0^t\left|\nabla f_n(s)\right|_\mu^*ds\\
&\leq\limsup_{n\to\infty}\int_\Omega\left|\nabla f_n\right|d\mu\\
&=\mu^+(A).
\end{aligned}\tag{3.14}$$

As is well known, $f_n \underset{L^1}{\rightarrow} \chi_A$ implies that (cf. [54, Lemma 2.1]):

$$\begin{aligned}
&(f_n)_\mu^{**}(t)\to(\chi_A)_\mu^{**}(t),\text{ uniformly for } t\in[0,1],\text{ and}\\
&(f_n)_\mu^*(t)\to(\chi_A)_\mu^*(t)\text{ at all points of continuity of }(\chi_A)_\mu^*.
\end{aligned}$$

Let $r=\mu(A)$, and observe that $(\chi_A)_\mu^{**}(t)=\min(1,\frac{r}{t})$, then, we deduce that for all $t>r$, $(f_n)_\mu^{**}(t)\to\frac{r}{t}$, and $(f_n)_\mu^*(t)\to(\chi_A)_\mu^*(t)=\chi_{(0,r)}(t)=0$. Inserting this information back in (3.14), we get

$$\frac{r}{t}I(t)\leq\mu^+(A),\ \forall t>r.$$

Now, since $I(t)$ is continuous, we may let $t\to r$ and we find that

$$I(\mu(A))\leq\mu^+(A),$$

as we wished to show. □

Remark 2. *In connection with inequality (3.2) see also Remark 14 below.*

Proposition 1. *Let $I:[0,1]\to[0,\infty)$ be an isoperimetric estimator on (Ω,d,μ). Suppose that there exists a constant $c>0$ such that*

$$\int_t^1 \frac{I(s)}{s}\frac{ds}{s}\leq c\frac{I(t)}{t},\quad t\in(0,1). \tag{3.15}$$

Then, $\forall f\in Lip(\Omega)$,

$$\int_0^t\left(\frac{I(\cdot)}{(\cdot)}[f_\mu^{**}(\cdot)-f_\mu^{*}(\cdot)]\right)^* ds\leq 4c\int_0^t \left|\nabla f\right|_\mu^*(s)ds. \tag{3.16}$$

Proof. We will first show that

$$\int_0^t (f_\mu^{**}(s)-f_\mu^{*}(s))\frac{I(s)}{s}ds\leq c\int_0^t \left|\nabla f\right|_\mu^*(s)ds. \tag{3.17}$$

As we have seen before

$$t(f_\mu^{**}(t)-f_\mu^{*}(t))=\int_0^t s\left(-f_\mu^*\right)'(s)ds.$$

Therefore, the left hand side of (3.17) can be rewritten as

$$B(t)=\int_0^t\left(\int_0^s x\left(-f_\mu^*\right)'(x)dx\right)\frac{I(s)}{s^2}ds.$$

Using our current assumptions, and Fubini's theorem, we find

$$\begin{aligned}
B(t)&=\int_0^t s\left(-f_\mu^*\right)'(s)\int_s^t\frac{I(s)}{s^2}ds\\
&\leq\int_0^t s\left(-f_\mu^*\right)'(s)\int_s^1\frac{I(s)}{s^2}ds\\
&\leq c\int_0^t s\left(-f_\mu^*\right)'(s)\frac{I(s)}{s}ds\\
&\leq c\int_0^t ((-f_\mu^*)'(.)I(.))^*(s)ds\\
&\leq c\int_0^t \left|\nabla f\right|_\mu^*(s)ds\quad\text{(by (3.4))}.
\end{aligned}$$

The proof of (3.17) is complete. By Theorem 1 we also have

$$(f_\mu^{**}(t)-f_\mu^{*}(t))\leq\frac{t}{I(t)}\left|\nabla f\right|_\mu^{**}(t),$$

therefore, by Lemma 2 of [82], we see that (3.16) holds. □

Remark 3. *Suppose that there exists $\alpha>1$, such that the isoperimetric estimator I^α is concave. Then, condition (3.15) holds. In fact, since the function $I(s)/s^{1/\alpha}$*

is decreasing, it follows that

$$\begin{aligned}\int_t^1 \frac{I(s)}{s}\frac{ds}{s} &= \int_t^1 \frac{I(s)}{s^{1/\alpha}}\frac{ds}{s^{2-1/\alpha}} \\ &\leq \frac{I(t)}{t^{1/\alpha}}\int_t^1 \frac{ds}{s^{2-1/\alpha}} \\ &\leq \frac{\alpha}{\alpha+1}\frac{I(t)}{t}.\end{aligned}$$

Remark 4. *We note for future use that if (3.15) holds then Proposition 3.16 implies that for all r.i. spaces X (cf. the discussion in Section 2.1 below) we have*

$$\left\| \left(\frac{I(t)}{t}[f_\mu^{**}(t) - f_\mu^*(t)] \right) \right\|_{\bar{X}} \leq \| |\nabla f| \|_X .$$

4. Pólya-Szegö

The theme of this section is that, under the presence of more symmetry, we can chose a special rearrangement such that the general Pólya-Szegö inequality takes a more familiar form, to wit: “there is a special symmetrization that does not increase the norm of the gradient”. As an application, in the next section we shall show sharp Poincaré-Sobolev inequalities for these model cases.

4.1. **Model Case 1: log concave measures.**

We consider product measures on $\mathbb{R}^n$ constructed using measures on $\mathbb{R}$ defined by

$$\mu_\Phi(x) = Z_\Phi^{-1}\exp\left(-\Phi(|x|)\right)dx = \varphi(x)dx, \qquad x \in \mathbb{R},$$

where Φ is convex, $\sqrt{\Phi}$ concave and where Z_Φ^{-1} is chosen to ensure that $\mu_\Phi(\mathbb{R})$ =1. It is known that the isoperimetric problem is solved by half-lines (cf. [28] and [24]) and the isoperimetric profile is given by

$$I_{\mu_\Phi}(t) = \varphi\left(H^{-1}(\min(t, 1-t)\right) = \varphi\left(H^{-1}(t)\right), \qquad t \in [0,1],$$

where H is the distribution function of μ_Φ, i.e. $H : \mathbb{R} \to (0,1)$ is the increasing function given by

$$H(r) = \int_{-\infty}^{r} \varphi(x)dx.$$

In what follows we will, furthermore, assume that $\Phi(0) = 0$, and that Φ is $\mathcal{C}^2$ on $[\Phi^{-1}(1), +\infty)$; then it is known (see [12]) that there exist constants $c_{1,}, c_2$ such that, for all $t \in [0,1]$,

$$c_1 L_\Phi(t) \leq I_{\mu_\Phi}(t) \leq c_2 L_\Phi(t), \tag{4.1}$$

where

$$L_\Phi(t) = \min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log \frac{1}{\min(t,1-t)}\right).$$

We consider the product probability measures $\mu_\Phi^{\otimes n}$ on $\mathbb{R}^n$. Their isoperimetric profiles $I_{\mu_\Phi^{\otimes n}}$ are dimension free (see [12]): there exists a universal constant $c(\Phi)$ such that

$$I_{\mu_\Phi}(t) \geq \inf_{n\geq 1} I_{\mu_\Phi^{\otimes n}}(t) \geq c(\Phi) I_{\mu_\Phi}(t). \tag{4.2}$$

In what follows we shall write $\mu=\mu_{\Phi}^{\otimes n}$. For a Borel set $\Omega\subset\mathbb{R}^n$, the perimeter is given by

$$\mu^{+}(\Omega)=\int_{\partial\Omega}\varphi(x_1)\cdots\varphi(x_n)dH_{n-1}(x),$$

where $dH_{n-1}(x)$ denotes the Hausdorff $(n-1)$ dimensional measure. The isoperimetric inequality now reads

$$\mu^{+}(\Omega)\geq I_{\mu}(\mu(\Omega)).$$

For a measurable set $\Omega\subset\mathbb{R}^n$, we let Ω° be the half space defined by

$$\Omega^{\circ}=H_r=\{x=(x_1,.....x_n):x_1<r\},\quad r\in\mathbb{R},$$

where $r\in\mathbb{R}$ is selected so that

$$\mu_{\Phi}(H_r)=\mu(\Omega),\text{ or more explicitly } r=H^{-1}(\mu(\Omega)).$$

It follows from (4.2) that

$$\begin{aligned}\mu^{+}(\Omega)&\geq I_{\mu}(\mu(\Omega))\\&\geq c(\Phi)I_{\mu_{\Phi}}(\mu_{\Phi}(H_r))\\&=c(\Phi)\varphi\left(H^{-1}(\mu_{\Phi}^{\otimes n}(\Omega))\right)\\&=c(\Phi)\mu^{+}(\Omega^{\circ}).\end{aligned}$$

There is a natural rearrangement associated with the symmetrization operation $\Omega\to\Omega^{\circ}$. For $f:\mathbb{R}^n\to\mathbb{R}$ we let

$$f^{\circ}(x)=f_{\mu}^{*}(H(x_1)).$$

Remark 5. *Note that, as in the Euclidean case, f° is equimeasurable with f :*

$$\begin{aligned}\mu_{f^{\circ}}(t)&=\mu\{x:f^{\circ}(x)>t\})=\mu\{x:f_{\mu}^{*}(H(x_1))>t\}\\&=\mu\{x:H(x_1)\leq\mu_f(t)\}=\mu\{x:x_1\leq H^{-1}(\mu_f(t))\}\\&=\mu_{\Phi}(-\infty,H^{-1}(\mu_f(t)))\\&=\mu_f(t).\end{aligned}$$

We can now show the following generalization of the Pólya-Szegö principle.

Theorem 2. *Consider the probability space $(\mathbb{R}^n,\mu)$, with $\mu=\mu_{\Phi}^{\otimes n}$. The following Pólya-Szegö inequality holds: $\forall f\in Lip(\mathbb{R}^n)$,*

$$\int_0^t\left|\nabla f^{\circ}\right|_{\mu}^{*}(s)ds\leq\frac{1}{c(\Phi)}\int_0^t\left|\nabla f\right|_{\mu}^{*}(s)ds. \tag{4.3}$$

In fact, (4.3) is equivalent to all the inequalities listed in Theorem 1 above.

Proof. Let A be an arbitrary Young's function A. Let $s=H(x_1)$. Then,

$$\begin{aligned}\int_0^1 A\left((-f_{\mu}^{*})'(s)I_{\mu_{\Phi}}(s)\right)ds&=\int_{\mathbb{R}}A(\left(-f_{\mu}^{*}\right)'(H(x_1))I_{\mu_{\Phi}}(H(x_1))\left|H'(x_1)\right|dx\\&=\int_{\mathbb{R}^n}A(\left(-f_{\mu}^{*}\right)'(\Phi(x_1))I_{\mu_{\Phi}}(\Phi(x_1))d\mu(x)\\&=\int_{\mathbb{R}^n}A(\left|\nabla f^{\circ}(x)\right|)d\mu(x),\end{aligned}$$

where in the last step we have used the fact that

$$(-f_{\mu}^{*})'(H(x_1))I_{\mu_{\Phi}}(H(x_1))=(f_{\mu}^{*})'(H(x_1))H'(x_1)=\left|\nabla f^{\circ}(x)\right|.$$

Since A is increasing, then by [20, exercise 3 pag. 88], we have

$$\int_{\mathbb{R}^n} A(|\nabla f^{\circ}(x)|)d\mu(x) = \int_0^1 A\left(|\nabla f^{\circ}|_{\mu}^{*}(s)\right) ds.$$

Thus,

$$\int_0^1 A\left((-f_{\mu}^{*})'(s)I_{\mu_\Phi}(s)\right) ds = \int_0^1 A\left(|\nabla f^{\circ}|_{\mu}^{*}(s)\right) ds.$$

Therefore, by [20, exercise 5 pag. 88], we have

$$\int_0^t ((-f_{\mu}^{*})'(\cdot)I_{\mu_\Phi}(\cdot))^{*}(s)ds = \int_0^t |\nabla f^{\circ}|_{\mu}^{*} ds. \tag{4.4}$$

Combining (4.4) with (4.2) and (3.4) we find

$$\begin{aligned}\int_0^t |\nabla f^{\circ}|_{\mu}^{*} ds &= \int_0^t ((-f_{\mu}^{*})'(\cdot)I_{\mu_\Phi}(\cdot))^{*}(s)ds \\ &\leq \frac{1}{c(\Phi)}\int_0^t ((-f_{\mu}^{*})'(\cdot)I_{\mu}(\cdot))^{*}(s) \\ &\leq \frac{1}{c(\Phi)}\int_0^t |\nabla f|_{\mu}^{*}(s)ds,\end{aligned}$$

as we wished to show. □

Remark 6. *If μ_Φ is the Gaussian measure, then $c(\Phi) = 1$, and we recover the classical Gaussian Pólya-Szegö principle (see* [49]*).*

4.2. **Model Case 2: the $n-$sphere.** Let $n \geq 2$ be an integer and $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere. For each $n \geq 2$, let $\omega_n = 2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$ be the $n-$dimensional Hausdorff measure of $\mathbb{S}^n$. On $\mathbb{S}^n$ we consider the geodesic distance d and the uniform probability measure σ_n. For $\theta \in [-\pi/2, \pi/2]$, let

$$\varphi_n(\theta) = \frac{\omega_{n-1}}{\omega_n}\cos^{n-1}\theta \quad \text{ and } \quad \Phi_n(\theta) = \int_{-\pi/2}^{\theta}\varphi_n(s)ds.$$

The spherical cap

$$C_\theta = \{(\theta_1, ..., \theta_n) \in \mathbb{S}^n : \theta_1 < \theta\}$$

has σ_n-measure $\Phi_n(\theta)$ and boundary measure $\varphi_n(\theta)$. Thus, by the Lévy-Schmidt theorem, the isoperimetric function of the sphere $I_{\mathbb{S}^n}$ coincides with $I_n = \varphi_n \circ \Phi_n^{-1}$ (see [11]). This function is continuous on $[0, 1]$ and symmetric with respect to $1/2$, and $I_n(0) = I_n(1) = 0$. Moreover, $(I_n)^{\frac{n}{n-1}}$ is concave.

Given a measurable set $\Omega \subset \mathbb{S}^n$, we let Ω° be the spherical cap defined by

$$\Omega^{\circ} = \{(\theta_1, ..., \theta_n) \in \mathbb{S}^n : \theta_1 < \theta\},$$

where $\theta \in [-\pi/2, \pi/2]$ is selected so that

$$\Phi_n(\theta) = \sigma_n(\Omega).$$

In other words, θ is defined by

$$\theta = \Phi^{-1}(\sigma_n(\Omega)).$$

Since spherical caps are the subsets of $\mathbb{S}^n$ which yield the equality in the isoperimetric inequality, we get

$$\sigma_n^{+}(\Omega) \geq I_n(\sigma_n(\Omega)) = \sigma_n^{+}(\Omega^{\circ}).$$

Let $f:\mathbb{S}^n\to\mathbb{R}$, associated with the operation $\Omega\to\Omega^\circ$ we define the rearrangement f° by

$$f^\circ(\theta_1,.....,\theta_n)=f^*_{\sigma_n}(\Phi_n(\theta_1)).$$

Theorem 3. *Consider the space $(\mathbb{S}^n,d,\sigma_n)$. The following Pólya-Szegö inequality holds, $\forall f\in Lip(\mathbb{S}^n)$,*

$$\int_0^t \left|\nabla f^\circ\right|^*_{\sigma_n}(s)ds\leq\int_0^t\left|\nabla f\right|^*_{\sigma_n}(s)ds. \tag{4.5}$$

Moreover, (4.5) is equivalent to any of inequalities stated in Theorem 1 above.

Proof. The proof is almost identical to the one of Theorem 2. First of all notice that, by considering spherical coordinates, we have

$$\omega_n=\int_{(-\pi/2,\pi/2)^{n-1}\times(-\pi,\pi)}\prod_{i=1}^{n-1}\cos^{n-i}\theta_i d\theta_1\cdots d\theta_n=\int_{\mathbb{S}^n}s_n(\theta)d\theta.$$

Therefore,

$$d\sigma_n=\frac{1}{\omega_n}s_n(\theta)d\theta.$$

Let A be a Young's function, and let $s=\Phi_n(\theta_1)$. For notational convenience we let $I=\int_0^1 A\left((-f^*_{\sigma_n})'(s)I_n(s)\right)ds$. Then,

$$\begin{aligned}I&=\int_{-\pi/2}^{\pi/2}A(\left(-f^*_{\sigma_n}\right)'(\Phi_n(\theta_1))I_n(\Phi_n(\theta_1))\left|\Phi_n'(\theta_1)\right|d\theta_1\\&=\int_{-\pi/2}^{\pi/2}A(\left(-f^*_{\sigma_n}\right)'(\Phi_n(\theta_1))I_n(\Phi_n(\theta_1))\frac{\omega_{n-1}}{\omega_n}\cos^{n-1}\theta_1 d\theta_1\\&=\int_{\mathbb{S}^{n-1}}s_{n-1}(\theta)d\theta\int_{-\pi/2}^{\pi/2}A(\left|\nabla f^\circ(\theta_1,.....,\theta_n)\right|)\frac{1}{\omega_n}\cos^{n-1}\theta_1 d\theta_1\\&=\int_{\mathbb{S}^n}A(\left|\nabla f^\circ(\theta_1,.....,\theta_n)\right|)\frac{1}{\omega_n}s_n(\theta)d\theta\\&=\int_{\mathbb{S}^n}A(\left|\nabla f^\circ(\theta_1,.....,\theta_n)\right|)d\sigma_n,\end{aligned}$$

where we have used the fact that

$$\begin{aligned}(-f^*_{\sigma_n})'(\Phi_n(\theta_1))I_n(\Phi_n(\theta_1))&=(f^*_{\sigma_n})'(\Phi_n(\theta_1))\Phi_n'(\theta_1)\\&=\left|\nabla f^\circ(\theta_1,.....,\theta_n)\right|.\end{aligned}$$

At this point we proceed in the same way as in the proof of Theorem 2. □

Remark 7. *Since $(I_n)^{\frac{n}{n-1}}$ is concave, then by Remark 3 we have that $\forall f\in Lip(\mathbb{S}^n)$*

$$\int_0^t\left(\frac{I(\cdot)}{(\cdot)}[f^{**}_{\sigma_n}(\cdot)-f^*_{\sigma_n}(\cdot)]\right)^*ds\leq 4c\int_0^t\left|\nabla f\right|^*_{\sigma_n}(s)ds. \tag{4.6}$$

Therefore, (4.6) is equivalent to any of the inequalities stated in Theorem 1 above. We also have (cf. Remark 4 above)

$$\left\|\frac{I(t)}{t}[f^{**}_{\sigma_n}(t)-f^*_{\sigma_n}(t)]\right\|_{\bar{X}}\leq\left\|\left|\nabla f\right|\right\|_X,$$

without any restrictions on the indices of X.

4.3. **Model Case 3: Model Riemannian manifolds.** Ros [107] has constructed general class of spaces that abstract some of the characteristics of the model spaces considered in this Section, and thus are particularly suited for our analysis. In this direction in [86] we showed that Ros's spaces have the isoperimetric Hardy type property (see Definition 6.2 below). In this section we complete the analysis of model spaces by showing that the Pólya-Szegö inequality holds for Ros's spaces.

