This is a preprint of: "The geometry of the real planar polynomial differential systems having their orbits imbedded in conics", Jaume Llibre, Adam Mahdi, Joaquim Roé, *Dyn. Syst.*, vol. 26(3), 287–321, 2011.
DOI: [10.1080/14689367.2011.588593]

THE GEOMETRY OF THE REAL PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS HAVING THEIR ORBITS IMBEDDED IN CONICS

JAUME LLIBRE, ADAM MAHDI AND JOAQUIM ROÉ

ABSTRACT. We classify and provide the global phase portraits in the Poincaré disc of all real planar polynomial differential systems having their orbits embedded in conics. This is achieved via the real affine classification of the pencils of conics. All such polynomial vector fields have degree less than or equal to 3. Also, when the degree is three, infinity is filled with singular points.

1. Introduction and statement of the main results

We consider *real planar polynomial differential systems* or simply *polynomial systems*, i.e. differential systems of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where $P(x,y)$, $Q(x,y)$ are real polynomials in x and y. The *degree* m of (1) is the maximum of the degrees of the polynomials P and Q, i.e. $m = \max\{\deg P, \deg Q\}$. We say that the polynomial differential system (1) is *non–degenerate* if P and Q are coprime, otherwise we say that it is *degenerate*.

Real planar polynomial differential systems appear in many areas of applied mathematics. Only on quadratic polynomial differential systems more than one thousand articles have been written but the understanding of the dynamics of these systems is far from complete. In this paper we study the global dynamics of the polynomial differential systems having all their orbits imbedded in conics. To be more precise we say that system (1) has the orbit γ *imbedded in a conic* if there exists a polynomial of degree two $F(x,y) \in \mathbb{R}[x,y]$ such that $\gamma \subset \{F(x,y) = 0\}$.

Although real conics are very simple curves and there are only nine different types of them up to an affine transformation, the differential polynomial systems having their orbits contained in conics give rise to a rich dynamics as is shown in the following theorem which is one of our main results.

Theorem 1. *Any polynomial differential system having its orbits embedded in conics, after an affine change of coordinates and rescaling of the independent variable (if necessary) can be written as one of the systems given in tables 1 and 2. All*

1991 *Mathematics Subject Classification.* Primary 34C05; 34A34; 34C14.

Key words and phrases. polynomial vector field,phase portraits, rational function, pencil of conics.

The first two authors are partially supported by a MEC/FEDER grant number MTM2008-03437 and by a CICYT grant number 2005SGR 00550; the third author is partially supported by MEC/FEDER grants number MTM2005-01518 and MTM2008-11391 and by a CICYT grant number 2005SGR 00787.

Type	Differential system	
$\mathbf{III}^{\ell^2}$	$\dot{x} = -2x,$	$\dot{y} = -y,$
$\mathbf{III}^{\ell^{22}}$	$\dot{x} = -x,$	$\dot{y} = y,$
$\mathbf{III}_a^{\ell^{22}}$	$\dot{x} = -2y,$	$\dot{y} = 2x,$
$\mathbf{V}^{\ell^4}$	$\dot{x} = -1,$	$\dot{y} = -2x.$
$\mathbf{I}^{\ell}$	$\dot{x} = x - x^2,$	$\dot{y} = -y + y^2,$
$\mathbf{I}_a^{\ell}$	$\dot{x} = -2xy,$	$\dot{y} = -1 + x^2 - y^2,$
$\mathbf{I}_b^{\ell}$	$\dot{x} = 2xy,$	$\dot{y} = -1 + x^2 + y^2,$
$\mathbf{I}_b^{\ell c}$	$\dot{x} = -2xy,$	$\dot{y} = 1 + x^2 - y^2,$
$\mathbf{II}^{\ell}$	$\dot{x} = x^2,$	$\dot{y} = -y^2,$
$\mathbf{II}^{\ell^2}$	$\dot{x} = x - 2xy,$	$\dot{y} = y - y^2,$
$\mathbf{II}^{\ell^{12}}$	$\dot{x} = -x + x^2,$	$\dot{y} = y,$
$\mathbf{II}_a^{\ell}$	$\dot{x} = -2xy,$	$\dot{y} = x^2 - y^2,$
$\mathbf{II}_a^{\ell^2}$	$\dot{x} = -2xy,$	$\dot{y} = -1 - y^2,$
$\mathbf{III}^0$	$\dot{x} = x(-1 + x - y),$	$\dot{y} = -y(-1 - x + y),$
$\mathbf{III}^{b^2}$	$\dot{x} = x(x - 1),$	$\dot{y} = y(x + 1),$
$\mathbf{III}^n$	$\dot{x} = 2x(1 - x),$	$\dot{y} = y(1 - 2x),$
$\mathbf{III}_a^0$	$\dot{x} = y + x^2 - xy,$	$\dot{y} = -x + xy - y^2,$
$\mathbf{III}_a^i$	$\dot{x} = 2xy,$	$\dot{y} = 2 + 2y^2,$
$\mathbf{IV}^{\ell^3}$	$\dot{x} = -x,$	$\dot{y} = -x^2 - y,$
$\mathbf{IV}^{\ell^{13}}$	$\dot{x} = -x^2,$	$\dot{y} = 1,$
$\mathbf{V}^0$	$\dot{x} = 2xy,$	$\dot{y} = -x + 2y^2,$
$\mathbf{V}^{b^4}$	$\dot{x} = -x^2,$	$\dot{y} = 2 - xy.$

TABLE 1. Linear and quadratic differential systems with orbits imbedded in conics.

polynomial differential systems of tables 1 and 2 have degree at most 3. The corresponding phase portraits are given in Figure 1. Note that all the phase portraits corresponding to polynomial differential systems of degree 3 have the infinity filled with singular points.

Our proof of Theorem 1 is based on the real affine classification of the pencils of conics, and each *type* corresponds to an equisingularity type of pencil. For some equisingularity types there is a continuous family of affine types, depending on parameters given in table 3. For types $\mathbf{I}^0$, $\mathbf{I}_b^0$, $\mathbf{II}^0$, $\mathbf{II}^{b^2}$ and $\mathbf{IV}^0$ the parameter space

Type	Differential system		Parameters
$\mathbf{I}^0$	$\dot{x}=-abx-a(1+b)x^2+ax^3-bxy^2$, $\dot{y}=-aby+b(1+a)y^2+ax^2y-by^3$,		$a\in(0,1)$ $b\in[-1,2-\frac{1}{a}]$ $b\neq 0$
$\mathbf{I}^b$	$\dot{x}=ax-ax^2-xy^2$, $\dot{y}=-ay+(1+a)y^2-y^3$,		$a\in[1/2,1)$
$\mathbf{I}^n$	$\dot{x}=ax-(1+a)x^2+x^3-xy^2$, $\dot{y}=-ay+(1+a)y^2+x^2y-y^3$,		$a\in(0,1)$
$\mathbf{I}^d$	$\dot{x}=-x+x^3-xy^2$, $\dot{y}=y+x^2y-y^3$,		
$\mathbf{I}_a^0$	$\dot{x}=(1-a-b)x+2(a-1)x^2+x^3+(b-a)xy^2$ $\dot{y}=(-1+a+b)y+x^2y-2by^2+(b-a)y^3$		$b<a(1-a)$ $b\geq 0$
$\mathbf{I}_a^i$	$\dot{x}=2(1-a)x+2(a-1)x^2+x^3-xy^2$ $\dot{y}=2(a-1)y+x^2y+2(1-a)y^2-y^3$		$a\in(-1,1)$
$\mathbf{I}_a^n$	$\dot{x}=2xy-2x^2y$ $\dot{y}=1-2x+a(2-a)x^2+y^2-2xy^2$		$a\in(-1,1)$
$\mathbf{I}_a^{d_1}$	$\dot{x}=2xy-2x^2y$, $\dot{y}=1-2x+y^2-2xy^2$,		
$\mathbf{I}_a^{d_2}$	$\dot{x}=2x^2-2x^3+2xy-2x^2y$ $\dot{y}=1-2x-2x^2y+y^2-2xy^2$		
$\mathbf{I}_b^0$	$\dot{x}=bx-(1+b)x^2+x^3-bxy^2$, $\dot{y}=-by+x^2y+aby^2-by^3$,		$a\in(-2,2)$ $\lvert b\rvert\in(0,1]$
$\mathbf{I}_b^b$	$\dot{x}=-x+x^2+xy^2$, $\dot{y}=y-ay^2+y^3$,		$a\in(-2,2)$
$\mathbf{I}_b^n$	$\dot{x}=2xy-2x^2y$, $\dot{y}=-1+2x-ax^2+y^2-2xy^2$,		$a\in(1,\infty)$
$\mathbf{II}^0$	$\dot{x}=-x^2+x^3-axy^2$,	$\dot{y}=x^2y+ay^2-ay^3$,	$\lvert a\rvert\in(0,1]$
$\mathbf{II}^b$	$\dot{x}=-x^2+x^3$,	$\dot{y}=x^2y+y^2$,	
$\mathbf{II}^{b^2}$	$\dot{x}=ax-(1+a)x^2+x^3$,	$\dot{y}=-ay+x^2y$	$\lvert a\rvert\in(0,1]$
$\mathbf{II}^n$	$\dot{x}=-x^2+x^3-xy^2$,	$\dot{y}=x^2y+y^2-y^3$,	
$\mathbf{II}_a^0$	$\dot{x}=x-ax^2+x^3-xy^2$,	$\dot{y}=-y+x^2y+2y^2-y^3$,	$\lvert a\rvert\in[0,2)$
$\mathbf{II}_a^{b^2}$	$\dot{x}=x+x^3$,	$\dot{y}=-y+x^2y$,	
$\mathbf{II}_a^n$	$\dot{x}=2xy-2x^2y$,	$\dot{y}=-x^2+y^2-2xy^2$,	
$\mathbf{IV}^0$	$\dot{x}=px^3+xy^2$,	$\dot{y}=px^2y-y^2+y^3$,	$p=\pm 1$
$\mathbf{IV}^b$	$\dot{x}=-x^3$,	$\dot{y}=-x^2y-y^2$,	
$\mathbf{IV}^{b^3}$	$\dot{x}=x^2-x^3$,	$\dot{y}=-1+2x-x^2y$,	

TABLE 2. Cubic differential systems with orbits imbedded in conics.

has two connected components, and it turns out that each connected component gives one topological equivalence class of non–degenerate planar polynomial differential system. Thus the classification shows that two such differential systems are topologically equivalent if and only if they are connected by a continuous, topologically trivial, family of differential systems, result for which we do not know an a priori reason.

Corollary 2. *Let* (1) *be a non–degenerate polynomial system having its orbits imbedded in conics and let*

$$\frac{dx}{dt} = P'(x,y), \qquad \frac{dy}{dt} = Q'(x,y) \tag{2}$$

be another polynomial differential system having its orbits imbedded in conics. Then (1) *and* (2) *are topologically equivalent if and only if there is a continuous family* $(P_t, Q_t)_{t\in[0,1]}$ *of pairs of polynomials such that the differential systems given by* P_t, Q_t *are topologically equivalent for all* t*, and coincide with* (1) *for* $t = 0$ *and with* (2) *for* $t = 1$*, modulo an affine change of coordinates.*

All quadratic polynomial differential systems having a rational first integral of degree two were classified in [5] and [3]. These systems have their orbits contained in conics. The cubic polynomial systems of Lotka-Voltera type having a rational first integral of degree two were characterized in [4]. Finally all cubic differential systems having a rational first integral of degree two were classified in [14]. All these results are particular cases of Theorem 1.

The main goal of Section 2 is to prove that the degree of non–degenerate system having all its orbits embedded in algebraic curves of degree d is at most $2d-1$ (see Theorem 6). In Section 3 we introduce some basic results on singular points of vector fields, the Poincaré compactification and the topological equivalence. Moreover, we recall the main result which allows us to distinguish topological non–equivalent phase portraits (see Theorem 10). In Section 4 we recall the projective classification of the pencils of conics depending on the base points, including, for the sake of completeness, the case of degenerate pencils. Theorem 1 is proved in Section 5, where the affine classification of pencils of conics is also provided.

2. Preliminaries

We associate system (1) with the *vector field*

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{3}$$

Let $f = f(x,y)$ be a complex polynomial in the variables x and y, i.e. $f \in \mathbb{C}[x,y]$. We say that $f = 0$ is an *invariant algebraic curve* of the vector field $\mathcal{X}$, or of system (1), if for some polynomial $K \in \mathbb{C}[x,y]$ we have $\mathcal{X}(f) = Kf$. The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. A $\mathcal{C}^1$ function $H : \mathbb{R}^2 \to \mathbb{R}$ we call the *first integral* of the system (1) or the vector field (3) if it is constant on all solution curves of (1). This is equivalent to $\mathcal{X}H \equiv 0$.

Let $H = f/g$ be a rational first integral of (1) where f and g are coprime. After Poincaré [18] we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable value of* H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x,y]$. Here, if $c = \infty$ then $f + cg$ denotes g.

