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Phase Portraits and Invariant Straight lines of Cubic Polynomial Vector Fields Having a Quadratic Rational First Integral

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Abstract

In this paper we classify all cubic polynomial differential systems having a first integral of degree two. In other words we characterize all the global phase portraits of the cubic polynomial differential systems having all their orbits contained in conics. We also determine their configurations of invariant straight lines. We show that there are exactly 36 topologically different phase portraits in the Poincaré disc associated to the family of cubic polynomial differential systems up to a reversed sense of their orbits.

1 Introduction and statement of the main results

We study *cubic polynomial vector fields* in $\mathbb{R}^2$ defined by the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) = P(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) = Q(x,y), \end{aligned} \qquad (1)$$

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where $p_0, q_0 \in \mathbb{R}$ and $p_i(x,y)$, $q_i(x,y)$ are homogenous polynomials of degree i $(i = 1,2,3)$ in x and y and $\big(p_3(x,y)\big)^2 + \big(q_3(x,y)\big)^2 \not\equiv 0$.

Our goal is to determine all phase portraits of systems (1) having a quadratic rational first integral H that is

$$H = \frac{c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2}{d_{00} + d_{10}x + d_{01}y + d_{20}x^2 + d_{11}xy + d_{02}y^2} = \frac{H_N}{H_D}, \tag{2}$$

with $c_{20}^2 + c_{11}^2 + c_{02}^2 + d_{20}^2 + d_{11}^2 + d_{02}^2 \neq 0$ and with the numerator and the denominator different from a constant, i.e. in this paper we do not allow that H or $1/H$ be a polynomial because in these cases the differential systems are essentially linear.

We remark that the quadratic vector fields having a rational first integral of degree 2 and their phase portraits have been characterized in [1, 2].

We note that the cubic polynomial differential systems having a rational first integral of degree 2 have all their orbits contained in conics. So, their orbits are very simple curves but this does not prevent their phase portrait from exhibiting a rich variety of dynamics as it is shown in our main result.

Theorem 1. *The phase portrait of a planar cubic polynomial differential system with a rational first integral of degree 2 is topologically equivalent to one of the* 38 *phase portraits described in Figure 1.*

The paper is organized as follows. In Section 2 we present some basic results of the systems studied in Theorem 1. The real cubic systems of Theorem 1 always have real or complex invariant straight lines of total multiplicity 6, see Section 3. Playing with the different configurations of the invariant straight lines we organize the proof of Theorem 1 in eights subsections inside Section 3.

2 Preliminaries

We note that the most general polynomial vector fields having a rational first integral (2) are $X = (P, Q)$ where

$$P(x,y) = -\frac{\partial H}{\partial y}\,(H_D)^2, \quad Q(x,y) = \frac{\partial H}{\partial x}\,(H_D)^2. \tag{3}$$

We denote by

$$\mathcal{M} = \begin{pmatrix} c_{00} & c_{10} & c_{01} & c_{20} & c_{11} & c_{02} \\ d_{00} & d_{10} & d_{01} & d_{20} & d_{11} & d_{02} \end{pmatrix}$$

the matrix of the coefficients of the polynomials H_N and H_D and by δ_{ij}, $1 \le i < j \le 6$, the minor of the matrix $\mathcal{M}$ constructed with the columns i and j. Then the differential systems corresponding to the vector fields (3) take the form:

$$\begin{aligned} \dot{x} &= \delta_{13} + (\delta_{15} + \delta_{23})\,x + 2\delta_{16}\,y + (\delta_{25} - \delta_{34})\,x^2 + 2\delta_{26}xy + \delta_{36}\,y^2 + x\,R_2(x,y), \\ \dot{y} &= -\,\delta_{12} - 2\delta_{14}\,x + (\delta_{23} - \delta_{15})\,y - \delta_{24}\,x^2 - 2\delta_{34}\,xy + (\delta_{26} - \delta_{35})y^2 + y\,R_2(x,y), \end{aligned} \tag{4}$$

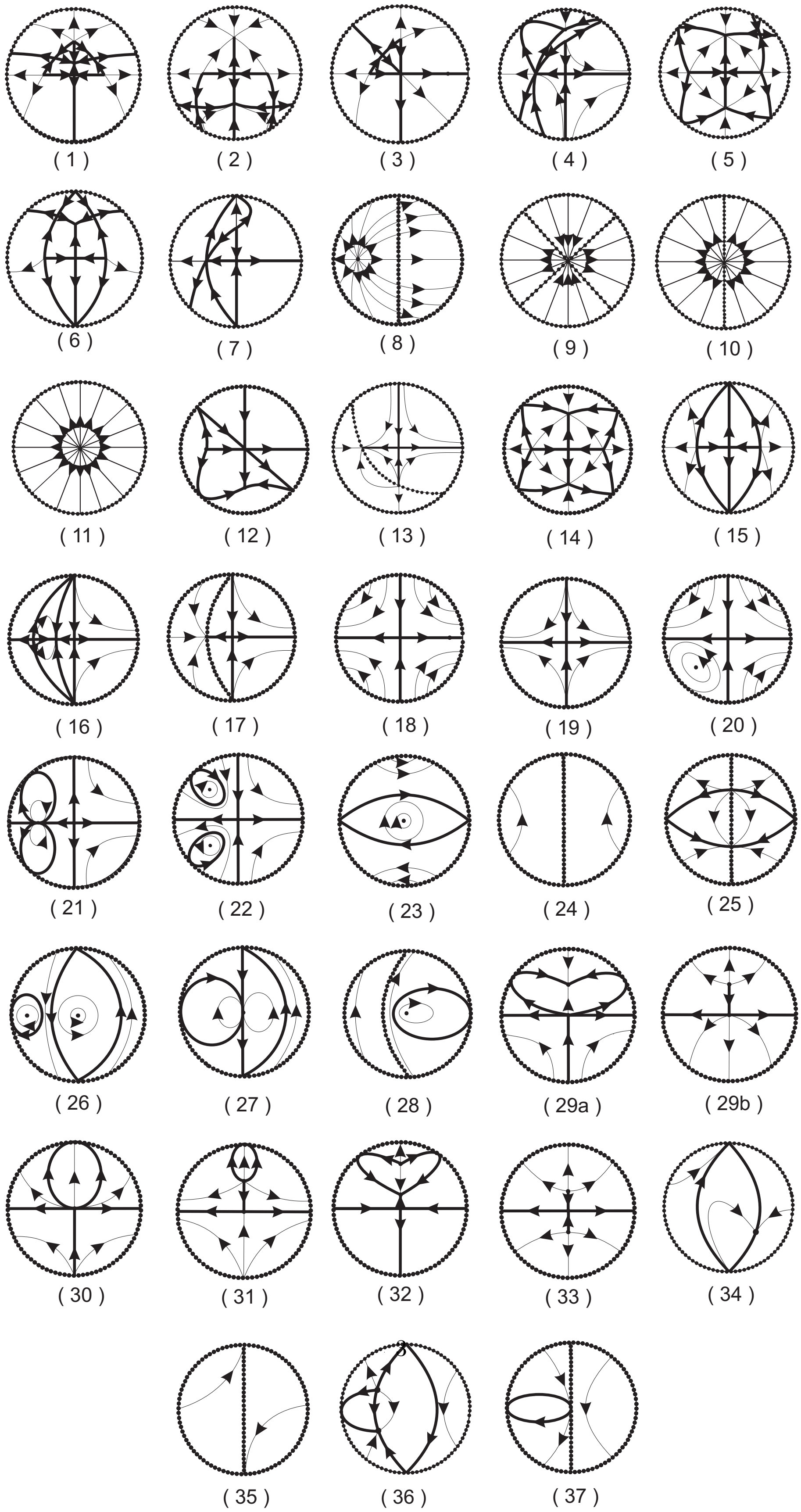

Figure 1: Phase portraits of cubic systems having a rational first integral of degree two.

where $R_2(x, y) = \delta_{45}\, x^2 + 2\delta_{46}\, xy + \delta_{56}\, y^2$.

We say that a polynomial differential system (1) is *degenerate* if P and Q have a common factor real or complex of degree ≥ 1. We say that the *infinity is degenerate* if it is full of singular points. In what follows sometimes instead of cubic polynomial differential system we will simply say *cubic system.*

Lemma 2. *If a cubic system* (1) *possesses a rational first integral of the form* (2) *then this system:*

(a) has a line of singularities at infinity;

(b) becomes a quadratic system if and only if the homogeneous quadratic part of the polynomials H_N and H_D from (2) *are proportional;*

(c) becomes homogenous cubic degenerate of the form $\dot{x} = x\, R_2(x, y)$, $\dot{y} = y\, R_2(x, y)$ if H_N and H_D are homogeneous quadratic forms.

Proof. As we said the most general form of the cubic vector fields having a rational first integral of the form (2) takes the form (4). The cubic homogeneous part of the vector field is denoted by $(\bar{P}, \bar{Q})$. It is clear that $(\bar{P}, \bar{Q}) = (xR_2, yR_2)$. Since $x\bar{Q} - y\bar{P} \equiv 0$ the statement (a) follows. Homogeneous quadratic parts of the polynomials H_N and H_D are proportional if and only if $\delta_{45} = \delta_{46} = \delta_{56} = 0$. This condition is fulfilled if and only if $R_2 \equiv 0$ and this shows statement (b). If we assume that

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & c_{20} & c_{11} & c_{02} \\ 0 & 0 & 0 & d_{20} & d_{11} & d_{02} \end{pmatrix}.$$

that it is clear that (4) is as in statement (c).

□

The vector field $\mathcal{X}$ *associated* to system (1) is defined by

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

Let $f \in \mathbb{C}[x, y]$, i.e. f is a polynomial with complex coefficients in the variables x and y. The complex algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of the real vector field $\mathcal{X}$ if for some polynomial $K \in \mathbb{C}[x, y]$ we have

$$\mathcal{X}f := P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that if a polynomial system has degree m then every cofactor has at most degree $m - 1$.