We recall briefly the construction and refer to [107] and [86] for more details. Let M_0 be an n_0-dimensional Riemannian manifold with distance d. A probability measure μ^0 on M that is absolutely continuous with respect to the volume $dVol_M$ will be called a **model measure**, if there exists a continuous family (in the sense of the Hausdorff distance on compact subsets) $\mathcal{D} = \{D^t : 0 \leq t \leq 1\}$ of closed subsets of M_0 satisfying the following conditions:

1. $D^s \subset D^t$, for $0 \leq s < t \leq$ and $\mu^0(D^t) = t$,
2. D^t is a smooth isoperimetric domain of μ^0 and $I_{\mu^0}(t) = \mu^0(D^t)$ is positive and smooth for $0 < t < 1$, where I_{μ^0} denotes the isoperimetric profile of M_0,
3. The r-enlargement of D^t, defined by $(D^t)_r = \{x \in M_0 : d(x, D^t) \leq r\}$ verifies $(D^t)_r = D^s$ for some $s = s(t,r)$, $0 \leq t \leq 1$,
4. $D^1 = M_0$ and D^0 is either a point or the empty set.

Let $f : M_0 \to \mathbb{R}$. The rearrangement $f^\circ : M_0 \to \mathbb{R}$, is defined by

$$f^\circ(x) = f^*_{\mu^0}(p(x)),$$

where

$$\begin{aligned} p : M_0 &\to [0,1] \\ x \in \partial D^t &\to t. \end{aligned}$$

Since p is measure preserving (cf. [86]) it is easy to verify that f° is equimeasurable with f :

$$\begin{aligned} \mu^0_{f^\circ}(t) &= \mu^0\{x : f^\circ(x) > t\} \\ &= \mu^0\{x : f^*_{\mu^0}(p(x)) > t\} \\ &= \mu^0\{x : p(x) \leq \mu^0_f(t)\} \\ &= \mu^0\{x : p^{-1}(0, \mu^0_f(t))\} \\ &= \mu^0_f(t). \end{aligned}$$

Moreover, from (cf. [86])

$$|\nabla p(x)| = \left| I_{\mu^0}(p(x)) \right|$$

we see that

$$\begin{aligned} |\nabla f^\circ(x)| &= (-f^*_{\mu^0})'(p(x)) \, |\nabla p(x)| \\ &= \left| (-f^*_{\mu^0})'(p(x)) I_{\mu^0}(p(x)) \right|. \end{aligned}$$

Therefore the analysis of Theorem 2 can be repeated verbatim and yields

Theorem 4. *Let (M_0, d) be an n_0-dimensional Riemannian manifold endowed with a model measure μ_0. Then, the following Pólya-Szegö inequality holds: $\forall f \in Lip(M_0)$*

$$\int_0^t |\nabla f^\circ|^*_{\mu^0}(s) ds \leq \int_0^t |\nabla f|^*_{\mu^0}(s) ds.$$

5. Poincaré Inequalities

Let (Ω, d, μ) be a probability metric space, and let I be an isoperimetric estimator for (Ω, d, μ).

In this section we study Poincaré type inequalities of the form

$$\left\| g - \int_{\Omega} g d\mu \right\|_{Y} \preceq \left\| \nabla g \right\|_{X}, \quad g \in Lip(\Omega), \tag{5.1}$$

where X, Y are rearrangement-invariant spaces on Ω.

It is easy to see that when $X = Y = L^1(\Omega)$ the inequality (5.1) follows readily from Ledoux's inequality (3.2). Indeed, using (3.2) we can readily see that for all $f \in Lip(\Omega)$,

$$\int_{\Omega} |f(x) - m| \, d\mu \leq \frac{1}{2I(1/2)} \int_{\Omega} |\nabla f(x)| \, d\mu, \tag{5.2}$$

where m is a median of f. Indeed, set $f^+ = \max(f - m, 0)$ and $f^- = -\min(f - m, 0)$ so that $f - m = f^+ - f^-$. Then,

$$\begin{aligned} \int_{\Omega} |f - m| \, d\mu &= \int_{\Omega} f^+ d\mu + \int_{\Omega} f^- d\mu \\ &= \int_0^{\infty} \mu_{f^+}(s) ds + \int_0^{\infty} \mu_{f^-}(s) ds \\ &= (A), \text{ say.} \end{aligned}$$

Each of these integrals can be estimated using the properties of the isoperimetric estimator and Ledoux's inequality (3.2). First we use the fact that $\frac{I(s)}{s}$ is decreasing combined with the definition of median, to find that

$$2\mu_g(s) I\left(\frac{1}{2}\right) \leq I(\mu_g(s)), \text{ where } g = f^+ \text{ or } g = f^-.$$

Consequently,

$$\begin{aligned} (A) &\leq \frac{1}{2I(\frac{1}{2})} \left(\int_0^{\infty} I(\mu_{f^+}(s)) ds + \int_0^{\infty} I(\mu_{f^-}(s)) ds \right) \\ &\leq \frac{1}{2I(\frac{1}{2})} \left(\int_{\Omega} \left| \nabla f^+(x) \right| d\mu + \int_{\Omega} \left| \nabla f^-(x) \right| d\mu \right) \text{ (by (3.2))} \\ &= \frac{1}{2I(1/2)} \int_{\Omega} |\nabla f(x)| \, d\mu. \end{aligned}$$

Thus,

$$\int_{\Omega} |f(x) - m| \, d\mu \leq \frac{1}{2I(1/2)} \int_{\Omega} |\nabla f(x)| \, d\mu.$$

The isoperimetric Hardy operator Q_I is the operator defined on measurable functions on $(0, 1)$ by

$$Q_I f(t) = \int_t^1 f(s) \frac{ds}{I(s)},$$

where I is an isoperimetric estimator. We consider the possibility of characterizing Poincaré inequalities of the form (5.1) in terms of the of the boundedness of Q_I as an operator from $\bar{X}$ to $\bar{Y}$.

Theorem 5. *Let X, Y be two r.i. spaces on Ω. Suppose that there exists an absolute constant C, such for every positive function $f \in \bar{X}$, with $suppf \subset (0,1/2)$, we have*

$$\|Q_I f\|_{\bar{Y}} \leq C \|f\|_{\bar{X}}. \tag{5.3}$$

Then, for all $g \in Lip(\Omega)$,

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \|\nabla g\|_X.$$

Moreover:

(a) *Suppose that the operator $\tilde{Q}_I f(t) = \frac{I(t)}{t} \int_t^{1/2} f(s) \frac{ds}{I(s)}$ is bounded on $\bar{X}$. Then, for all $g \in Lip(\Omega)$, we have*

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| \left(g - \int_\Omega g d\mu \right)_\mu^* (t) \frac{I(t)}{t} \right\|_{\bar{X}} \preceq \|\nabla g\|_X.$$

(b) *If $\overline{\alpha}_X < 1$, or if the isoperimetric estimator I satisfies (3.15), then, for all $g \in Lip(\Omega)$ we have,*

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| g - \int_\Omega g d\mu \right\|_{LS(X)} \preceq \|\nabla g\|_X. \tag{5.4}$$

Proof. Let $g \in Lip(\Omega)$. Write

$$g_\mu^*(t) = \int_t^{1/2} \left(-g_\mu^*\right)'(s) ds + g_\mu^*(1/2), \ t \in (0, 1/2].$$

Thus,

$$\begin{aligned}
\|g\|_Y = \left\|g_\mu^*\right\|_{\bar{Y}} &\preceq \left\|g_\mu^* \chi_{[0,1/2]}\right\|_{\bar{Y}} \\
&\preceq \left\| \int_t^{1/2} \left(-g_\mu^*\right)'(s) ds \right\|_{\bar{Y}} + g_\mu^*(1/2) \|1\|_{\bar{Y}} \\
&\leq \left\| \int_t^{1/2} \left(-g_\mu^*\right)'(s) I(s) \frac{ds}{I(s)} \right\|_{\bar{Y}} + 2 \|1\|_{\bar{Y}} \|g\|_{L_1} \\
&\preceq \left\| \left(-g_\mu^*\right)'(s) I(s) \right\|_{\bar{X}} + \|g\|_{L_1} \quad \text{(by (5.3))} \\
&\preceq \|\nabla g\|_X \ + \|g\|_{L_1} \text{ (by (3.4))}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\| g - \int_\Omega g d\mu \right\|_Y &\preceq \|\nabla g\|_X \ + \left\| g - \int_\Omega g d\mu \right\|_{L_1} \\
&\preceq \|\nabla g\|_X + \|\nabla g\|_{L_1} \text{ (by (5.2))} \\
&\preceq \|\nabla g\|_X \text{ (by (2.3))}.
\end{aligned}$$

Part (a) It will be convenient to let $\bar{X}_I$ be the r.i. space on $(0,1)$ defined by the condition

$$\|h\|_{\bar{X}_I} = \left\| h(t) \frac{I(t)}{t} \right\|_{\bar{X}} < \infty.$$

Let us start by proving that

$$\|f\|_{\bar{Y}} \preceq \left\|f_\mu^*\right\|_{\bar{X}_I}. \tag{5.5}$$

Indeed, let $0<t<1/2$. From

$$f_\mu^*(t)\ln 2 \leq \int_{t/2}^{t} f_\mu^*(s)\frac{ds}{s} \leq \int_{t/2}^{1/2} f_\mu^*(s)\frac{I(s)}{s}\frac{ds}{I(s)},$$

we see that for $t\in(0,1/2)$,

$$f_\mu^*(t) \preceq \int_{t/2}^{1/2} f_\mu^*(s)\frac{I(s)}{s}\frac{ds}{I(s)} + f_\mu^*(1/2).$$

Consequently,

$$\begin{aligned}
\left\| f_\mu^*(t)\chi_{(0,1/2)}(t)\right\|_{\bar{Y}} &\preceq \left\| \int_{t/2}^{1} \left(f_\mu^*(s)\frac{I(s)}{s}\right)\chi_{(0,1/2)}(s)\frac{ds}{I(s)}\right\|_{\bar{Y}} + \|f\|_1 \\
&\leq 2\left\| Q_I\left(f_\mu^*(s)\frac{I(s)}{s}\chi_{(0,1/2)}(s)\right)\right\|_{\bar{Y}} + \|f\|_1 \quad \text{(by (2.8))} \\
&\preceq \left\| f_\mu^*(t)\frac{I(t)}{t}\right\|_{\bar{X}} + \|f\|_1 \\
&\preceq \left\| f_\mu^*\right\|_{\bar{X}_I},
\end{aligned}$$

where in the last step we estimated $\|f\|_1$ as follows

$$\begin{aligned}
\|f\|_1 = \int_0^1 f_\mu^*(t)dt &\leq 2\int_0^{1/2} f_\mu^*(t)dt \\
&= \int_0^{1/2} f_\mu^*(t)\frac{I(t)}{t}\frac{t}{I(t)}dt \\
&\leq \frac{2}{I(1/2)}\int_0^1 f_\mu^*(t)\frac{I(t)}{t}dt \\
&\preceq \left\| f_\mu^*(t)\frac{I(t)}{t}\right\|_{\bar{X}} \quad \text{(by (2.3))}.
\end{aligned}$$

From the previous discussion we see that

$$\begin{aligned}
\|f\|_{\bar{Y}} &\preceq \left\| f_\mu^*(t)\chi_{(0,1/2)}(t)\right\|_{\bar{Y}} \\
&\preceq \left\| f_\mu^*(t)\frac{I(t)}{t}\right\|_{\bar{X}} \\
&= \left\| f_\mu^*\right\|_{\bar{X}_I}.
\end{aligned}$$

Now, we show that for all $f\in\bar{X}$, with $\operatorname{supp} f\subset(0,1/2)$,

$$\|Q_I f\|_{\bar{X}_I} \preceq \|f\|_{\bar{X}}.$$

Indeed, this is equivalent to the boundedness of the operator $\tilde{Q}_I$:

$$\begin{aligned}
\|Q_I f\|_{\bar{X}_I} &= \left\| \int_t^1 f(s)\frac{ds}{I(s)}\right\|_{\bar{X}_I} \\
&= \left\| \frac{I(t)}{t}\int_t^1 f(s)\frac{ds}{I(s)}\right\|_{\bar{X}} \\
&= \left\| \tilde{Q}_I f\right\|_{\bar{X}} \\
&\preceq \|f\|_{\bar{X}}.
\end{aligned}$$

Consequently, by the first part of the theorem we have that for all $g \in Lip(\Omega)$

$$\left\| \left(g - \int_{\Omega} gd\mu \right)_{\mu}^{*} (t) \frac{I(t)}{t} \right\|_{\bar{X}} = \left\| \left(g - \int_{\Omega} gd\mu \right)_{\mu}^{*} \right\|_{\bar{X}_I} \preceq \|\nabla g\|_X . \tag{5.6}$$

Finally, combining (5.6) and (5.5) we obtain

$$\begin{aligned} \left\| g - \int_{\Omega} gd\mu \right\|_{Y} &= \left\| \left(g - \int_{\Omega} gd\mu \right)_{\mu}^{*} (t) \right\|_{\bar{Y}} \\ &\preceq \left\| \left(g - \int_{\Omega} gd\mu \right)_{\mu}^{*} (t) \frac{I(t)}{t} \right\|_{\bar{X}} \\ &\leq \|\nabla g\|_X . \end{aligned}$$

Part (b) Let us start by proving that,

$$\|f\|_Y \preceq \|f\|_{LS(X)} + \|f\|_{L^1}. \tag{5.7}$$

Since $\left(f_\mu^{**}\right)'(t) = -\frac{1}{t}\left(f_\mu^{**}(t) - f_\mu^{*}(t)\right)$, by the fundamental theorem of Calculus, we have

$$f_\mu^{**}(t) = \int_t^{1/2} \left(f_\mu^{**}(s) - f_\mu^{*}(s)\right) \frac{ds}{s} + f_\mu^{**}(1/2), \quad 0 < t \leq 1/2.$$

Therefore,

$$\begin{aligned} \left\| f_\mu^*(t)\chi_{(0,1/2)}(t) \right\|_{\bar{Y}} &\leq \left\| \int_t^{1/2} \left(f_\mu^{**}(s) - f_\mu^{*}(s)\right) \frac{ds}{s} \right\|_{\bar{Y}} + f_\mu^{**}(1/2) \left\|1\right\|_{\bar{Y}} \\ &\preceq \left\| \int_t^{1} \frac{I(s)}{s} \left(f_\mu^{**}(s) - f_\mu^{*}(s)\right) \chi_{(0,1/2)}(s) \frac{ds}{I(s)} \right\|_{\bar{Y}} + \|f\|_{L_1} \\ &\preceq \left\| (f_\mu^{**}(t) - f_\mu^{*}(t))\chi_{(0,1/2)}(t) \frac{I(t)}{t} \right\|_{\bar{X}} + \|f\|_{L_1} \\ &\preceq \left\| (f_\mu^{**}(t) - f_\mu^{*}(t)) \frac{I(t)}{t} \right\|_{\bar{X}} + \|f\|_{L_1} . \end{aligned}$$

We have obtained

$$\begin{aligned} \left\| f_\mu^* \right\|_{\bar{Y}} \preceq \left\| f_\mu^*(t)\chi_{(0,1/2)}(t) \right\|_{\bar{Y}} &\preceq \\ &\left\| (f_\mu^{**}(t) - f_\mu^{*}(t)) \frac{I(t)}{t} \right\|_{\bar{X}} + \|f\|_{L_1} \\ &= \|f\|_{LS(X)} + \|f\|_{L_1} . \end{aligned}$$

Assume that $\overline{\alpha}_X < 1$. We are going to prove (5.4). Let $g \in Lip(\Omega)$. Applying successively (5.7), (2.10), (5.2), (2.3), and the fact that P is a bounded operator on

$\bar{X}$, we have

$$\begin{aligned}\left\|g-\int_\Omega gd\mu\right\|_Y &= \left\|\left(g-\int_\Omega gd\mu\right)_\mu^*\right\|_{\bar{Y}}\\ &\preceq \left\|g-\int_\Omega gd\mu\right\|_{LS(X)}+\left\|g-\int_\Omega gd\mu\right\|_{L_1}\\ &\preceq \left\|P\left(\left|\nabla\left(g-\int_\Omega gd\mu\right)\right|_\mu^*\right)\right\|_{\bar{X}}+\|\nabla g\|_{L_1}\\ &\preceq \left\|P\left(|\nabla g|_\mu^*\right)\right\|_{\bar{X}}+\|\nabla g\|_{\bar{X}}\\ &\preceq \|\nabla g\|_X .\end{aligned}$$

Finally, suppose that I satisfies (3.15). Then, by Remark 7,

$$\|g\|_{LS(X)} \preceq \|\nabla g\|_X ,$$

as we wished to show. □

5.1. **Model Case 1.** In what follows our ambient space will be the probability space $(\mathbb{R}^n, d\mu_\Phi^{\otimes n})$. We refer to Section 4.1 for notation and background information.

Theorem 6. *Let $X = X(\mathbb{R}^n)$, $Y = Y(\mathbb{R}^n)$ be r.i. spaces. Then, the following statements are equivalent*

(1)

$$\left\|f-\int_{\mathbb{R}^n} fd\mu_\Phi^{\otimes n}\right\|_Y \preceq \|\nabla f\|_X , \quad \forall f\in Lip(\mathbb{R}^n). \tag{5.8}$$

(2)

$$\left\|\int_t^1 f(s)\frac{ds}{I_{\mu_\Phi}(s)}\right\|_{\bar{Y}} \preceq \|f\|_{\bar{X}} , \quad \forall 0\le f\in \bar{X},\ \text{with } supp(f)\subset(0,1/2). \tag{5.9}$$

Proof. $(\mathbf{1})\to(\mathbf{2})$. Let us write $\mu=\mu_\Phi^{\otimes n}$. Given a positive measurable function f with $suppf\subset(0,1/2)$, consider

$$F(t)=\int_t^1 f(s)\frac{ds}{I_{\mu_\Phi}(s)}, \qquad t\in(0,1),$$

and define

$$u(x)=F(H(x_1)), \qquad x\in\mathbb{R}^n.$$

Then,

$$|\nabla u(x)|=\left|\frac{\partial}{\partial x_1}u(x)\right|=\left|-f(H(x_1))\frac{H'(x_1)}{I_{\mu_\Phi}(H(x_1))}\right|=f(H(x_1)).$$

Let A be a Young's function and let $s=H(x_1)$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} A(f(H(x_1)))d\mu(x) &= \int_{\mathbb{R}} A(f(H(x_1)))d\mu_\Phi(x_1)\\ &= \int_0^1 A(f(s))ds.\end{aligned}$$

Therefore,

$$|\nabla u|_\mu^*(t)=f^*(t), \tag{5.10}$$

and

$$u_\mu^*(t) = \int_t^1 f(s) \frac{ds}{I_{\mu_\Phi}(s)}. \tag{5.11}$$

By Lemma 2, (5.8) is equivalent to

$$\|u - m\|_Y \preceq \|\nabla u\|_X \,,$$

where m is a median of u. Now, since $\mu\{u = 0\} \geq 1/2$, it follows that 0 is a median of u, and we get

$$\|u\|_Y \preceq \|\nabla u\|_X \,. \tag{5.12}$$

From (5.10) and (5.11) it follows that

$$\|u\|_Y = \left\|u_\mu^*\right\|_{\bar{Y}} \text{ and } \|\nabla u\|_X = \left\| |\nabla u|_\mu^* \right\|_{\bar{X}} = \|f\|_{\bar{X}} \,,$$

therefore, inserting this information back in (5.12), we obtain (5.9).