Let $H = f/g$ be a rational first integral of (1). We say that it has a *degree* n if it is the maximum of the degrees of f and g. We also say that the degree n of the

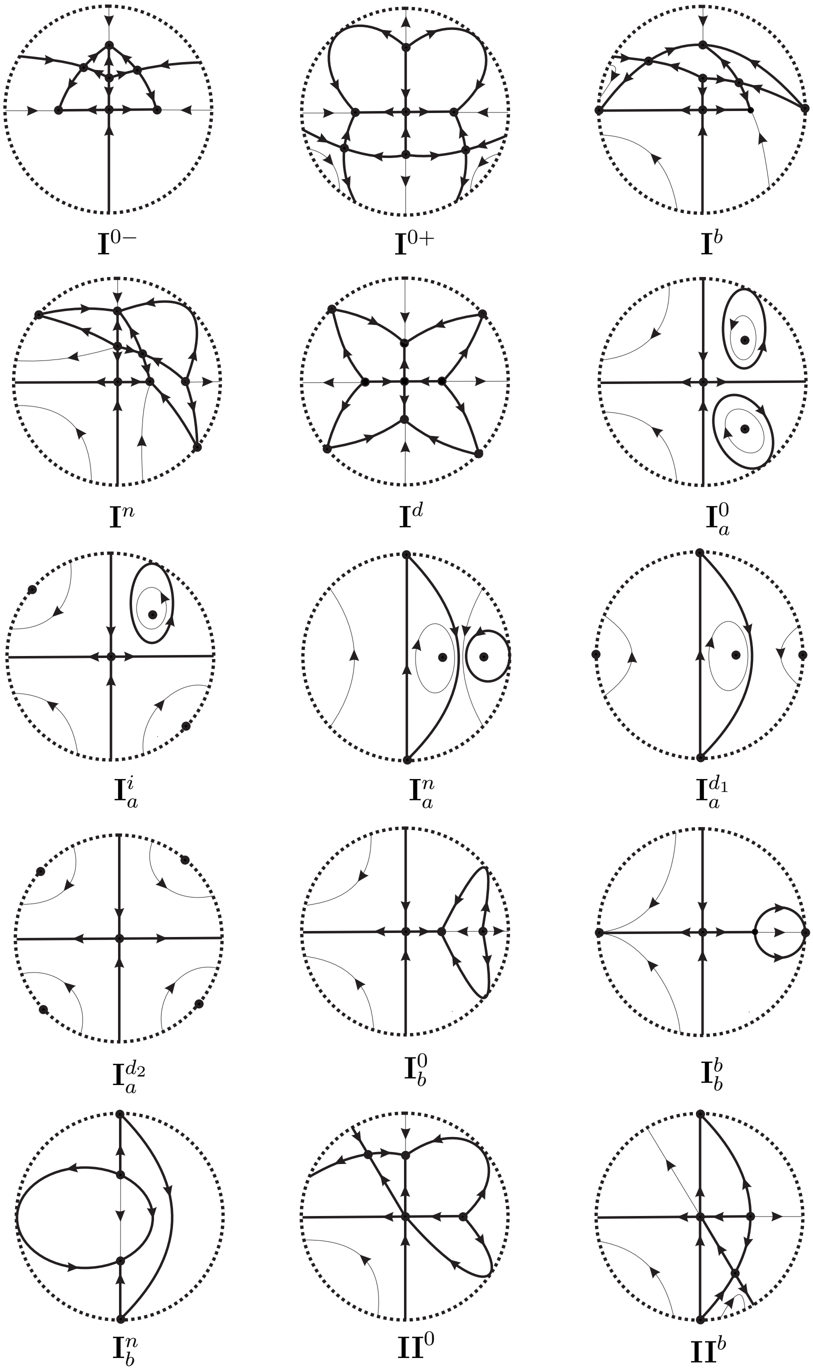


FIGURE 1. Non–equivalent phase portraits of non–degenerate planar polynomial systems having their orbits embedded in conics.

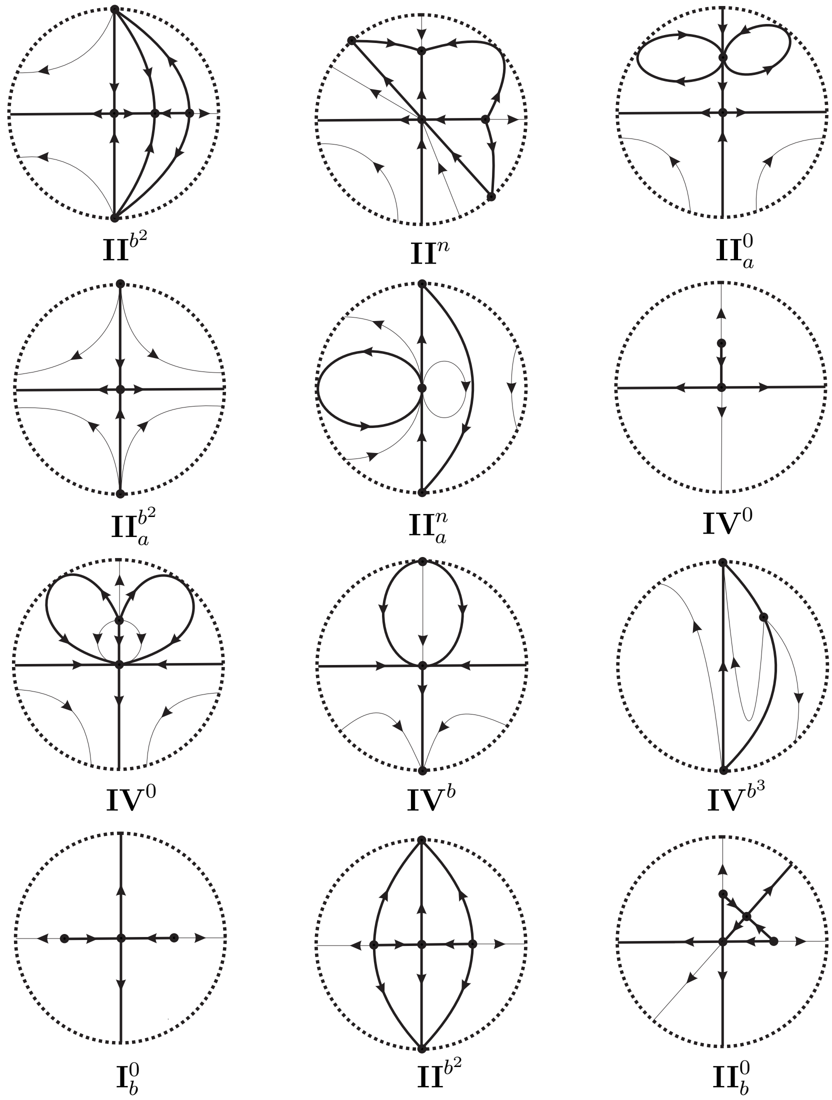


FIGURE 2. Continuation of Figure 1.

first integral H is *minimal* if any other rational first integral of (1) has a degree greater than or equal to n.

The following result is proved in [7, p.128].

Theorem 3. *Assume that for the non–degenerate polynomial system* (1) *the first integral H is rational and minimal. Then H has finitely many remarkable values.*

The Darboux theory of integrability says that sufficient number of invariant algebraic curves implies the existence of a first integral. In [11] Jouanolou gives a sufficient condition for the existence of a rational first integral, for a shorter proof see [8].

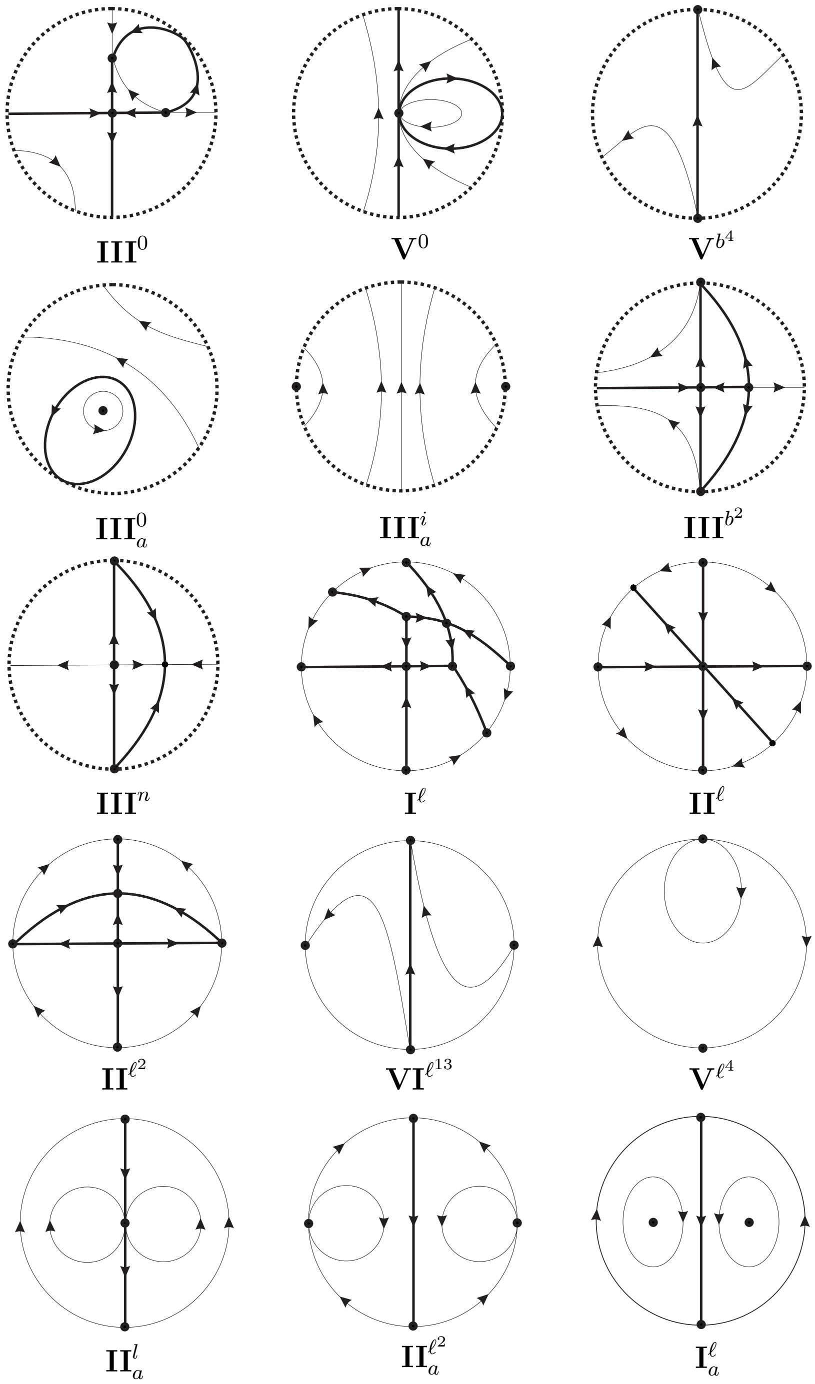


FIGURE 3. Continuation of Figure 2.

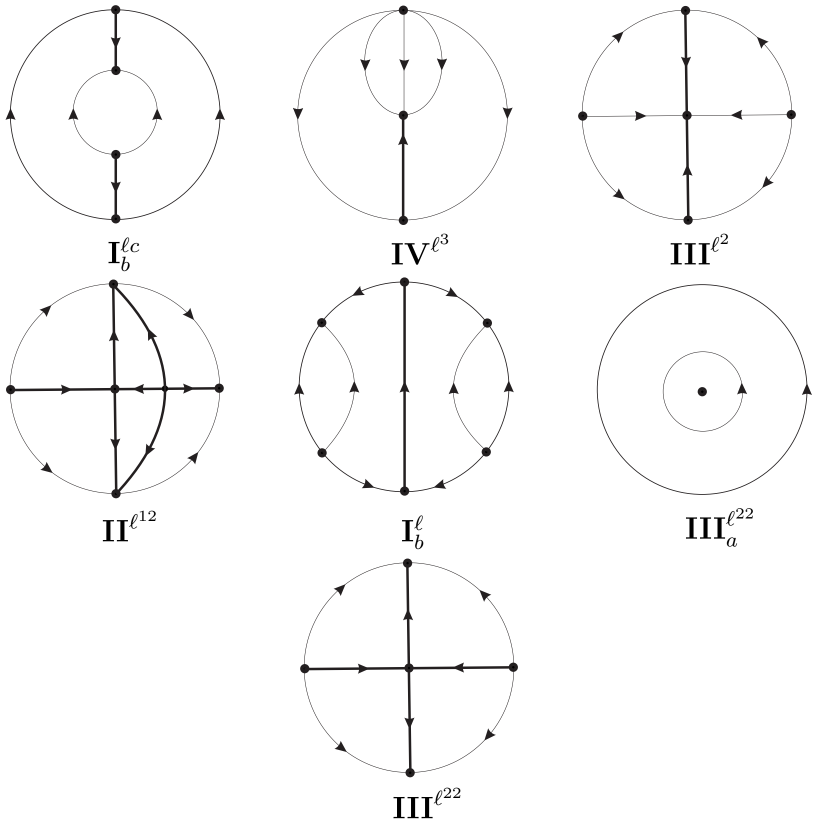


FIGURE 4. Phase portraits of linear, quadratic and cubic polynomial systems having their orbits embedded in straight lines.

Theorem 4 (Jouanolou). *If a polynomial vector field of degree m has at least $\frac{1}{2}m(m+1)+2$ algebraic irreducible invariant curves, then it has a rational first integral.*

Finally we recall the Unique Factorization Theorem, see for example [12, IV, 2.4].

Theorem 5. *Any non–constant polynomial $f(x_1,\ldots,x_n)$ over a field $\mathbb{R}$ or $\mathbb{C}$ can be written uniquely (up to order and non–zero scalar) in the form $f = f_1^{r_1}...f_s^{r_s}$ where $f_1,\ldots,f_s$ are irreducible (and pairwise distinct) in $\mathbb{C}[x,y]$.*

The polynomials $f_1,\ldots,f_s$ are the *irreducible components* of f, and the numbers $r_1,\ldots,r_s$ are their *multiplicities.* If $f, f_1,\ldots,f_s$ have degrees $d, d_1,\ldots,d_s$ respectively, then $d = r_1d_1 + \cdots + r_sd_s$. Note that Unique Factorization implies, given two polynomials f and g, the existence of a Greatest Common Factor (GCF) ϕ, determined up to non–zero scalar, such that f/ϕ and g/ϕ are coprime polynomials.

Theorem 6. *Assume that system* (1) *is non–degenerate and all its orbits are contained in algebraic curves of degree d. Then the degree of system* (1) *is at most* $2d-1$.