Lemma 3. *Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^2$. If the polynomial functions f and g are relatively prime, then f/g is a rational first integral of $\mathcal{X}$ if and only if f and g are both invariant algebraic curves with the same cofactor.*

Proof. Let $H = f/g$ be a first integral with f and g two non–constant coprime polynomials. So $\mathcal{X}(f/g) = 0$, or equivalently $f(\mathcal{X}g) = g(\mathcal{X}f)$. Since f does not divide g, we have that $\mathcal{X}f = Kf$ for some polynomial $K \in \mathbb{C}[x, y]$. Therefore, we also have $\mathcal{X}g = Kg$. Similarly if we take two algebraic curves f

and g with the same cofactor K, then we have $K = (\mathcal{X}f)/f$ and $K = (\mathcal{X}g)/g$. Then $(\mathcal{X}f)g - f(\mathcal{X}g) = 0$. Since

$$\mathcal{X}\left(\frac{f}{g}\right) = \frac{(\mathcal{X}f)g - f(\mathcal{X}g)}{g^2} = 0$$

the lemma follows. □

Corollary 4. *If f/g is a rational first integral of $\mathcal{X}$ then $(\alpha f+\beta g)/(\gamma f+\delta g)$ is also a rational first integral of $\mathcal{X}$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ verifying the condition $\alpha\delta - \beta\gamma \neq 0$.*

Proof. Suppose that f/g is the first integral of $\mathcal{X}$ and $a, b, c, d \in \mathbb{R}$. Then $(af)/g+b = (af+bg)/g$ is another first integral of $\mathcal{X}$ and so is $cg/(af+bg)+d = (ad\,f+(c+bd)g)/(af+bg)$. Putting $\alpha = ad$, $\beta = c+bd$, $\gamma = a$ and $\delta = b$ and by the assumption $\alpha\delta - \beta\gamma \neq 0$ so the first integral is never a constant we prove the corollary. □

3 Classification of the configurations of the invariant straight lines

We say that the invariant straight line $\mathcal{L}(x, y) = ux + vy + w = 0$, where $u, v, w \in \mathbb{C}$ for the cubic vector field $\mathcal{X}$ has *multiplicity* m if there exists a sequence of real cubic vector fields $\mathcal{X}_k$ tending to $\mathcal{X}$, such that each $\mathcal{X}_k$ has exactly m distinct (complex) invariant straight lines $\mathcal{L}_k^1 = 0, \ldots, \mathcal{L}_k^m = 0$, tending to $\mathcal{L} = 0$ as $k \to \infty$ (with the topology of their coefficients).

In what follows we construct the necessary and sufficient conditions for a cubic system (4) to have an invariant straight line. We consider a cubic system (1) and the following associated four polynomials:

$$C_i(x, y) = yp_i(x, y) - xq_i(x, y) \in \mathbb{R}[a, x, y],\ \ i = 0, 1, 2, 3.$$

Remark 5. *For system (4) it follows immediately that $C_3(x, y) \equiv 0$.*

We denote by $\mathrm{Res}_z(f(z), g(z))$ the resultant of the polynomials $f(z)$ and $g(z)$. Following [6] we shall prove the next results.

Proposition 6. *The straight line $\tilde{L}(x, y) = ux + vy = 0$ is invariant for a cubic system (4) with $p_0^2 + q_0^2 \neq 0$ if and only if for $i = 1, 2$ the following relations hold:*

$$\mathrm{Res}_\gamma(C_i, C_0) = 0 \quad (\gamma = \frac{y}{x} \quad or \quad \gamma = \frac{x}{y}). \tag{5}$$

Proof. The line $\tilde{L}(x, y)=0$ is invariant for system (4) if and only if

$$u(p_0 + p_1 + p_2 + p_3) + v(q_0 + q_1 + q_2 + q_3) = (ux + vy)(S_0 + S_1 + S_2),$$

for some homogeneous polynomials $S_i(x, y)$ of degree i in x and y. The last equality is equivalent to

$$\begin{aligned}
&up_0 + vq_0 = 0,\\
&up_1(x, y) + vq_1(x, y) = (ux + vy)S_0,\\
&up_2(x, y) + vq_2(x, y) = (ux + vy)S_1(x, y),\\
&up_3(x, y) + vq_3(x, y) = (ux + vy)S_2(x, y).
\end{aligned}$$

If $x = -v, y = u$, then the left–hand sides of the previous equalities become $C_0(-v, u)$, $C_1(-v, u)$, $C_2(-v, u))$ and $C_3(-v, u))$ respectively, and the last polynomial vanishes (see Remark 5). At the same time the right–hand sides of these identities vanish. Thus we obtain equations $C_i(-v, u) = 0$ $(i = 0, 1, 2)$ in which C_0 (respectively, C_1 and C_2) is a homogeneous polynomial of degree 1 (respectively 2 and 3) in the parameters u and v, and $C_0(x, y) \neq 0$ because $p_0^2 + q_0^2 \neq 0$. Hence by the properties of the resultant the necessary and sufficient conditions for the existence of a common solution of this system of equations are conditions (5). □

Let $(x_0, y_0) \in \mathbb{R}^2$ be an arbitrary point on the phase plane of systems (4). Consider a translation τ bringing the origin of coordinates to the point (x_0, y_0). We denote by (4^τ) the system obtained after applying the transformation τ, and by $\widetilde{\boldsymbol{a}} = \boldsymbol{a}(x_0, y_0) \in \mathbb{R}^{20}$ the 20–tuple of its coefficients. If $\gamma = y/x$ or $\gamma = x/y$ then, for $i = 1, 2$ we denote

$$\begin{aligned} \Omega_i(\boldsymbol{a}, x_0, y_0) &= \mathrm{Res}_\gamma\Big(C_i\big(\widetilde{\boldsymbol{a}}, x, y\big), C_0\big(\widetilde{\boldsymbol{a}}, x, y\big)\Big) \in \mathbb{R}[\boldsymbol{a}, x_0, y_0], \\ \mathcal{E}_i(\boldsymbol{a}, x, y) &= \Omega_i(\boldsymbol{a}, x_0, y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[\boldsymbol{a}, x, y]. \end{aligned} \qquad (6)$$

Remark 7. *For $j = 1, 2$ the polynomials $\mathcal{E}_j(\boldsymbol{a}, x, y)$ are affine comitants (for more details see* [6]*), homogeneous in the coefficients of system* (4) *and non–homogeneous in the variables x and y.*

The geometrical meaning of these affine comitants is given by the following lemma.

Lemma 8. *The straight line $L(x, y) = ux + vy + w = 0$ is invariant for a cubic system* (4) *if and only if the polynomial $L(x, y)$ is a common factor of the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathbb{C}$.*

Proof. Let $(x_0, y_0) \in \mathbb{R}^2$ be a non–singular point of system (4) (i.e. $P(x_0, y_0)^2 + Q(x_0, y_0)^2 \neq 0$) which lies on the line $L(x, y) = 0$, i.e. $ux_0 + vy_0 + w = 0$. Denote by $\tilde{L}(x, y) = (L \circ \tau)\ (x, y) = ux + vy$ (τ is a translation) and consider the line $ux + vy = 0$. By Proposition 6 the straight line $\tilde{L}(x, y) = 0$ will be an invariant line of systems (4^τ) if and only if the conditions (5) are satisfied for these systems, i.e. for $i = 1, 2$, $\Omega_i(\boldsymbol{a}, x_0, y_0) = 0$ for each point (x_0, y_0) on the line $L(x, y) = ux + vy + w = 0$. Thus from Nullstellensatz we have $\Omega_i(\boldsymbol{a}, x_0, y_0) = (ux_0 + vy_0 + w)\tilde{\Omega}_i(\boldsymbol{a}, x_0, y_0)$. Taking into account relations (6) the lemma follows. □

Proposition 9. *Every non–degenerate cubic system having a rational first integral of the form* (2) *possesses invariant affine straight lines (real and/or complex) of total multiplicity six.*

Proof. Calculating for systems (4) the affine invariant polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ we obtain

$$\mathcal{E}_1 = W(c_{ij}, d_{ij}, x, y), \quad \mathcal{E}_2 = W(c_{ij}, d_{ij}, x, y)\widetilde{W}(c_{ij}, d_{ij}, x, y),$$

where $W(c_{ij}, d_{ij}, x, y)$ (respectively $\widetilde{W}(c_{ij}, d_{ij}, x, y)$) is a homogenous polynomial of degree 6 (respectively of degree 2) in the parameters c_{ij}, d_{ij} and a non-homogenous polynomial of degree 6 (respectively of degree 2) in the variables x and y. As the polynomial $W(c_{ij}, d_{ij}, x, y)$ is a common factor of the affine comitants $\mathcal{E}_1$ and $\mathcal{E}_2$ by Lemma 8 the polynomial $W(c_{ij}, d_{ij}, x, y)$ is a product of six invariant affine straight lines, which could be real and/or complex, distinct and/or coinciding. This completes the proof of the proposition. □

Note that the multiplicity of an invariant straight line $ux + vy + w = 0$ is given by the number of times that $ux + vy + w$ divides $\mathcal{E}_1$, for more details see [3].

From the proof of the above lemma the next result follows immediately.

Corollary 10. *For a non–degenerate cubic system having a rational first integral of the form* (2) *the affine invariant polynomial* $\mathcal{E}_1(x, y)$ *gives six invariant straight lines taking into account their multiplicity.*

The information about the existence of invariant straight lines of total multiplicity six will be crucial in determining all the phase portraits of non–degenerate systems (4). Before we describe the way to achieve it we recall some notions about algebraic curves and say what we mean by a configuration of invariant straight lines.

If $F = F(x, y) \in \mathbb{R}[x, y]$ is a real polynomial of degree two then it is known that this polynomial can be brought to one of the nine normal forms (see Proposition 22 of the Appendix). In the Appendix we recall two main invariants Δ and δ of a conic. In this article we will often use the following terminology. We say that a conic $F = 0$ is of *hyperbolic type* if $\delta < 0$, is of *parabolic type* if $\delta = 0$, and is of *elliptic type* if $\delta > 0$.

We say that a real conic $F = 0$ is *reducible* if it factorizes in the complex domain, i.e. if $F = (ax + by)(cx + dy)$ for some $a, b, c, d \in \mathbb{C}$. If $F = 0$ is not reducible we say that it is *irreducible.* Thus a reducible conic of hyperbolic type is a real conic that factorizes in two real intersecting straight lines; a reducible conic of parabolic type is a real conic that factorizes in either two distinct real or complex parallel straight lines or one real straight line of multiplicity two; finally, reducible conic of elliptic type is a real conic that factorizes in two complex conjugate straight lines (which intersect in a real point). Therefore reducible conic of hyperbolic, elliptic and parabolic type via an affine transformation it can be brought respectively to $xy = 0$, $x^2 + y^2$; $x^2 + \alpha$, where $\alpha \in \{0, \pm 1\}$.