(**2**) $\rightarrow$ (**1**) was proved in Theorem 5. □

5.2. **Examples.** Let $\alpha \geq 0$, $p \in [1, 2]$, $\gamma = \exp(2\alpha/(2 - p))$, and

$$\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1} \exp\left(-|x|^p (\log(\gamma + |x|)^\alpha\right) dx, \qquad x \in \mathbb{R}.$$

Using estimate (4.1) (see [12] and [13]) we get

(5.13)
$$I_{\mu_{p,\alpha}^{\otimes n}}(s) \simeq s \left(\log \frac{1}{s}\right)^{1-\frac{1}{p}} \left(\log\log\left(e + \frac{1}{s}\right)\right)^{\frac{\alpha}{p}} = s\beta_{p,\alpha}(s), \qquad 0 < s \leq 1/2,$$

moreover the constants that appear in equivalence (5.13) are independent of n.

Remark that for $p = 2$ and $\alpha = 0$, we obtain Gaussian measure. So in some sense these probabilities form a scale between exponential ($p = 1$ and $\alpha = 0$) and Gaussian measure.

In what follows we write $\mu = \mu_{p,\alpha}^{\otimes n}$.

The corresponding operators Q_{I_μ} and $\tilde{Q}_{I_\mu}$ associated with μ are given by

$$Q_{I_\mu} f(t) \simeq \int_t^{1/2} f(s) \frac{ds}{s\beta_{p,\alpha}(s)} \text{ and } \tilde{Q}_{I_\mu} f(t) \simeq \beta_{p,\alpha}(t) \int_t^{1/2} f(s) \frac{ds}{s\beta_{p,\alpha}(s)}.$$

Notice that if X is a r.i. space such that $\underline{\alpha}_X > 0$, then the operator $\tilde{Q}_{I_\mu} f$ is bounded on X. Indeed, pick $\underline{\alpha}_X > a > 0$, then since $t^a \beta_{p,\alpha}(t)$ is increasing near zero, we get

$$\tilde{Q}_{I_\mu} f(t) \simeq \frac{t^a \beta_{p,\alpha}(t)}{t^a} \int_t^{1/2} f(s) \frac{s^a ds}{s s^a \beta_{p,\alpha}(s)} \preceq \frac{1}{t^a} \int_t^{1/2} s^a f(s) \frac{ds}{s} = Q_a f(t),$$

we conclude noting that Q_a is bounded on X on account of the fact that $\underline{\alpha}_X > a$ (see Remark 2.9).

Theorems 5 and 6 now yield

Theorem 7. *Let X, Y be two r.i. spaces on $(\mathbb{R}^n, \mu)$.*

Part I.

The following statements are equivalent:

(i) *For every Lipschitz function* f *on* $\mathbb{R}^n$

$$\left\| f-\int_{\mathbb{R}^n} f d\mu \right\|_Y \preceq \|\nabla f\|_X . \tag{5.14}$$

(ii) *For every positive function* $f\in X$ *with supp*$f\subset(0,1/2)$,

$$\left\| \int_t^1 f(s)\frac{ds}{s\beta_{p,\alpha}(s)} \right\|_{\bar{Y}} \preceq \|f\|_{\bar{X}} .$$

Part II.

(1) *If* $\underline{\alpha}_X>0$, *then the r.i. space defined by*

$$\left\{ f\in L^1 : \left\| f_\mu^*(t)\beta_{p,\alpha}(t)\right\|_{\bar{X}}<\infty \right\},$$

is optimal among all r.i. spaces Y *that satisfy (5.14) in the sense that for all* $f\in Lip(\mathbb{R}^n)$

$$\left\| f-\int_{\mathbb{R}^n} f d\mu \right\|_Y \preceq \left\| \left(f-\int_{\mathbb{R}^n} f d\mu\right)_\mu^*(s)\beta_{p,\alpha}(s) \right\|_{\bar{X}} \preceq \|\nabla f\|_X .$$

(2) *If* $0=\underline{\alpha}_X<\overline{\alpha}_X<1$, *then the r.i. set defined by*

$$\left\{ f\in L^1 : \left\| \left(\left(f-\int_{\mathbb{R}^n} f d\mu\right)_\mu^{**}(s) - \left(f-\int_{\mathbb{R}^n} f d\mu\right)_\mu^{*}(s) \right)\beta_{p,\alpha}(s) \right\|_{\bar{X}} <\infty \right\},$$

is optimal, among all r.i. spaces Y *that satisfy (5.14), in the sense that*

$$\begin{aligned}\left\| f-\int f \right\|_Y &\preceq \left\| \left(\left(f-\int_{\mathbb{R}^n} f d\mu\right)_\mu^{**}(s) - \left(f-\int_{\mathbb{R}^n} f d\mu\right)_\mu^{*}(s) \right)\beta_{p,\alpha}(s) \right\|_{\bar{X}} \\ &\preceq \|\nabla f\|_X , \quad f\in Lip(\mathbb{R}^n).\end{aligned}$$

5.3. **Feissner type inequalities.** Theorem 7 readily improves upon Feissner's inequalities (cf. [1], [7], [8], [51]). Indeed, for the particular choice $X=L^q$ $(1\leq q<\infty)$, Theorem 7 yields

$$\int_0^1 \left(\left(f-\int_{\mathbb{R}^n} f d\mu\right)_\mu^*(s)\frac{I_\mu(s)}{s} \right)^q ds \preceq \int |\nabla f(x)|^q d\mu.$$

In particular, using the asymptotics of I_μ given by (5.13), we get

$$\int_0^1 f^*(s)^q\left(\beta_{p,\alpha}(s)\right)^q ds \preceq \int_{\mathbb{R}^n} |\nabla f(x)|^q d\mu + \int_{\mathbb{R}^n} |f(x)|^q d\mu.$$

Moreover, the space $L^q(LogL)^{q\left(1-\frac{1}{p}\right)}(LogLogL)^{\frac{\alpha q}{p}}$ is best possible among r.i. spaces Y for which the Poincaré inequality $\left\|f-\int_{\mathbb{R}^n} f d\mu\right\|_Y \preceq \|\nabla f\|_{L^q}$ holds. More precisely, there exist constants c_1 and c_2 such that

$$\left\| f-\int_{\mathbb{R}^n} f d\mu \right\|_Y \leq c_1 \left\| f-\int_{\mathbb{R}^n} f d\mu \right\|_{L^q(LogL)^{q\left(1-\frac{1}{p}\right)}(LogLogL)^{\frac{\alpha q}{p}}} \leq c_2\|\nabla f\|_{L^q}.$$

Notice that, since the equivalence (5.13) does not depend on n, c_1 and c_2 are independent of n.

The case $X = L^{\infty}$, has not been treated before. Note that since $I_\mu(t)/t$ decreases and $\lim_{t\to 0} I_\mu(t)/t = \infty$,

$$\sup_{0<t<1} f_\mu^*(t)\frac{I_\mu(t)}{t} < \infty \;\Leftrightarrow\; f = 0.$$

But Theorem 7 ensures that

$$\left\| \left(\left(f - \int_{\mathbb{R}^n} f d\mu \right)_\mu^{**}(t) - \left(f - \int_{\mathbb{R}^n} f d\mu \right)_\mu^{*}(t) \right) \beta_{p,\alpha}(t) \right\|_{L^\infty} \preceq \|\nabla f\|_{L^\infty}. \tag{5.15}$$

Furthermore, for every r.i. space Y such that

$$\left\| f - \int_{\mathbb{R}^n} f d\mu \right\|_Y \preceq \|\nabla f\|_{L^\infty},$$

the following embedding holds

$$\|f\|_Y \preceq \left\| \left(f_\mu^{**}(t) - f_\mu^*(t) \right) \beta_{p,\alpha}(t) \right\|_{L^\infty} + \|f\|_1.$$

Notice that due to the cancelation provided by $f_\mu^{**}(t) - f_\mu^*(t)$, the corresponding space $LS(L^\infty)$ is nontrivial. The relation to concentration inequalities follows from (5.15) using the method developed in [83].

Let us finally consider Sobolev embeddings into L^∞. Notice that from inequality (3.5) we get

$$\begin{aligned}
\|f\|_\infty - 2\int_0^{1/2} f_\mu^*(t)dt &= \int_0^{1/2} \left(f_\mu^{**}(t) - f_\mu^*(t) \right) \frac{dt}{t} \\
&\leq \int_0^{1/2} \left(\frac{1}{t} \int_0^t |\nabla f|_\mu^*(s)ds \right) \frac{dt}{I_\mu(t)} \\
&= \int_0^{1/2} |\nabla f|_\mu^*(s) \int_s^{1/2} \frac{ds}{sI_\mu(s)}.
\end{aligned}$$

Using the asymptotics of I_μ combined with the Poincaré inequality (5.2) yields

$$\|f - m\|_\infty \preceq \int_0^{1/2} |\nabla f|_\mu^*(s) \frac{ds}{s\left(\log\frac{1}{s}\right)^{1-\frac{1}{p}} \left(\log\log\left(e + \frac{1}{s}\right)\right)^{\frac{\alpha}{p}}}.$$

5.4. **Model Case 2:** In this section we work with the probability space $(\mathbb{S}^n, d, \sigma_n)$. We refer to Section 4.2 for notation and background information.

Theorem 8. *Let X, Y be two r.i. spaces on the $n-$sphere $\mathbb{S}^n$.*

Part I.

The following are equivalent:

(i)

$$\left\| g - \int_{\mathbb{S}^n} g d\sigma_n \right\|_Y \preceq \|\nabla g\|_X, \quad g \in Lip(\mathbb{S}^n). \tag{5.16}$$

(ii)

$$\left\| \int_t^1 f(s) \frac{ds}{I_{\sigma_n}(s)} \right\|_{\bar{Y}} \preceq \|f\|_{\bar{X}}, \quad \forall 0 \leq f \in \bar{X},\ \textit{with } supp(f) \subset (0, 1/2).$$

Part II.

(1) *If* $\underline{\alpha}_X > 1/n$, *then the r.i. space defined by the condition*

$$\left\{f \in L^1(\mathbb{S}^n) : \left\| f^*_{\sigma_n}(t)t^{-1/n} \right\|_{\bar{X}} < \infty \right\}$$

is optimal among all r.i. spaces Y *that satisfy (5.16) in the sense that*

$$\left\| f - \int_{\mathbb{S}^n} f d\sigma_n \right\|_Y \preceq \left\| \left(f - \int_{\mathbb{S}^n} f d\sigma_n \right)^*_{\sigma_n} (t)t^{-1/n} \right\|_{\bar{X}} \preceq \|\nabla f\|_X , \quad f \in Lip(\mathbb{S}^n).$$

(2) *If* $\underline{\alpha}_X \leq 1/n$, *then the r.i. set defined by*

$$\left\{ f \in L^1(\mathbb{S}^n) : \left\| \left(\left(f - \int_{\mathbb{S}^n} f d\sigma_n \right)^{**}_{\sigma_n} (t) - \left(f - \int_{\mathbb{S}^n} f d\sigma_n \right)^*_{\sigma_n} (t) \right) t^{-1/n} \right\|_{\bar{X}} < \infty \right\},$$

is optimal among all r.i. spaces Y *that satisfy (5.16) in the sense that*

$$\begin{aligned} \left\| f - \int_{\mathbb{S}^n} f d\sigma_n \right\|_Y &\preceq \left\| \left(\left(f - \int_{\mathbb{S}^n} f d\sigma_n \right)^{**}_{\sigma_n} (s) - \left(f - \int_{\mathbb{S}^n} f d\sigma_n \right)^*_{\sigma_n} (s) \right) t^{-1/n} \right\|_{\bar{X}} + \|f\|_{L^1} \\ &\preceq \|\nabla f\|_X , \quad f \in Lip(\mathbb{S}^n). \end{aligned}$$

Proof. $(\mathbf{1}) \Leftrightarrow (\mathbf{2})$. The argument given in Theorem 6 can be repeated verbatim with the following changes: Given a positive measurable function f with $supp f \subset (0, 1/2)$, let

$$F(t) = \int_t^1 f(s) \frac{ds}{I_{\sigma_n}(s)}, \qquad t \in (0, 1),$$

and define u (in spherical coordinates) by

$$u(\theta_1, ..., \theta_n) = F(\Phi(\theta_1)), \qquad (\theta_1, ..., \theta_n) \in \mathbb{S}^n.$$

Part II follows readily from the fact that (see [26])

$$I_{\sigma_n}(t) \simeq t^{1-1/n}, \ 0 < t \leq 1/2.$$

□

6. Poincaré Inequalities and Cheeger's inequality

6.1. **Poincaré inequalities and Hardy operators.** In general it is not possible to reduce the validity of the sharp Poincaré inequalities to the boundedness of the Hardy operators Q_I without requiring extra properties of the metric spaces. In fact (cf. [86] for the details), for a given $0 < \beta < 1/2$, consider

$$I(s) = s^{1-\beta}, \quad 0 \leq s \leq 1/2.$$

Let Ω be a $2(1-\beta)-$John domain on $\mathbb{R}^2$, $(|\Omega| = 1)$. The isoperimetric profile $I_\Omega(s)$ of Ω satisfies (cf. [58])

$$I_\Omega(s) \simeq I(s), \quad 0 \leq s \leq 1/2,$$

and (cf. [66])

$$\left\| g - \int_\Omega g \right\|_{L^{\frac{4}{1-2\beta}}} \preceq \|\nabla g\|_{L^2} .$$

However, the operator

$$Q_{I_\Omega} f(t) = \int_t^{1/2} f(u) \frac{du}{I_\Omega(u)}$$

is not bounded from L^2 to $L^{\frac{4}{1-2\beta}}$. In fact, the extra properties required on the metric spaces are not related with the form of the isoperimetric profile. Indeed, it is possible to build a compact surface of revolution M such that there exists a constant c depending only of I such that

$$cI(s) \leq I_M(s) \leq I(s), \quad 0 \leq s \leq 1/2,$$

and, such that for any pair of r.i. spaces X, Y on M, the Poincaré inequality

$$\left\| g - \int_M g dVol_M \right\|_Y \preceq \|\nabla g\|_X, \quad g \in Lip(M).$$

is equivalent to

$$Q_{I_M} : \bar{X} \to \bar{Y} \text{ is bounded.}$$

6.2. **Isoperimetric Hardy condition.** In his recent work (cf. [93], [94], [95]) E. Milman has considered convexity conditions that imply the equivalence of a hierarchy of progressively weaker Poincaré type inequalities and Cheeger's inequality. More precisely, among other results, Milman has shown that[12]

Theorem 9. *(E. Milman) Let (Ω, d, μ) be a space satisfying E. Milman's convexity conditions[13]. Then following statements are equivalent*
(E1) Cheeger's inequality

$$\exists C > 0 \ s.t. \quad I_{(\Omega,d,\mu)} \geq Ct, \quad t \in (0, 1/2].$$

(E2) Poincaré's inequality

$$\exists P > 0 \ s.t. \quad \|f - m\|_{L^2(\Omega)} \leq P \|f\|_{L^2(\Omega)}.$$

(E3) Exponential concentration: for all $f \in Lip(\Omega)$ with $\|f\|_{Lip(\Omega)} \leq 1$,

$$\exists c_1, c_2 > 0 \ s.t. \quad \mu\{|f - m| > t\} \leq c_1 e^{-c_2 t}, \quad t \in (0, 1).$$

(E4) First moment inequality: for all $f \in Lip(\Omega)$ with $\|f\|_{Lip(\Omega)} \leq 1$,

$$\exists F > 0 \ s.t. \quad \|f - m\|_{L^1(\Omega)} \leq F.$$

In fact, E. Milman also shows

Theorem 10. *Let (Ω, d, μ) be a space satisfying E. Milman's convexity conditions. Let $1 \leq q \leq \infty$, and let N be a Young's function such that $\frac{N(t)^{1/q}}{t}$ is non-decreasing, and there exists $\alpha > \max\{\frac{1}{q} - \frac{1}{2}, 0\}$ such that $\frac{N(t^\alpha)}{t}$ non-increasing. Then, the following statements are equivalent:*
(E5) (L_N, L^q) Poincaré inequality holds

$$\exists P > 0 \ s.t. \quad \|f - m\|_{L_N(\Omega)} \leq P \|f\|_{L^q(\Omega)}.$$

(E6) Any isoperimetric profile estimator I satisfies: there exists a constant $c > 0$ such that

$$I(t) \geq c \frac{t^{1-1/q}}{N^{-1}(1/t)}, \quad t \in (0, 1/2].$$

[12] We refer to E. Milman's papers for an account of the history of the problem.
[13] All our model spaces above satisfy these conditions.

Milman approaches these results using a variety of different tools including the semigroup approach of Ledoux ([71], [72], [73]). In this section we study the self improving problem in a more general context via Poincaré inequalities. From our experience with the model spaces (cf. Theorem 6 and Theorem 8 above) we were led to the following condition

Definition 2. *We shall say that a probability metric space (Ω, d, μ) is of isoperimetric Hardy type if for any given isoperimetric estimator I, the following are equivalent for all r.i. spaces $X = X(\Omega)$, $Y = Y(\Omega)$: there exists a constant $c = c(X, Y)$ such that*

(1)
$$\left\| f - \int_{\Omega} f d\mu \right\|_{Y} \leq c \left\| \nabla f \right\|_{X}, \quad \forall f \in Lip(\Omega).$$

(2) *There exists a constant $c_1 = c_1(X, Y) > 0$ such that*
$$\left\| Q_I f \right\|_{\bar{Y}} \leq c_1 \left\| f \right\|_{\bar{X}}, \quad 0 \leq f \in \bar{X}, \text{ with } supp(f) \subset (0, 1/2),$$
where Q_I is the isoperimetric Hardy operator

$$Q_I f(t) = \int_t^1 f(s) \frac{ds}{I(s)}. \tag{6.1}$$

We now show the following extensions of E. Milman's Theorems.