Proof. Assume that system (1) has all its orbits embedded in algebraic curves of degree d. Since the system has infinitely many invariant algebraic curves by Jounalou's theorem there exists a minimal rational first integral $H = f/g$. This means that all the orbits of this system are contained in a level curves $f/g = h$ for some $h \in \mathbb{R} \cup \{\infty\}$, or equivalently in $f - hg = 0$ for $h \in \mathbb{R}$ or $g = 0$ for $h = \infty$. According to Theorem 3 there are finitely many values h such that $f - hg$ is reducible in $\mathbb{C}[x, y]$, and by assumptions all the orbits of the system are contained in algebraic curves of degree d. Therefore it follows that generically the polynomials $f - hg$ have degree d. Consequently H is a rational first integral of degree d. Since H is a first integral we have

$$\frac{f_x g + f g_x}{g^2} P + \frac{f_y g + f g_y}{g^2} Q \equiv 0. \tag{4}$$

Let ϕ be the GCF of $f_x g + f g_x$ and $f_y g + f g_y$. By the Unique Factorization Theorem we get unique (up to order and non–zero scalar) expressions $(f_x g + f g_x)/\phi = u_1^{p_1} \dots u_s^{p_s}$ and $(f_y g + f g_y)/\phi = w_1^{q_1} \dots w_r^{q_r}$, where u_i and w_j are irreducible, and moreover $u_1^{p_1} \dots u_s^{p_s}$ and $w_1^{q_1} \dots w_r^{q_r}$ are coprime. Since (1) is non–degenerate, P and Q are coprime too and hence condition (4) is fulfilled if and only if $P = -c w_1^{q_1} \dots w_r^{q_r}$ and $Q = c^{-1} u_1^{p_1} \dots u_s^{p_s}$ for some non–zero constant c. Since the polynomials $u_1^{p_1} \dots u_s^{p_s}$ and $w_1^{q_1} \dots w_r^{q_r}$ are of degree at most $2d - 1$ the theorem follows. □

From Theorem 6 the next result follows immediately.

Corollary 7. *The degree of a non–degenerate polynomial differential system* (1) *having its orbits embedded in conics is at most three.*

Corollary 7 and Theorem 4 imply that in order to classify all polynomial differential systems (or associated polynomial vector fields) having their orbits imbedded in conics it is enough to consider systems of degree at most three having a rational first integral of degree two. Thus we consider polynomial differential system having a rational first integral $H = H_N/H_D$, defined by

$$\dot{x} = -\frac{\partial H}{\partial y} (H_D)^2, \quad \dot{y} = \frac{\partial H}{\partial x} (H_D)^2, \tag{5}$$

where $H_N, H_D \in \mathbb{R}[x, y]$ are polynomials in two variables of degree at most 2. We notice that non–degenerate polynomial differential systems having a rational first integral of degree two are of the form (5) or a quotient of it. It is clear that for any $h \in \mathbb{R} \cup \{\infty\}$ the set $H_N/H_D = h$ is invariant. Equivalently for $\alpha, \beta \in R$ any conic that belongs to the *pencil of conics*

$$\alpha H_N + \beta H_D = 0, \tag{6}$$

is also invariant. The polynomials H_N and H_D are called *generators* of the pencil of conics (6). In Section 5 we associate with pencils of conics their *normal forms*, i.e. equations such that every pencil with the given type can be transformed by an affinity into one and only one of the given equations; this amounts to giving a complete affine classification of pencils of real affine conics. The determination of the normal forms involves some arbitrary choices. Whenever possible, we chose reducible fibers (for definition see Section 5) as generators of the pencil, and coordinates in which the axes are (or are related to) components of the reducible fibers.

We have seen that with a polynomial vector field having a rational first integral $H = H_N/H_D$ of degree 2 we can associate the pencil of conics (6). On the other hand, with a pencil of conics (6) we can associate a polynomial differential system (5) of at most degree three having a rational first integral of the form $H = H_N/H_D$.

Now we prove a property regarding the points at infinity of the cubic systems having their orbits embedded in conics.

Lemma 8. *If a cubic system possesses all its orbits embedded in conics, then it has a line of singular points at infinity.*

Proof. We denoted the cubic homogeneous part of these vector fields by $(\bar{P}, \bar{Q})$. It is enough to notice that they can be written as $(\bar{P}, \bar{Q}) = (xR_2, yR_2)$, where R_2 is a homogeneous polynomial of degree 2. Since $x\bar{Q} - y\bar{P} \equiv 0$ the lemma follows. □

3. Singular points, Poincaré compactification and topological equivalence

In this section we recall basic definitions and results on the local phase portraits of a singular point. We compactify a polynomial vector field extending it to the infinity. We introduce also, among other things, the definition of the topological equivalence. Then we state the Markus-Neumann-Peixoto Theorem that allows us to determine all topologically equivalent system by restricting ourselves to studying the flow of the system on the set of their separatrix configurations.

3.1. **Singular points.** A point $u = (x_0, y_0) \in \mathbb{R}^2$ is said to be a *singular point* of the vector field $\mathcal{X}$ if $P(u) = Q(u) = 0$. We recall first some results which hold when P and Q are analytic functions in a neighborhood of u. As usual P_x denotes the partial derivative of P with respect to the variable x.

If $\Delta = P_x(u)Q_y(u) - P_y(u)Q_x(u)$ and $T = P_x(u) + Q_y(u)$, then the singular point u is said to be *non–degenerate* if $\Delta \neq 0$. Then u is an isolated singular point. Moreover, u is a *saddle* if $\Delta < 0$, a *node* if $T^2 \geq 4\Delta > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a *focus* if $4\Delta > T^2 > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *center* if $T = 0 < \Delta$; for more details see [2], p. 183.

A degenerate singular point u (i.e. $\Delta = 0$) with $T \neq 0$ is called *semi–hyperbolic*, and u is isolated in the set of all singular points. In the next proposition we summarize the results on semi–hyperbolic singular points that we shall need in this paper. For a proof see Theorem 65 of [2] and Theorem 2.19 of [9].

Proposition 9. *Let* $(0,0)$ *be an isolated point of the vector field* $(F(x,y),\ y + G(x,y))$*, where* F *and* G *are analytic functions in a neighborhood of the origin starting with quadratic terms in the variables* x *and* y*. Let* $y = g(x)$ *be the solution of the equation* $y + G(x,y) = 0$ *in a neighborhood of* $(0,0)$*. Assume that the development of the function* $f(x) = F(x, g(x))$ *is of the form* $f(x) = \mu x^m + HOT$ *(Higher Order Terms), where* $m \geq 2$ *and* $\mu \neq 0$*. When* m *is odd, then* $(0,0)$ *is either an unstable node, or a saddle depending if* $\mu > 0$*, or* $\mu < 0$*, respectively. In the case of the saddle the stable separatrices are tangent to the* x*–axis. If* m *is even, then* $(0,0)$ *is a saddle–node, i.e. the singular point is formed by the union of two hyperbolic sectors with one parabolic sector. The stable separatrix is tangent to the positive (respectively negative)* x*–axis at* $(0,0)$ *according to* $\mu < 0$ *(respectively* $\mu > 0$*). The two unstable separatrices are tangent to the* y*–axis at* $(0,0)$.

The singular points which are non–degenerate or semi–hyperbolic are called *elementary.*

When $\Delta = T = 0$ but the Jacobian matrix at u is not the zero matrix and u is isolated in the set of all singular points, we say that u is *nilpotent.* Now we summarize the results on nilpotent singular points that we shall need. For details see [1], or Theorem 3.5 of [9].

3.2. **Poincaré compactification.** Let $\mathcal{X} \in P_n(\mathbb{R}^2)$ be a planar polynomial vector field of degree n. The *Poincaré compactified vector field* $p(\mathcal{X})$ *corresponding to* $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows (see, for instance [10] or Chapter 5 of [9]). Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point y. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified with the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X} \in P_n(\mathbb{R}^2)$, $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2\backslash\mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ in a neighborhood of the infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc,* and it is denoted by $\mathbb{D}^2$. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

In the rest of this work we say that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$, preserving or reversing simultaneously the sense of all orbits.

As $\mathbb{S}^2$ is a differentiable manifold, for computing the expression for $p(\mathcal{X})$, we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so z represents different things according to the local charts under consideration), then some easy computations give for $p(\mathcal{X})$ the following expressions:

$$z_2^n\Delta(z)\left(Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right)\right) \text{ in } U_1, \tag{7}$$

$$z_2^n\Delta(z)\left(P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)\right) \text{ in } U_2, \tag{8}$$

$$\Delta(z)\left(P(z_1, z_2), Q(z_1, z_2)\right) \text{ in } U_3,$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbb{S}^1$. *In what follows we omit the factor* $\Delta(z)$ *by rescaling the vector field* $p(\mathcal{X})$. Thus we obtain a polynomial vector field in each local chart. The singular points of $p(X)$ contained in $\mathbb{S}^1$ are called the *infinite singular points* of $\mathcal{X}$.

3.3. **Topological equivalence.** Let φ be a C^k local flow with $k \geq 0$ on the 2–dimensional manifold M. Of course, for $k = 0$ the flow is continuous. We say that (M_1, φ_1) and (M_2, φ_2) are C^k*–equivalent* if there is a C^k diffeomorphism of M_1 onto M_2 which takes orbits of φ_1 onto orbits φ_2 preserving or reversing sense (but not necessarily the parametrization). Of course a C^0 diffeomorphism is a homeomorphism.

We say that (M, φ) is C^k*–parallel* if it is C^k–equivalent to one of the following flows:

(i) $\mathbb{R}^2$ with the flow defined by $\dot{x} = 1$, $\dot{y} = 0$ (*strip flow*);
(ii) $\mathbb{R}^2 \setminus \{0\}$ with the flow defined by $\dot{r} = 0$, $\dot{\theta} = 1$ (*annular flow*);
(iii) $\mathbb{R}^2 \setminus \{0\}$ with the flow defined by $\dot{r} = r$, $\dot{\theta} = 0$ (*radial flow*);
(iv) $\mathbb{S}^1 \times \mathbb{S}^1$ with rational flow (*toral flow*).

Let $p \in M$, we denote by $\gamma(p)$ the *orbit* of the flow φ on M through p, more precisely $\gamma(p) := \{\varphi_p(t) : t \in I_p\}$, where I_p is the maximal open interval of the solution of φ_p. The *positive semiorbit* of $p \in M$ is $\gamma^+(p) = \{t \in I_p, t \geq 0\}$. In a similar way we define the *negative semiorbit* $\gamma^-(p)$ of $p \in M$. We define the α–limit and ω–limit of $p \in M$ as

$$\alpha(p) = \overline{\gamma^-(p)} - \gamma^-(p), \quad \omega(p) = \overline{\gamma^+(p)} - \gamma^+(p).$$

Let $\gamma(p)$ be an orbit of the flow φ defined on M. A *parallel neighborhood* of the orbit $\gamma(p)$ is an open neighborhood N of γ such that (N, φ) is C^k–equivalent to a parallel flow for some $k \geq 0$.

We say that $\gamma(p)$ is a *separatrix* of φ if $\gamma(p)$ is not contained in a parallel neighborhood N satisfying the following two assumptions:

(1) for every $q \in N$, $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$,
(2) $\overline{N} \setminus N$ consists of $\alpha(p)$, $\omega(p)$ and exactly two orbits $\gamma(a)$, $\gamma(b)$ of φ, with $\alpha(a) = \alpha(p) = \alpha(b)$ and $\omega(a) = \omega(p) = \omega(b)$.

For polynomial vector fields the separatrices are the singular points, the limit cycles and the boundaries of any hyperbolic sector.

We denote by Σ the union of all separatrices of φ. Then Σ is a closed invariant subset of M. A component of the complement of Σ in M, with the restricted flow, is called a *canonical region* of φ.

Let (φ, M) be a continuous flow on the 2–manifold M and let Σ be the set of all separatrices of (φ, M). In every canonical region U of (φ, M) we choose an orbit γ_U. Then a *separatrix configuration* of (φ, M) is formed by the union of the set Σ and the set of all orbits γ_U.

Theorem 10 (Markus[15], Neumann[16], Peixoto[17])**.** *Let (φ_1, M_1) and (φ_2, M_2) be two continuous flows on the 2-manifolds M_1 and M_2. Then two flows are topologically equivalent if and only if there exists a homeomorphism $h : M_1 \to M_2$, which takes the orbits of the separatrix configuration of (φ_1, M_1) into the orbits of the separatrix configuration of (φ_2, M_2).*

For more details on the results presented in this section see Section 1.9 of [9].

4. Pencils of conics

We recall next the basic definitions on pencils of conics and their well-known projective classification, which will be used in our affine classification. A *pencil*

Type	Base points
I	4, all simple, all real
$\mathbf{I}_a$	4, all simple, none real
$\mathbf{I}_b$	4, all simple, two of them real
II	3, one of them double, two simple, all real
$\mathbf{II}_a$	3, one of them real and double, two non-real and simple
III	2, both double, both real
$\mathbf{III}_a$	2, both double, both non–real base
IV	2, one of them simple, the other one triple, both real
V	1, quadruple, real

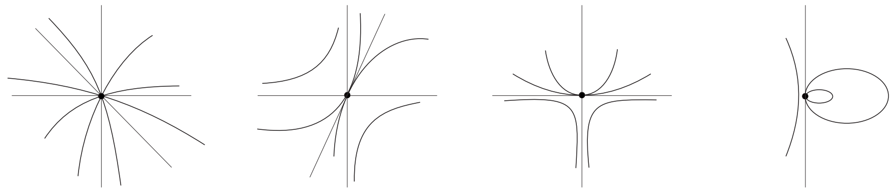
Table 3. Projective types of real non–degenerate pencils

of conics is a 1-dimensional linear system of plane curves of degree 2. Given two distinct conics $f = 0$, $g = 0$ there is a unique pencil containing them, formed by all conics $f + cg = 0$, $c \in \mathbb{R} \cup \infty$, where, if $c = \infty$ then $f + cg$ denotes g. Equivalently, one may write $\lambda f + \mu g = 0$, $(\lambda : \mu) \in \mathbb{P}^1$, where $\mathbb{P}^n$ denotes the n–dimensional projective space. Each conic in the pencil is called a *fiber*.

We shall be dealing with pencils of affine conics defined over the real numbers. Any given affine pencil naturally determines a projective pencil in the projective closure of the affine plane. Note that we allow affine conics to degenerate into lines and empty conics, defined by equations of degree 1 or 0; embedding the affine plane in the projective plane, this corresponds to the line at infinity being a component of one conic of the pencil.

The projective classification (up to automorphisms of $\mathbb{P}^2$) of pencils of conics can be found in the literature on projective geometry, both over the complex numbers (see for instance [19, VII], [6, 8.6]) and over the reals (see [13]). We say that a conic is *irreducible* if it is either an ellipse, or a hyperbola, or a parabola; otherwise we say that it is *reducible*. A pencil of conics is *degenerate* when all the conics of the pencil are reducible; otherwise the pencil is called *non–degenerate*. Each of these has a few subtypes, depending on the nature of the base locus.