We will also say that a differential system (4) having a rational first integral $H = H_N/H_D$ *has* or *possesses* a real conic $F = 0$ if there exist α, $\beta \in \mathbb{R}$ such that

$$F = \alpha H_N + \beta H_D. \tag{7}$$

That is to say the set $F = 0$ coincides with $H = -\beta/\alpha$ for $\alpha \neq 0$, and $F = H_D$ for $\alpha = 0$.

Lemma 11. *If system* (4) *has at least two non–proportional reducible conics (say* $H_1 = 0$ *and* $H_2 = 0$*) then the following statements hold.*

(a) The rational function H_1/H_2 *also is a first integral for this system.*

(b) The system becomes quadratic if and only if the homogeneous quadratic parts of the conics H_1 *and* H_2 *are proportional.*

(c) *The system becomes homogenous cubic degenerate if H_1 and H_2 are homogeneous polynomials.*

Proof. From our assumptions and the fact that system (4) has the first integral H_N/H_D we have

$$H_1 = \alpha_1 H_N + \beta_1 H_D, \qquad H_2 = \alpha_2 H_N + \beta_2 H_D,$$

so we have

$$\frac{H_1}{H_2} = \frac{\alpha_1 H_N + \beta_1 H_D}{\alpha_2 H_N + \beta_2 H_D}.$$

Since H_1 and H_2 are not proportional we have $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ and now (a) follows from Corollary 4.

Calculating $\mathcal{X} = (P, Q)$ as in (3) for $H = H_N/H_D$ and choosing H_N and H_D with a proportional quadratic homogeneous parts we obtain (b), and for H_N and H_D homogeneous we have (c). □

Lemma 12. *If a pencil of conics $\Phi_{\alpha,\beta}(x, y) = \alpha H_N + \beta H_D$ possesses a single reducible conic, then this conic could be either of hyperbolic or of parabolic type.*

Proof. Suppose the contrary, that the single reducible conic of the pencil $\Phi_{\alpha,\beta}$ is of elliptic type. Then clearly we can assume that H_N is such a conic (i.e. $H_N = H_{\mathbf{e}}$) due to a re-parametrization of the pencil of conics. On the other hand, due to an affine transformation we may consider $H_{\mathbf{e}} = x^2 + y^2$. Moreover, if the conic H_D contains the term γy^2 we may assume $\gamma = 0$ due to the substitution $\alpha \to \alpha - \beta\gamma$.

Thus we get the pencil of conics

$$\begin{aligned}
\Phi_{\alpha,\beta}(x, y) &= \alpha(x^2 + y^2) + \beta(ax^2 + bxy + cx + dy + e) \\
&= (\alpha + a)x^2 + \beta bxy + \alpha y^2 + cx + dy + e \\
&= s_{11}x^2 + 2s_{12}xy + s_{22}(\alpha, \beta)y^2 + 2s_{13}x + 2s_{23}y + s_{33},
\end{aligned}$$

and setting $s_{ij}(\alpha, \beta, a, b, c, d, e) = s_{j\,i}(\alpha, \beta, a, b, c, d, e)$ we calculate the respective determinant:

$$\begin{aligned}
\Delta(\alpha, \beta) &= 4 \det ||s_{ij}||_{i,j\in\{1,2,3\}} \\
&= \beta \left[4e\alpha^2 + (c^2 + d^2 - 4ae)\alpha\beta + (bcd - ad^2 - b^2e)\beta^2\right] \equiv \beta W.
\end{aligned}$$

So, in order to have a single reducible conic (and namely, $H_{\mathbf{e}} = x^2 + y^2$ which corresponds to the solution $\beta = 0$ of the equation $\Delta(\alpha, \beta) = 0$) it is necessary that the Discriminant$\big(W(\alpha, \beta)\big) < 0$ or Discriminant$\big(W(\alpha, \beta)\big) = 0$ but in this case the double solution of the equation $W(\alpha, \beta) = 0$ must coincide with $\beta = 0$.

Calculation yields: Discriminant$\big(W(\alpha, \beta)\big) = (4ae - c^2 + d^2)^2 + 4(cd - 2be)^2 \geq 0$ and therefore, the unique solution could be $\beta = 0$, which must be triple for the equation $\Delta(\alpha, \beta) = 0$. However in order to reach this situation the conditions $e = c = d = 0$ have to be satisfied and we get that the considered pencil of conics possess also another reducible conic $H_D = x(ax + by)$. The obtained contradiction proves the lemma. □

We call the *configuration of invariant straight lines* (or simply the *configuration*) of system (1) the set of all its invariant straight lines (real or complex), each endowed with its multiplicity and together with all the real singular points of (1) located on these lines endowed with their multiplicity.

Using this information we divide the problem of finding all topologically nonequivalent phase portraits of systems (4) into 6 cases. We denote by $\mathbf{h}$, $\mathbf{p}$ and $\mathbf{e}$ the reducible conic of hyperbolic, parabolic and elliptic type respectively. By $\mathcal{S}(\mathbf{m},\mathbf{n})$ we denote the class of systems (4) having at least two different reducible conics of the type $\mathbf{m}$ and $\mathbf{n}$, where $\mathbf{m},\mathbf{n} \in \{\mathbf{h},\mathbf{e},\mathbf{p}\}$.

First we consider the family $\mathcal{S}(\mathbf{h},\mathbf{h})$ that consists of all system (3) having two nonproportional conics of hyperbolic type, say H_1 and H_2. By Lemma 11(a) H_1/H_2 is a first integral of the system and by Corollary 10 calculating $\mathcal{E}_1 = H_1H_2\hat{H}$, we find another reducible invariant conic $\hat{H} = 0$.

Then we study the family $\mathcal{S}(\mathbf{h},\mathbf{p})$ of all system having two reducible conics one of hyperbolic type $H_1 = 0$ and the other of parabolic type $H_2 = 0$. Calculating $\mathcal{E}_1 = H_1H_2\hat{H}$ we determine the third reducible conic $\hat{H} = 0$. Of course if $\hat{H} = 0$ is of hyperbolic type then the system belongs to $\mathcal{S}(\mathbf{h},\mathbf{h})$ since there are two reducible conics of hyperbolic type, namely H_2 and $\hat{H}$. Thus we exclude this case when studying family $\mathcal{S}(\mathbf{h},\mathbf{p})$.

We are going to consider the following families $\mathcal{S}(\mathbf{h},\mathbf{h})$, $\mathcal{S}(\mathbf{h},\mathbf{p})$, $\mathcal{S}(\mathbf{h},\mathbf{e})$, $\mathcal{S}(\mathbf{p},\mathbf{p})$, $\mathcal{S}(\mathbf{p},\mathbf{e})$, $\mathcal{S}(\mathbf{e},\mathbf{e})$ in this order. At each step we exclude the cases that have been studied before. As a first step we shall construct all topologically distinct configurations of invariant straight lines occurring for each of the mentioned classes of systems (4). Then we consider systems having a reducible conic either of hyperbolic or elliptic type and no reducible conic of other type. At each step we again exclude the cases that lead to phase portraits that have been studied before. By Lemma 12 systems having only one reducible conic of elliptic type do not exist.

3.1 The subfamily of systems of type $\mathcal{S}(\mathbf{h},\mathbf{h})$

Assume that a system (4) possesses two distinct reducible conics of hyperbolic type, say $H_{\mathbf{h}}^{(i)} = L_1^{(i)}L_2^{(i)}$ $(i = 1, 2)$. We shall consider two geometrical distinct possibilities:

(i) either the centers of the conics $H_{\mathbf{h}}^{(1)}$ and $H_{\mathbf{h}}^{(2)}$ (i.e. the intersection points of the lines $L_1^{(i)}$ and $L_2^{(i)}$ $(i = 1, 2)$) are distinct,

(ii) or they coincide.

Proposition 13. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{h}}^{(1)}/H_{\mathbf{h}}^{(2)}$ where $H_{\mathbf{h}}^{i}$ is a reducible conic of hyperbolic type for $i = 1, 2$ and the centers of the conics $H_{\mathbf{h}}^{(1)}$ and $H_{\mathbf{h}}^{(2)}$ are distinct. Then this system can be written in the form*

$$\dot{x} = x(b + (1 + b)x + x^2 + ay^2), \quad \dot{y} = y(-b + (b - a)y + x^2 + ay^2), \qquad (8)$$

where $a, b \in \mathbb{R}$ and $a \neq -1$ having the first integral $H = (x - y + 1)(x + ay + b)/(xy)$. Moreover, the configurations of invariant straight lines of this *system are*

$$\begin{array}{lcl} \textit{Config.}1 & \Leftrightarrow & ab(b-1)(a+b)(a+b^2)\neq 0,\ a<0;\\ \textit{Config.}2 & \Leftrightarrow & ab(b-1)(a+b)(a+b^2)\neq 0,\ a>0;\\ \textit{Config.}3 & \Leftrightarrow & b(a+b)(b-1)=0,\ a>0;\\ \textit{Config.}4 & \Leftrightarrow & b(a+b)(b-1)=0,\ a<0;\\ \textit{Config.}5 & \Leftrightarrow & a=-b^2\neq 0;\\ \textit{Config.}6 & \Leftrightarrow & a=0,\ b(b-1)\neq 0;\\ \textit{Config.}7 & \Leftrightarrow & a=0,\ b=1;\\ \textit{Config.}8 & \Leftrightarrow & a=0,\ b=0. \end{array}$$

Proof. In order to have a cubic system, according to Lemma 11, we shall consider that the quadratic homogeneous parts of $H_{\mathbf{h}}^{(1)}$ and $H_{\mathbf{h}}^{(2)}$ are not proportional. Therefore, since the centers of these conics are distinct, there exists a component of a conic which intersects both components of the other one in two distinct points. So without loss of generality we can assume that $H_{\mathbf{h}}^{(2)} = xy$ (due to the affine transformation $x_1 = L_1^{(2)}$ and $y_1 = L_2^{(2)}$), and that the line $L_1^{(1)}$ intersects both lines $x = 0$ and $y = 0$ in two distinct points, say $(0, \alpha)$ and $(\beta, 0)$. Then we can assume that this line is $x - y + 1 = 0$ due to the rescaling $(x, y) \mapsto (x/\beta, -y/\alpha)$.

In short, we obtain the first integral $H = (x-y+1)(cx+ay+b)/(xy)$ and since $a^2+c^2 \neq 0$ we can consider $c \neq 0$ (due to the change $(x, y) \mapsto (-y, -x)$). Finally without loss of generality we can assume $c = 1$ (multiplying by $1/c$ this being equivalent to the time rescaling $t \to t/c$ for systems (4)).