Theorem 11. *Suppose that (Ω, d, μ) is of isoperimetric Hardy type then*
$$(E1) \Leftrightarrow (E2) \Leftrightarrow (E3) \Leftrightarrow (E4)$$

Proof. Suppose that Cheeger's inequality (E1) holds, $I(s) \succeq s$, $s \in (0, 1/2)$. Therefore, for all $f \geq 0$, with $supp(f) \subset (0, 1/2)$, we have

$$Q_I f(t) = \int_t^1 f(s) \frac{ds}{I(s)} \preceq Qf(t) = \int_t^1 f(s) \frac{ds}{s}. \tag{6.2}$$

In particular, since $Q : L^2(0, 1) \to L^2(0, 1)$, we see that
$$\left\| Q_I f \right\|_{L^2} \leq C \left\| f \right\|_{L^2}, \text{ for all } f \ \geq 0, \text{ such that } supp(f) \subset (0, 1/2).$$
Consequently, by the isoperimetric Hardy property, the (L^2, L^2) Poincaré inequality (E2) holds. Conversely, if the (L^2, L^2) Poincaré inequality holds, then
$$\left\| Q_I f \right\|_{L^2} \leq C \left\| f \right\|_{L^2}, \text{for all } f \text{ such that } supp(f) \subset (0, 1/2).$$
Moreover, since $L^2 \subset L^{2,\infty}$, we have
$$\left\| Q_I f \right\|_{L^{2,\infty}} \leq C \left\| f \right\|_{L^2} \text{ for all } f \geq 0 \text{ such that } supp(f) \subset (0, 1/2).$$
Let $f = \chi_{(0,r)}$, with $r \leq 1/2$. Then, the previous inequality readily gives
$$\sup_t t^{1/2} \int_t^r \frac{ds}{I(s)} \leq C r^{1/2},$$
and, since $I(t)$ increases on $(0, 1/2)$, we get
$$\frac{1}{I(r)} \sup_t t^{1/2} (r - t) \preceq C r^{1/2}.$$
Moreover, since on the other hand
$$\sup_{t<r} t^{1/2} (r - t) \geq \left(\frac{r}{2}\right)^{1/2} \frac{r}{2}$$

we see that

$$I(t) \succeq t, \quad t \in (0, 1/2].$$

It is also elementary to see that the operator Q defined above is a bounded operator $Q : L^{\infty} \mapsto \exp L$. Indeed, using an equivalent norm for $\exp L$ (cf. [61]) we compute

$$\left\| \int_t^1 f(s) \frac{ds}{s} \right\|_{\exp(L)} = \sup_{0<t<1} \frac{\int_t^1 f(s) \frac{ds}{s}}{1 + \log \frac{1}{t}} \leq \|f\|_{L_\infty}.$$

Therefore, if (E1) holds then, by (6.2),

$$Q_I : L^{\infty} \to \exp L,$$

and therefore, by the isoperimetric Hardy property, we see that for $f \in Lip(\Omega)$ we have

$$\|f - m\|_{\exp(L)} \preceq \|\nabla f\|_{\infty}. \tag{6.3}$$

In other words, the exponential concentration inequality (E3) holds. Conversely, suppose that (6.3) holds. Then, by the isoperimetric Hardy property, we have,

$$\sup_t \frac{\int_t^{1/2} f(s) \frac{ds}{I(s)}}{1 + \log \frac{1}{t}} \preceq \|f\|_{\infty}. \tag{6.4}$$

Insert the function $f(s) = \chi_{(0,1/2)}(s) \in L^{\infty}$ in (6.4); then, using the fact that $s/I(s)$ increases, we see that for all $t \in (0, 1/2)$ we have

$$\begin{aligned} c &\succeq \sup_{t<1/2} \frac{\int_t^{1/2} \frac{s}{s} \frac{ds}{I(s)}}{1 + \log \frac{1}{t}} \\ &\succeq \frac{t}{I(t)} \frac{\int_t^{1/2} \frac{ds}{s}}{1 + \log \frac{1}{t}} \\ &\succeq \frac{t}{I(t)} \frac{\log \frac{1}{t} + \log \frac{1}{2}}{1 + \log \frac{1}{t}} \\ &\succeq \frac{t}{I(t)}. \end{aligned}$$

In other words, we see that Cheeger's inequality (E1) holds. Finally, (E3) combined with the trivial embedding

$$\|f - m\|_{L^1(\Omega)} \leq c \|f - m\|_{\exp L(\Omega)}$$

implies

$$\|f - m\|_{L^1(\Omega)} \preceq \|\nabla f\|_{L^{\infty}(\Omega)}.$$

Therefore (E4) holds. Conversely, if (E4) holds then

$$\|Q_I f\|_{L^1} \leq C \|f\|_{L^{\infty}} \text{ for all } f \geq 0 \text{ such that } supp(f) \subset (0, 1/2).$$

A familiar calculation using $f = \chi_{(0,r)}$, with $r \leq 1/2$, shows that

$$I(t) \succeq t^2, \ t \in (0, 1/2].$$

However (here we use an argument in [93]), we know that $I(t)/t$ is decreasing and $I(t)$ is symmetric about $1/2$ so by a convexity argument we can deduce that

$$I(t) \succeq t, \ t \in (0, 1/2]$$

concluding the proof. □

In general it is not possible to improve on (2.7) unless we have more information about X. On the other hand, when dealing with Orlicz spaces, then assuming some extra growth properties of the Young's functions allow us to improve upon (2.7). More specifically, if $\frac{N(t)}{t^q}$ is increasing then

$$\|f\|_{L_N} \preceq \|f\|_{\Lambda(\phi_{L_N},q)} = \left\{ \int_0^1 [f^*(s)\phi_{L_N}(s)]^q \frac{ds}{s} \right\}^{1/q}, \tag{6.5}$$

while the opposite inequality holds if $\frac{N(t)}{t^q}$ decreases (cf. [96, pag 43]).

Theorem 12. *Suppose that* (Ω, d, μ) *is of isoperimetric Hardy type. Then* $(E5) \Leftrightarrow (E6)$*. In fact,* $(E6) \Rightarrow (E5)$ *is true without the assumption that* (Ω, d, μ) *is of isoperimetric Hardy type.*

Proof. If (E5) holds then, in view of (2.7), and the fact that $\Lambda(L^q) = L^{q,1}$, we have

$$\|Q_I f\|_{M(L_N(\Omega))} \preceq \|f\|_{L^{q,1}}.$$

Therefore, there exists a constant $C > 0$ such that for $f = \chi_{(0,r)}$, $0 < r < 1/2$, we have

$$\sup_{t<r} \left\{ \phi_{L_N}(t) \int_t^r \frac{ds}{I(s)} \right\} \le Cr^{1/q}.$$

Thus,

$$\begin{aligned} \sup_{t<r} \phi_{L_N}(t) \frac{1}{I(r)}(r-t) &\ge \frac{1}{2}\phi_{L_N}(r/2)\frac{r}{I(r)} \\ &\ge \frac{1}{4}\phi_{L_N}(r)\frac{r}{I(r)} \text{ (since } \phi_{L_N}(t)/t \text{ decreases).} \end{aligned}$$

Summarizing, we have

$$I(r) \succeq r^{1-1/q}\phi_{L_N}(r), \ 0 < r < 1/2.$$

Consequently, recalling (2.6) we obtain (E6).

Suppose now that (E6) holds. We will show below that

$$\|Q_I f\|_{\Lambda(\phi_{L_N},q)} \preceq \|f\|_{L^q}. \tag{6.6}$$

This given, in view of (6.5), we see that

$$\|Q_I f\|_{L_N} \preceq \|f\|_{L^q}.$$

Therefore (E5) follows by the isoperimetric Hardy property. To prove (6.6) we use (E6) in order to estimate Q_I by

$$Q_I f(t) \preceq \int_t^{1/2} \frac{f(s)s^{1/q-1}}{\phi_{L_N}(s)} s \frac{ds}{s} \le Q\left(\frac{f(s)s^{1/q-1}}{\phi_{L_N}(s)} s\right)(t).$$

Thus, since $Q\left(\frac{f(s)s^{1/q-1}}{\phi_{L_N}(s)}s\right)(t)$ is decreasing, and using a suitable version of Hardy's inequality (cf. (6.7) below) we get

$$\begin{aligned}
\|Q_I f\|_{\Lambda(\phi_{L_N},q)} &\preceq \left\{\int_0^1 \left(\int_t^1 \frac{f(s)s^{1/q-1}}{\phi_{L_N}(s)}s\frac{ds}{s}\right)^q (\phi_{L_N}(t))^q \frac{dt}{t}\right\}^{1/q} \\
&\preceq \left\{\int_0^1 \left(\frac{f(t)t^{1/q}}{\phi_{L_N}(t)}t\frac{1}{t}\right)^q (\phi_{L_N}(t))^q \frac{dt}{t}\right\}^{1/q} \\
&= \|f\|_q ,
\end{aligned}$$

as we wished to show. To justify the application of Hardy's inequality we need to verify (see [88, Page 45]) that

$$\sup_{0<r<1}\left(\int_0^r (\phi_{L_N}(t))^q \frac{dt}{t}\right)^{1/q}\left(\int_r^1 \left(\frac{(\phi_{L_N}(t))^q}{t}\right)^{\frac{-1}{q-1}} \frac{dt}{t^{\frac{q}{q-1}}}\right)^{\frac{q-1}{q}} \leq c. \tag{6.7}$$

To this end observe that, under our current assumptions on the growth of N, we have

$$\frac{N(t)^{1/q}}{t} \text{ increasing} \Rightarrow \frac{[\phi_{L_N}(t)]^q}{t}\text{decreasing},$$

$$\frac{N(t^\alpha)}{t} \text{ decreasing} \Rightarrow \frac{(\phi_{L_N}(t))^{1/\alpha}}{t} \text{ increasing} \Rightarrow \frac{\phi_{L_N}(t)}{t^\alpha} \text{ increasing.}$$

Therefore,

$$\begin{aligned}
\frac{1}{r}\int_0^r (\phi_{L_N}(t))^q \frac{dt}{t} &= \frac{1}{r}\int_0^r (\phi_{L_N}(t))^{q-1} \frac{\phi_{L_N}(t)}{t^\alpha}\frac{t^\alpha dt}{t} \\
&\leq \frac{\phi_{L_N}(r)}{r^\alpha} (\phi_{L_N}(t))^{q-1} \frac{1}{r}\int_0^r \frac{t^\alpha dt}{t} \\
&= \frac{\phi_{L_N}(r)}{r^\alpha} (\phi_{L_N}(t))^{q-1} \frac{1}{r}\frac{r^\alpha}{\alpha} \\
&= \frac{1}{\alpha}\frac{(\phi_{L_N}(r))^q}{r}. \qquad (6.8)
\end{aligned}$$

To estimate the second integral in (6.7) let $w(s) = \frac{(\phi_{L_N}(t))^q}{t}$, then

$$\begin{aligned}
\int_r^1 (w(t))^{\frac{-1}{q-1}} \frac{dt}{t^{\frac{q}{q-1}}} &= \int_r^1 \frac{w(t)}{(tw(t))^{\frac{q}{q-1}}}dt \\
&\leq \frac{1}{\alpha}\int_r^1 \frac{w(t)}{\left(\int_0^t w(s)ds\right)^{\frac{q}{q-1}}}dt \quad \text{(by (6.8))} \\
&\leq \frac{1}{\alpha}\frac{1}{\left(\int_0^r w(s)ds\right)^{\frac{q}{q-1}}}\int_r^1 w(t)dt \\
&= \frac{1}{\alpha}\left(\int_0^r w(s)ds\right)^{\frac{-1}{q-1}}.
\end{aligned}$$

Thus,

$$\left(\int_0^r (\phi_{L_N}(t))^q \frac{dt}{t}\right)^{1/q} \left(\int_r^1 \left(\frac{(\phi_{L_N}(t))^q}{t}\right)^{\frac{-1}{q-1}} \frac{dt}{t^{\frac{q}{q-1}}}\right)^{\frac{q-1}{q}} \leq \frac{1}{\alpha},$$

and (6.7) holds. □

Remark 8. *In the particular case when* $L_N(\Omega) = L^p$ $(p \geq q)$, *then* $\Lambda(\phi_{L_N}, q) = L^{p,q}$ *and we obtain*

$$\left\|f - \int_\Omega f d\mu\right\|_{L^{p,\infty}} \preceq \|\nabla f\|_{L^q} \Rightarrow \left\|f - \int_\Omega f d\mu\right\|_{L^{p,q}} \preceq \|\nabla f\|_{L^q}.$$

For more on this type of self improvement for Poincaré inequalities see [82].

Remark 9. *The fact that Cheeger's inequality implies concentration can be also had readily from (3.5). To see this observe that if* $I(t) \succeq t$, *and* f *is* $1 - Lip(\Omega)$ *then from (3.5) we get*

$$f^{**}(t) - f^*(t) \preceq c,$$

in other words $f \in L(\infty, \infty)$, *the weak class of Bennett-De Vore-Sharpley* [19]. *Since it is known (cf.* [20]*) that* $L(\infty,\infty) \subset e^L$ *(cf. also* [83] *for more general results) we see that Cheeger's inequality indeed implies*

$$f \in Lip(\Omega) \Rightarrow f \in e^L,$$

i.e. Cheeger's inequality $\Rightarrow$*concentration.*

Theorem 13. *All the model spaces introduced in Section 4 are of isoperimetric Hardy type.*

Proof. This follows from Theorem 6, Theorem 8 and finally the case of Ros's model spaces was treated in [86]. □

7. Transference Principle

A very useful property of symmetrization methods is to reduce complicated problems to simpler model problems where symmetry can be used to find a solution. In this section we show how to use symmetrization to transfer inequalities[14] from one metric space to another. As we shall see the isoperimetric Hardy property plays an important role in this process.

Theorem 14. *Suppose that* (Ω, d, μ) *is of isoperimetric Hardy type. Suppose that* (Ω_1, d_1, μ_1) *is a probability metric space such that there exists* $c > 0$ *such that*

$$I_{(\Omega_1,d_1,\mu_1)}(t) \geq c I_{(\Omega,d,\mu)}(t), \quad t \in (0, 1/2]. \tag{7.1}$$

Let $X(\Omega), Y(\Omega)$ *be r.i. spaces for which there exists a constant* $c > 0$ *such that the following Poincaré inequality holds*

$$\left\|g - \int_\Omega g d\mu\right\|_{Y(\Omega)} \leq c \||\nabla g|\|_{X(\Omega)}, \quad \text{for all } g \in Lip(\Omega). \tag{7.2}$$

[14] This circle of ideas of course is well known in the theory of semigroups, and one can use the symmetrization inequalities in this context as well (cf [30], [70]). We hope to return to this point elsewhere.

Then, there exists a constant $c_1 > 0$ such that

$$\left\| g - \int_{\Omega_1} g d\mu_1 \right\|_{Y(\Omega_1)} \leq c \left\| |\nabla g| \right\|_{X(\Omega_1)}, \quad \textit{for all } g \in Lip(\Omega_1).$$

Proof. Since (Ω, d, μ) is of isoperimetric Hardy type the Poincaré inequality (7.2) implies the existence of a constant $c_1 > 0$ such that

$$\left\| Q_{I_{(\Omega,d,\mu)}} f \right\|_{\bar{Y}(0,1)} \leq c_1 \left\| f \right\|_{\bar{X}(0,1)}, \text{ for all } f \geq 0, \text{ with supp} f \subset (0, 1/2). \tag{7.3}$$

In view of (7.1) we have

$$\int_t^1 f(s) \frac{ds}{I_{(\Omega_1,d_1,\mu_1)}(s)} \preceq \int_t^1 f(s) \frac{ds}{I_{(\Omega,d,\mu)}(s)}, \text{ for all } f \geq 0, \text{ with supp} f \subset (0, 1/2).$$

Therefore, (7.3) can be lifted to

$$\left\| Q_{I_{(\Omega_1,d_1,\mu_1)}} f \right\|_{\bar{Y}(0,1)} \preceq \left\| f \right\|_{\bar{X}(0,1)}, \text{ for all } f \geq 0, \text{ with supp} f \subset (0, 1/2).$$

Therefore we conclude by Theorem 5. □

Corollary 1. *Let M be a (compact) connected Riemannian manifold of dimension $n \geq 2$, with Ricci curvature bounded from below by $\rho > 0$. Let σ be the normalized volume on M. Let $\bar{X}(0,1)$, $\bar{Y}(0,1)$ be two r.i. spaces for which the following Poincaré inequality holds in the probability space $(\mathbb{S}^n, d, \sigma_n)$*

$$\left\| g - \int_{\mathbb{S}^n} g d\sigma_n \right\|_{Y(\mathbb{S}^n)} \preceq \left\| |\nabla g| \right\|_{X(\mathbb{S}^n)}, \quad g \in Lip(\mathbb{S}^n).$$

holds. Then,

$$\left\| g - \int_{M} g d\sigma \right\|_{Y(M)} \preceq \left\| |\nabla g| \right\|_{X(M)}, \quad g \in Lip(M).$$

Proof. The Lévy-Gromov isoperimetric inequality (see [75], [55], [53]) yields (recall $I_n = I_{\mathbb{S}^n}$, see Section 4.2 above)

$$I_M \geq \sqrt{\frac{\rho}{n-1}} I_n.$$

Therefore,

$$\left\| \int_t^1 f(s) \frac{ds}{I_M(s)} \right\|_{\bar{Y}} \preceq \left\| f \right\|_{\bar{X}}, \quad \forall 0 \leq f \in \bar{X}, \text{ with } supp(f) \subset (0, 1/2),$$

and the result follows from Theorem 13 and Theorem 14. □

Finally, let us now present our last example.

Let $1 < p \leq 2$, $\mu_p(x) = Z_p^{-1} \exp\left(-|x|^p\right) dx$, $x \in \mathbb{R}$, and let $\mu = \mu_p^{\otimes n}$. Every log-concave probability measure ν on $\mathbb{R}^d$ such that $\exp(\varepsilon |x|^p) \in L^1(\nu)$ for some $\varepsilon > 0$ and $p \in [1, 2]$ satisfies up to a constant the same isoperimetric inequality as μ_p (see [25], and [10]). This result was extended in [14] to the setting of Riemannian manifolds under appropriate curvature conditions. Using these results we get

Corollary 2. *Let M be a smooth, complete, connected Riemannian manifold without boundary. Let $d\nu(x) = e^{-V(x)}d\sigma(x)$ be a probability measure on M, (σ normalized volume on M) with a twice continuously differentiable potential V. Let $1 < p \leq 2$, and suppose that there exists $x_0 \in M$ and $\varepsilon > 0$ such that*

$$\exp(\varepsilon d(x_0, x)^p) \in L^1(\mu),$$

and, moreover, suppose that

$$HessV + Ric \geq 0.$$

Let $\bar{X}$, $\bar{Y}$ be two r.i. spaces on $(0,1)$ for which the following Poincaré inequality holds

$$\left\| \left(g - \int_{\mathbb{R}^n} g d\mu \right)^*_\mu (t) \right\|_{\bar{Y}} \preceq \left\| |\nabla g|^*_\mu \right\|_{\bar{X}}, \quad g \in Lip(\mathbb{R}^n).$$

Then,

$$\left\| \left(f - \int_M f d\nu \right)^*_\nu \right\|_{\bar{Y}} \preceq \left\| |\nabla g|^*_\nu \right\|_{\bar{X}}, \quad g \in Lip(M).$$

Proof. By ([14, Theorem 7.2]) there exists $\kappa > 0$ such that

$$I_M(t) \geq \kappa s \left(\log \frac{1}{s} \right)^{1-\frac{1}{p}} \simeq I_{\mu_p}(s), \qquad 0 < s \leq 1/2,$$

and we finish using Theorem 14. □

Remark 10. *Let $M = M_1 \times M_2$ be the product of Riemannian manifolds with volume 1. Then, the isoperimetric profile of I_M, can be estimated in terms of the isoperimetric profiles of I_{M_i} as follows (see [100])*

$$I_M(s) \geq \frac{1}{\sqrt{2}} \inf \left\{ s_1 I_{M_1}(s_2) + s_2 I_{M_2}(s_1) : s_1 s_2 = s \quad or \quad 1 - s \right\}.$$

For example, if $I_{M_i}(s) \geq c_i s^{1-1/p_i}$, ($p_i > 1$), then

$$I_M(s) \geq c s^{1-1/(p_1+p_2)}.$$

Using this estimate, Theorems 14 and 5, we can easily derive Poincaré inequalities on M.

7.1. **Gaussian Isoperimetric type and a question of Triebel.** When we were revising an earlier version of our manuscript we received a query from Professor Hans Triebel concerning certain Sobolev inequalities with dimension free constants (cf. [117]). In this section we provide a positive answer to Prof. Triebel's question using the transference principle.