Two conics in a non–degenerate pencil intersect in four points (real or complex) counted with multiplicity. Any other conic in the pencil passes through these points, which are called the *base points* of the pencil. A base point is said to be *simple*, *double*, *triple* or *quadruple* when the intersection multiplicity of two conics there is 1, 2, 3, or 4 respectively. Through any other point of the plane there goes exactly one conic of the pencil (thus a pencil of conics can be identified to a rational map $\mathbb{P}^2 \to \mathbb{P}^1$ given by polynomials of degree 2, modulo automorphisms of $\mathbb{P}^1$, whence the name of fibers for the conics in a pencil). The complex projective classification of pencils of conics depends only on the multiplicities of the base points, whereas the real classification must distinguish real base points from (pairs of) complex conjugate base points. It is given in table 3, using the nomenclature of Levy's book [13]; roman numerals refer to the complex classification, with subindices to specify the refinement due to considering pencils over the real numbers.



(a) (b) (c) (d)

Figure 5. Radial and non–radial base points.

All real projective pencils of conics contain *singular fibers.* They can be pairs of distinct real lines, pairs of complex conjugate lines (which show up as isolated singular points in the real plane), or double (real) lines; the corresponding singular points are ordinary double points with two real branches (or simply real double points), ordinary double points with two complex conjugated branches (or simply isolated double points), and non–isolated singularities respectively (every point on a double line is singular). These are essential elements of the pencil.

Given four points in the (projective) plane, no three of which are aligned, the set of lines joining pairs of points is called a *complete quadrangle.* Each such line is a *side* of the quadrangle, and two opposite sides (in the sense that they join disjoint pairs of the given points) meet in a *diagonal point* of the quadrangle. Every complete quadrangle has six sides and three non-aligned diagonal points. The four given points plus the three diagonal points are *vertices* of the complete quadrangle. For a pencil of type **I**, no three of the four base points are ever aligned and so they determine a complete quadrangle (some elements of which are complex if the pencil has type $\mathbf{I}_a$ or $\mathbf{I}_b$). It is easy to see that the singular fibers of the pencil are the real pairs of opposite sides of the quadrangle. The diagonal points of the quadrangle are the singular points of the singular fibers. The triangle formed by diagonal points has important properties as well: the tangent line to a conic of the pencil at a point on a side of this triangle passes through the opposite diagonal point. For this reason the triangle of diagonal points is called *self–polar* with respect to the pencil.

At a simple base point, no two conics of the pencil have the same tangent line and no conic of the pencil is singular there. Moreover, there are conics going in all tangent directions there. So the picture of the pencil near a base point looks like Fig. 5(a) and the corresponding differential system has a radial node there.

At a double base point all irreducible fibers of the pencil are mutually tangent, and one fiber is singular there. So the picture of the pencil near a double base point looks like Fig. 5(b). If the singular fiber is a pair of lines, then the corresponding differential system has a point with two hyperbolic and two parabolic sectors; if the singular fiber is a pair of conjugated complex lines, then the differential system has a point with two elliptic and two parabolic sectors; finally, if the singular fiber is a double line, then the differential system has a non-radial node.

At a triple base point all irreducible fibers are mutually tangent with the same curvature, and one fiber is singular there, which is necessarily a pair of lines. So the picture of the pencil near a base triple point looks like Fig. 5(c) and the corresponding differential system has a point with two hyperbolic and two parabolic sectors.

Finally at a quadruple base point all irreducible fibers share three terms in the Taylor expansion, and one fiber is singular there, which is necessarily a double line. So the picture of the pencil near a quadruple base point looks like Fig. 5(d) and the corresponding differential system has a point with an elliptic, a hyperbolic and two parabolic sectors.

Pencils of type **II** have two singular fibers, both are pairs of lines. One has the double point at the double base point and the second goes simply through it. Thus their geometry is described by the position of four points, three of which are aligned (the two simple base points and the non–base double point), and the lines joining them.

Pencils of type **III** have two singular fibers: the double line through the two double base points, and a pair of lines, each of which is the tangent line to the nonsingular fibers at one base point. Thus their geometry is described by a triangle, two vertices of which are the base double points and the third is the only non–base double point.

Pencils of type **IV** have a unique singular fiber, which is a pair of lines. The intersection point of both lines is the triple base point, the remaining base point belongs to one of the lines. All other fibers are irreducible, tangent to the other line at the intersection point and with fixed curvature there. Thus their geometry is described by two lines and a point on one of the lines.

Pencils of type **V** have a unique singular fiber which is a double line $2L$ containing the (quadruple) base point p. All other fibers are irreducible, tangent to L at p (they intersect each other with multiplicity four there). Thus their geometry is described by a line and a point on it.

For the sake of completeness we recall also the projective classification of degenerate pencils of conics. Since the differential systems arising from these have their orbits imbedded in straight lines, we do not consider them in this article. Degenerate pencils consist entirely of singular conics, i.e., pairs of lines, and their geometry is deduced from that of pencils of lines. It is elementary that a pencil of lines consists of the lines through a given point. We shall see that in the affine setting, this splits in two cases, depending whether the base point is affine or sits at infinity (in which case the pencil consists of the lines parallel to a given one).

If two singular conics with no common component intersect in more than one point then the pencil they generate is clearly of one of the types above, so it is non–degenerate. Therefore, if two fibers of a degenerate pencil of conics have no common component, then they intersect at the commont singular point, and hence all fibers are pairs of lines going through that point. These pencils are called *involutions of lines*. There are two kinds of such, depending on whether they contain double lines or not (over the complex field they always contain two double lines, but if they are complex conjugate then they don't show up in the real pencil). On the other hand, if two fibers of the pencil have a common component, then this must be a *fixed component* of the pencil, the moving part being a pencil of lines. Depending on whether the base point of the moving part belongs to the fixed component or not, this gives rise to two projective types.

Type	Degenerate pencil
VI	Involution of lines, no double lines
$\mathbf{VI_a}$	Involution of lines, 2 double lines
VII	Fixed line + pencil based off the line
$\mathbf{VII_a}$	Fixed line + pencil based on line

Table 4. Projective types of real degenerate pencils

5. Classification of non–degenerate pencils

Any given affine pencil naturally determines a projective pencil in the projective closure of the affine plane. It then belongs to one of the types above, and the affine classification follows by considering the relative position of the line at infinity with respect to the pencil.

For our purposes the *singular elements* of a given pencil are its base points, the singular fibers and the singular points of the fibers. We shall be interested only in the real singular elements. Two projective pencils are real *equisingular* if there is a bijection between the sets of real singular elements (real base points, real singular points and real components of fibers) preserving the types; the classification of non–degenerate pencils by equisingularity coincides with the projective classification given above. We say that two affine pencils are *affine real equisingular* when their projective closures are real equisingular and the equisingularity maps points at infinity to points at infinity. So the classification by affine equisingularity is discrete and it can be easily deduced from the projective classification. We denote each affine equisingularity type by a roman numeral with a subindex and a superindex; the number and the subindex correspond to the real projective type as above, and the superindex is a shorthand description of the singularities at infinity.

For each affine real non–degenerate equisingularity type we shall give *normal forms*, i.e. equations such that every pencil with the given type can be transformed by an affinity into one and only one of the given equations; this amounts to giving a complete affine classification of pencils of real affine conics. The normal forms involve parameters, which are in fact continuous affine invariants of the classification. It is clear that continuous invariants must appear: consider for instance type I. It is possible to choose a projective reference given by the four base points; then the Plücker coordinates of the line at infinity (modulo permutations of the base points, which form a finite group) are continuous affine invariants.

The determination of the normal forms involves some arbitrary choices. Whenever possible, we have chosen singular fibers as generators of the pencil, and coordinates in which the axes are (or are related to) components of the singular fibers.

5.1. **Pencils of type I.** The geometry of pencils of types **I**, $\mathbf{I}_a$ and $\mathbf{I}_b$ (also known as *general pencils*) is easily described in terms of the complete quadrangle determined by their four base points. (In case $\mathbf{I}_b$ there is only one real degenerate fiber and only one real double point, the other two being complex conjugate).

The classification by affine equisingularity consists in determining the possible incidence positions of the line at infinity ℓ_∞ with respect to the seven vertices of a complete quadrangle, which gives rise to the following five types:

Type	Description
$\mathbf{I}^0$	ℓ_∞ does not go through any singularity.
$\mathbf{I}^b$	ℓ_∞ goes through a base point.
$\mathbf{I}^n$	ℓ_∞ goes through a double point.
$\mathbf{I}^d$	ℓ_∞ goes through two double points (side of the diagonal triangle).
$\mathbf{I}^\ell$	ℓ_∞ is contained in a fiber (side of the complete quadrangle).

It is clear that there can be no more types, since a line containing a base point and a double point of the pencil is a side of the complete quadrangle (hence a component of a singular fiber) and the three diagonal points are not aligned. Three of these types have continuous affine invariants and the other two correspond to a unique affine type each.

Proposition 11. *The following are normal forms for type* **I** *pencils:*

Type	Normal form	Parameters
$\mathbf{I}^0$	$(xy,(x+y-1)(ax+by-ab))$	$0<a<1, -1\le b\le 2-1/a, b\neq 0$
$\mathbf{I}^b$	$(xy,(x+y-1)(y-a))$	$1/2\le a<1.$
$\mathbf{I}^n$	$(xy,(x+y-1)(x+y-a)$	$0<a<1.$
$\mathbf{I}^d$	$(xy,(x+y-1)(x+y+1))$	
$\mathbf{I}^\ell$	$(xy,(x-1)(y-1))$	

Proof. Type $\mathbf{I}^0$. The line at infinity does not go through any of the seven vertices. Choose an ordering for the four base points p_1,p_2,p_3,p_4, and let $l_{ij}=p_i\vee p_j$. Then $p=l_{12}\cap l_{34}$ is one of the double points, and there is a unique affine reference with $p=(0,0)$, $p_1=(1,0)$, $p_3=(0,1)$. This forces $p_2=(b,0)$ and $p_4=(0,a)$ for some real numbers $a,b\neq 0,1$, and the two generators can be taken as stated in the normal form. The pair $(a,b)\in(\mathbb{R}\setminus\{0,1\})^2$ is not uniquely determined by the pencil, but depends on the choice of an ordering. Thus the group of permutations of $\{p_1,p_2,p_3,p_4\}$ acts on $(\mathbb{R}\setminus\{0,1\})^2$ and a fundamental domain of this action has to be taken as parameter space Γ for the normal form. The computation of $\Gamma=\{(a,b)\in\mathbb{R}^2\,|\,0<a<1, b\ge -1, b\le 2-1/a, b\neq 0\}$ is elementary (compute the coordinate change induced by each permutations of the p_i) but somewhat lengthy, and is left to the reader.

Type $\mathbf{I}^b$. If the line at infinity contains one of the base points, the situation is similar to the previous one, except that only three of the points can be permuted. Let p_1 be the point at infinity, choose an ordering for the three affine base points p_2,p_3,p_4 let $l_{ij}=p_i\vee p_j$. Then $p=l_{12}\cap l_{34}$ is one of the double points, and there is a unique affine reference with $p=(0,0)$, $p_2=(1,0)$, $p_3=(0,1)$. This forces $p_4=(0,a)$ for some real number $a\neq 0,1$, and the two generators can be taken as stated in the normal form. The number $a\in\mathbb{R}\setminus\{0,1\}$ is not uniquely determined by the pencil, but depends on the choice of an ordering. Thus the group of permutations of $\{p_2,p_3,p_4\}$ acts on $(\mathbb{R}\setminus\{0,1\})^2$ and again a fundamental domain of this action, whose computation is elementary, has to be taken as parameter space Γ for the normal form. Remark that these normal forms can be obtained as limits of the case $\mathbf{I}^0$ when $a/b \mapsto 0$.

Type $\mathbf{I}^n$. If the line at infinity contains one of the double points, again not all permutations are allowed. Assume as before $\{p_1, p_2, p_3, p_4\}$ are the base points, ordered in such a way that, denoting $l_{ij} = p_i \vee p_j$, $p_\infty = l_{13} \cap l_{24}$ sits at infinity. Choose a reference as before, such that $p = l_{12} \cap l_{34} = (0,0)$, $p_1 = (1,0)$, $p_3 = (0,1)$. This forces $p_2 = (b,0)$ and $p_4 = (0,a)$ for some real numbers $a, b \neq 0, 1$, but since p_∞ must sit at infinity we are forced to have then $b = a$ and the two generators can be taken as stated in the claimed normal form. The fact that this parameter can be taken between 0 and 1 follows again by analyzing the action of the permutations, but now we are restricted to the subgroup formed by the permutations mapping $\{p_1, p_3\}$ either to itself or to $\{p_2, p_4\}$. These are not limits of normal forms of type $\mathbf{I}^0$ even though they are indeed limits of pencils of type $\mathbf{I}^0$.

Type $\mathbf{I}^d$. If the line at infinity contains two of the double points then it is a side of the diagonal triangle. Assume $\{p_1, p_2, p_3, p_4\}$ are the base points, ordered in such a way that, denoting $l_{ij} = p_i \vee p_j$, $p^1_\infty = l_{13} \cap l_{24}$ and $p^2_\infty = l_{14} \cap l_{23}$ sit at infinity. Choose a reference as before, such that $p = l_{12} \cap l_{34} = (0,0)$, $p_1 = (1,0)$, $p_3 = (0,1)$. By the theorem of the complete quadrangle, on each of the lines $p = l_{12}$ and l_{34}, the two base points and the pair formed by p and the point at infinity divide each other harmonically. This forces $p_2 = (-1,0)$ and $p_4 = (0,-1)$ and the two generators can be taken as stated. Observe that a coordinate change induced by one of the permissible permutations, which are exactly those mapping $\{p_1, p_2\}$ either to itself or to $\{p_3, p_4\}$, does not affect the normal form.