Thus having this first integral systems (4) can be written in the form (8) and according to Corollary 10 they have the following real invariant straight lines

$$\mathcal{E}_1 = xy(x-y+1)(x+ay+b)(bx+ay+b)(x-by+b) = L_1L_2L_3L_4L_5L_6 = 0, \quad (9)$$

where $a, b \in \mathbb{R}$ and $a \neq -1$. It would be convenient to represent these six lines in the matrix form

$$(L_1, L_2, L_3, L_4, L_5, L_6) = (x, y)M + (0, 0, 1, b, b, b),$$

where

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 & b & 1 \\ 0 & 1 & -1 & a & a & -b \end{pmatrix}.$$

It is known that two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are parallel if and only $A_1B_2 - A_2B_1 = 0$. Thus in order to have six invariant straight lines in six different directions all 2×2 minors of matrix M have to be different from zero. We denote by d_{ij}, $1 \leq i < j \leq 6$ the minors of the matrix M constructed using the columns i and j. Then we have

$$\begin{array}{lllll} d_{12}=1, & d_{13}=-1, & d_{14}=a, & d_{15}=a, & d_{16}=-b,\\ d_{23}=-1, & d_{24}=-1, & d_{25}=-b, & d_{26}=-1, & d_{34}=a+1,\\ d_{35}=a+b, & d_{36}=-b+1, & d_{45}=a(1-b), & d_{46}=-b^2-a, & d_{56}=-b^2-a. \end{array}$$

Taking into consideration that $a \neq -1$ (which must be fulfilled for systems (8)) we conclude that the minors $d_{ij} \neq 0$ for all $1 \leq i < j \leq 6$ if and only if the condition

$$ab(b-1)(a+b)(a+b^2) \neq 0 \quad (10)$$

holds. So we conclude that in this generic case there are six different straight lines in six different directions and we shall show that there exist two distinct configurations depending on the sign of the parameter a.

Indeed since all the invariant straight lines $L_i = 0$, $i = 1, \dots, 6$ of system (8) are real, then clearly their intersection points are finite singularities of these systems. In the generic case (10) we shall see that the system has four star points, which are intersections of three invariant straight lines.

Denoting by $\text{Int}(L_i, L_j, L_k)$ the intersection point of the straight lines L_i, L_j and L_k, it is easy to determine the coordinates of the four star points:

$$M_1 := \text{Int}(L_1, L_3, L_6) = (0, 1), \qquad M_2 := \text{Int}(L_2, L_3, L_5) = (-1, 0),$$
$$M_3 := \text{Int}(L_2, L_4, L_6) = (-b, 0), \qquad M_4 := \text{Int}(L_1, L_4, L_5) = (0, -b/a).$$

We observe that two star points have fixed coordinates ($M_1 = (0,1)$ and $M_2 = (-1, 0)$) as well as the singular point $M_0(0,0)$. On the other hand the star points $M_3(x_0, 0)$ and $M_4(0, y_0)$ (where $x_0 = -b, y_0 = -b/a$) are moving on the axes when the parameters a and b vary.

It is easy to see that the quadrilateral formed by the points M_1, M_2, M_3 and M_4 is convex if $x_0y_0 < 0$, and is concave if $x_0y_0 > 0$. Since $\text{sign}\,(x_0y_0) = \text{sign}\,(b^2/a)$, then we get **Config. 1** if $a < 0$ and **Config. 2** if $a > 0$.

In what follows we assume that the condition $ab(b-1)(a+b)(a+b^2) = 0$ is fulfilled. Then considering all the different possibilities in order that this expression be zero, we obtain the remaining six configurations of the statement of the proposition. □

3.1.1 Phase portraits of systems of type $\mathcal{S}(\mathbf{h}, \mathbf{h})$

The goal of this subsection is to determine the phase portraits of the systems of type $\mathcal{S}(\mathbf{h}, \mathbf{h})$. In other words to determine all topologically non–equivalent phase portraits for this family of systems. Thus first we introduce among other things the definition of the topological equivalence. Then we enunciate the Marcus-Neumann-Peixoto theorem that allows us to determine all topologically equivalent system by restricting ourselves mainly to studying the flow of the system on set of their separatrices.

Let φ be a C^k local flow with $k \geq 0$ on the 2–dimensional manifold M. Of course, for $k = 0$ the flow is continuous. We say that (M_1, φ_1) and (M_2, φ_2) are C^k*–equivalent* if there is a C^k diffeomorphism of M_1 onto M_2 which takes orbits of φ_1 onto orbits φ_2 preserving or reversing sense (but not necessarily the parametrization). Of course, a C^0 diffeomorphism is a homeomorphism.

We say that (M, φ) is C^k*–parallel* if it is C^k–equivalent to one of the following flows:

(i) $\mathbb{R}^2$ with the flow defined by $\dot{x} = 1$, $\dot{y} = 0$ (*strip flow*);

(ii) $\mathbb{R}^2 \setminus \{0\}$ with the flow defined by $\dot{r} = 0$, $\dot{\theta} = 1$ (*annular flow*);

(iii) $\mathbb{R}^2 \setminus \{0\}$ with the flow defined by $\dot{r} = r$, $\dot{\theta} = 0$ (*spiral flow*);

(iv) $\mathbb{S}^1 \times \mathbb{S}^1$ with rational flow (*toral flow*).

Let $p \in M$, we denote by $\gamma(p)$ the *orbit* of the flow φ on M through p, more precisely $\gamma(p) := \{\varphi_p(t) : t \in I_p\}$. The *positive semiorbit* of $p \in M$ is

$\gamma^+(p) = \{t \in I_p, t \geq 0\}$. In a similar way we define the *negative semiorbit* $\gamma^-(p)$ of $p \in M$. We define the α–limit and ω–limit of $p \in M$ as

$$\alpha(p) = \overline{\gamma^-(p)} - \gamma^-(p), \quad \omega(p) = \overline{\gamma^+(p)} - \gamma^+(p).$$

Let $\gamma(p)$ be an orbit of the flow φ defined on M. A *parallel neighborhood* of the orbit $\gamma(p)$ is an open neighborhood N of γ such that (N, φ) is C^k–equivalent to a parallel flow for some $k \geq 0$.

We say that $\gamma(p)$ is a *separatrix* of φ if $\gamma(p)$ is not contained in a parallel neighborhood N satisfying the following two assumptions:

(1) for every $q \in N$, $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$,

(2) $\overline{N} \setminus N$ consists of $\alpha(p)$, $\omega(p)$ and exactly two orbits $\gamma(a)$, $\gamma(b)$ of φ, with $\alpha(a) = \alpha(p) = \alpha(b)$ and $\omega(a) = \omega(p) = \omega(b)$.

We denote by Σ the union of all separatrices of φ. Then Σ is a closed invariant subset of M. A component of the complement of Σ in M, with the restricted flow, is called a *canonical region* of φ.

Let (φ, M) be a continuous flow on the 2–manifold M and let Σ be the set of all separatrices of (φ, M). In every canonical region U of (φ, M) we choose an orbit γ_U. Then a *separatrix configuration* of (φ, M) is formed by the union of the set Σ and the set of all orbits γ_U.

Theorem 14 (Marcus-Neumann-Peixoto)**.** *Let (φ_1, M_1) and (φ_2, M_2) be two continuous flows on the 2-manifolds M_1 and M_2. Then two flows are topologically equivalent if and only if there exists a homeomorphism $h : M_1 \to M_2$, which takes the orbits of the separatrix configuration of (φ_1, M_1) into the orbits of the separatrix configuration of (φ_2, M_2).*

Now we shall determine the phase portraits of system (8).

First we focus on the generic case, i.e. when the condition (10) is fulfilled. In this case we have seven singular points M_i, $i = 1, \ldots, 7$. In Table 1 we show the coordinates of M_i, and the determinant Δ and the trace T of the Jacobian matrix at M_i.

We have always three saddles M_2, M_5 and M_7 since Δ is negative at these three singular points. The rest of the singular points M_1, M_3, M_4 and M_6 are nodes since the equation $T^2 = 4\Delta$ holds for all of them. Now we consider the stability of the nodes. The conditions for the stability and instability of the nodes are given in Table 2. Thus when the condition (10) is fulfilled we get two topologically distinct phase portraits. For $a < 0$ we get a phase portrait that is topologically equivalent to **Picture 1** and for $a > 0$ we have a phase portrait that is topologically equivalent to **Picture 2**, see Figure 2.

Now we analyze the phase portraits of system (8) in a non-generic case, i.e. when condition (10) is not fulfilled.

First we determine the phase portraits of system (8) having the configurations of the invariant straight lines **Config. 3** and **4**. Thus we consider system (8) when $b(a + b)(b - 1) = 0$ for $a > 0$ and $a < 0$.

Assume that $b = 0$ and $a(a + 1) \neq 0$ then system (8) takes the form

$$\dot{x} = x(x + x^2 + ay^2), \qquad \dot{y} = y(-ay + x^2 + ay^2). \tag{11}$$

Table 1: The values of the determinant Δ and of the trace T at the singular points of system (8).

Singular point	Δ	T
$M_1 = (-1, 0)$	$(b-1)^2$	$2(1-b)$
$M_2 = (0, 0)$	$-b^2$	0
$M_3 = (0, 1)$	$(a+b)^2$	2(a+b)
$M_4 = (0, -b/a)$	$b^2(a+b)^2/a^2$	$2b(a+b)/a$
$M_5 = (-\frac{a+b}{a+1}, \frac{1-b}{a+1})$	$-(b-1)^2(a+b)^2/(a+1)^2$	0
$M_6 = (-b, 0)$	$b^2(b-1)^2$	$2b(b-1)$
$M_7 = (-\frac{b(a+b)}{a+b^2}, \frac{b(b-1)}{a+b^2})$	$-b^2(b-1)^2(a+b)^2/(a+b^2)^2$	0.

Table 2: The stability of the singular points of system (8).