We consider Triebel's notation. Let $Q^n = (0,1)^n$, the unit cube in $\mathbb{R}^n$. Triebel asks for a treatment of dimension free Sobolev inequalities for the space $\mathring{\mathrm{W}}^1_1(Q^n) = \overline{C_0^\infty(Q^n)}^{W_1^1(Q^n)}$. More specifically, Triebel asks (in our notation) if one can prove dimension free inequalities of the form

$$\left(\int_0^1 [f^*(t)]^q (1 + \log \frac{1}{t})^\alpha dt \right)^{1/q} \preceq \|\nabla f\|_{L^q(Q^n)} + \|f\|_{L^q(Q^n)}, \tag{7.4}$$

for a suitable power $\alpha =?$ of the logarithm. To resolve this question, we first need to understand the "correct" power of the logarithm that is needed here. For this we consider the isoperimetry of Q^n. It is known that (cf. [107, Theorem 7])

$$I_{Q^n} \geq I_\gamma.$$

Therefore, since $(\mathbb{R}^n, \gamma_n)$ is of Hardy isoperimetric type (cf. [83]), we can use Theorem 14 to transfer to Q^n the Gaussian Poincaré inequalities. By the asymptotic behavior of I_{γ_n} it follows that, for $1 < q < \infty$, we have

$$\left(\int_0^1 \left[\left(f - \int_{Q^n} f \right)^{**} (t) \right]^q \left(1 + \log \frac{1}{t} \right)^{q/2} dt \right)^{1/q} \preceq \|\nabla f\|_{L^q(Q^n)},$$

with constants independent of the dimension. Finally, an application of the triangle inequality yields

$$\left(\int_0^1 f^{**}(t)^q \left(1 + \log \frac{1}{t} \right)^{q/2} dt \right)^{1/q} \preceq \|\nabla f\|_{L^q(Q^n)} + \|f\|_{L^q(Q^n)},$$

and the constants are independent of the dimension. This statement proves (7.4) with $\alpha = q/2$, thus providing a positive answer to Professor Triebel's conjecture.

Let us consider a similar result for the $p-$unit ball, i.e. let

$$B_p^n = \left\{ x = (x_1, \cdots, x_n) : \|x\|_p^p = |x_1|^p + \cdots + |x_n|^p \leq 1 \right\}, \qquad 1 \leq p \leq 2,$$

and consider on B_p^n the normalized volume measure

$$V_p^n = \frac{vol\ |_{B_p^n}}{vol(B_p^n)}.$$

In the recent paper [110], S. Sodin proves that,

$$I_{V_p^n}(\tilde{a}) \geq c n^{1/p} \tilde{a} \log^{1-1/p} \frac{1}{\tilde{a}}; \qquad \tilde{a} = \min(a, 1-a);\ 0 < a < 1,$$

where c is an absolute constant; in particular, since $n \geq 2$, we get

$$I_{V_p^n}(\tilde{a}) \geq c 2^{1/p} \tilde{a} \log^{1-1/p} \frac{1}{\tilde{a}}.$$

At this point we can use again Theorem 14 to transfer to V_p^n the Poincaré inequalities. Indeed, let $1 \leq p \leq 2$ and consider the measure

$$\mu_p(x) = Z_p^{-1} \exp\left(-|x|^p\right) dx, \qquad x \in \mathbb{R}.$$

Since $(\mathbb{R}^n, \mu_p^{\otimes n})$ is of Hardy isoperimetric type (see Example 5.2 above) and by the asymptotic properties of $I_{\mu_p^{\otimes n}}$ (see (5.13)), there exist constants c_1 and c_2, that do not depend on n, such that

$$c_1 \tilde{a} \log^{1-1/p} \frac{1}{\tilde{a}} \leq I_{\mu_p^{\otimes n}}(\tilde{a}) \leq c_2 \tilde{a} \log^{1-1/p} \frac{1}{\tilde{a}}.$$

By Theorem 14 it follows that, for $1 < q < \infty$, we have

$$\left(\int_0^1 \left[\left(f - \int_{B_p^n} f dV_p^n \right)^{**} (t) \right]^q \left(1 + \log \frac{1}{t} \right)^{q(1-1/p)} dt \right)^{1/q} \preceq \|\nabla f\|_{L^q(B_p^n)}.$$

Consequently,

$$\left(\int_0^1 f^{**}(t)^q \left(1+\log\frac{1}{t}\right)^{q(1-1/p)} dt\right)^{1/q} \preceq \|\nabla f\|_{L^q(B_p^n)} + \|f\|_{L^q(B_p^n)},$$

with constants that are independent of the dimension.

Remark 11. *In the particular case* $p=2$, $q=2$ *and* $f\in \mathring{W}_1^2(B_2^n)=\overline{C_0^\infty(Q^n)}^{W_1^2(B_2^n)}$ *this result was obtained in* [64]. *For* $p=2$ *and* $1<q<n/3$ *and other related results see* [65].

One could also approach other questions posed by Triebel using our techniques but this would take us too far away from the main topics of this paper.

On the other hand the ideas discussed in this section can be pushed further. Let (M,d) be a Riemannian manifold endowed with a probability measure μ on M which is absolutely continuous with respect the volume $dVol_M$. We say that M admits a **Gaussian isoperimetric inequality**, if there is a positive constant $c(\mu)$ such that

$$I_\mu(t)\geq c(\mu)I_\gamma(t)$$

(where I_γ denotes the Gaussian isoperimetric profile). It is known that this family includes any compact manifold (with or without boundary) endowed with its Riemannian probability (see [107] an the references quoted therein).

Corollary 3. *Let* γ_n *be the Gaussian measure on* $\mathbb{R}^n$. *Let* (M,d) *be a Riemannian manifold which admits a Gaussian isoperimetric inequality. Suppose that* $\bar{X}, \bar{Y}$ *are r.i. spaces on* $(0,1)$, *for which the Gaussian Poincaré inequality holds:*

$$\left\|g-\int_{\mathbb{R}^n} g d\gamma_n\right\|_{Y(\mathbb{R}^n,\gamma_n)} \preceq \||\nabla g|\|_{X(\mathbb{R}^n,\gamma_n)}, \quad g\in Lip(\mathbb{R}^n).$$

Then,

$$\left\|g-\int_M g d\mu\right\|_{Y(M,d)} \preceq \||\nabla g|\|_{X(M,d)}, \quad g\in Lip(M).$$

In particular, if $1<p<\infty$, *there exists a constant* c_p *such that*

$$\int_0^1 f^*(t)^p\left(1+\log\frac{1}{t}\right)^{p/2} d\mu \leq c_p\left(\int_M |\nabla f(x)|^p\, d\mu+\int_M |f(x)|^p\, d\mu\right), \quad f\in Lip(M).$$

Remark 12. *Concerning concentration, it was proved in* [83] *that for* $f\in Lip(\mathbb{R}^n)$

$$\left(f_{\gamma_n}^{**}(t)-f_{\gamma_n}^{*}(t)\right)\preceq |\nabla f|_{\gamma_n}^{**}(t)\left(\log\frac{1}{t}\right)^{-1/2}, \; 0<t<\frac{1}{2},$$

therefore if (M,d) *is a Riemannian manifold admitting a Gaussian isoperimetric inequality then: for all* $f\in Lip(M)$ *we have*

$$\left(f_{\mu}^{**}(t)-f_{\mu}^{*}(t)\right)\preceq |\nabla f|_{\mu}^{**}(t)\left(\log\frac{1}{t}\right)^{-1/2}, \; 0<t<\frac{1}{2}.$$

As a consequence (cf. [83]*)* $f\in e^{L^2}$.

8. Estimating isoperimetric profiles via semigroups

In this section we discuss an extension of the approach in [93], [94] to the self improving results in Section 6.2. In the case of connected Riemannian manifolds, whose Ricci curvature is bounded from below, E. Milman using methods of Ledoux ([71], [72], [73]) has developed a semigroup approach which produces isoperimetric estimates starting from the Poincaré inequalities

$$\left\| g - \int_{\Omega} g d\mu \right\|_{X} \preceq \|\nabla g\|_{L^{q}}, \quad g \in Lip(\Omega),$$

where X is an L^p space or an Orlicz space. In this section we show that the analysis can be streamlined and extended to r.i. spaces.

Let $\Omega = (M, g)$ be a smooth complete connected Riemannian manifold equipped with a probability measure μ, with density $d\mu = exp(-\psi)dVol_M$, $\psi \in C^2(M, \mathbb{R})$. Let

$$\Delta_{(\Omega,\mu)} = \Delta_{\Omega} - \nabla\psi \cdot \nabla,$$

be the associated Laplacian (Δ_Ω is the usual Laplace-Beltrami operator on Ω). Let $(P_t)_{t\geq 0}$ denote the semi-group associated to the diffusion process with infinitesimal generator $\Delta_{(\Omega,\mu)}$ (see [42], [72]) characterized by the second order system

$$\frac{\partial}{\partial t} P_t(f) = \Delta_{(\Omega,\mu)}(P_t(f)), \quad P_0(f) = f,$$

where $f \in \mathcal{B}(\Omega)$ (the space of bounded smooth[15] real functions on Ω).

For each $t \geq 0$, $p \geq 1$, $P_t : L^p(\Omega) \to L^p(\Omega)$ is a bounded linear operator. We list a few elementary properties of these operators

- $P_t 1 = 1$.
- $f \geq 0 \Rightarrow P_t f \geq 0$.
- $\int (P_t f)\, g d\mu = \int f\, (P_t g)\, d\mu$.
- $(P_t f)^{\alpha} \leq P_t f^{\alpha}$, $\forall \alpha \geq 1$.
- $P_t \circ P_s = P_{s+t}$.
- $P_t : X(\Omega) \to X(\Omega)$ is bounded on any r.i. space $X(\Omega)$.

Moreover, if the Barry-Émery curvature-dimension condition holds (cf. [6]):

$$Ric_g + Hess_g \psi \geq 0, \tag{8.1}$$

then, for all $t \geq 0$ and $f \in \mathcal{B}(\Omega)$, we have the pointwise inequality

$$2t \left|\nabla P_t f\right|^2 \leq P_t f^2 - (P_t f)^2 . \tag{8.2}$$

Theorem 15. *Let $\Omega = (M, g)$ be a smooth complete connected Riemannian manifold which satisfies the convexity assumption (8.1). Let X, Y be two r.i. spaces on Ω such that conditions (a) and (b) hold:*

Condition (a): either (i) X is q concave for some $q \geq 2$;
or
(ii) $\bar{\alpha}_X < 1/2$.

Condition (b): There exists $c = c(X, Y)$ such that the (Y, X) Poincaré inequality holds

$$\left\| g - \int_{\Omega} g d\mu \right\|_{Y} \leq c \|\nabla g\|_{X}, \qquad \forall g \in Lip(\Omega). \tag{8.3}$$

[15] we could use C^∞ functions here.

Then, there exists a constant $c_1 > 0$ such that

$$I_{(M,g,\mu)}(t) \geq c_1 t(1-t)\frac{\varphi_Y(t(1-t))}{\varphi_X(t(1-t))},$$

where φ_X and φ_Y are the fundamental functions of the r.i. spaces X and Y.

Proof. We shall follow closely Milman's proof of Theorem 2.9 in [93]. Let A denote an arbitrary Borel set in Ω with $\mu^+(A) < \infty$. We need to show

$$\mu^+(A) \geq c_1 \mu(A)(1-\mu(A))\frac{\varphi_X((1-\mu(A))\mu(A))}{\varphi_Y((1-\mu(A))\mu(A))}. \tag{8.4}$$

Using a standard approximation argument (cf. [93]) we get

$$\sqrt{2t}\mu^+(A) \geq \int |\chi_A - P_t\chi_A|\, d\mu.$$

Rewrite the right hand side as follows

$$\begin{aligned}\int |\chi_A - P_t\chi_A|\, d\mu &= \int_A (1-P_t\chi_A)\, d\mu + \int_{\Omega\setminus A} P_t\chi_A d\mu = 2\left(\mu(A) - \int_A P_t\chi_A d\mu\right)\\ &= 2\left(\mu(A)\,(1-\mu(A)) - \int_\Omega (P_t\chi_A - \mu(A))\,(\chi_A - \mu(A))\, d\mu\right).\end{aligned}$$

Using the fact that X satisfies condition (a) we will show that there exists a constant $c > 0$ such that

$$\begin{aligned}J(t) &= \int_\Omega (P_t\,(\chi_A - \mu(A)))\,(\chi_A - \mu(A))\, d\mu\\ &\leq \frac{4c}{\sqrt{2t}}\varphi_X((1-\mu(A))\mu(A))\frac{(1-\mu(A))\mu(A)}{\varphi_Y((1-\mu(A))\mu(A))}.\end{aligned} \tag{8.5}$$

This given, we deduce that

$$\begin{aligned}\mu^+(A) &\geq \frac{\mu(A)\,(1-\mu(A)) - J(t)}{\sqrt{2t}}\\ &\geq (1-\mu(A))\mu(A)\left(\frac{1}{\sqrt{2t}} - \frac{2c}{t}\frac{\varphi_X((1-\mu(A))\mu(A))}{\varphi_Y((1-\mu(A))\mu(A))}\right).\end{aligned}$$

Choosing

$$t_0 = 16\left(c\frac{\varphi_X((1-\mu(A))\mu(A))}{\varphi_Y((1-\mu(A))\mu(A))}\right)^2,$$

we obtain (8.4). It remains to prove (8.5). By Hölder's inequality, (8.3) and (8.2), we find

$$\begin{aligned}J(t) &= \int_\Omega (P_t\,(\chi_A - \mu(A)))\,(\chi_A - \mu(A))\, d\mu\\ &\leq \|P_t\,(\chi_A - \mu(A))\|_Y\,\|\chi_A - \mu(A)\|_{Y'}\\ &\leq \frac{c}{\sqrt{2t}}\,\|\nabla P_t\,(\chi_A - \mu(A))\|_X\,\|\chi_A - \mu(A)\|_{Y'}\\ &\leq \frac{c}{\sqrt{2t}}\left\|\sqrt{P_t\,(\chi_A - \mu(A))^2}\right\|_X\,\|\chi_A - \mu(A)\|_{Y'}.\end{aligned} \tag{8.6}$$

If X is q concave, then $X^{(\frac{1}{q})}$ is an r.i. space and, therefore, P_t is bounded on $X^{(\frac{1}{q})}$. Consequently,

$$\begin{aligned}
\left\| \sqrt{P_t\left(\chi_A-\mu(A)\right)^2} \right\|_X &= \left\| \left(P_t\left(\chi_A-\mu(A)\right)^2\right)^{\frac{q}{2}} \right\|^q_{X^{(\frac{1}{q})}} \\
&= \left\| P_t\left(\chi_A-\mu(A)\right)^q \right\|^q_{X^{(\frac{1}{q})}} \qquad \text{(since } q/2\geq 1\text{)} \\
&\leq \left\| \left(\chi_A-\mu(A)\right)^q \right\|^q_{X^{(\frac{1}{q})}} \\
&= \left\| \chi_A-\mu(A) \right\|_X . \qquad (8.7)
\end{aligned}$$

On the other hand, suppose now that $\bar{\alpha}_X < 1/2$ holds. Then,

$$\begin{aligned}
\left\| \sqrt{P_t\left(\chi_A-\mu(A)\right)^2} \right\|_X &\leq \left\| \left(\frac{1}{r}\int_0^r \left[P_t\left(\chi_A-\mu(A)\right)\right]^*(s)^2 ds \right)^{1/2} \right\|_{\bar{X}} \\
&\leq c\left\| P_t\left(\chi_A-\mu(A)\right) \right\| \qquad \text{(since } \bar{\alpha}_X<1/2\text{)} \\
&\leq c\left\| \chi_A-\mu(A) \right\|_X . \qquad (8.8)
\end{aligned}$$

To estimate the right hand side of (8.7) and (8.8) we note that for any r.i space $Z=Z(\Omega)$ we have,

$$\begin{aligned}
\left\| \chi_A-\mu(A) \right\|_Z &\leq (1-\mu(A))\left\|\chi_A\right\|_Z+\mu(A)\left\|\chi_{\Omega\setminus A}\right\|_Z \\
&= (1-\mu(A))\varphi_Z(\mu(A))+\mu(A)\varphi_Z(1-\mu(A)) \\
&\leq 2\varphi_Z((1-\mu(A))\mu(A)), \qquad (8.9)
\end{aligned}$$

where in the last inequality we have used the concavity of φ_Z.

Combining (8.9), (8.8), (8.7) and (8.6) yields

$$\begin{aligned}
J(t) &\leq \frac{c}{\sqrt{2t}}\left\|\chi_A-\mu(A)\right\|_X\left\|\chi_A-\mu(A)\right\|_{Y'} \\
&\leq \frac{4c}{\sqrt{2t}}\varphi_X((1-\mu(A))\mu(A))\varphi_{Y'}((1-\mu(A))\mu(A)) \\
&= \frac{4c}{\sqrt{2t}}\varphi_X((1-\mu(A))\mu(A))\frac{(1-\mu(A))\mu(A)}{\varphi_Y((1-\mu(A))\mu(A))} \quad \text{(by (2.5))}.
\end{aligned}$$

Therefore, (8.5) holds and the desired result follows. □

Remark 13. *Note that for any r.i. space $Z=Z(\Omega)$, we have $Z^{(2)}\subset Z$, and $Z^{(2)}$ is $2-$concave. It follows from the previous result that for any smooth complete connected Riemannian manifold that satisfies the convexity assumption (8.1) the isoperimetric estimate*

$$I_{(M,g,\mu)}(t)\geq c_1 t\frac{\varphi_Y(t)}{\sqrt{\varphi_X(t)}},\quad 0<t\leq 1/2$$

follows from

$$\left\| g-\int_\Omega g d\mu \right\|_Y \leq c\left\|\nabla g\right\|_X, \qquad \forall g\in Lip(\Omega).$$

9. Higher order Sobolev inequalities

In this section we consider the higher order versions of Theorem 1. Since the setting of metric spaces is not adequate to deal with higher order derivatives in this section we work on Riemannian manifolds.

Let $\Omega = (M, g)$ be a smooth complete connected Riemannian manifold equipped with a probability measure μ. Under the presence of smoothness we can give more precise formulae. The next result is essentially given in [52], we provide a detailed proof for the sake of completeness.

Proposition 2. *Let I be an isoperimetric estimator. Suppose that $f \in C^{\infty}(\Omega)$ is a positive function, and denote by $dH_{n-1}(x)$ the corresponding $(n-1)$ dimensional measure on $\{f = t\}$ associated with $d\mu$. Moreover, suppose that f has no degenerate critical points. Then,*

(i) For all regular values of f (therefore a.e. $t > 0$)

$$\frac{d}{dt}(\mu_f(t)) = \frac{1}{\left(f_\mu^*\right)'(\mu_f(t))} = -\int_{\{f=t\}} \frac{1}{|\nabla f(x)|} dH_{n-1}(x). \tag{9.1}$$

(ii) For almost all t

$$\int_{\{f=t\}} |\nabla f(x)|^{q-1} dH_{n-1}(x) \geq (I(\mu_f(t)))^q \left(\left(-f_\mu^*\right)'(\mu_f(t))\right)^{q-1}. \tag{9.2}$$

In particular, for all almost all $t \in [0, ess \sup f)$,

$$\int_{\{f=f_\mu^*(t)\}} |\nabla f(x)|^{q-1} dH_{n-1}(x) \geq (I(t))^q \left(\left(-f_\mu^*\right)'(t)\right)^{q-1}.$$

(iii) ($q-$Ledoux inequality)

$$\int |\nabla f(x)|^q d\mu \geq \int_0^{\infty} I(\mu_f(t))^q \left(\left(-f_\mu^*\right)'(\lambda_f(t))\right)^{q-1} dt. \tag{9.3}$$

Proof. (i) The co-area formula implies (cf. [38, pag 157])

$$\mu_f(t) = \mu\left(\{f > t\} \cap \{|\nabla f| = 0\}\right) + \int_t^{\infty} \int_{\{f=s\}} \frac{1}{|\nabla f(x)|} dH_{n-1}(x) ds.$$

Our assumptions on f imply that

$$\mu\left(\{f > t\} \cap \{|\nabla f| = 0\}\right) = 0, \ a.e.$$

Consequently,

$$\frac{d}{dt}(\mu_f(t)) = -\int_{\{f=t\}} \frac{1}{|\nabla f(x)|} dH_{n-1}(x), \ a.e.$$

Since f_μ^* and μ_f restricted to $[0, ess \sup |f|]$ are inverses (cf. [115, pag 935]), we get

$$f_\mu^*(\mu_f(t)) = t,$$

and therefore the remaining formula in (9.1) follows.