Type $\mathbf{I}^\ell$. If the line at infinity is one of the sides of the complete quadrangle, assume $\{p_1, p_2, p_3, p_4\}$ are the base points, ordered in such a way that $p_2, p_4 \in \ell_\infty$. Denote $l_{ij} = p_i \vee p_j$, $p^1_\infty = l_{13} \cap l_{24}$ and choose a reference such that $p = l_{12} \cap l_{34} = (0,0)$, $p_1 = (1,0)$, $p_3 = (0,1)$. Then $p_2 = (\infty, 0)$ and $p_4 = (0, \infty)$ and the two generators can be taken as stated. Observe again that a coordinate change induced by one of the permissible permutations does not affect the normal form. □

Proposition 12. *Any polynomial differential system of type* $\mathbf{I}$ *after an affine change of coordinates can be written as*

Type	Differential system	Parameters
$\mathbf{I}^0$	$\dot{x} = -abx - a(1+b)x^2 + ax^3 - bxy^2$, $\dot{y} = -aby + b(1+a)y^2 + ax^2y - by^3$,	$a \in (0,1)$ $-1 \le b \le 2 - 1/a$, $b \neq 0$
$\mathbf{I}^b$	$\dot{x} = ax - ax^2 - xy^2$, $\dot{y} = -ay + (1+a)y^2 - y^3$,	$a \in [1/2, 1)$
$\mathbf{I}^n$	$\dot{x} = ax - (1+a)x^2 + x^3 - xy^2$, $\dot{y} = -ay + (1+a)y^2 + x^2y - y^3$,	$a \in (0,1)$
$\mathbf{I}^d$	$\dot{x} = -x + x^3 - xy^2$, $\dot{y} = y + x^2y - y^3$,	
$\mathbf{I}^\ell$	$\dot{x} = x - x^2$, $\dot{y} = -y + y^2$.	

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 6.

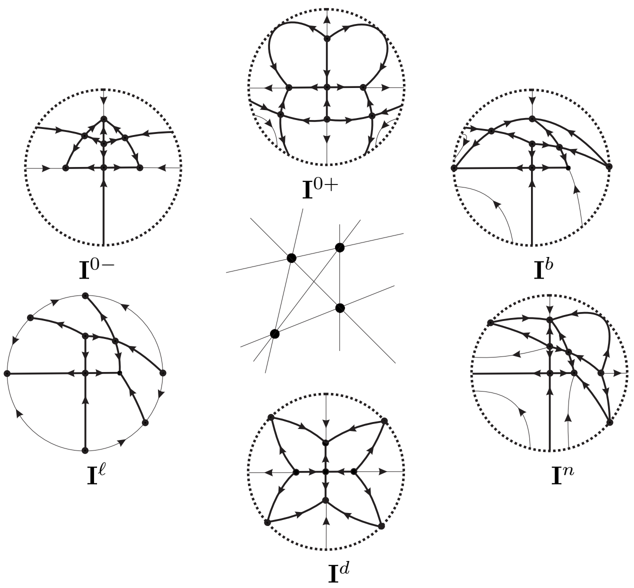


FIGURE 6. Phase portraits of systems corresponding to the case of four different real base points.

Proof. In Proposition 11 we characterized giving the five normal forms $\mathbf{I}^0$, $\mathbf{I}^b$, $\mathbf{I}^n$, $\mathbf{I}^d$ and $\mathbf{I}^\ell$ for each of the type I pencil of conics. Now associating polynomial differential system (5) with each pencil the first part of the proposition follows.

System $\mathbf{I}^0$. This system is affinely equivalent to the system

$$\dot{x} = x(b + (1+b)x + x^2 + ay^2),$$
$$\dot{y} = y(-b + (b-a)y + x^2 + ay^2),$$

for $ab(b-1)(a+b)(a+b^2) \neq 0$ and was studied in [14, p.10]. This system has three saddles and four nodes. It was proved that whenever the quadrilateral formed by the nodes of the system is convex we have a phase portrait topologically equivalent to $\mathbf{I}^{0+}$, if it is concave we get phase portrait equivalent to $\mathbf{I}^{0-}$, see Figure 6.

System $\mathbf{I}^b$. First we analyze the finite singular points. The system has six singular points $M_1 = (0,0)$, $M_2 = (0,1)$, $M_3 = (0,a)$, $M_4 = (1,0)$, $M_5 = (1-a, a)$ and $M_6 = ((a-1)/a, 1)$. We denote by $\Delta(M_i)$ and $T(M_i)$ respectively the determinant and the trace of the Jacobian matrix of system $\mathbf{I}^b$ at the singular point M_i. We get $-\Delta(M_1) = \Delta(M_4) = a^2$, $\Delta(M_2) = -\Delta(M_6) = (a-1)^2$ and $\Delta(M_3) = -\Delta(M_5) = a^2(a-1)^2$, and the trace of the Jacobian matrix at each singular point $T(M_1) = T(M_5) = T(M_6) = 0$, $T(M_2) = 2(a-1)$, $T(M_3) = 2a(1-1)$ and $T(M_4) = -2a$. Using basic results of Section 3.1 we get that there are three saddles M_1, M_5 and M_6, two stable nodes M_3 and M_4 and one unstable node M_2.

Now using results of Section 3.2 we study the stability of infinite singular points. In chart U_1 system $\mathbf{I}^b$ writes

$$\dot{z}_1 = z_1 z_2(a + (a+1)z_1 - 2az_2), \qquad \dot{z}_2 = z_2(az_2 + z_1^2 - az_2^2),$$

so the infinity $z_2 = 0$ is a line of singularities. After removing the factor z_2 from the system we get the system

$$\dot{z}_1 = z_1(a + (a+1)z_1 - 2az_2), \qquad \dot{z}_2 = (az_2 + z_1^2 - az_2^2),$$

having $(0,0)$ as a singular point. Simple calculations for this system yield $\Delta(0,0) = a^2$ and $T(0,0) = 2a$ so $(0,0)$ is an unstable node. In the chart U_2 the reduced system takes the form

$$\dot{z}_1 = -z_1(1 + a + az_1 - 2az_2), \qquad \dot{z}_2 = (z_2 - 1)(az_2 - 1),$$

which has no singular points on $z_2 = 0$. Taking into account all the information on the singular points we get the phase portrait $\mathbf{I}^b$ in Figure 6.

System $\mathbf{I}^n$. The system has six singular points: $M_1 = (0,0)$, $M_2 = (0,1)$, $M_3 = (0,a)$, $M_4 = (a/(1+a), a/(1+a))$, $M_5 = (1,0)$ and $M_6 = (a,0)$. Similar calculations as in the previous case yield $\Delta(M_1) = -a^2$, $\Delta(M_1) = \Delta(M_5) = (a-1)^2$, $\Delta(M_3) = \Delta(M_6) = a^2(a-1)^2$ and $\Delta(M_4) = -(a^2(a-1)^2)/(a+1)^2$. Now we calculate the trace $T(M_i)$ of the Jacobian matrix at each of the singular points: $T(M_1) = T(M_4) = 0$, $T(M_2) = T(M_5) = 2(1-a)$ and $T(M_3) = T(M_6) = 2(1-a)$. So we have two saddles M_1 and M_4 and two stable nodes M_2 and M_6 and two unstable nodes M_3 and M_5.

Now we analyze the stability of infinite singular points. In chart U_1 system $\mathbf{I}^n$ writes

$$\dot{z}_1 = z_1z_2(1 + a + (a+1)z_1 - 2az_2), \qquad \dot{z}_2 = -z_2(1 - (a+1)z_2 - z_1^2 + az_2^2),$$

having a line of singularities at infinity $z_2 = 0$. The rescaled system by z_2 has one singular point at $(-1,0)$. The determinant of the Jacobian matrix is equal to $\Delta(-1,0) = (1+a)^2)$, $T(-1,0) = 2(1+a)$ so the singular point is an unstable node. In chart U_2 the rescaled system writes

$$\dot{z}_1 = -z_1(1 + a + (a+1)z_1 - 2az_2), \qquad \dot{z}_2 = (1 - (a+1)z_2 - z_1^2 + az_2^2),$$

having no singular point at the origin. We get the phase portrait $\mathbf{I}^n$ of Figure 6.

System $\mathbf{I}^d$. The system has five singular points $M_1 = (-1,0)$, $M_2 = (0,0)$, $M_3 = (1,0)$, $M_4 = (0,-1)$ and $M_5 = (0,0)$. Performing similar analysis to the previous case we get one saddle M_2, two unstable nodes M_1 and M_3, and two stable nodes M_4 and M_5.

In chart U_1 system $\mathbf{I}^d$ writes

$$\dot{z}_1 = 2z_1z_2^2, \qquad \dot{z}_2 = z_2(-1 + z_1^2 + z_2^2),$$

having a line of singularities at infinity $z_2 = 0$. After rescaling we get a system having two singular points $(-1,0)$ and $(1,0)$ both saddles. In chart U_2 system $\mathbf{I}^d$, after rescaling by z_2 has no singular point at the origin. We get the phase portrait $\mathbf{I}^d$ of Figure 6.

System $\mathbf{I}^\ell$. There are are four finite singular points $M_1 = (0,0)$, $M_2 = (0,1)$, $M_3 = (1,0)$ and $M_4 = (1,1)$. Two saddles M_1 and M_4, an unstable node M_2 and a stable node M_3.

In chart U_1 system $\mathbf{I}^l$ writes

$$\dot{z}_1 = z_1(1 + z_1 - 2z_2), \qquad \dot{z}_2 = -z_2(z_2 - 1),$$

and there are two singular points: an unstable node $(0,0)$ and a saddle $(-1,0)$. In chart U_2 the system writes

$$\dot{z}_1 = -z_1(1 + z_1 - 2z_2), \qquad \dot{z}_2 = z_2(z_2 - 1),$$

and the origin is a stable node. Finally we get phase portrait $\mathbf{I}^\ell$ of Figure 6.

□

5.2. **Type $\mathbf{I}_a$.** The four distinct base points are now complex and pairwise conjugate. Thus of the three degenerate fibers, one is a pair of lines (each of which goes through a pair of conjugate base points) and two are pairs of complex conjugate lines, with a unique real point. The classification by affine equisingularity consists again in determining the possible incidences of the line at infinity ℓ_∞ (which is real) with respect to the seven vertices of a complete quadrangle, but now the four angles are complex conjugate points. This gives rise to the following six types:

Type	Description
$\mathbf{I}_a^0$	ℓ_∞ does not go through any singularity.
$\mathbf{I}_a^i$	ℓ_∞ goes through an isolated double point.
$\mathbf{I}_a^n$	ℓ_∞ goes through the real double point.
$\mathbf{I}_a^{d1}$	ℓ_∞ goes through the real double point and an isolated double point.
$\mathbf{I}_a^{d2}$	ℓ_∞ goes through both isolated double points.
$\mathbf{I}_a^\ell$	ℓ_∞ is contained in a fiber (a side of the complete quadrangle).

It is clear that there can be no more types, since the three double points are not aligned. Three of these types have continuous affine invariants and the other three correspond to a unique affine type each.

Proposition 13. *The following are normal forms for type $\mathbf{I}_a$ pencils:*

Type	Normal form	Parameters
$\mathbf{I}_a^0$	$(xy, (x-1)^2 - 2a(x-1)(y-1) + (a-b)(y-1)^2)$	$0 \le b < a(1-a)$.
$\mathbf{I}_a^i$	$(xy, (x-1)^2 - 2a(x-1)(y-1) + (y-1)^2)$	$-1 < a < 1$.
$\mathbf{I}_a^n$	$(x(x-1), y^2 + (ax-1)^2)$	$0 < a < 1$.
$\mathbf{I}_a^{d1}$	$(x(x-1), y^2 + 1)$.	
$\mathbf{I}_a^{d2}$	$(xy, (x+y)^2 + 1)$.	
$\mathbf{I}_a^\ell$	$(x, x^2 + y^2 + 1)$.	

Proof. Type $\mathbf{I}_a^0$. Choose an ordering for the four base points p_1, p_2, p_3, p_4 such that by complex conjugation one has $\bar{p}_1 = p_2$, $\bar{p}_3 = p_4$, and let $l_{ij} = p_i \vee p_j$. Then $p = l_{12} \cap l_{34}$ is the real double point. Let $q = l_{13} \cap l_{24}$ be one of the isolated double points, and choose the unique affine reference with $p = (0,0)$, $q = (1,1)$, $l_{12} = \{x = 0\}$ and $l_{34} = \{y = 0\}$. This forces one of the degenerate conics to be defined by a homogeneous quadratic polynomial in $x-1$, $y-1$ with negative a discriminant, and hence the two generators can be taken as stated in the normal form. The parameter b must be strictly less than $a(1-a)$ for the discriminant to be negative. The pair $(a,b) \in \mathbb{R}^2$ is not uniquely determined by the pencil, but depends on the choice of an ordering. Thus the group of permutations of $\{p_1, p_2, p_3, p_4\}$ mapping $\{p_1, p_3\}$ either to itself or to $\{p_2, p_4\}$ acts on $\{(a,b) | b < a(1-a)\}$, and a fundamental domain of this action (which is easily seen to be given by $b \geq 0$) has to be taken as parameter space for the normal form.

Type $\mathbf{I}_a^i$. If the line at infinity contains one of the isolated double points, the situation is similar to the previous one, except that l_{14} and l_{23} are now parallel, which forces the coefficients of $(x-1)^2$ and $(y-1)^2$ to be equal in the second generator, and no permutations are allowed.

Type $\mathbf{I}_a^n$. If the line at infinity contains the real double point, take notations as before and assume again that $\bar{p}_1 = p_2$, $\bar{p}_3 = p_4$, so $p_\infty = l_{12} \cap l_{34}$ sits at infinity. Choose a reference such that l_{12} is $\{x=0\}$, l_{34} is $\{x=1\}$ and the line through the isolated double points is $\{y=0\}$. Thus again the y-coordinates of each pair of base points are not only conjugate but also opposite; multiplication of y by a constant allows us to assume that they are $(0,i)$, $(0,-i)$, $(1,ti)$, $(1,-ti)$ for suitable $t \in \mathbb{R}$. The fact that $a = 1/t+1$ can be taken between 0 and 1 follows again by analyzing the action of the permutations, restricted to the subgroup formed by permutations mapping $\{p_1,p_2\}$ either to itself or to $\{p_3,p_4\}$.

Type $\mathbf{I}_a^{d1}$. If the line at infinity contains the real double point and one isolated double point, then it is a side of the diagonal triangle as in $\mathbf{I}^d$. Thus using coordinates as in the previous case the isolated double point becomes the point at infinity of $\{y=0\}$, and the parameter t is forced to be 1, hence the normal form follows.

Type $\mathbf{I}_a^{d2}$. If the line at infinity contains both isolated double points, then the coordinates of each pair of base points, taking a reference as in case $\mathbf{I}_a^i$, are not only conjugate but also opposite, and by a suitable coordinate change they can be assumed to be $(0,i)$, $(0,-i)$, $(i,0)$, $(-i,0)$ which justifies the normal form follows.