Singular point	Type	Stable	Unstable
$M_1 = (-1, 0)$	Node	$b > 1$	$b < 1$
$M_2 = (0, 0)$	Saddle	—	—
$M_3 = (0, 1)$	Node	$a + b < 0$	$a + b > 0$
$M_4 = (0, -b/a)$	Node	$(a+b)b/a < 0$	$(a+b)b/a > 0$
$M_5 = (-\frac{a+b}{a+1}, \frac{1-b}{a+1})$	Saddle	—	—
$M_6 = (-b, 0)$	Node	$b \in (0, 1)$	$b \in (-\infty, 0) \cup (1, +\infty)$
$M_7 = (-\frac{b(a+b)}{a+b^2}, \frac{b(b-1)}{a+b^2})$	Saddle	—	—

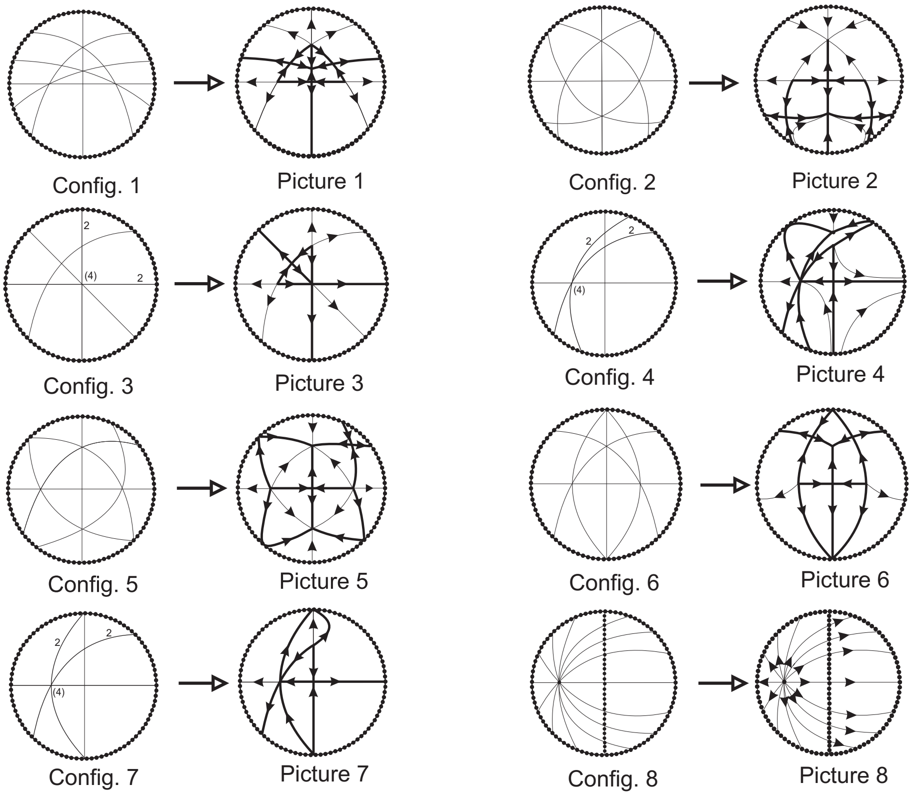

Figure 2: Configurations of the invariant straight lines of system (8) and their corresponding phase portraits.

The system has four singular points: two nodes $M_1 = (-1, 0)$ and $M_2 = (0, 1)$ one saddle $(-\frac{a}{a+1}, \frac{1}{a+1})$, and one degenerate singularity at the origin. The trace T of the Jacobian matrix of system (11) at M_1, denoted by $T(M_1)$, is equal to 2 and at M_2 we have $T(M_2) = 2a$. So the first node M_1 is unstable and the second is stable for $a < 0$ and unstable for $a > 0$. Thus we get respectively a pictures that are topologically equivalent to **Pictures 3** and **4** of Figure 2.

Now assume that $b = 1$ and $a(a-1) \neq 0$, then system (8) becomes

$$\dot{x} = x(1 + 2x + x^2 + ay^2), \qquad \dot{y} = y(-1 + (1-a)y + x^2 + ay^2). \tag{12}$$

The system has four singular points: two nodes $M_1 = (0, 1)$, $M_2 = (0, -1/a)$, one saddle at the origin, and one degenerate singularity at $(-1, 0)$. Similarly as in the previous case for system (12) we have $T(M_1) = 2(a+1)$ and $T(M_2) = 2(a+1)/a$. We consider the product $T(M_1)T(M_2)) = 4(a+1)^2/a$. So the stability of the two nodes M_1 and M_2 are distinct for $a < 0$ and we get the phase portrait that is topologically equivalent to **Picture 3**. The stability of the nodes coincide for $a > 0$ and we get a phase portrait that is topologically equivalent to **Picture 4**.

Now assume that $b = -a$ and $a(a+1) \neq 0$, then system (8) becomes

$$\dot{x} = x(-a + (1-a)x + x^2 + ay^2), \quad \dot{y} = y(a - 2ay + x^2 + ay^2). \tag{13}$$

System (13) has four singular points: two nodes $M_1 = (-1, 0)$ and $M_2 = (a, 0)$, one saddle at the origin and one degenerate singularity $(0, 1)$. In this case we also get **Picture 3** for $a > 0$ and **Picture 4** for $a < 0$, see Figure 2. We showed that if $b(b+1)(a+b) = 0$, $a > 0$ then system (8) has the phase portrait of **Picture 3**, and if $b(b+1)(a+b) = 0$ and $a < 0$ then **Picture 4**, see Figure 2.

Now we determine the phase portrait of system (8) having the configuration **Config. 5**. Thus we consider system (8) when $a = -b^2$ and $b \neq 0, 1$, i.e.

$$\dot{x} = x(b + (1+b)x + x^2 - b^2y^2), \quad \dot{y} = y(-b + b(1+b)y + x^2 - b^2y^2).$$

The system has six singular points: four nodes $M_1 = (-1, 0)$, $M_2 = (0, 1)$, $M_3 = (0, 1/b)$ and $M_4 = (-b, 0)$, and two saddles one at the origin and the other at $(-\frac{b}{b+1}, \frac{b-1}{b})$. Since we have $T(M_1) = 2(1-b)$, $T(M_2) = 2b(1-b)$, $T(M_3) = -2(1-b)$ and $T(M_4) = 2b(b-1)$. We determine the stability of the nodes and we get the phase portraits that are all topologically equivalent to the one of **Picture 5** in Figure 2.

To determine the phase portrait of system (8) having the configuration **Config. 6** we consider (8) when $a = 0$ and $b(b-1) \neq 0$, i.e.

$$\dot{x} = x(b + (1+b)x + x^2), \quad \dot{y} = y(-b + by + x^2).$$

Our system has six singular points: three nodes $M_1 = (-1, 0)$, $M_2 = (0, 1)$ and $M_3 = (-b, 0)$ and three saddles one at the origin, $(-b, -b+1)$ and $(-1, (b-1)/b)$. We also have $T(M_1) = 2(1-b)$, $T(M_2) = 2b$ and $T(M_3) = 2b(b-1)$. We obtain the unique phase portraits that are topologically equivalent to the one of **Picture 6**, see Figure 2.

We determine the phase portrait of system (8) having the configuration **Config. 7**. Thus we consider (8) when $a = 0$ and $b = 1$, i.e.

$$\dot{x} = x(1 + 2x + x^2), \quad \dot{y} = y(-1 + y + x^2).$$

Then system (8) has three singular points: a degenerate one $(-1, 0)$, a saddle $(0, 0)$ and a node $(0, 1)$. We get the phase portrait **Picture 7** in Figure 2.

Finally we determine the phase portrait of system (8) having the configuration **Config. 8**. Thus we consider system (8) when $a = 0$ and $b = 0$, i.e.

$$\dot{x} = x^2(1 + x), \quad \dot{y} = yx^2. \tag{14}$$

This system is degenerate since it has the common factor x^2. Thus it has the line of singularities $x = 0$. We have the unique phase portrait, see **Picture 8** in Figure 2.

Proposition 15. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{h}}^{(1)}/H_{\mathbf{h}}^{(2)}$ where $H_{\mathbf{h}}^{(i)}$ is a reducible conic of hyperbolic type for $i = 1, 2$, and that the centers of the conics $H_{\mathbf{h}}^{(1)}$ and $H_{\mathbf{h}}^{(2)}$ coincide. Then this system can be written in the form*

$$\dot{x} = x(x^2 + ay^2), \quad \dot{y} = y(x^2 + ay^2), \tag{15}$$

where $a \in \mathbb{R} \setminus \{-1\}$ having the first integral $H = (x - y)(x + ay)/(xy)$. Moreover, the configurations of invariant straight lines of this system are

$$\begin{array}{lcl} Config.9 & \Leftrightarrow & a < 0, \\ Config.10 & \Leftrightarrow & a = 0, \\ Config.11 & \Leftrightarrow & a > 0. \end{array}$$

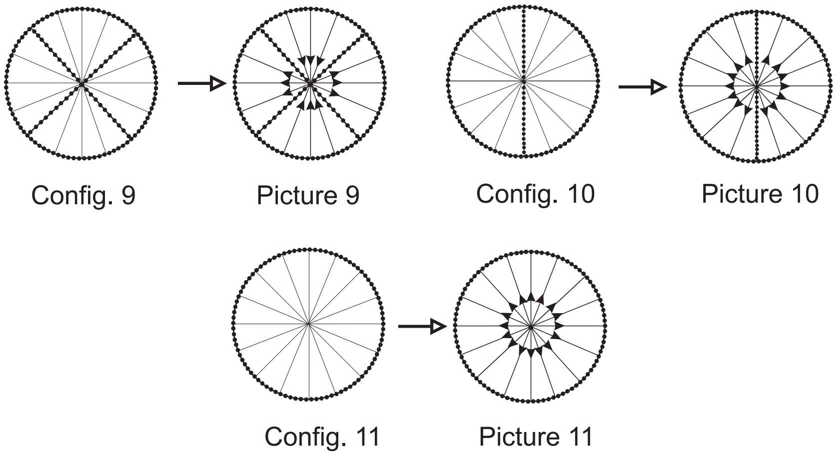

Figure 3: Configurations of invariant straight lines of system (15) and their corresponding phase portraits.

Proof. In this case providing that $H_{\mathbf{h}}^{(2)} = xy$ we obtain that all four lines pass through the origin. By Lemma 11 there exist at least three directions. Then without loss of generality we can assume $H_{\mathbf{h}}^{(1)} = (x - y)(bx + ay)$, where $a^2 + b^2 \neq 0$, and we can consider $b = 1$ due to the change $(x, y) \mapsto (-y, -x)$ and $a \neq 1$ otherwise $H_{\mathbf{h}}^{1}$ would not be of hyperbolic type. Thus we get one–parameter family of systems (15). Then studying all the possible configurations of the system varying the parameter a, we obtain $Config.9$ if $a < 0$, $Config.10$ if $a = 0$ and $Config.11$ if $a > 0$.