(ii) By the definition of isoperimetric profile

$$I(\mu_f(t)) \leq \int_{\{f=t\}} dH_{n-1}(x).$$

We estimate the right hand side using Hölder's inequality,

$$\begin{aligned}\int_{\{f=t\}} dH_{n-1}(x) &= \int_{\{f=t\}} |\nabla f(x)|^{1/q'} \frac{1}{|\nabla f(x)|^{1/q'}} dH_{n-1}(x) \\ &\leq \left(\int_{\{f=t\}} |\nabla f(x)|^{q-1} dH_{n-1}(x)\right)^{1/q} \left(\int_{\{f=t\}} \frac{1}{|\nabla f(x)|} dH_{n-1}(x)\right)^{1/q'}.\end{aligned}$$

Combining these inequalities we obtain

$$I(\mu_f(t))^q \leq \left(\int_{\{f=t\}} |\nabla f(x)|^{q-1} dH_{n-1}(x)\right)\left(\int_{\{f=t\}} \frac{1}{|\nabla f(x)|} dH_{n-1}(x)\right)^{q-1}.$$

Therefore, by (9.1)

$$I(\mu_f(t))^q \left(\left(-f_\mu^*\right)'(\mu_f(t))\right)^{q-1} \leq \int_{\{f=t\}} |\nabla f(x)|^{q-1} dH_{n-1}(x).$$

(iii) The co-area formula implies

$$\int_0^\infty \left(\int_{\{f=t\}} |\nabla f(x)|^{q-1} dH_{n-1}(x)\right) dt = \int_\Omega |\nabla f(x)|^q d\mu,$$

consequently (9.3) follows by integrating (9.2). □

Remark 14. *In particular if $q = 1$ then (9.3) becomes Ledoux's inequality (cf. (3.2) above)*

$$\int_0^\infty I(\mu_f(t))dt \leq \int_\Omega |\nabla f(x)| \, d\mu.$$

Remark 15. *Formulae (9.1) appears in several places in the literature (cf.* [112, (1), pag 709], [22, pag 81], [9, pag 52]*) with different degrees of generality. In concrete applications when the "correct" symmetrization f° is available (e.g. $\mathbb{R}^n$, with Lebesgue or Gaussian measure), then for $f \in W_1^1(\Omega)$, we have for a.e. t,*

$$\mu\left(\{f^\circ > t\} \cap \{|\nabla f^\circ| = 0\}\right) = 0$$

and

$$\frac{d}{dt}(\mu_f(t)) = -\int_{\{f^\circ=t\}} \frac{1}{|\nabla f^\circ(x)|} dH_{n-1}(x), \text{ a.e.}$$

follows.

Remark 16. *To extend these inequalities we can use Morse theory. Indeed, it is well known (cf.* [98, pag 37]*) that bounded smooth functions can be uniformly approximated (together with their derivatives) by smooth functions with non degenerate critical points.*

Our objective is to extend the first order estimates (3.3) and (3.5) of Theorem 1. The corresponding results are given by our next theorem

Theorem 16. *Suppose that the assumptions of Proposition 2 hold. Then,*

(i) *Maz'ya-Talenti second order inequality*

$$-I(t)^2 \left(-f_\mu^*\right)'(t) \leq \int_0^t |\Delta f|_\mu^*(s)ds, \text{ a.e.} \tag{9.4}$$

(ii) *Oscillation inequality*

$$f_\mu^{**}(t) - f_\mu^*(t) \leq \frac{1}{t}\int_0^t \left(\frac{s}{I(s)}\right)^2 \left|\Delta f\right|_\mu^{**}(s)ds \tag{9.5}$$

Proof. (i) In preparation to use Green's formula we write

$$\Delta f = -div(\nabla f).$$

Note that the level surface $\{f = t\} = \partial\{f > t\}$ and that the formula for the inner unit normal to $\{f = t\}$ at a point x is given by

$$\nu(x) = \frac{\nabla f(x)}{|\nabla f(x)|}.$$

Therefore, by Green's theorem,

$$\begin{aligned} -\int_{\{f>t\}} \Delta f(x)d\mu &= \int_{\{f>t\}} div(\nabla f) \\ &= \int_{\{f=t\}} \frac{|\nabla f(x)|^2}{|\nabla f(x)|} dH_{n-1}(x) \\ &\geq I(\mu_f(t))^2 \left(-f_\mu^*\right)'(\mu_f(t)) \text{ (by (9.2))}. \end{aligned}$$

Consequently for a.e. t,

$$\begin{aligned} I(t)^2 \left(-f_\mu^*\right)'(t) &\leq \int_{\{f>f_\mu^*(t)\}} \left|\Delta f(x)\right| d\mu \\ &\leq \int_0^t \left|\Delta f(x)\right|_\mu^*(s)ds, \end{aligned}$$

as we wished to show.

(ii) We start with the familiar (cf. Theorem 1 above, specially the proof of (3) $\Rightarrow$ (5)),

$$f_\mu^{**}(t) - f_\mu^*(t) = \frac{1}{t}\int_0^t s\left(-f_\mu^*\right)'(s)ds.$$

We work with the right hand side as follows,

$$\begin{aligned} \frac{1}{t}\int_0^t s\left(-f_\mu^*\right)'(s)ds &= \frac{1}{t}\int_0^t \frac{s}{I(s)^2} I(s)^2 \left(-f_\mu^*\right)'(s)ds \\ &\leq \frac{1}{t}\int_0^t \frac{s}{I(s)^2}\left(\frac{s}{s}\int_0^s \left|\Delta f\right|_\mu^*(u)du\right) ds \text{ (by (9.4))} \\ &= \frac{1}{t}\int_0^t \left(\frac{s}{I(s)}\right)^2 \left|\Delta f\right|_\mu^{**}(s)ds. \end{aligned}$$

□

Remark 17. *Since in this paper we assume that I(s) is concave then we see that (9.5) implies the more suggestive inequality*

$$f_\mu^{**}(t) - f_\mu^*(t) \leq \left(\frac{t}{I(t)}\right)^2 \frac{1}{t}\int_0^t \left|\Delta f\right|_\mu^{**}(s)ds. \tag{9.6}$$

By a routine iteration procedure we can extend Theorem 16 to higher order derivatives (cf. [80]*).*

We discuss briefly some examples. It follows from (9.6) and a routine approximation that for r.i. spaces away from L^1 (i.e. $\bar{\alpha}_X < 1$) we have

$$\left\| \left(f_\mu^{**}(t) - f_\mu^*(t)\right) \left(\frac{I(t)}{t}\right)^2 \right\|_{\bar{X}} \preceq \left\| |\Delta f| \right\|_X , \; f \in C^\infty(\Omega). \tag{9.7}$$

In the Euclidean case (9.7) can be used to extend the results in [97], while in the Gaussian case they provide an extension of the results in [51], [7], [8], [109] to the context of r.i. For comparison we note that the method of proof used in these references is completely different.

For example to recover the higher order Gaussian L^p Sobolev results in these references, we just need to observe that in this case

$$\left\| \left(f_\mu^{**}(t) - f_\mu^*(t)\right) \left(\frac{I(t)}{t}\right)^2 \right\|_{L^p} \simeq \|f\|_{L^p(LogL)^p} .$$

Our inequalities also apply to the measures

$$\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1} \exp\left(- |x|^p \left(\log(\gamma + |x|)^\alpha\right) dx,\right.$$

discussed in Examples 5.2 above, moreover, the corresponding inequalities can be readily obtained since we have precise estimates of the isoperimetric profiles $I_{\mu_{p,\alpha}^{\otimes n}}(s)$.

In the next section we shall see a considerable extension of these results, as well as applications to the study of non-linear elliptic equations.

10. Integrability of solutions of elliptic equations

The techniques discussed in this paper also have applications to the study of the integrability and regularity of the solutions of non-linear elliptic equations of the form

$$\begin{cases} -div(a(x,u,\nabla u)) = fw & \text{in } \Delta, \\ u = 0 & \text{on } \partial\Delta, \end{cases} \tag{10.1}$$

where Δ is domain of $\mathbb{R}^n$ $(n \geq 2)$, such that $\mu = w(x)dx$ is a probability measure on $\mathbb{R}^n$, or Δ has Lebesgue measure 1 if $w = 1$, and $a(x,\eta,\xi) : \Delta \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that for some fixed $p > 1$,

$$a(x,t,\xi).\xi \geq w(x) \left|\xi\right|^p, \quad \text{for a.e. } x \in \Delta \subset \mathbb{R}^n, \quad \forall \eta \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^n. \tag{10.2}$$

In what follows to fix ideas and simplify the presentation we take

$$p = 2,$$

but were appropriate we shall indicate the necessary changes to deal with the general case (cf. Remark 18 below).

To see what results are possible consider the special case, $w = 1, a(x,t,\xi) = \xi$. Then (10.1) becomes

$$\begin{cases} \tilde{\Delta} u = f & \text{in } \Delta, \\ u = 0 & \text{on } \partial\Delta. \end{cases}$$

In this case we can derive *apriori* sharp integrability of the solutions directly from the results in Section 9 to find that

$$\left(-u_\mu^*\right)'(t) \left(\frac{I(t)}{t}\right)^2 \leq \frac{1}{t} \int_0^t f_\mu^{**}(s)ds,$$

where $I=I_{(\mathbb{R}^n;\mu)}$ is the isoperimetric profile of $(\mathbb{R}^n;\mu)$. These estimates lead to the following *apriori* sharp integrability result

$$\left\|\left(u_\mu^{**}(t)-u_\mu^{*}(t)\right)\left(\frac{I(t)}{t}\right)^2\right\|_{\bar{X}}\preceq\left\|f_\mu^{**}\right\|_X.$$

In this section we shall extend these estimates to solutions of (10.1) (cf. Theorem 17). Moreover, we also obtain results on the regularity of $|\nabla f|$. For example, we will show that

$$\left|\nabla u\right|_\mu^*(t)\leq\left(\frac{2}{t}\int_{t/2}^{\mu(\Delta)}\left(\frac{I(s)}{s}f_\mu^{**}(s)\right)^2ds\right)^{1/2}.$$

These estimates can be used to obtain, under suitable assumptions on $\bar{X}$ (cf. Theorem 19 below),

$$\left\|\frac{I(t)}{t}\left|\nabla u\right|_\mu^*(t)\right\|_{\bar{X}}\preceq\left\|f_\mu^{**}\right\|_{\bar{X}}.$$

As with most other results in this paper, our estimates incorporate the isoperimetric profile and thus are valid for different geometries. In particular, our results are valid for domains on $\mathbb{R}^n$ provided with Lebesgue or Gaussian measure, and in both instances our *apriori* integrability results are sharp. In fact, the integrability results that we obtain for u contain all the known results (previously known for specific r.i. spaces like Orlicz or Lorentz spaces), and, furthermore, are new and sharper on the borderline cases. The integrability of the gradient is a more difficult problem for these methods, and here our results are not definitive even though, for a certain range of values of the parameters, we extend and improve on the classical results (cf. [3], [21], [46], for more on this point as well as an extensive list of references).

To proceed we needed an adequate notion of solution. Indeed, in the literature one can find a number of different definitions of what is "a" solution for problem (10.1). However, under fairly general conditions it is well known that many of these definitions coincide (cf. [3]). We adopt the definition of entropy (or entropic) solution[16] since it is better adapted for our techniques. We recall that a measurable function u is an entropy solution of (10.1) if, for all $t>0$, $\max\{|u|,t\}\mathrm{sign}\{u\}$ belongs to $W_0^{1,2}(w,\Delta)$[17], and

$$\int_{|u-\psi|<t}a(x,u,\nabla u)(\nabla u-\nabla\psi)dx\leq\int_{|u-\psi|<t}fwdx,$$

for every $\psi\in W_0^{1,2}(w,\Delta)\cap L^\infty(\Delta)$, where the weighted Sobolev space $W_0^{1,2}(w,\Delta)$ is the closure of $C_0^\infty(\Delta)$ under the norm

$$\|u\|_{W_0^{1,2}(w,\Delta)}^2=\int_\Delta|u(x)|^2\,w(x)dx+\int_\Delta|\nabla u(x)|^2\,w(x)dx.$$

[16]For example, in the classical case (i.e. $w(x)=1$ and Δ bounded), under further assumptions on $a(x,t,\xi)$, it has been proved that an entropy solution of (10.1) exists (see, for example, [21] and the references therein).

[17]One could start with more general $u's$ but it can be showed that if $f\in L^1(w,\Delta)$, then an entropy solution will automatically belong to $W_0^{1,2}(w,\Delta)$. If $p>1$, then one requires $p>2-1/n$, in order to gurantee that entropy solutions belong to $W_0^{1,p}(w,\Delta)$.

It is known, for example, that if $f \in W^{-1,2}(w,\Delta)$, the notion of entropy solution coincides with the usual definition of weak solution (cf. [3]).

The relation between, isoperimetry and the rearrangements of entropic solutions is given by the following:

Theorem 17. *Let $u \in W_0^1(w,\Delta)$ be a solution of (10.1). Let $\mu = w(x)dx$, and let $I = I_{(\mathbb{R}^n;\mu)}$ be the isoperimetric profile of $(\mathbb{R}^n;\mu)$. Then, the following inequalities hold*

(1)

$$\left(-u_\mu^*\right)'(t)I(t)^2 \leq \int_0^t f_\mu^*(s)ds, \ a.e. \tag{10.3}$$

(2)

$$\int_t^{\mu(\Delta)} \left(|\nabla u|^2\right)_\mu^*(s)ds \leq \int_t^{\mu(\Delta)} \left(\left(-u_\mu^*\right)'(s)\int_0^s f_\mu^*(z)dz\right)ds. \tag{10.4}$$

Proof. Let u be an entropy solution of (10.1). Let $0 < t < t+h < \infty$. Consider the test function given by [18]

$$u_t^{t+h}(x) = \begin{cases} h sign(u) & \text{if } |u(x)| > t+h, \\ (|u(x)| - t)\, sign(u) & \text{if } t < |u(x)| \leq t+h, \\ 0 & \text{if } |u(x)| \leq t. \end{cases}$$

Then, by the definition of entropic solution, we get

$$\begin{aligned} J(t,h) &= \frac{1}{h}\int_{\{t<|u(x)|\leq t+h\}} |\nabla u(x)|^2 d\mu \\ &\leq \frac{1}{h}\int_{\{t<|u(x)|\leq t+h\}} |f(x)|\,[u(x)-t]d\mu + \int_{\{|u(x)|>t+h\}} |f(x)|\,d\mu \\ &\leq \int_{\{t<|u(x)|\leq t+h\}} |f(x)|\,d\mu + \int_{\{|u(x)|>t+h\}} |f(x)|\,d\mu. \end{aligned} \tag{10.5}$$

Combining Hölder's inequality,

$$\begin{aligned} \left(\frac{1}{h}\int_{\{t<|u(x)|\leq t+h\}} |\nabla u(x)|\,d\mu\right)^2 &\leq J(t,h)\frac{\mu_u\{t < |u(x)| < t+h\}}{h} \\ &\leq J(t,h)\left(\frac{\mu_u(t) - \mu_u(t+h)}{h}\right), \end{aligned}$$

with (10.5), we find that

$$\left(\frac{1}{h}\int_{\{t<|u(x)|\leq t+h\}} |\nabla u(x)|\,d\mu\right)^2 \leq$$

$$\left(\int_{\{t<|u(x)|\leq t+h\}} |f(x)|\,d\mu + \int_{\{|u(x)|>t+h\}} |f(x)|\,d\mu\right)\left(\frac{\mu_u(t) - \mu_u(t+h)}{h}\right).$$

[18]This is a standard procedure which has been used by many authors see for example, [112], [115], [21], [3] and the references therein.

Letting $h \to 0$ we readily see that

$$\left(-\frac{d}{dt}\int_{\{|u(x)|>t\}}|\nabla u(x)|\,d\mu\right)^2 \leq -\frac{d\mu_u}{dt}(t)\int_{\{|u(x)|>t\}}|f(x)|\,d\mu.$$

In the previous inequality replace t by $u_\mu^*(t)$. Note that

$$\int_{\left\{|u(x)|>u_\mu^*(t)\right\}}|f(x)|\,d\mu \leq \int_0^t f_\mu^*(s)ds,$$

moreover, by Theorem 1, (3.3), and the chain rule, we have

$$\left(\frac{d}{dt}\int_{\{|u(x)|>\cdot\}}|\nabla u(x)|\,d\mu\bigg|_{u_\mu^*(t)}\right)^2 \geq (-u_\mu^*)'(t)\,[I(t)]^2\,.$$

Combining these observations we see that

$$(-u_\mu^*)'(t)\,[I(t)]^2 \leq -\frac{d\mu_u}{dt}(u_\mu^*(t))\int_0^t f_\mu^*(s)ds.$$

On the other hand, as shown in [114, pag 936, discussion in (iii)],

$$-\frac{d\mu_u}{dt}(u_\mu^*(t)) \leq 1, \text{ a.e. }.$$

Therefore we arrive at

$$(-u_\mu^*)'(t)\,[I(t)]^2 \leq \int_0^t f_\mu^*(s)ds,$$

as we wished to show.

Following [3] we consider the function

$$\Phi(t) = \int_{\{|u(x)|\leq t\}}|\nabla u(x)|^2\,d\mu, \quad t\in(0,\infty).$$

Is is plain that Φ is increasing, moreover, by a suitable change of notation, (10.5) yields that, for $0<t_1<t_2$,

$$\begin{aligned}\Phi(t_1)-\Phi(t_2) &= \int_{\{t_1<|u(x)|\leq t_2\}}|\nabla u(x)|^2\,d\mu \\ &\leq (t_2-t_1)\left(\int_{\{t_1<|u(x)|\leq t_2\}}|f(x)|\,d\mu + \int_{\{|u(x)|>t_2\}}|f(x)|\,d\mu\right) \\ &\leq 2\,(t_2-t_1)\,\|f\|_1\,.\end{aligned}$$

Consequently, Φ is a Lipschitz continuous function. Pick $t_1 = u_\mu^*(s+h)$ and $t_2 = u_\mu^*(s)$, then, upon dividing both sides of the previous inequality by h, we find that

$$\begin{aligned}&\frac{\Phi(u_\mu^*(s+h))-\Phi(u_\mu^*(s))}{h} \\ &\leq \left(\frac{u_\mu^*(s)-u_\mu^*(s+h)}{h}\right)\left(\int_{\left\{u_\mu^*(s+h)<|u(x)|\leq u_\mu^*(s)\right\}}|f(x)|\,d\mu + \int_{\left\{|u(x)|>u_\mu^*(s)\right\}}|f(x)|\,d\mu\right).\end{aligned}$$

Letting $h\to 0$ we obtain

$$-\frac{\partial}{\partial s}\left(\Phi(u_\mu^*(s))\right) \leq \left(-u_\mu^*\right)'(s)\int_0^s f_\mu^*(r)dr. \tag{10.6}$$

Integrating (10.6) from t to $\mu(\Delta)$ we get

$$\Phi(u_\mu^*(t))-\Phi(u_\mu^*(\mu(\Delta))\leq\int_t^{\mu(\Delta)}\left(\left(-u_\mu^*\right)'(s)\int_0^s f_\mu^*(r)dr\right)ds.$$

Now, since $u=0$ on $\partial\Delta$, it follows that $\mu\left(|u|\leq u_\mu^*(\mu(\Delta)\right)=0$. Thus $\Phi(u_\mu^*(\mu(\Delta))=0$, and consequently the previous inequality becomes

$$\int_{\left\{|u|\leq u_\mu^*(t)\right\}}|\nabla u(x)|^2\,d\mu\leq\int_t^{\mu(\Delta)}\left(\left(-u_\mu^*\right)'(s)\int_0^s f_\mu^*(r)dr\right)ds. \tag{10.7}$$

On the other hand, by the definition of decreasing rearrangement (see [68, Page 70]), we have

$$\begin{aligned}\int_{\left\{|u|\leq u_\mu^*(t)\right\}}|\nabla u(x)|^2\,d\mu&\geq\inf_{\mu(E)=\mu\left\{|u|\leq u_\mu^*(t)\right\}}\int_E|\nabla u(x)|^2\,d\mu\\&=\int_{\mu\left\{|u|>u_\mu^*(t)\right\}}^{\mu(\Delta)}\left(|\nabla u|^2\right)_\mu^*(s)ds\\&\geq\int_t^{\mu(\Delta)}\left(|\nabla u|^2\right)_\mu^*(s)ds. \qquad (10.8)\end{aligned}$$

Combining (10.7) and (10.8) we obtain (10.4). □

We now make explicit the sharp *apriori* integrability conditions for solutions of (10.1) that are implied by our analysis. It is here that the isoperimetric profile pays a crucial role in determining the correct nature of the estimates: e.g. in the Gaussian case it automatically leads to $L^p(LogL)^q$ integrability conditions, etc.