Type $\mathbf{I}_a^\ell$. If the line at infinity is a component of the real degenerate fiber, take coordinates such that p_1 and $\bar{p}_1 = p_2$ lie at infinity, p_3 and $\bar{p}_3 = p_4$ lie on $\{x=0\}$ and $\{y=0\}$ is the line through the two isolated singularities. In this way the coordinates of the four base points are pairwise opposite and conjugate, and multiplying by a constant they are all 0, i and $-i$. Hence the normal form. □

Proposition 14. *Any polynomial differential system of type* $\mathbf{I}_a$ *after an affine change of coordinates can be written as*

Type	Differential system	Parameters
$\mathbf{I}_a^0$	$\dot{x} = (1-a-b)x + 2(a-1)x^2 + x^3 + (b-a)xy^2$ $\dot{y} = (-1+a+b)y + x^2y - 2by^2 + (b-a)y^3$	$b < a(1-a)$ $b \geq 0$
$\mathbf{I}_a^i$	$\dot{x} = 2(1-a)x + 2(a-1)x^2 + x^3 - xy^2$ $\dot{y} = 2(a-1)y + x^2y + 2(1-a)y^2 - y^3$	$a \in (-1,1)$
$\mathbf{I}_a^n$	$\dot{x} = 2xy - 2x^2y$ $\dot{y} = 1 - 2x + a(2-a)x^2 + y^2 - 2xy^2$	$a \in (-1,1)$
$\mathbf{I}_a^{d_1}$	$\dot{x} = 2xy - 2x^2y$ $\dot{y} = 1 - 2x + y^2 - 2xy^2$	
$\mathbf{I}_a^{d_2}$	$\dot{x} = 2x^2 - 2x^3 + 2xy - 2x^2y$ $\dot{y} = 1 - 2x - 2x^2y + y^2 - 2xy^2$	
$\mathbf{I}_a^\ell$	$\dot{x} = -2xy$ $\dot{y} = -1 + x^2 - y^2$	

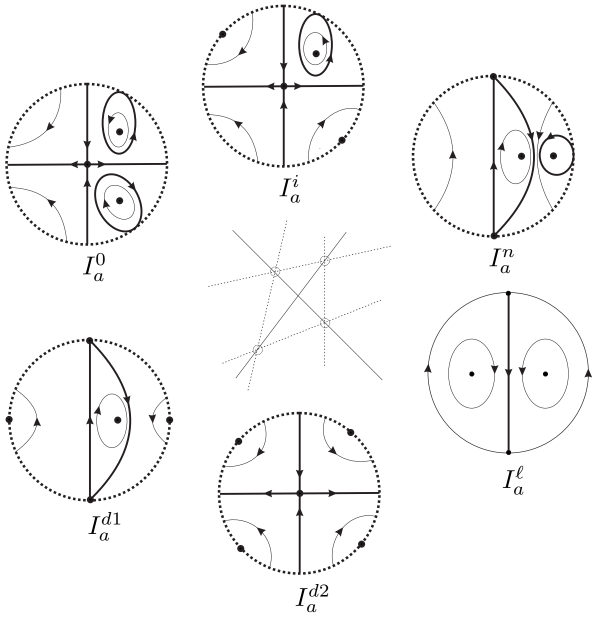


FIGURE 7. Phase portraits of systems corresponding to the case of four non–real base points.

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 7.

Proof. The pencils of conics of type I_a were characterized in Proposition 13. There are six affinely non–equivalent types of these pencils $\mathbf{I}_a^0$, $\mathbf{I}_a^i$, $\mathbf{I}_a^n$, $\mathbf{I}_a^{d_1}$, $\mathbf{I}_a^{d_2}$, and $\mathbf{I}_a^\ell$. Associating a polynomial differential system (5) to each of these types of pencil the first part of the proposition follows.

The phase portraits for each of these systems is determined in the same way as in Proposition 12. □

5.3. **Type $\mathbf{I}_b$.** There are now two real and two non real base points. Only two sides of the complete quadrangle (giving rise to the singular fiber) and only one vertex of the self–polar triangle (and its polar line) are real. The classification by affine equisingularity consists of determining the possible incidences of the line at infinity ℓ_∞ with respect to the real singular elements. This gives rise to the following five types:

Type	Description
$\mathbf{I}_b^0$	ℓ_∞ does not go through any singularity.
$\mathbf{I}_b^b$	ℓ_∞ goes through a real base point.
$\mathbf{I}_b^n$	ℓ_∞ goes through the double point.
$\mathbf{I}_b^\ell$	ℓ_∞ goes through the two real base points.
$\mathbf{I}_b^{\ell c}$	ℓ_∞ goes through the two non–real base points.

The differences with the previous cases are that there are only three real singular points (so there are clearly no more cases) and that the two components of the special fiber are intrinsically distinct, depending on whether they contain real base points or not. Three of these types have continuous affine invariants and the other two correspond to a unique affine type each.

Proposition 15. *The following are normal forms for type* $\mathbf{I}_b$ *pencils:*

Type	Normal form	Parameters
$\mathbf{I}_b^0$	$(xy, (x-1)(x-b)+by(y-a)),$	$-2 < a < 2, 0 < \lvert b\rvert \le 1, b \ne 1.$
$\mathbf{I}_b^b$	$(xy, x-1-y(y-a)),$	$-2 < a < 2.$
$\mathbf{I}_b^n$	$(x(x-1), y^2+ax-1),$	$a > 1.$
$\mathbf{I}_b^c$	$(xy, 1-x^2+y^2).$	
$\mathbf{I}_b^\ell$	$(x, x^2-y^2+1).$	
$\mathbf{I}_b^{\ell c}$	$(x, x^2+y^2-1).$	

Proof. Type $\mathbf{I}_b^0$. Choose an ordering for the four base points p_1, p_2, p_3, p_4 such that by complex conjugation one has $\bar{p}_1 = p_2$, and let $l_{ij} = p_i \vee p_j$. Then $p = l_{12} \cap l_{34}$ is the real double point. Choose the unique affine reference with $p = (0,0)$, $p_3 = (1,0)$, and $p_1 = (0,\alpha)$ with $\lvert\alpha\rvert = 1$. Then $xy = 0$ is the degenerate fiber and $(x-1)(x-b)+by(y-a)$ belongs to the pencil for $p_4 = (b,0)$ and $a = \alpha+\bar{\alpha} \in (-2,2)$. Clearly $b \in \mathbb{R}^2$ is not uniquely determined by the pencil, but depends on the ordering. The only admissible permutation which would alter the normal form is the transposition of p_3 and p_4, so b can be (uniquely) taken with $0 < \lvert b\rvert < 1$ (i.e., the fundamental domain of the action is $(-1,1) \setminus \{0\}$) and the normal form is as claimed.

Type $\mathbf{I}_b^b$. If the line at infinity contains one of the base points (which we may assume is p_4), the situation is similar to the previous one, except that there is no parameter b and no permutations are allowed.

Type $\mathbf{I}_b^n$. If the line at infinity contains the double point then, setting as before $\bar{p}_1 = p_2$, we choose the affine reference which has $p_1 = (1, ti)$, $t \in \mathbb{R}^*$, $p_3 = (0,1)$, $p_4 = (0,-1)$. Note that the x axis is then the polar of the double point and the normal form (with $a = 1+t^2$) is unaffected by permissible permutations.

Type $\mathbf{I}_b^\ell$. The line at infinity is a component of the degenerate fiber; hence one generator of the pencil will be a line (through the two non–real base points); assume it is $\{x = 0\}$. Choose the polar of the double point (which lies at infinity) as the other axis. Again multiplying x and y by suitable constants the normal form follows.

Type $\mathbf{I}^{\ell c}$. Proceed as in the previous case. □

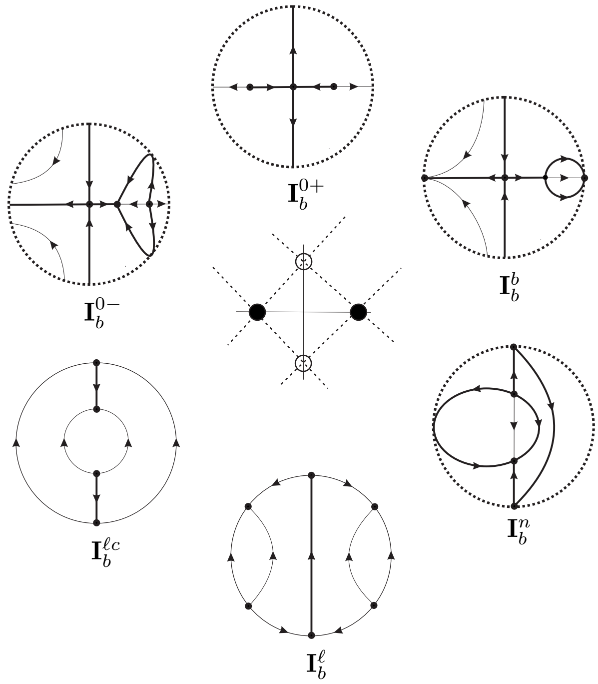


FIGURE 8. Phase portraits of systems corresponding to the case of two non–real and two real base points.

Proposition 16. *Any polynomial differential system of type* $\mathbf{I}_b$ *after an affine change of coordinates can be written as*

Type	Differential system	Parameters
$\mathbf{I}_b^0$	$\dot{x} = bx - (1+b)x^2 + x^3 - bxy^2,$ $\dot{y} = -by + x^2y + aby^2 - by^3,$	$a \in (-2,2)$ $\lvert b\rvert \in (0,1]$
$\mathbf{I}_b^b$	$\dot{x} = -x + x^2 + xy^2,$ $\dot{y} = y - ay^2 + y^3,$	$a \in (-2,2)$
$\mathbf{I}_b^n$	$\dot{x} = 2xy - 2x^2y,$ $\dot{y} = -1 + 2x - ax^2 + y^2 - 2xy^2,$	$a \in (1,\infty)$
$\mathbf{I}_b^\ell$	$\dot{x} = 2xy, \quad \dot{y} = -1 + x^2 + y^2,$	
$\mathbf{I}_b^{\ell c}$	$\dot{x} = -2xy, \quad \dot{y} = 1 + x^2 - y^2.$	

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 8.

5.4. **Type II.** These pencils have two degenerate fibers, both are pairs of lines. One has the double point at the double base point and the second goes simply through it. Thus we are dealing with the incidences of the line at infinity with respect to four points, three of which are aligned, and the lines joining them. This gives rise to the following seven types:

Type	Description
$\mathbf{II}^0$	ℓ_∞ does not go through any singularity.
$\mathbf{II}^b$	ℓ_∞ goes through a simple base point.
$\mathbf{II}^{b^2}$	ℓ_∞ goes through the double base point.
$\mathbf{II}^n$	ℓ_∞ goes through the non–base double point.
$\mathbf{II}^\ell$	ℓ_∞ goes through both simple base points.
$\mathbf{II}^{\ell^2}$	ℓ_∞ goes through both double points.
$\mathbf{II}^{\ell^{12}}$	ℓ_∞ goes through the double and a simple base point.

Two of these types have a continuous affine invariant and the other five correspond to a unique affine type each.

Proposition 17. *The following are normal forms for type* **II** *pencils:*

Type	Normal form	Parameters
$\mathbf{II}^0$	$(xy, (x+y-1)(x+ay)),$	$0 < \lvert a\rvert \leq 1, a \neq 1.$
$\mathbf{II}^b$	$(xy, (x-1)(x+y)).$	
$\mathbf{II}^{b^2}$	$(xy, (x-1)(x-a)),$	$0 < \lvert a\rvert \leq 1, a \neq 1.$
$\mathbf{II}^n$	$(xy, (x+y-1)(x+y)).$	
$\mathbf{II}^\ell$	$(xy, x+y).$	
$\mathbf{II}^{\ell^2}$	$(x, y(y-1)).$	
$\mathbf{II}^{\ell^{12}}$	$(xy, x-1).$	

Proof. Type $\mathbf{II}^0$. Let p_1 be the double base point and choose an ordering for the simple base points p_2 and p_3. Choose the unique affine reference given by these three points, so that $p_1 = (0,0)$, $p_2 = (1,0)$ and $p_3 = (0,1)$. Then the two degenerate fibers must have equations as in the stated normal form. The parameter a might take any real value different from 0 and 1; quotienting by the permutation of p_2 and p_3 one gets the stated parameter space.

Type $\mathbf{II}^b$. If the line at infinity contains one of the simple base points (which we may assume is p_2), the situation is similar to the previous one, except that p_2 being the point at infinity on $\{y = 0\}$, the affine reference is not unique; multiplying x by a suitable constant one can assume that the second degenerate fiber is $\{(x-1)(x+y) = 0\}$.

Type $\mathbf{II}^{b^2}$. If the line at infinity contains the double base point p_1 we pick a different reference. Let p be the non–base double point, and choose a reference with $p = (0,0)$, p_1 at infinity on $\{x = 0\}$, $p_2 = (1,0)$. Then $p_3 = (a,0)$ and the normal form follows by considering the action of permutations of p_2 and p_3.

Type $\mathbf{II}^n$. If the line at infinity contains the non–base double point then, setting coordinates as in case $\mathbf{II}^0$ the normal form follows. Note that it is unaffected by permutations.

Type $\mathbf{II}^\ell$. Choose an affine reference so that $p_1 = (0,0)$ is the double base point, the coordinate lines go through the two base points at infinity, and $(1,-1)$ belongs to the second degenerate fiber.

Type $\mathbf{II}^{\ell^2}$. Choose an affine reference so that the two simple base points are $p_2 = (0,0)$ and $p_3 = (0,1)$, and the double base point p_1 is the point at infinity of $\{y = 0\}$.