Evidently the configurations determined above lead to the respective phase portraits of Figure 3. □

3.2 The subfamily of systems of type $\mathcal{S}(\mathbf{h}, \mathbf{p})$

Assume that system (4) possess only one reducible conic of hyperbolic type, say $H_{\mathbf{h}}$ and at least one reducible conic of parabolic type $H_{\mathbf{p}}$. In this subsection we show all the configurations of the invariant straight lines that the system can have and their corresponding phase portraits.

Proposition 16. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{p}}/H_{\mathbf{h}}$ where $H_{\mathbf{h}}$ and $H_{\mathbf{p}}$ are reducible conics of hyperbolic and parabolic type respectively. Then this system can be written in the form*

$$\begin{aligned} \dot{x} &= x(cd + (c+d)x + x^2 - b^2y^2), \\ \dot{y} &= y(-cd - b(c+d)y + x^2 - b^2y^2), \end{aligned} \tag{16}$$

where $b \in \mathbb{R}$, and either $c, d \in \mathbb{R}$ or $d = \bar{c} \in \mathbb{C}\backslash\mathbb{R}$, having the first integral $H = (x + by + c)(x + by + d)/(xy)$. Moreover, all the possible configurations of the invariant straight lines of this system which have not appeared in Propositions 13 and 15 are given in Figure 4.

Proof. We can consider $H_{\mathbf{h}} = xy$ and the factorization over $\mathbb{C}$ of the parabolic conic will be $H_{\mathbf{p}} = (ax + by + c)(ax + by + d)$. As the conic $H_{\mathbf{p}}$ must be real we have $a = \bar{a}$ and $b = \bar{b}$, i.e. $a, b \in \mathbb{R}$. Moreover, since $a^2 + b^2 \neq 0$ we can consider $a \neq 0$ due to the change $(x, y) \mapsto (y, x)$, and then via the rescaling $x \to x/a$ we can assume $a = 1$.

In short, we obtain the first integral $H = (x + by + c)(x + by + d)/(xy)$ and hence system (4) becomes of the form (16). To determine the type of the third reducible conic (say $\widehat{H}$) of this system, by Corollary 10, we calculate

$$\mathcal{E}_1 = -2xy(x + by + c)(x + by + d)(dx + bcy + cd)(cx + bdy + cd). \tag{17}$$

Therefore $\widehat{H} = (dx + bcy + cd)(cx + bdy + cd) = 0$. Since all systems with two different hyperbolic conics were considered in the previous section we assume that $\widehat{H}$ is either non–hyperbolic or is exactly $xy = 0$. We are in the latter case if and only if $cd = 0$. Suppose that $c = 0$, then we have the first integral $H = (x + by)(x + by + d)/(xy)$. We assume that $bd \neq 0$ otherwise we get degenerate systems that were already considered. So by the rescaling we get the first integral of the system $H = (x + y)(x + y + 1)/(xy)$

$$\dot{x} = x(x + x^2 - y^2), \qquad \dot{y} = y(x^2 - y - y^2). \tag{18}$$

Calculating

$$\mathcal{E}_1 = x^2y^2(x + y)(1 + x + y),$$

we get the configuration of invariant straight lines **Config. 12**.

We assume now that the third reducible conic $\widehat{H}$ is not of hyperbolic type. Then the condition $\delta \geq 0$ (for the definition see the appendix.) must hold. That is

$$b^2(c - d)^2(c + d)^2 \leq 0. \tag{19}$$

Without loss of generality we can assume $b \in \{0, 1\}$ due to the rescaling $y \to y/b$ if $b \neq 0$, and we shall consider these two cases.

Case: $b^2(c-d)^2(c+d)^2 < 0$. Then $c, d \in \mathbb{C} \setminus \mathbb{R}$ and $b \neq 0$. More precisely $c = r + is$ and $d = r - is$ such that $rs \neq 0$ and $b = 1$. So system (16) can be written as

$$\dot{x} = x[(r^2 + s^2) + 2rx + x^2 - y^2], \qquad \dot{y} = y[-(r^2 + s^2) - 2ry + x^2 - y^2].$$

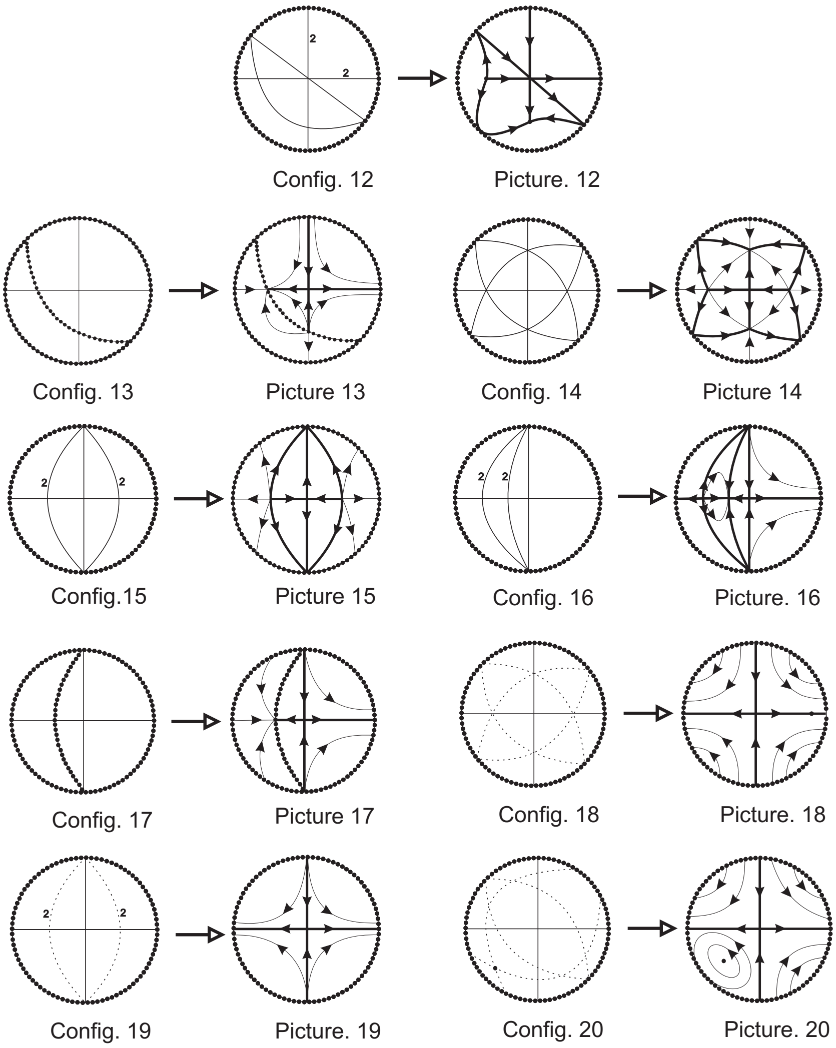

Figure 4: Configurations of invariant straight lines and corresponding phase portraits of system (16).

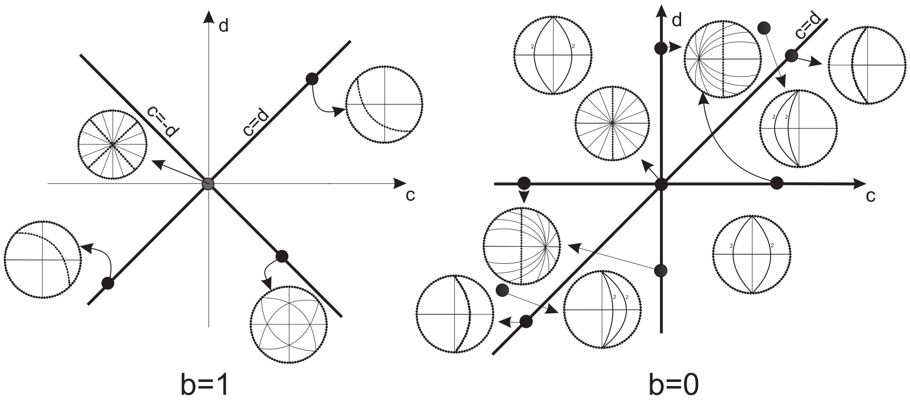

Figure 5: Bifurcation diagram of the configurations of invariant straight lines for system (16) in the case when the invariant straight lines are real.

Since $rs \neq 0$ we can assume that $r = 1$ due to the rescaling $(x, y, t) \longmapsto (rx, ry, t/r^2)$. Finally we arrive at the one parameter family of systems

$$\dot{x} = x[(1 + s^2) + 2x + x^2 - y^2], \qquad \dot{y} = y[-(1 + s^2) - 2y + x^2 - y^2].$$

having the configuration $\mathcal{E}_1$ of the invariant straight lines

$$2xy(x+y+1+si)(x+y+1-si)[(s-i)x-(s+i)y-i(1+s^2)][(s+i)x-(s-i)y+i(1+s^2)],$$

that corresponds to **Config. 20**, see Figure 4.

Case: $b^2(c - d)^2(c + d)^2 = 0$. In this case the third conic is of parabolic type.

1) Assuming $b = 1$ we have $(c - d)(c + d) = 0$.

1a) If $d = c$ then this parameter must be real and we get the following degenerate systems

$$\dot{x} = x(x - y + c)(x + y + c), \qquad \dot{y} = y(x - y - c)(x + y + c), \tag{20}$$

where $c \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c^2)$ if $c \neq 0$. So if $c = 1$ we get **Config. 13** and if $c = 0$ we get a configuration topologically equivalent to $Config.10$ (Figure 3).

1b) If $d = -c$ then we obtain the systems

$$\dot{x} = x(-c^2 + x^2 - y^2), \qquad \dot{y} = y(c^2 + x^2 - y^2), \tag{21}$$

where either $c \in \mathbb{R}$, or $0 \neq c = ir \in \mathbb{C}$. In the first case we can assume $c \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c^2)$ if $c \neq 0$, whereas in the second case we can assume $c = i$ due to the rescaling $(x, y, t) \mapsto (rx, ry, t/r^2)$. The system (21) possesses the following invariant straight lines

$$xy(x + y + c)(x + y - c)(x - y + c)(x - y - c) = 0.$$

We skip the case $c = 0$, because this case we get the degenerate system belonging to the $\mathcal{S}(\mathbf{h}, \mathbf{h})$ family. We obtain then **Config. 14** if $c = 1$, and **Config. 18** if $c = i$, see Figure 4. The bifurcation of the configurations of the invariant straight lines of system (16) for $b = 1$ and $(c - d)(c + d) = 0$ when the invariant straight lines are real is described in Figure 5.