The analysis that follows is natural extension of the one given in Section 5. Consequently, there is a natural Hardy type operator associated with the isoperimetric profile that we shall use to study the integrability of solutions of (10.1), namely the operator R_I (compare with the operator Q_I defined by (6.1) above),

$$R_I(h)(t)=\int_t^{\mu(\Delta)}\left(\frac{s}{I(s)}\right)^2h(s)\frac{ds}{s}.$$

Theorem 18. *Let X,Y be two r.i. spaces on Δ such that,*

$$\|R_I(h)\|_{\bar Y}\preceq\|h\|_{\bar X}\,, \tag{10.9}$$

and, suppose that $\overline{\alpha}_X<1$ (cf. Remark 2.9). Then, if u is a solution of (10.1) with datum $f\in X(\Delta)$, we have

$$\left\|u_\mu^*\right\|_{\bar Y}\preceq\left\|f_\mu^*\right\|_{\bar X}\,. \tag{10.10}$$

and

$$\left\|u_\mu^*\right\|_{\bar Y}\preceq\left\|\left(\frac{I(t)}{t}\right)^2\left(u_\mu^{**}(t)-u_\mu^*(t)\right)\right\|_{\bar X}+\left\|u_\mu^*\right\|_{L^1}\preceq\left\|f_\mu^*\right\|_{\bar X}\,. \tag{10.11}$$

Moreover, in the case that the operator $\tilde R_I(h)(t)=\left(\frac{I(s)}{s}\right)^2\int_t^{\mu(\Delta)}\left(\frac{s}{I(s)}\right)^2h(s)\frac{ds}{s}$ is bounded on $\bar X$, then if u is the solution of (10.1) with datum $f\in X(\Delta)$, we have

$$\left\|u_\mu^*\right\|_{\bar Y}\preceq\left\|\left(\frac{I(t)}{t}\right)^2u_\mu^*(t)\right\|_{\bar X}\preceq\left\|f_\mu^*\right\|_{\bar X}\,. \tag{10.12}$$

Proof. Using the fundamental theorem of calculus, the fact that $u_\mu^*(\mu(\Delta)) = 0$, and (10.3), we get

$$\begin{aligned} u_\mu^*(t) &= \int_t^{\mu(\Delta)} \left(-u_\mu^*\right)'(s)ds \\ &\leq \int_t^{\mu(\Delta)} \left(\frac{s}{I(s)}\right)^2 f_\mu^{**}(s)\frac{ds}{s} \\ &= R_I(f_\mu^{**})(t). \end{aligned}$$

Therefore (10.10) follows from (10.9).

We shall now prove (10.11). First we shall prove

$$\left\|u_\mu^*\right\|_{\bar{Y}} \preceq \left\|\left(\frac{I(t)}{t}\right)^2 \left(u_\mu^{**}(t) - u_\mu^*(t)\right)\right\|_{\bar{X}} + \left\|u_\mu^*\right\|_{L^1}.$$

By the fundamental theorem of calculus we have

$$\begin{aligned} u_\mu^{**}(t) &\leq \int_t^1 \left(\frac{s}{I(s)}\right)^2 \left\{\left(\frac{I(s)}{s}\right)^2 \left(u_\mu^{**}(s) - u_\mu^*(s)\right)\right\}\frac{ds}{s} + \left\|u_\mu^*\right\|_{L^1} \\ &= R_I(\{\cdot\cdot\})(t) + \left\|u_\mu^*\right\|_{L^1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\|u_\mu^*\right\|_{\bar{Y}} &\leq \left\|u_\mu^{**}\right\|_{\bar{Y}} \\ &\leq \|R_I(\{\cdot\cdot\})\|_{\bar{Y}} + \left\|u_\mu^*\right\|_{L^1} \\ &\leq \left\|\left(\frac{I(s)}{s}\right)^2 \left(u_\mu^{**}(s) - u_\mu^*(s)\right)\right\|_{\bar{X}} + \left\|u_\mu^*\right\|_{L^1}. \end{aligned}$$

Now, we prove the remaining inequality of (10.11). Suppose that u is a solution of (10.1). Then, since $u \in W_0^{1,1}(w;\Omega)$, we get that

$$\begin{aligned} \left(\frac{I(t)}{t}\right)^2 \left(u_\mu^{**}(t) - u_\mu^*(t)\right) &= \left(\frac{I(t)}{t}\right)^2 \frac{1}{t}\int_0^t s\left(-u_\mu^*\right)'(s)ds \\ &\leq \frac{1}{t}\int_0^t I(s)^2 \frac{1}{s}\left(-u_\mu^*\right)'(s)ds \text{ (since } I(t)/t \text{ decreases)} \\ &\leq \frac{1}{t}\int_0^t f_\mu^{**}(s)ds \quad \text{(by (10.3))}. \end{aligned}$$

Therefore,

$$\left\|\left(\frac{I(t)}{t}\right)^2 \left(u_\mu^{**}(t) - u_\mu^*(t)\right)^2\right\|_{\bar{X}} \preceq \left\|f_\mu^*\right\|_{\bar{X}} \quad (\text{since } \overline{\alpha}_X < 1).$$

Finally, to prove (10.12) it will be convenient to define the r.i. space on $(0,1)$,

$$\bar{X}_{I^2} = \{h : \|h\|_{\bar{X}_{I^2}} < \infty\},$$

where

$$\|h\|_{\bar{X}_{I^2}} = \left\|h(t)\left(\frac{I(t)}{t}\right)^2\right\|_{\bar{X}}.$$

Using the same argument given in the proof of Theorem 5 part (a), we can prove that

$$\|f\|_{\bar{Y}} \preceq \left\|f_\mu^*(t)\right\|_{\bar{X}_{I^2}}.$$

Now, we show that for all $f \in \bar{X}$,

$$\|R_I(f)\|_{\bar{X}_{I^2}} \preceq \|f\|_{\bar{X}}.$$

Indeed, this is equivalent to the boundedness of the operator $\tilde{R}_I$:

$$\begin{aligned}
\|R_I(f)\|_{\bar{X}_{I^2}} &= \left\| \int_t^{\mu(\Delta)} \left(\frac{s}{I(s)}\right)^2 f(s)\frac{ds}{s} \right\|_{\bar{X}_{I^2}} \\
&= \left\| \left(\frac{I(s)}{s}\right)^2 \int_t^{\mu(\Delta)} \left(\frac{s}{I(s)}\right)^2 f(s)\frac{ds}{s} \right\|_{\bar{X}} \\
&= \left\| \tilde{R}_I f \right\|_{\bar{X}} \\
&\preceq \|f\|_{\bar{X}}.
\end{aligned}$$

Consequently, by the first part of the theorem we have that

$$\left\| \left(\frac{I(t)}{t}\right)^2 u_\mu^*(t) \right\|_{\bar{X}} = \left\|u_\mu^*\right\|_{\bar{X}_{I^2}} \preceq \left\|f_\mu^*\right\|_{\bar{X}}.$$

□

In view of (10.12), for a given datum $f \in X(\Delta)$, $\bar{X}_I$ is the "natural space" to measure the regularity of the gradient, in fact we have

Theorem 19. *Let u be any entropic solution of (10.1). Then, we have*

$$|\nabla u|_\mu^*(t) \leq \left(\frac{2}{t} \int_{t/2}^{\mu(\Delta)} \left(\frac{I(s)}{s} f_\mu^{**}(s) \right)^2 ds \right)^{1/2}. \tag{10.13}$$

Furthermore, suppose that f, the right hand side of (10.1), belongs to a r.i. space $X(\Delta)$, such that $1/2 < \underline{\alpha}_{\bar{X}_I}$. Then,

$$\left\| \frac{I(t)}{t} |\nabla u|_\mu^*(t) \right\|_{\bar{X}} \preceq \left\|f_\mu^{**}\right\|_{\bar{X}}. \tag{10.14}$$

Proof. Indeed, by (10.4), we know that

$$\begin{aligned}
\int_{t/2}^{\mu(\Delta)} \left(|\nabla u|^2\right)_\mu^*(s)ds &\leq \int_{t/2}^{\mu(\Delta)} \left(\left(-u_\mu^*\right)'(s) \int_0^s f_\mu^*(z)dz \right) ds \\
&\leq \int_{t/2}^{\mu(\Delta)} \left(\frac{s}{I(s)} f_\mu^{**}(s) \right)^2 ds.
\end{aligned}$$

Moreover,

$$\int_{t/2}^{\mu(\Delta)} \left(|\nabla u|^2\right)_\mu^*(s)ds \geq \int_{t/2}^{t} \left(|\nabla u|^2\right)_\mu^*(s)ds \geq \left(|\nabla u|^2\right)_\mu^*(t)\frac{t}{2}.$$

Thus

$$|\nabla u|_\mu^*(t) \leq \left(\frac{2}{t} \int_{t/2}^{\mu(\Delta)} \left(\frac{s}{I(s)} f_\mu^{**}(s) \right)^2 ds \right)^{1/2}.$$

Finally we prove (10.14):

$$
\begin{aligned}
\left\| |\nabla u|_\mu^* \right\|_{\bar{X}_I} &= \left\| \frac{I(t)}{t} \left|\nabla u\right|_\mu^* (t) \right\|_{\bar{X}} \\
&\leq \left\| \frac{I(t)}{t} \left(\frac{2}{t} \int_{t/2}^{\mu(\Delta)} \left(\frac{s}{I(s)} f_\mu^{**}(s) \right)^2 ds \right)^{1/2} \right\|_{\bar{X}} \\
&\leq \left\| \frac{I(t/2)}{t/2} \left(\frac{2}{t} \int_{t/2}^{\mu(\Delta)} \left(\frac{s}{I(s)} f_\mu^{**}(s) \right)^2 ds \right)^{1/2} \right\|_{\bar{X}} \\
&\leq 2 \left\| \frac{I(t)}{t} \left(\frac{1}{t} \int_{t}^{\mu(\Delta)} \left(\frac{s}{I(s)} f_\mu^{**}(s) \right)^2 ds \right)^{1/2} \right\|_{\bar{X}} \quad \text{(by (2.8))} \\
&= 2 \left\| \left(\frac{1}{t} \int_{t}^{\mu(\Delta)} \left(\frac{s}{I(s)} f_\mu^{**}(s) \right)^2 ds \right)^{1/2} \right\|_{\bar{X}_I} \\
&\preceq \left\| \frac{s}{I(s)} f_\mu^{**}(s) \right\|_{\bar{X}_I} \quad \text{(by Lemma 1, since } 1/2 < \underline{\alpha}_{\bar{X}_I}) \\
&= \left\| f_\mu^{**} \right\|_{\bar{X}}.
\end{aligned}
$$

□

Remark 18. *The results in this section can be easily adapted to the study of ellipticity conditions of the type*

$$
a(x,t,\xi).\xi \geq w(x) \left|\xi\right|^p, \quad \textit{for a.e. } x \in \Delta \subset \mathbb{R}^n, \quad \forall \eta \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^n,
$$

where $1 < p < \infty$. *In this case inequalities (10.3) and (10.4) became respectively*

$$
\left(-u_\mu^*\right)'(t) I(t)^{\frac{p}{p-1}} \leq \left(\int_0^t f_\mu^*(s) ds \right)^{\frac{1}{p-1}},
$$

$$
\int_t^{\mu(\Omega)} \left(|\nabla u|^p\right)_\mu^* (s) ds \leq \int_t^1 \left(\left(-u_\mu^*\right)'(s) \int_0^s f_\mu^*(z) dz \right) ds,
$$

and condition (10.9) needs to be replaced by

$$
\left\| \int_t^{\mu(\Omega)} \left(\left(\frac{s}{I(s)} \right)^p f_\mu^{**}(s) \right)^{\frac{1}{p-1}} \frac{ds}{s} \right\|_{\bar{Y}} \preceq \left\| f^* \right\|_{\bar{X}}^{\frac{1}{p-1}}.
$$

We omit the details and refer to [85] for more details.

Remark 19. *To fix ideas in this paper we have only considered elliptic equations in divergence form on domains of* $\mathbb{R}^n$. *However, the proof of Theorem 17 can be easily adapted to the setting of* $n-$*dimensional Riemannian manifolds* M *with finite volume (say* $\mathrm{vol}(M) = 1$*) as considered by Cianchi in* [36]. *Indeed, mutatis mutandi Theorem 18 can be easily reformulated and is valid in this more general setting (cf.* [85]*).*

10.1. **Sharpness of the results.** We comment briefly on the sharpness of the results in this section and refer to [85] for a more detailed analysis. In the classical papers of Talenti and his school (cf. [112], [113], [114], [115] and the many references therein) the sharpness of the estimates is obtained, roughly speaking, by comparing solutions of the Dirichlet problems for suitable classes of elliptic equations linear operators in divergence form, with radial solutions of the Laplace equation on a ball, whose measure is equal to the measure of the original domain.

Under sufficient symmetry (for example in the case model cases discussed in Section 4, and in particular the abstract model of Section 4.3), one can construct comparison equations and show the sharpness of the results. We do not pursue this matter further in this long paper but it is appropriate to mention that the natural extremal functions for comparison in the model cases have rearrangements given an explicit formula, namely functions v such that

$$v_{\mu}^{*}(t)=\int_{t}^{\mu(\Delta)}\left(\frac{s}{I(s)}\right)^{2}f_{\mu}^{**}(s)\frac{ds}{s}.$$

In fact note that, by Theorem 18, any entropic solution u of (10.1) must satisfy

$$u_{\mu}^{*}(t)\preceq v_{\mu}^{*}(t).$$

This is the pointwise domination is captured in the papers mentioned earlier. Moreover, a suitable oscillation of u is also controlled by the oscillation of v!. Indeed, the oscillation under control is none other than $u_{\mu}^{**}(t)-u_{\mu}^{*}(t):$

$$\begin{aligned}
u_{\mu}^{**}(t)-u_{\mu}^{*}(t) &= \frac{1}{t}\int_{0}^{t}s\left(-u_{\mu}^{*}\right)'(s)ds \\
&\leq \frac{1}{t}\int_{0}^{t}\left(\frac{s}{I(s)}\right)^{2}f_{\mu}^{**}(s)ds \quad \text{(by (10.3))} \\
&= v_{\mu}^{**}(t)-v_{\mu}^{*}(t).
\end{aligned}$$

Furthermore, the analysis of the proof of Theorem 18 shows that, if $\tilde{R}_{I}$ is bounded on $\bar{X}$,

$$\left\|\left(\frac{I(t)}{t}\right)^{2}v_{\mu}^{*}(t)\right\|_{\bar{X}}\simeq\left\|f_{\mu}^{*}\right\|_{\bar{X}}.$$

Therefore, if $\overline{\alpha}_{X}<1$,

$$\left\|\left(\frac{I(t)}{t}\right)^{2}\left(v_{\mu}^{**}(t)-v_{\mu}^{*}(t)\right)\right\|_{\bar{X}}+\left\|v_{\mu}^{*}\right\|_{L^{1}}\simeq\left\|f_{\mu}^{*}\right\|_{\bar{X}}.$$

10.2. **Examples.** To fix ideas we discuss a concrete set of examples.

10.2.1. *The Euclidian case.* Consider

$$\left\{\begin{array}{ll} -div(a(x,u,\nabla u))=f & \text{in } \Delta, \\ u=0 & \text{on } \partial\Delta, \end{array}\right. \tag{10.15}$$

with ellipticity condition,

$$a(x,t,\xi).\xi\succeq|\xi|^{2},\quad \text{for a.e.}\quad x\in\Delta,\ \ \forall\eta\in\mathbb{R},\ \ \forall\xi\in\mathbb{R}^{n}.$$

where $\Delta\subset\mathbb{R}^{n}$ is domain with $|\Delta|=1$. Theorems 18 and 19, yield:

Theorem 20. *Let $X(\Omega)$ be an r.i. space such that $\bar{\alpha}_{\bar{X}} < 1$. Let u be a solution of (10.15) with datum $f \in X(\Omega)$. Then*

(1) *If $\underline{\alpha}_{\bar{X}} > 2/n$,*
$$\left\| s^{-\frac{2}{n}} u^*(s) \right\|_{\bar{X}} \preceq \|f\|_{\bar{X}}.$$

(2) *If $\underline{\alpha}_{\bar{X}} \leq 2/n$,*
$$\left\| s^{-\frac{2}{n}} (u^{**}(s) - u^*(s)) \right\|_{\bar{X}} + \|u\|_{L^1} \preceq \|f\|_{\bar{X}}. \tag{10.16}$$

(3) *If $\underline{\alpha}_{\bar{X}} > \frac{1}{2} + \frac{1}{n}$,*
$$\left\| s^{-\frac{1}{n}} |\nabla u|^* (s) \right\|_{\bar{X}} \preceq \|f\|_X.$$

Proof. The isoperimetric function of $(\mathbb{R}^n, |\cdot|, dx)$ is
$$I(s) = nc_n^{1/n} s^{(n-1)/n},$$
where c_n is the volume of the unit ball. Therefore
$$R_I h(t) \simeq t^{-2/n} \int_t^{\mu(\Delta)} s^{2/n} f^{**}(s) \frac{ds}{s},$$
defines a bounded operator on X since $\underline{\alpha}_{\bar{X}} > 2/n$. The second statement follows similarly. Finally to see the third statement, observe that the corresponding space $\bar{X}_I$ is defined (up to constants) by the condition
$$\|f\|_{\bar{X}_I} = \left\| t^{-1/n} f \right\|_{\bar{X}} < \infty.$$
It follows readily from the definitions of Boyd indices that
$$\underline{\alpha}_{\bar{X}_I} = \underline{\alpha}_{\bar{X}} - \frac{1}{n}.$$
Consequently, since by assumption $\underline{\alpha}_{\bar{X}} > \frac{1}{2} + \frac{1}{n}$, we get
$$\underline{\alpha}_{\bar{X}_I} > \frac{1}{2}.$$
The desired result now follows from Theorem 19. □

Recall that given f a measurable function, its symmetric spherical decreasing rearrangement, f°, is defined by
$$f^\circ(x) = f^*\left(c_n |x|^n\right), \quad x \in \mathbb{R}^n. \tag{10.17}$$

Let $\Delta^\bigstar$ be the ball centered at the origin with $\left|\Delta^\bigstar\right| = |\Delta| = 1$. Let v the (radial) solution to the problem
$$\begin{cases} -div\,(\nabla v) = f^\circ & \text{in } \Delta^\bigstar, \\ v = 0 & \text{on } \partial\Delta^\bigstar. \end{cases} \tag{10.18}$$
It is easy to see that ν is given by
$$v(x) = \kappa \int_{c_n|x|^n}^1 \left(s^{-\frac{2(n-1)}{n}} \int_0^s f^*(r)dr \right) ds, \qquad x \in \Delta^\bigstar.$$
with $\kappa = \left(\frac{1}{nc_n^{1/n}}\right)^2$. Obviously,
$$v^*(t) = \kappa \int_t^1 \left(s^{-\frac{2(n-1)}{n}} \int_0^s f^*(r)dr \right) ds,$$

This analysis shows that for these problems our methods provide sharp results (cf. [112], [113], [114], [115] and the many references therein).