Type $\mathbf{II}^{\ell^{12}}$. Choose an affine reference so that the non–base double point is $p = (0,0)$ and the unique affine base point is $p_2 = (1,0)$. Then necessarily the other simple base point is the point at infinity of $\{y = 0\}$ and the double base point is the point at infinity of $\{x = 0\}$. The normal form follows. □

Proposition 18. *Any polynomial differential system of type* **II** *after an affine change of coordinates can be written as*

Type	Differential system		Parameters
$\mathbf{II}^0$	$\dot{x} = -x^2 + x^3 - axy^2,$	$\dot{y} = x^2y + ay^2 - ay^3,$	$\|a\| \in (0,1]$
$\mathbf{II}^b$	$\dot{x} = -x^2 + x^3,$	$\dot{y} = x^2y + y^2,$	
$\mathbf{II}^{b^2}$	$\dot{x} = ax - (1+a)x^2 + x^3,$	$\dot{y} = -ay + x^2y$	$\|a\| \in (0,1]$
$\mathbf{II}^n$	$\dot{x} = -x^2 + x^3 - xy^2,$	$\dot{y} = x^2y + y^2 - y^3,$	
$\mathbf{II}^\ell$	$\dot{x} = x^2,$	$\dot{y} = -y^2,$	
$\mathbf{II}^{\ell^2}$	$\dot{x} = x - 2xy,$	$\dot{y} = y - y^2$	
$\mathbf{II}^{\ell^{12}}$	$\dot{x} = -x + x^2,$	$\dot{y} = y.$	

9.

5.5. **Type $\mathbf{II}_a$.** These pencils have two degenerate fibers, one is a pair of complex conjugated lines with the isolated double point at the double base point and the other is a pair of lines (with a real double point) one of which goes simply through it. Thus we are now dealing with the incidences of the line at infinity with respect to two lines and a point (the double base point) on one of them. This gives rise to the following five types:

Type	Description
$\mathbf{II}_a^0$	ℓ_∞ does not go through any singularity.
$\mathbf{II}_a^{b^2}$	ℓ_∞ goes through the double base point.
$\mathbf{II}_a^n$	ℓ_∞ goes through the double point.
$\mathbf{II}_a^\ell$	ℓ_∞ goes through the conjugate simple base points.
$\mathbf{II}_a^{\ell^2}$	ℓ_∞ goes through both double points.

These cases are the same as in type **II**, except that $\mathbf{II}_a^{b^2}$ and $\mathbf{II}_a^{\ell^{12}}$ can not exist because the simple base points and lines containing only one simple base point are not real. One of these types has a continuous affine invariant and the other four correspond to a unique affine type each.

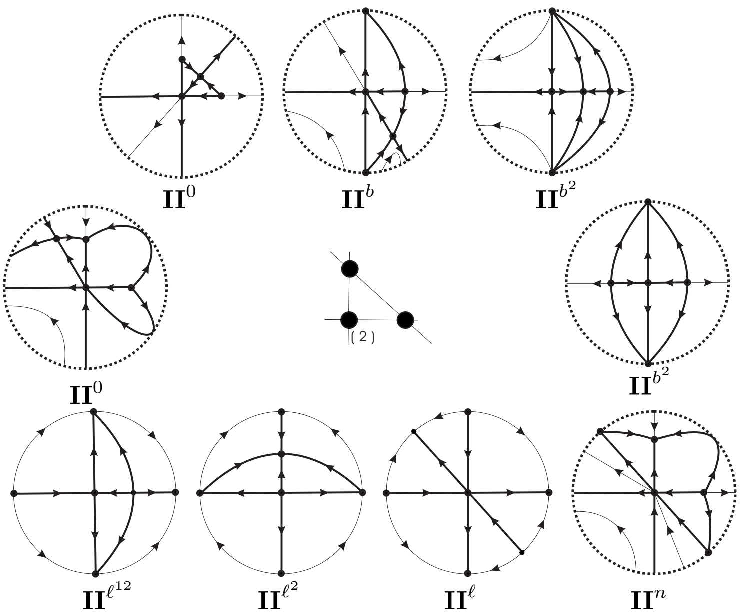


FIGURE 9. Phase portraits of systems corresponding to the case of three real different base points: two simple and one double.

Proposition 19. *The following are normal forms for type* $\mathbf{II}_a$ *pencils:*

Type	Normal form	Parameters
$\mathbf{II}_a^0$	$(xy, x^2 + ax(y-1) + (y-1)^2)$,	$\lvert a\rvert < 2$.
$\mathbf{II}_a^{b^2}$	$(xy, x^2 + 1)$.	
$\mathbf{II}_a^n$	$(x(x-1), x^2 + y^2)$.	
$\mathbf{II}_a^\ell$	$(x, x^2 + y^2)$.	
$\mathbf{II}_a^{\ell^2}$	$(x, y^2 + 1)$.	

Proof. Type $\mathbf{II}_a^0$. Choose an affine reference where the double point sits at $(0,0)$, the isolated double point at $(0,1)$ and the real pair of lines in the pencil is $\{xy = 0\}$. Then the two complex conjugate base points are $(\alpha, 0)$ and $(\bar{\alpha}, 0)$, where we may further assume that $|\alpha| = 1$. The normal form follows, with $a = \alpha + \bar{\alpha} = 2\,\mathrm{Re}(\alpha)$.

Type $\mathbf{II}_a^{b^2}$. If the line at infinity contains the double base point, the situation is very similar to the previous one, except that the two conjugate lines of the second degenerate fiber are parallel to the y axis instead of meeting at (0,1).

Type $\mathbf{II}_a^n$. Now it is the real double point which sits at infinity. Choose an affine reference where the isolated double point sits at $(0,0)$, the direction of the real double point is that of the y axis, the x axis is the polar of the real double point and the non–real base points are $(1,\alpha)$ and $(1,\bar{\alpha})$. By the choice of the x axis, $\bar{\alpha} = -\alpha$, and hence modulo an affine coordinate change we may assume $\alpha = i$, which justifies the normal form.

Type $\mathbf{II}_a^\ell$. Similar to the previous case, put the origin at the double base point and the simple base points on the lines $y = \pm ix$.

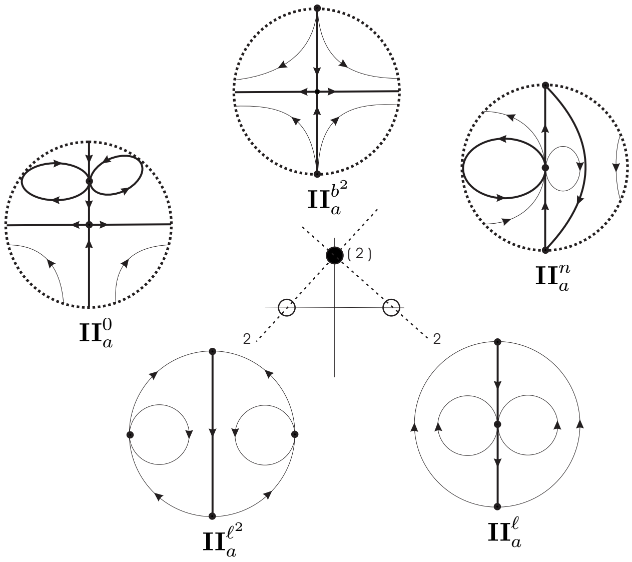


FIGURE 10. Phase portraits of systems corresponding to the case of one double real and two non–real base points. points.

Type $\mathbf{II}_a^{\ell^2}$. Choose a reference where the y axis contains the simple base points and the origin is its mean point, and furthermore the double base point lies on the x axis. Then modulo an affine coordinate change we may assume the simple base points are $(i, 0)$ and $(-i, 0)$, and the normal form follows. □

Proposition 20. *Any polynomial differential system of type* $\mathbf{II}_a$ *after an affine change of coordinates can be written as*

Type	Differential system		Parameters
$\mathbf{II}_a^0$	$\dot{x} = x - ax^2 + x^3 - xy^2,$	$\dot{y} = -y + x^2y + 2y^2 - y^3,$	$\lvert a\rvert \in [0, 2)$
$\mathbf{II}_a^{b^2}$	$\dot{x} = x + x^3,$	$\dot{y} = -y + x^2y,$	
$\mathbf{II}_a^n$	$\dot{x} = 2xy - 2x^2y,$	$\dot{y} = -x^2 + y^2 - 2xy^2,$	
$\mathbf{II}_a^{\ell}$	$\dot{x} = -2xy,$	$\dot{y} = x^2 - y^2,$	
$\mathbf{II}_a^{\ell^2}$	$\dot{x} = -2xy,$	$\dot{y} = -1 - y^2.$	

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 10.

5.6. **Type III.** These pencils have two degenerate fibers: the double line through the two base points, and a pair of lines, each of which goes through exactly one base point. All other fibers are non–degenerate, and are tangent to the pair of lines at the base points. Thus we are dealing with the relative positions of the line at infinity with respect to three nonconcurrent lines. This gives rise to the following five equisingularity types:

Type	Description
$\mathbf{III}^0$	ℓ_∞ does not go through any base point nor the double point.
$\mathbf{III}^{b^2}$	ℓ_∞ goes through one (double) base point.
$\mathbf{III}^n$	ℓ_∞ goes through the double point.
$\mathbf{III}^{\ell^2}$	ℓ_∞ goes through the double point and a base point.
$\mathbf{III}^{\ell^{22}}$	ℓ_∞ is the double line.

Each of these types corresponds to a unique affine type.

Proposition 21. *The following are normal forms for type* **III** *pencils:*

Type	Normal form
$\mathbf{III}^0$	$(xy, (x+y-1)^2)$.
$\mathbf{III}^{b^2}$	$(xy, (x-1)^2)$.
$\mathbf{III}^n$	$(x(x-1), y^2)$.
$\mathbf{III}^{\ell^2}$	(x, y^2).
$\mathbf{III}^{\ell^{22}}$	$(xy, 1)$.

Proof. Type $\mathbf{III}^0$. Generically the line at infinity does not meet any of the intersection points of the three given lines. We can thus choose an affine reference given by the three points of intersection, in any desired order. We choose $(0,0)$ to be the double point of the pair of lines; the normal form follows.

Type $\mathbf{III}^{b^2}$. If the line at infinity contains one of the base points, the situation is very similar to the previous one, except that the double line is parallel to one of the axes.

Type $\mathbf{III}^n$. Similar to the above, the double point sits at infinity hence the pair of lines is a pair of parallel lines.

Type $\mathbf{III}^{\ell^2}$. The line at infinity is one of the pair of lines.

Type $\mathbf{III}^{\ell^{22}}$. The double line coincides with the line at infinity. □

Proposition 22. *Any polynomial differential system of type* **III** *after an affine change of coordinates can be written as*

Type	Differential system	
$\mathbf{III}^0$	$\dot{x} = x(-1+x-y)$,	$\dot{y} = -y(-1-x+y)$,
$\mathbf{III}^{b^2}$	$\dot{x} = x(x-1)$,	$\dot{y} = y(x+1)$
$\mathbf{III}^n$	$\dot{x} = 2x(1-x)$,	$\dot{y} = y(1-2x)$,
$\mathbf{III}^{\ell^2}$	$\dot{x} = -2x$,	$\dot{y} = -y$,
$\mathbf{III}^{\ell^{22}}$	$\dot{x} = -x$,	$\dot{y} = y$.

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 11.

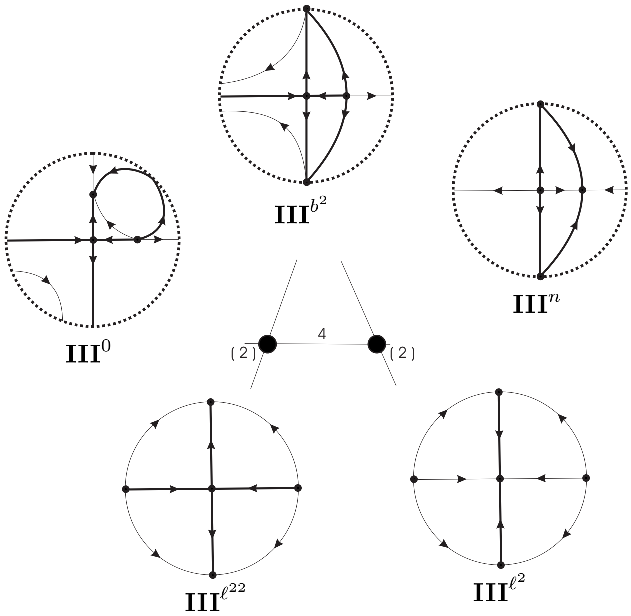


FIGURE 11. Phase portraits of systems corresponding to the case of two real double base points.

5.7. **Type $\mathbf{III}_a$.** These pencils have two degenerate fibers: the double of the line through the two base points, and a pair of complex conjugated lines. All other fibers are non–degenerate. Thus we are dealing with the relative positions of the line at infinity with respect to a line and a point not on this line. This gives rise to the following three equisingularity types:

Type	Description
$\mathbf{III}^0$	ℓ_∞ does not go through any base point nor the double point.
$\mathbf{III}^i$	ℓ_∞ goes through the isolated double point.
$\mathbf{III}^{\ell^{22}}$	ℓ_∞ is the double line.

Each of these types corresponds to a unique affine type, in a similar fashion to Type III.

Proposition 23. *The following are normal forms for type* $\mathbf{III}_a$ *pencils:*

Type	Normal form
$\mathbf{III}_a^0$	$(x^2+y^2,(x+y-1)^2)$.
$\mathbf{III}_a^i$	(x^2,y^2+1).
$\mathbf{III}_a^{\ell^{22}}$	$(x^2+y^2,1)$.

Proof. Type $\mathbf{III}_a^0$. Generically the line at infinity does not go through the isolated double point and is not a double line. We can thus choose an affine reference in

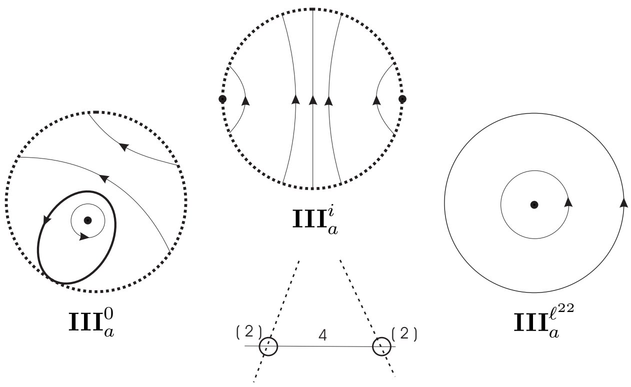


FIGURE 12. Phase portraits of systems corresponding to the case of two non–real double base points.

which $p = (0,0)$ is the isolated double point and the two complex conjugate lines are $x = iy$ and $x = -iy$. With a suitable coordinate change we may further assume that the double line cuts the axes at the points $(0,1)$ and $(1,0)$. The normal form follows.