2) Assume now that $b = 0$. Then we get the family of systems

$$\dot{x} = x(x + c)(x + d), \qquad \dot{y} = y(-cd + x^2), \tag{22}$$

where either $c, d \in \mathbb{R}$ or $d = \bar{c} \in \mathbb{C} \setminus \mathbb{R}$. Using (17) we obtain the following invariant straight lines

$$xy(x + c)^2(x + d)^2 = 0.$$

2a) Assume first that $c, d \in \mathbb{R}$. Due to the rescaling $(x, y, t) \mapsto (cx, y, t/c^2)$ if $c \neq 0$ we may assume $c \in \{0, 1\}$. If $c = 1$ then it is easy to observe that the configurations of invariant straight lines of systems (22) are given by **Config. 15** if $d < 0$, by **Config. 16** if $d > 0, d \neq 1$, and by **Config. 17** if $d = 1$. In the case $c = 0$, by the same reasons as above we can assume, that $d \in \{0, 1\}$. For $d = 0$ we get the configuration that is topologically equivalent to **Config. 10**. Finally, when $c = 1$, $d = 0$, and $c = 0$, $d = 1$ we get exactly system (14) that was considered in the previous section.

2b) Suppose now that $d = \bar{c} \in \mathbb{C} \setminus \mathbb{R}$ and assume $c = r + is$ ($s \neq 0$ and we can consider $s = 1$ due to the rescaling $(x, y, t) \mapsto (sx, y, t/s^2)$). So we obtain the following one–parameter family of systems

$$\dot{x} = x[1 + (x + r)^2], \qquad \dot{y} \ = y[-1 - r^2 + x^2],$$

having the configuration of invariant straight lines corresponding to **Config. 19**, see Figure 4. The bifurcation of the configurations of the invariant straight lines of system (16) when $b = 0$ and when the invariant straight lines are real is described in Figure 5. □

3.2.1 Phase portraits of systems of type $\mathcal{S}(\mathbf{h}, \mathbf{p})$

In this subsection we determine the phase portraits for each of the configurations of invariant straight lines of system (16).

First we determine the phase portrait of system (16) having the configuration of invariant straight lines **Config. 12**, thus we consider system (18). The system has three singular points: one degenerate at the origin, one stable node at $(0, -1)$ and one unstable node at $(-1, 0)$.

We determine the phase portrait of system (16) having the configuration **Config. 13** of the invariant straight lines. Thus we consider system (16) whith $b = 1$ and $c = d = 1$, i.e.

$$\dot{x} = x(x - y + 1)(x + y + 1), \qquad \dot{y} = y(x - y - 1)(x + y + 1).$$

The system has a common factor $x + y + 1$ hence the phase portraits will contain the line of singular points $x+y+1 = 0$. There is one isolated singular point at the origin which is a saddle. For the phase portrait see **Picture 13** in Figure 4.

To determine the phase portrait of system (16) having the configuration **Config. 14** we consider system (16) with $b = 1$ and $c = -d = 1$, i.e.

$$\dot{x} = x(-1 + x^2 - y^2), \qquad \dot{y} = y(1 + x^2 - y^2).$$

The system has five singular points which are two stable nodes $(0, 1)$ and $(0, -1)$, two unstable nodes $(1, 0)$ and $(-1, 0)$, and one saddle at the origin, see **Picture 14**.

We determine the phase portrait of system (16) having the configuration **Config. k**, for $k = 15, 16, 17$. So we consider system (16) with $b = 0$, $c = 1$, i.e.

$$\dot{x} = x(x + 1)(x + d), \qquad \dot{y} = y(-d + x^2),$$

and respectively $d < 0$, $1 \neq d > 0$ and $d = 1$. Suppose that $d \in \mathbb{R} \setminus \{0, 1\}$. Then the system has three singular points: a saddle $M_1 = (0, 0)$, and two nodes $M_2 = (-1, 0)$ and $M_3 = (-d, 0)$. M_2 is a stable node for $d > 1$ and unstable for $d < 1$. Finally M_3 is stable for $d \in (0, 1)$, and unstable for $d \in (-\infty, 0) \cup (1, +\infty)$. This leads to two topologically distinct phase portraits see **Picture 15** for $d < 0$, and see Picture 15 for $d > 0$.

If $d = 1$ then the above system has a line of singular points $x + 1 = 0$. The only isolated singularity is the origin which is a saddle, see **Picture 17**.

We determine the phase portrait of system (16) having the configuration **Config. 20**. Thus we consider system (16) with $b = 1$, $\bar{c} = \bar{d} = 1 + si \in \mathbb{C}$ and $s \neq 0$ i.e.

$$\dot{x} = x[(1 + s^2) + 2x + x^2 - y^2], \quad \dot{y} = y[-(1 + s^2) - 2y + x^2 - y^2].$$

The system has two singular points: a saddle at the origin and a center at $(-1/2[1 + s^2], -1/2[1 + s^2])$, see **Picture 20**.

We determine the phase portrait of system (16) having the configuration **Config. 18**. Thus we consider system (16) for $b = 1$ and $c = i$, i.e.

$$\dot{x} = x(1 + x^2 - y^2), \qquad \dot{y} \ = y(-1 + x^2 - y^2).$$

The system has only one singular point at the origin which is a saddle, see **Picture 18**.

We determine the phase portrait of system (16) having the configuration **Config. 19**. Thus we consider system (16) for $b = 1$ and $d = \bar{c} = r + i \in \mathbb{C}$, i.e.

$$\dot{x} = x[1 + (x + r)^2], \qquad \dot{y} = y[-1 - r^2 + x^2].$$

The system has only one singular point at the origin which is saddle, see **Picture 19**.

3.3 The subfamily of systems of type $\mathcal{S}(\mathbf{h}, \mathbf{e})$

Assume that system (4) possesses only one reducible conic of hyperbolic type, say $H_{\mathbf{h}}$ and at least one reducible conic of elliptic type $H_{\mathbf{e}}$.

Proposition 17. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{e}}/H_{\mathbf{h}}$ where $H_{\mathbf{h}}$ and $H_{\mathbf{e}}$ are reducible conics of* *hyperbolic and elliptic type respectively. Then this system can be written in the form*

$$\begin{aligned} \dot{x} &= x\big[b^2 + d^2 + 2(b + cd)x + (c^2 + 1)x^2 - y^2\big], \\ \dot{y} &= y\big[-b^2 - d^2 - 2dy + (c^2 + 1)x^2 - y^2\big], \end{aligned} \tag{23}$$

where $d \in \{0, 1\}$, having the first integral $H = \big((x+b)^2 + (cx+y+d)^2\big)/(xy)$. Moreover, all the possible configurations of invariant straight lines of this system which have not appeared in Propositions 13, 15 and 16 are given in Figure 6.

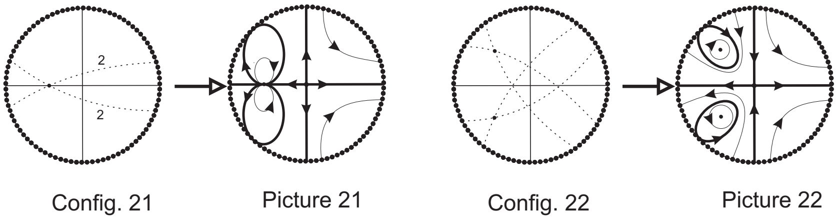

Figure 6: Configurations of the invariant straight lines and corresponding phase portraits of system (23).

Proof. Since the conic $H_{\mathbf{e}}$ is reducible we may assume $H_{\mathbf{e}} = x^2 + y^2$ due to an affine transformation. On the other hand we have $H_{\mathbf{h}} = L_1 L_2$, and since a rotation keeps the form of $H_{\mathbf{e}}$, we may consider $L_1 = ax + b$ and $L_2 = cx + ey + d$, where $ae \neq 0$ (as $H_{\mathbf{h}}$ is of hyperbolic type). Then via the affine transformation $x_1 = L_1$, $y_1 = L_2$ we obtain $H_{\mathbf{h}} = x_1 y_1$ and $H_{\mathbf{e}} = (x_1 - b)^2 + (cx_1/e - ay_1/e + (ad - bc)/e)^2$. Since $ae \neq 0$ applying the change $(x_1, y_1) \mapsto (x, ey/a)$ and renaming the parameters we arrive to the first integral $H = \big((x + b)^2 + (cx + y + d)\big)^2/(xy)$.

In short, systems (4) become of the form (23) where we may assume $d \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (dx, dy, t/d^2)$ if $d \neq 0$. To determine the type of the third reducible conic (say $\widehat{H}$) of these systems according to Corollary 10 we calculate

$$\mathcal{E}_1 = -2xy\big[(x+b)^2 + (cx+y+d)^2\big]\big[(bcx-dx-by)^2 + (b^2+d^2+bx+cdx+dy)^2\big].$$

Hence the third reducible conic $\widehat{H} = \big[(bcx-dx-by)^2 + (b^2+d^2+bx+cdx+dy)^2\big]$ is of elliptic type if

$$b^2 + 2bcd - d^2 \neq 0, \tag{24}$$

for the details see Appendix 1. Otherwise $\widehat{H} = 0$ are two complex parallel lines so we do not analyze this case since we have already considered this type of systems in the previous section. Calculating the resultant of the quadratic homogenous parts of the conics $H_{\mathbf{e}}$ and $\widehat{H}$ with respect to the variable y we obtain

$$\mathrm{Res}_y[H_{\mathbf{e}}^{(2)}, \widehat{H}^{(2)}] = 16b^2(bc - d)^2(c^2 + 1)x^4.$$

Hence the components (i.e. complex lines) of these conics are parallel if and only if the condition $b(bc - d) = 0$ holds. In fact we show that they coincide when this condition holds.

To show this we notice that $(b^2+d^2)H_e - \widehat{H} = 4b(bc-d)xy$. This means that if $b(bc-d)=0$ then the following equality holds $(b^2+d^2)H_e = \widehat{H}$.

First we consider system (23) when $d=0$. We exclude $b=0$ because of condition (24). As we showed before the two conics $H_{\mathbf{e}}=0$ and $\widehat{H}=0$ in this case coincide if and only if $c=0$ and we get the configuration **Config. 21**, see Figure 6. If $c\neq 0$ then all reducible conics are different and we get the unique configuration **Config. 22**.