Remark 20. *In Theorem 20 the hypothesis $\bar{\alpha}_{\bar{X}} < 1$ cannot be omitted. For example let $\bar{X} = L^1$. The solution of (10.18), satisfies*

$$v^{**}(t) - v^{*}(t) = \frac{\kappa}{t}\int_0^t s^{\frac{2}{n}} f^{**}(s)ds.$$

Thus

$$\begin{aligned}\left\| t^{-\frac{2}{n}}(v^{**}(t) - v^{*}(t))\right\|_{L_1} &\simeq \int_0^1 t^{-\frac{2}{n}}\left(\frac{1}{t}\int_0^t s^{\frac{2}{n}} f^{**}(s)ds\right)dt \\ &\simeq \int_0^1 f^{**}(t)dt = \|f^*\|_{L\log L}.\end{aligned}$$

Therefore (10.16) does not hold.

10.2.2. *Between exponential and Gaussian measure.* Now we consider a set of elliptic problems associated with Gaussian measures and explain how they fit our models. Let $\alpha \geq 0,\ p \in [1,2]$ and $\gamma = \exp(2\alpha/(2-p))$, and let

$$\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1}\exp\left(-|x|^p\left(\log(\gamma + |x|)^{\alpha}\right)dx = \varphi_{\alpha,p}(x)dx, \qquad x \in \mathbb{R},\right.$$

and

$$\varphi_{\alpha,p}^n(x) = \varphi_{\alpha,p}(x_1)\cdots\varphi_{\alpha,p}(x_n), \quad \text{and } \mu = \mu_{p,\alpha}^{\otimes n}.$$

Consider

$$\left\{\begin{array}{ll} -div(a(x,u,\nabla u)) = f\varphi_{\alpha,p}^n & \text{in } \Delta, \\ u = 0 & \text{on } \partial\Delta, \end{array}\right. \tag{10.19}$$

with the ellipticity condition,

$$a(x,t,\xi).\xi \succeq \varphi_{\alpha,p}^n(x)\,|\xi|^2, \quad \text{for a.e.} \quad x \in \Delta, \ \ \forall \eta \in \mathbb{R}, \ \ \forall \xi \in \mathbb{R}^n,$$

where $\Delta \subset \mathbb{R}^n$ is an open set such that $\mu(\Delta) < 1$. Theorems 18 and 19, yield

Theorem 21. *Let $\Delta \subset \mathbb{R}^n$ be an open set such that $\mu(\Delta) < 1$. Let u be a solution of (10.19) with datum $f \in X(\Delta)$. Assume that $\overline{\alpha}_{\bar{X}} < 1$. Then,*

1. *If $0 < \underline{\alpha}_{\bar{X}}$,*
$$\left\| \left(\log\frac{1}{s}\right)^{2\left(1-\frac{1}{p}\right)}\left(\log\log\left(e + \frac{1}{s}\right)\right)^{2\frac{\alpha}{p}} u_\mu^*(s)\right\|_{\bar{X}} \preceq \|f\|_X.$$
2. *If $0 = \underline{\alpha}_{\bar{X}}$,*
$$\left\| \left(\log\frac{1}{s}\right)^{2\left(1-\frac{1}{p}\right)}\left(\log\log\left(e + \frac{1}{s}\right)\right)^{2\frac{\alpha}{p}} (u_\mu^{**}(s) - u_\mu^*(s))\right\|_{\bar{X}} + \|u\|_{L^1} \preceq \|f\|_X.$$
3. *If $\underline{\alpha}_{\bar{X}} > 1/2$,*
$$\left\| \left(\log\frac{1}{s}\right)^{\left(1-\frac{1}{p}\right)}\left(\log\log\left(e + \frac{1}{s}\right)\right)^{\frac{\alpha}{p}} |\nabla u|_\mu^*(s)\right\|_{\bar{X}} \preceq \|f\|_X.$$

Proof. Since $\mu(\Delta)<1$, it follows from (5.13) that

$$I_{\mu_{p,\alpha}^{\otimes n}}(s)\simeq s\left(\log\frac{1}{s}\right)^{1-\frac{1}{p}}\left(\log\log\left(e+\frac{1}{s}\right)\right)^{\frac{\alpha}{p}},\quad 0<s<\mu(\Delta).$$

Therefore,

$$R_Ih(s)\simeq\int_t^{\mu(\Delta)}\left(\frac{1}{\left(\log\frac{1}{s}\right)^{1-\frac{1}{p}}\left(\log\log\left(e+\frac{1}{s}\right)\right)^{\frac{\alpha}{p}}}\right)^2 f_\mu^{**}(s)\frac{ds}{s}.$$

The proof given in example 5.2 can be easily adapted to see that $\tilde{R}_I$ is bounded on $\bar{X}$, if $0<\underline{\alpha}_{\bar{X}}$ and $\overline{\alpha}_{\bar{X}}<1$. The second statement follows similarly. Finally to see the third statement, notice that

$$\|f\|_{\bar{X}_{I_\mu}}\simeq\left\|\left(\log\frac{1}{s}\right)^{\left(1-\frac{1}{p}\right)}\left(\log\log\left(e+\frac{1}{s}\right)\right)^{\frac{\alpha}{p}}f(s)\right\|_{\bar{X}}$$

and an easy computation shows that

$$\underline{\alpha}_{\bar{X}}=\underline{\alpha}_{\bar{X}_{I_\mu}}.$$

Hence, Theorem 19 applies. □

Example 1. *Let $q>1$, $m\geq 1$ and let $\lambda\in\mathbb{R}$. Consider the Lorentz-Zygmund spaces $L^{q,m}(\log L)^\lambda$ defined using the quasi-norms*

$$\begin{aligned}\|f\|_{L^{q,m}(\log L)^\lambda}&=\left\|t^{\frac{1}{q}-\frac{1}{m}}\left(1+\log\frac{1}{t}\right)^\lambda f^{**}(t)\right\|_{L^m}\\&\simeq\left\|t^{\frac{1}{q}-\frac{1}{m}}\left(1+\log\frac{1}{t}\right)^\lambda f^{*}(t)\right\|_{L^m}.\end{aligned}$$

It is well known that

$$\underline{\alpha}_{L^{q,m}(\log L)^\lambda}=\bar{\alpha}_{L^{q,m}(\log L)^\lambda}=\frac{1}{q}.$$

Suppose that $f\in L^{q,m}(\log L)^\lambda, 1<q<\infty$, and u is a solution of (10.19). Then,

$$\left(\int_0^{\mu(\Delta)}\left(s^{\frac{1}{q}}\left(1+\log\frac{1}{s}\right)^{2\left(1-\frac{1}{p}\right)+\lambda}\left(\log\log\left(e+\frac{1}{s}\right)\right)^{2\frac{\alpha}{p}}u_\mu^*(s)\right)^m\frac{ds}{s}\right)^{\frac{1}{m}}$$
$$\preceq\|f\|_{L^{q,m}(\log L)^\lambda}.$$

Moreover, if $2<q$, then by Theorem 19 we get,

$$\left(\int_0^{\mu(\Delta)}\left(s^{\frac{1}{q}}\left(1+\log\frac{1}{s}\right)^{\left(1-\frac{1}{p}\right)+\lambda}\left(\log\log\left(e+\frac{1}{s}\right)\right)^{\frac{\alpha}{p}}\left|\nabla u\right|_\mu^*(s)\right)^m\frac{ds}{s}\right)^{\frac{1}{m}}$$
$$\preceq\|f\|_{L^{q,m}(\log L)^\lambda}.$$

In particular, if $p=2$ and $\alpha=0$ we recover the results of [43], [44] *for the Gaussian case.*

In this context (see section 4.1) there is a suitable rearrangement $f^{\circ}:\mathbb{R}^n\to\mathbb{R}$ defined by

$$f^{\circ}(x)=f^{*}(H(x_1)).$$

where $H:\mathbb{R}\to(0,1)$ is given by

$$H(r)=\int_{-\infty}^{r}\varphi_{\alpha,p}(x)dx.$$

Therefore one is led to compare (10.19) with

$$\left\{\begin{array}{ll} -\left(\varphi_{\alpha,p}^{n}v_{x_1}\right)_{x_1}=f^{\circ}\varphi_{\alpha,p}^{n} & \text{in } \Delta^{\bigstar}, \\ v=0 & \text{on } \partial\Delta^{\bigstar}, \end{array}\right. \tag{10.20}$$

where $\Delta^{\bigstar}$ is the half space defined by

$$\Delta^{\bigstar}=H_r=\{x=(x_1,.....x_n):x_1<r\},$$

and $r\in\mathbb{R}$ is selected so that $H(r)=\mu(\Delta)$. The solution of (10.20) is given by inspection:

$$v(x_1)=\int_{x_1}^{r}\left(Z_{p,\alpha}^{-1}\exp\left(|t|^{p}\left(\log(\gamma+|t|)^{\alpha}\right)\int_{-\infty}^{t}f^{\circ}(s)\varphi_{\alpha,p}(s)ds\right)dt,\qquad x_1\in\Delta^{\bigstar}.$$

Note that since

$$\begin{aligned} v^{\circ}(x)&=\int_{H(x_1)}^{r}Z_{p,\alpha}^{-1}\exp\left(|t|^{p}\left(\log(\gamma+|t|)^{\alpha}\right)\int_{-\infty}^{t}f^{\circ}(s)\varphi_{\alpha,p}(s)dsdt\right. \\ &=\int_{x_1}^{\mu(\Delta)}Z_{p,\alpha}^{-1}\exp\left(\left|H^{-1}(t)\right|^{p}\left(\log(\gamma+\left|H^{-1}(t)\right|)^{\alpha}\right)\int_{-\infty}^{H^{-1}(s)}f^{\circ}(s)\varphi_{\alpha,p}(s)ds\frac{\partial H^{-1}}{\partial t}(t)dt\right. \\ &=\int_{x_1}^{\mu(\Delta)}\left(\frac{s}{I_{\mu_{p,\alpha}}(s)}\right)^{2}\frac{1}{s}\int_{0}^{s}f_{\mu}^{*}(z)dzds, \end{aligned}$$

and

$$v_{\mu}^{*}=(v^{\circ})_{\mu}^{*},$$

we have

$$v_{\mu}^{*}(t)\simeq\int_{t}^{\mu(\Delta)}\left(\frac{s}{I_{\mu_{p,\alpha}}(s)}\right)^{2}f_{\mu}^{**}(s)ds.$$

This shows that for this set of problems our methods also provide sharp results (compare with [43], [44], [45].)

11. Connection with some capacitary inequalities due to Maz'ya

We comment briefly, and somewhat informally, on a connection between what we have termed the Maz'ya-Talenti inequality (3.3) and some of Maz'ya's capacitary inequalities (cf. [89], [90]). Indeed, we show explicitly how to derive symmetrization inequalities of the type discussed in this paper, from Maz'ya's capacitary inequalities.

Recall that (3.3) was originally formulated on $\mathbb{R}^n$ (cf. [113] and the references therein) with Lebesgue measure, where of course $I(t)=c_n t^{1-1/n}$, and we shall restrict ourselves to this setting[19]. Moreover, although this is an important point, and the constants can be made quite explicit, we shall not keep track of the absolute

[19] We note that one interesting aspect of the method of capacitary inequalities is that it can be implemented in very general settings. On the other hand we have to postpone a general discussion for another occassion.

constants in this discussion. We must also refer to [89], [90] for background and notation. In what follows we let Ω be an open set in $\mathbb{R}^n$, $|\cdot|$ =Lebesgue measure. Then, for a compact set $F \subset \Omega$, Maz'ya [89, cf. (8.7)] shows that, for $1 \leq p < n$,

$$cap_p(F,\Omega) \succeq \left| |\Omega|^{\frac{p-n}{n(p-1)}} - |F|^{\frac{p-n}{n(p-1)}} \right|^{1-p}, \ p<n, \tag{11.1}$$

while for $p = n$ we have

$$cap_n(F,\Omega) \succeq (\log|\Omega| - \log|F|)^{1-n}. \tag{11.2}$$

To develop the connection we shall compute capacities normalizing the smooth truncations as follows. Let $0 < t_1 < t_2 < \infty$, $f \in C_0^\infty(\Omega)$, then we define

$$N[f_{t_1}^{t_2}(x)] = \frac{f_{t_1}^{t_2}(x)}{t_2 - t_1} = \begin{cases} 1 & \text{if } |f(x)| > t_2, \\ \leq 1 & \text{if } t_1 < |f(x)| \leq t_2, \\ 0 & \text{if } |f(x)| \leq t_1. \end{cases}.$$

Therefore, by definition we can estimate

$$cap_p\left(\overline{\{|f(x)| > t_2\}},\ \{|f(x)| > t_1\}\right) \leq \frac{1}{(t_2-t_1)^p}\int_{\{t_1<|f|<t_2\}} |\nabla f(x)|^p\, dx.$$

Let $t_1 = f^*(t)$, $t_2 = f^*(t+h)$, $h > 0$. Then, we have

$$\begin{aligned} &cap_p\left(\{|f(x)| \geq f^*(t)\},\ \{|f(x)| \geq f^*(t+h)\}\right)[f^*(t+h) - f^*(h)]^p \\ &\leq \int_{\{f^*(t+h)<|f|<f^*(t)\}} |\nabla f(x)|^p\, dx. \end{aligned}$$

Combining with (11.1) we obtain,

$$cap_p\left(\{|f(x)| \geq f^*(t)\},\ \{|f(x)| \geq f^*(t+h)\}\right) \succeq \left| |t+h|^{\frac{p-n}{n(p-1)}} - |t|^{\frac{p-n}{n(p-1)}} \right|^{1-p},$$

and therefore.

$$[f^*(t+h) - f^*(h)]^p \left| |t+h|^{\frac{p-n}{n(p-1)}} - |t|^{\frac{p-n}{n(p-1)}} \right|^{1-p} \preceq \int_{\{f^*(t+h)<|f|<f^*(t)\}} |\nabla f(x)|^p\, dx,$$

and

$$\left(\frac{f^*(t+h) - f^*(h)}{h}\right)^p \left| \frac{|t+h|^{\frac{p-n}{n(p-1)}} - |t|^{\frac{p-n}{n(p-1)}}}{h} \right|^{1-p} \preceq \frac{1}{h}\int_{\{f^*(t+h)<|f|<f^*(t)\}} |\nabla f(x)|^p\, dx.$$

Now we let $h \to 0$, to find

$$\left(\frac{(p-n)}{n(p-1)}\right)^{1-p} [(-f^*)'(t)]^p \left(t^{\frac{p-n}{n(p-1)}-1}\right)^{1-p} \preceq \frac{d}{dt}\int_{\{|f|>f^*(t)\}} |\nabla f(x)|^p\, dx.$$

In particular, for $p = 1$ we actually get

$$s^{1-1/n}(-f^*)'(s) \preceq \frac{\partial}{\partial s}\int_{\{|f|>f^*(s)\}} |\nabla f(x)|\, dx.$$

Moreover, for $p = n$ the same argument, but using (11.2) instead, yields

$$\left(\frac{f^*(t+h) - f^*(h)}{h}\right)^n \left| \frac{\log|t+h| - \log|t|}{h} \right|^{1-n} \preceq \frac{1}{h}\int_{\{f^*(t+h)<|f|<f^*(t)\}} |\nabla f(x)|^n\, dx,$$

so that

$$s^{n-1}\left((-f^*)'(s)\right)^n \preceq \frac{\partial}{\partial s}\int_{\{|f|>f^*(s)\}} |\nabla f(x)|^n\, dx.$$

The previous argument can easily be made rigorous and extended to the more general setting of Section 3.

12. Appendix: A few (and only a few) bibliographical notes

It has not been out intention to provide a comprehensive bibliography. Indeed, the topics discussed in this paper have been intensively studied for a long time, with a variety of different approaches, and even though the bibliography we have collected is rather large it is by definition very incomplete and many times during the text we had to refer the reader to papers quoted within the quoted papers and books... Therefore, we must apologize in advance for oversights. With this important proviso we make a few (and only a few) bibliographical notes and add a few more references that were not mentioned in the main text. Moreover, we take the opportunity to very briefly comment on some results and correct some of our previous bibliographical oversights in earlier publications for which we must apologize yet again.

As was pointed in out in [15], the inequality (1.4), which in the Euclidean case takes the form

$$f^{**}(t) - f^{*}(t) \leq c_n t^{1/n} \left|\nabla f\right|^{**}(t), \tag{12.1}$$

is implicit in [4, Appendix]. However, it was not used in this form in [4], but rather as

$$f^{**}(t) \leq c_n t^{1/n} \left|\nabla f\right|^{**}(t) + f^{*}(t),$$

followed by the triangle inequality. This step however destroys the effect of the cancellation afforded by (12.1). In [67] one can find a similar inequality but with the left hand side $f^{**}(t) - f^{*}(t)$ replaced by $f^{*}(t) - f^{*}(2t)$. This leads to equivalent type of inequalities as it was shown, much later, in [15] and [102]. Neither of these papers uses isoperimetry explicitly and the proofs are ad-hoc. For yet another approach using maximal operators see [63] (and the references therein!).

Oscillation inequalities have a long history, for example they appear very prominently in the work of Herz [60] and Garsia-Rodemich [54]. A discrete version of Talenti's inequality was also recorded in [116, Proposition 4].

The role of the oscillation spaces as limiting spaces seems to have originated with the work of Bennett-De Vore and Sharpley [19]. At any rate $f^{**}(t) - f^{*}(t)$ has interesting interpretations in interpolation theory (cf. [19], [108] and for still a different interpretation see [62] and [81]). The role of oscillation spaces in the limiting cases of the Sobolev embedding theorem seems to have been noticed first by Tartar [116]. Using the notation of [77] it follows from [116, Proposition 4] that $W_n^1(\Omega) \subset W_n(\Omega)$. This result was also pointed out later in [77]. At the time we wrote [83] we were also unaware of the results in [52], we hope to have rectified this oversight with the discussion presented in Section 9.

Sobolev embeddings have a long history (for different perspectives cf. [88], [2], [47], just to name a few). The first complete treatment of embeddings of Sobolev spaces in the setting of rearrangement invariant spaces with necessary and sufficient conditions that we know is [41], and later extended in [48, in particular see the comments at the bottom of page 310]. A good deal of this work on r.i. spaces been inspired by the classical work of Moser-Trudinger and O'Neil (cf. [101], [31], [59] and the references therein).

We conclude mentioning that in this paper we have not considered compactness of embeddings. However, we believe that the methods of [104] and [87] can be generalized to the setting of this paper, and we hope to return to the matter elsewhere.

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