Type $\mathbf{III}_a^i$. Similar to the above, the isolated double point sits at infinity.

Type $\mathbf{III}_a^\ell$. Similar to type $\mathbf{III}_a^0$, the double line now coincides with the line at infinity. □

Proposition 24. *Any polynomial differential system of type* $\mathbf{III}_a$ *after an affine change of coordinates can be written as*

Type	Differential system	
$\mathbf{III}_a^0$	$\dot{x} = y + x^2 - xy,$	$\dot{y} = -x + xy - y^2,$
$\mathbf{III}_a^i$	$\dot{x} = 2xy,$	$\dot{y} = 2 + 2y^2,$
$\mathbf{III}_a^{\ell^{22}}$	$\dot{x} = -2y,$	$\dot{y} = 2x.$

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 12.

5.8. **Type IV.** These pencils have a unique degenerate fiber, which is a pair of lines. The intersection point of both lines is the triple base point, the remaining base point belongs to one of the lines. All other fibers are non–degenerate, tangent to the other line at the intersection point and with fixed curvature there. Thus we are dealing with the relative positions of the line at infinity with respect to two lines and a point on one of the lines. This gives rise to the following five equisingularity types:

Type	Description
$\mathbf{IV}^0$	ℓ_∞ does not go through any singularity.
$\mathbf{IV}^b$	ℓ_∞ goes through the simple base point.
$\mathbf{IV}^{b^3}$	ℓ_∞ goes through the triple base point.
$\mathbf{IV}^{\ell^3}$	ℓ_∞ is the singular component not through the simple base point.
$\mathbf{IV}^{\ell^{13}}$	ℓ_∞ goes through both base points.

One of these types corresponds to two affine types, and each other four correspond to a unique affine type each.

Proposition 25. *The following are normal forms for type* **IV** *pencils:*

Type	Normal form
$\mathbf{IV}^0$	$(xy, y - y^2 \pm x^2)$.
$\mathbf{IV}^b$	$(xy, y - x^2)$.
$\mathbf{IV}^{b^3}$	$(x(x-1), xy - 1)$.
$\mathbf{IV}^{\ell^3}$	$(x, y - x^2)$.
$\mathbf{IV}^{\ell^{13}}$	$(x, xy - 1)$.

Proof. Type $\mathbf{IV}^0$. Generically the line at infinity does not go through any of the base points. We can thus choose an affine reference in which the nodal triple base point is $(0,0)$, the simple base point is $(0,1)$ and the singular fiber is $xy = 0$. Another generator of the pencil will be of the form $y + ax^2 + bxy - y^2$, because it is tangent to $y = 0$ at $(0,0)$ and goes through the base point $(0,1)$. By combining with the first generator we may assume that $b = 0$, and multiplying x by a real constant we may assume that $|a| = 1$. Hence every affine pencil of projective type IV with both base points in the affine part can be affinely transformed into one of the two normal forms displayed.

These two normal forms are not affinely equivalent, thus follows from the (affine) invariance of the lines $x = 0$, $y = 0$ and the point $(0,1)$; an affinity sending one to the other would have to be of the form $(x,y) \mapsto (ax, y)$ and this forces $a^2 = -1$ which is impossible

Type $\mathbf{IV}^b$. If the line at infinity contains the simple base point, the situation is quite similar to the previous one, to the effect that $xy = 0$ can be taken as the equation of one of the generators, and the other generator will be of the form $y + ax^2 + bxy$, because it is tangent to $y = 0$ at $(0,0)$ and goes through the point at infinity of $x = 0$. Again by combining with the first generator we may assume that $b = 0$, and by multiplying x by a real constant we may assume that $|a| = 1$. Finally, by changing the sign of y we may assume that $a = 1$, hence the normal form is obtained.

Type $\mathbf{IV}^{b^3}$. If the nodal triple base point sits at infinity, but the degenerate fiber does not contain the line at infinity, then we may assume that this fiber is $\{x(x-1) = 0\}$. Then all other fibers of the pencil have the vertical asymptote $x = 0$ and go through a fixed point of $x = 1$, which we may assume is $(1,1)$. Thus for the second generator of the pencil we can assume that has the form $xL(x,y) - 1$, where

$L(x, y)$ is a linear affine function with $L(1, 1) = 1$ and effectively dependent on y (otherwise it would also be degenerate). Now it is obvious that an affine change of coordinates allows $L = y$ and thus the normal form is as stated.

Type $\mathbf{IV}^{\ell^3}$. The line at infinity is the component of the special fiber which does not contain the simple base point. We choose an affine reference so that the other component is $x = 0$ and the simple base point is the origin. All non–degenerate fibers will be parabolas with vertical axis going through the origin. The second generator must therefore be of the form $ay + bx - x^2$, with $a \neq 0$; by an affine change of coordinates we can set $a = 1, b = 0$, hence the normal form follows.

Type $\mathbf{IV}^{\ell^{13}}$. The line at infinity is the component of the special fiber through the simple base point. We choose an affine reference so that the other component is $x = 0$ and the simple base point belongs to the lines y =constant. All non–degenerate fibers will have $x = 0$ and a horizontal line as asymptotes. The second generator must therefore be of the form $x(ay+b)-1$, with $a \neq 0$; by an affine change of coordinates we can set $a = 1, b = 0$, hence the normal form is obtained. □

Proposition 26. *Any polynomial differential system of type* **IV** *after an affine change of coordinates can be written as*

Type	Differential system		Parameters
$\mathbf{IV}^0$	$\dot{x} = px^3 + xy^2$,	$\dot{y} = px^2y - y^2 + y^3$,	$p = \pm 1$
$\mathbf{IV}^b$	$\dot{x} = -x^3$,	$\dot{y} = -x^2y - y^2$,	
$\mathbf{IV}^{b^3}$	$\dot{x} = x^2 - x^3$,	$\dot{y} = -1 + 2x - x^2y$,	
$\mathbf{IV}^{\ell^3}$	$\dot{x} = -x$,	$\dot{y} = -x^2 - y$,	
$\mathbf{IV}^{\ell^{13}}$	$\dot{x} = -x^2$,	$\dot{y} = 1$.	

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 13.

5.9. **Type V.** These pencils have a unique degenerate fiber which is a double line $2L$ containing the (quadruple) base point p. All other fibers are non–degenerate, tangent to L at p (they intersect each other with multiplicity four there). Thus we are dealing with the incidences of the line at infinity with respect to a line and a point on it. This gives rise to the following three equisingularity types:

Type	Description
$\mathbf{V}^0$	ℓ_∞ does not go through the base point.
$\mathbf{V}^{b^4}$	ℓ_∞ goes through the base point.
$\mathbf{V}^{\ell^4}$	ℓ_∞ is the double line.

Each these types corresponds to a unique affine type.

Proposition 27. *The following are normal forms for type* **V** *pencils:*

Type	Normal form
$\mathbf{V}^0$	$(x^2, x - y^2)$.
$\mathbf{V}^b$	$(x^2, xy - 1)$.
$\mathbf{V}^\infty$	$(y - x^2, 1)$.

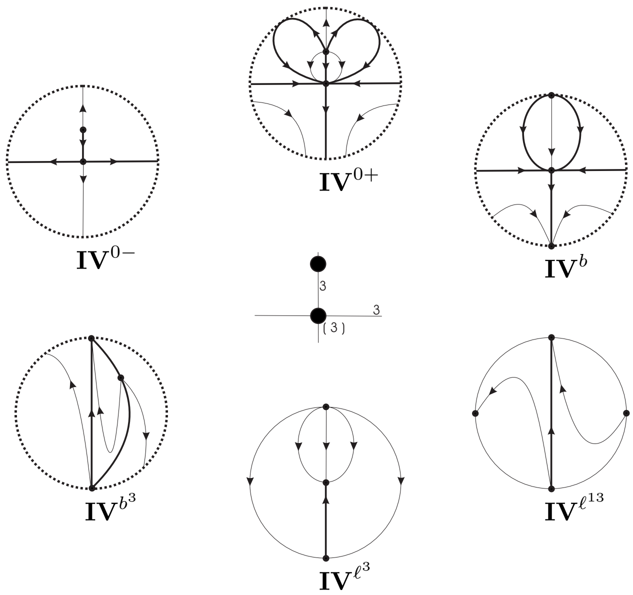


FIGURE 13. Phase portraits of systems corresponding to the case of one triple and one simple base point.

Proof. Type $\mathbf{V}^0$. Generically the line at infinity does not go through the base point. We can thus choose an affine reference with origin at p and such that the double line is $\{x = 0\}$. Choosing x^2 as the first generator, every other curve in the pencil must be non–degenerate and have equation $x + F_2(x, y) = 0$ where F_2 is a homogeneous form of degree 2 not divisible by x. By adding a suitable scalar multiple of the first generator, we may assume that $-F_2$ is the square of a linear form distinct from x. Then an affine change of coordinates allows $F_2 = -y^2$ as claimed.

Type $\mathbf{V}^{b^4}$. If the line at infinity contains the base point but is not a double line, the situation is quite similar to the previous one, to the effect that x^2 can be taken as equation of one of the generators; the other fibers must now have $x = 0$ as an asymptote, so the other generator will be of the form $x(ay + bx) - 1$. Again by combining with the first generator we may assume that $b = 0$, and by multiplying x or y by a real constant we may assume that $a = 1$, hence the normal form is obtained.

Type $\mathbf{V}^{\ell^4}$. If the double line is the line at infinity, then one of the generators can be taken to be 1. Assume that the base point belongs to the line $x = 0$. All non–degenerate fibers must be tangent to infinity there, i.e. must be parabolas with vertical axis. So the other generator will be of the form $ay + bx + c - x^2$ with $a \neq 0$. Again by combining with the first generator we may assume that $c = 0$, and by an affine change of coordinates we may assume that $a = 1, b = 0$, hence the normal form follows. □

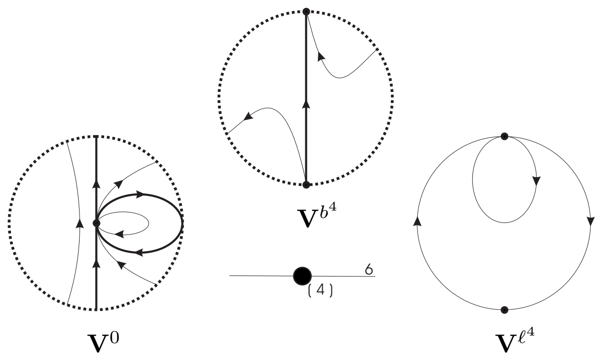


FIGURE 14. Phase portraits of systems corresponding to the case of one quadruple base point.

Proposition 28. *Any polynomial differential system of type* $\mathbf{V}$ *after an affine change of coordinates can be written as*

Type	Differential system
$\mathbf{V}^0$	$\dot{x} = 2xy, \quad \dot{y} = -x + 2y^2,$
$\mathbf{V}^{b^4}$	$\dot{x} = -x^2, \quad \dot{y} = 2 - xy,$
$\mathbf{V}^{\ell^4}$	$\dot{x} = -1, \quad \dot{y} = -2x.$

Moreover, any polynomial differential system of this type is topologically equivalent to one of the phase portraits of Figure 14.

REFERENCES

[1] A.F. Andreev, *Investigation of the behaviour of the integral curves of a system of two different equations in the neighborhood of a singular point*, Trans. Amer. Math. Soc. **8** (1958), 183–207.

[2] A.A. Andronov, Leontovich E.A., I.I. Gordon, and A.L. Maier, *Qualitative theory of second–order dynamical systems*, Wiley, New York, 1973.

[3] J.C. Artés, J. Llibre, and N. Vulpe, *Quadratic systems with a rational first integral of degree 2: a complete classification in the coefficient space* $\mathbb{R}^{12}$, Rend. Circ. Mat. Palermo (2) **56** (2002), 417–444.

[4] L. Cairó and J. Llibre, *Phase portraits of cubic polynomial vector fields of Lotka-Volterra type having a rational first integral of degree 2*, J. Phys. A **40** (2007), 6329–6348.

[5] L. Cairó and J. Llibre, *Phase portraits of quadratic polynomial vector fields having a rational first integral of degree 2*, Nonlinear Anal., Ser. A: Theory Methods **67** (2007), 327–348.

[6] E. Casas-Alvero, *Analytic projective geometry*, preliminary version, 2008.

[7] J. Chavarriga, H. Giacomini, J. Giné, and J. Llibre, *Darboux integrability and the inverse integrating factor*, J. Differential Equations **194** (2003), 116–139.

[8] C. Christopher and J. Llibre, *Integrability via invariant algebraic curves for planar polynomial differential systems*, Ann. Differential Equations **16** (2000), 5–19.

[9] F. Dumortier, J. Llibre, and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, Berlin, 2006.

[10] E.A. Gonzalez, *Generic properties of polynomial vector fields at infinity*, Trans. Amer. Math. Soc. **143** (1969), 201–222.

[11] J.P. Jouanolou, *Equations de pfaff algébriques*, Lectures Notes in Mathematics, Springer, Berlin, 1979.

[12] S. Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[13] H. Levy, *Projective and related geometries*, second ed., The Macmillan Co., New York, 1967.

[14] J. Llibre, A. Mahdi, and N. Vulpe, *Phase portraits and invariant straights lines of cubic polynomial vector fields having a quadratic rational first intergral*, preprint 2008.

[15] L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Math. Soc. **76** (1954), 127–148.

[16] D.A. Neumann, *Classification of continuous flows on* 2*-manifolds*, Proc. Amer. Math. Soc. **48** (1975), 73–81.

[17] M.M. Peixoto, *On the classification of flows on* 2*-manifolds*, Proc. Sympos. Univ. Bahia (Salvador 1971), 1973, pp. 389–419.

[18] H. Poincaré, *Sur lintégration des équations différentielles du premier ordre et du premier degré i and ii*, Rend. Circ. Mat. Palermo **5** (1891), 161–191.

[19] J.G. Semple and G.T. Kneebone, *Algebraic projective geometry*, Oxford Classic Texts in the Physical Sciences, The Clarendon Press Oxford University Press, New York, 1998, Reprint of the 1979 edition.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

E-mail address: jllibre@mat.uab.cat, jroe@mat.uab.cat, mahdi@mat.uab.cat