Now we consider system (23) for $d=1$. By the previous comment the two conics $H_{\mathbf{e}}$ and $\widehat{H}$ coincide if and only if $b(bc-1)=0$. If $b=0$ then the two elliptic conics $H_{\mathbf{e}}$ and $\widehat{H}$ coincide and this gives us a real point $(0,-d)$ of multiplicity four and we have the configuration **Config. 21**. If $bc-1=0$ then again the two conics coincide and this gives a point $(-b,0)$ of multiplicity four and we get the configuration homeomorphically equivalent to the previous one. Assume now that $b(bc-1)\neq 0$. Then all three conics $xy=0$, $H_{\mathbf{e}}=0$ and $\widehat{H}=0$ are distinct and their centers are real singular points. We get **Config. 22**. □

Thus we completed the examination of the invariant straight lines of system (4) in the case when among the three reducible conics there is one of hyperbolic type and another one of elliptic type.

3.3.1 Phase portraits of systems of type $\mathcal{S}(\mathbf{h},\mathbf{e})$

In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (23).

Consider first system (23) when $d=0$, i.e.

$$\dot{x} = x\big[b^2+2bx+(c^2+1)x^2-y^2\big], \quad \dot{y} = y\big[-b^2+(c^2+1)x^2-y^2\big]. \quad (25)$$

Because of the condition (24) we have $b\neq 0$.

Assume that $c\neq 0$. Then the system has three singular points: one saddle at the origin and two centers $(-b,-bc)$ and $(-b,bc)$, see **Picture 22** in Figure 6. The bifurcation diagram of system (25) is given in Figure 7.

Now assume that $c=0$ then system (25) has two singular points one saddle at the origin and one degenerate singular point $(-b,0)$, see **Picture 21**.

Consider now system (23) when $d=1$, i.e.

$$\begin{aligned} \dot{x} &= x\big[b^2+1+2(b+c)x+(c^2+1)x^2-y^2\big],\\ \dot{y} &= y\big[-b^2-1-2y+(c^2+1)x^2-y^2\big]. \end{aligned} \quad (26)$$

Because of the condition (24) we have $b^2+2bc-1\neq 0$.

If $b(bc-1)\neq 0$ then system (26) has three singular points: one saddle at the origin and two centers $(-b,bc-1)$ and $((b+b^3)/(1-b^2-2bc)),-((1+b^2)(bc-1))/(-1+b^2+2bc)$, see Picture 21.

If $b(bc-1)=0$ then system (26) has two singular points: one saddle at the origin and one degenerate singular point $(-(b+c)/(1+c^2),0)$, see **Picture 21**. The bifurcation diagram of system (26) is given in Figure 7.

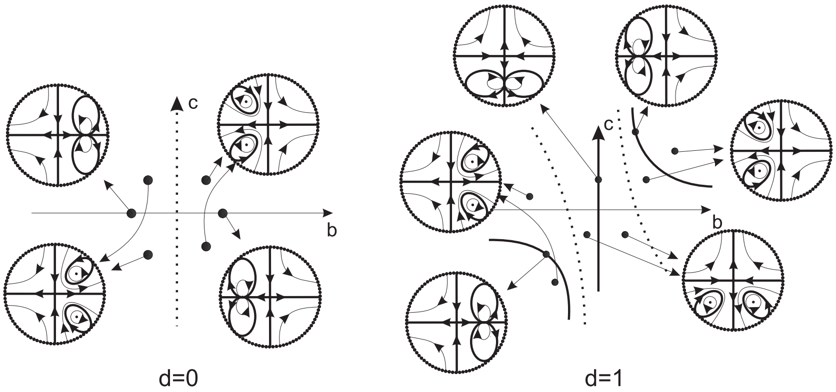

Figure 7: Bifurcation diagram of the phase portraits of system (23).

3.4 The subfamily of systems of type $\mathcal{S}(\mathbf{p}, \mathbf{p})$

Assume that system (4) possesses two different reducible conics of parabolic type, say $H_{\mathbf{p}}^{(1)}$ and $H_{\mathbf{p}}^{(2)}$. In this subsection we determine all the configurations of invariant straight lines that the system can have and their corresponding phase portraits.

Proposition 18. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{p}}^{(1)}/H_{\mathbf{p}}^{(2)}$ where $H_{\mathbf{p}}^{(1)}$ and $H_{\mathbf{p}}^{(2)}$ are reducible conics of parabolic type. Then this system can be written in the form*

$$\dot{x} = y(x^2 + a), \quad \dot{y} = x(y^2 + b), \tag{27}$$

where $a, b \in \mathbb{R}$, having the first integral $H = (x^2 + a)/(y^2 + b)$. Moreover, all the possible configurations of invariant straight lines of this system which have not appeared in Propositions 13, 15, 16 and 17 are given in Figure 8.

Proof. Assume that system (4) possesses two distinct reducible conics of parabolic type $H_{\mathbf{p}}^{(i)} = L_1^{(i)} L_2^{(i)}$ $(i = 1, 2)$. In order to have a cubic system, according to Lemma 11 we shall consider that the quadratic homogeneous parts of $H_{\mathbf{p}}^{(1)}$ and $H_{\mathbf{p}}^{(2)}$ are not proportional. This means that we have two couples of parallel lines crossing in two distinct directions, say the direction of the line $L_1 = ax + by = 0$ and $L_2 = cx + dy = 0$, with $ad - bc \neq 0$. Then via the linear transformation $x_1 = L_1$ and $y_1 = L_2$, we get the following first integral $H = (x^2 + a)/(y^2 + b)$. Therefore applying the time rescaling $t \to t/2$ we arrive to the family of systems (27) where $a, b \in \mathbb{R}$. Moreover, due to the rescaling $(x, y, t) \mapsto (|a|^{1/2}x, y, |a|^{-1}t)$ if $a \neq 0$ we may assume $a \in \{0, \pm 1\}$.

To determine the type of the third reducible conic (say $\widehat{H}$) of this system according to Corollary 10 we calculate

$$\mathcal{E}_1 = (a + x^2)(b + y^2)(ay^2 - bx^2). \tag{28}$$

So $\widehat{H} = ay^2 - bx^2$. If $ab > 0$ then the conic $\widehat{H} =$ is of hyperbolic type and system (27) is included in the family $\mathcal{S}(\mathbf{h}, \mathbf{p})$ that has been studied before. Thus we assume $ab \leq 0$.

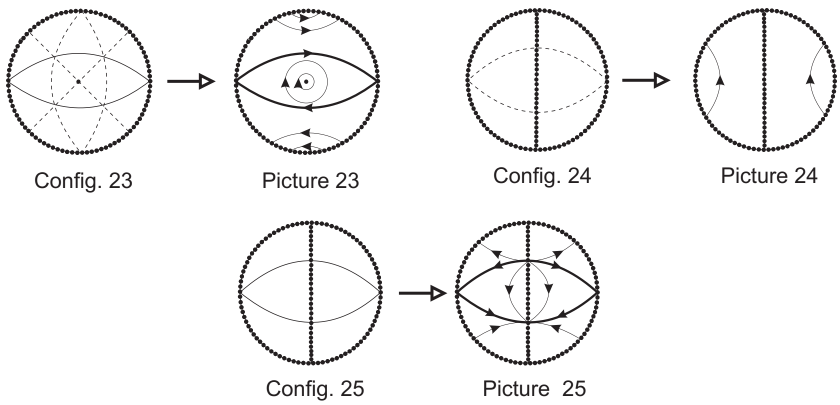

Figure 8: Configurations of invariant straight lines and their corresponding phase portraits of system (27).

If $ab < 0$ $\widehat{H}$ is of elliptic type and we get the configuration **Config. 23** of Figure 8.

If $ab = 0$ we my assume $a = 0$ (due to the change $(x, y) \mapsto (y, x)$). So if $b > 0$ we get **Config. 24** and if $b < 0$ we get **Config. 25**. If $b = 0$ then we get a degenerate system that already was examined (**Config. 9**). □

For the bifurcation of the configurations of invariant straight lines for system (27) see Figure 9.

3.4.1 Phase portraits of systems of type $\mathcal{S}(\mathbf{p}, \mathbf{p})$

In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (27) given in Figure 8.

Consider system (27) having the **Config. 23** of invariant straight lines, i.e. when parameters $ab < 0$. The system has only one singular point at the origin which is a center. There are also two parallel lines surrounding the origin either $x^2 + a = 0$ when $a < 0$ or $y^2 + b =$ when $b < 0$. Thus the phase portrait corresponding to **Config. 23** is **Picture 23**, see Figure 8.

Now consider system (27) having the configuration **Config. 24**, i.e.

$$\dot{x} = x^2 y, \quad \dot{y} = x(y^2 + b), \tag{29}$$

where $b > 0$. There is a line of singular points $x = 0$ and no isolated singular points. There are no real invariant straight lines because $\mathcal{E}_1 = -bx^4(b + y^2)$. We get the phase portrait **Picture 24**.

Finally we determine the phase portrait of system (27) having **Config. 25**, i.e. system (29) when $b < 0$. There are two straight invariant lines $y^2 + b = 0$ and a line of singular points $x = 0$. We get the phase portrait **Picture 25**.

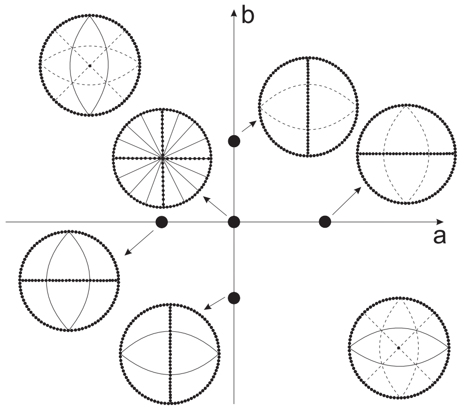

Figure 9: Bifurcation diagram of the configurations of the invariant straight lines for system (27).

3.5 The subfamily of systems of type $\mathcal{S}(\mathbf{p},\mathbf{e})$

Assume that system (4) possesses two different reducible conics one of parabolic type, say $H_{\mathbf{p}}$ and another one of elliptic type $H_{\mathbf{e}}$. In this subsection we determine all the configurations of invariant straight lines that these kind of systems can have and their corresponding phase portraits.

Proposition 19. *Assume that a cubic system has a rational first integral of degree two of the form $H_{\mathbf{p}}/H_{\mathbf{e}}$ where $H_{\mathbf{p}}$ and $H_{\mathbf{e}}$ are reducible conics of parabolic and elliptic type respectively. Then this system can be written in the form*

$$\begin{aligned} \dot{x} &= 2y\big(a^2 + b + 2ax + x^2\big), \\ \dot{y} &= -2\big((a^2 + b)x + ax^2 - ay^2 - xy^2\big), \end{aligned} \tag{30}$$

where $a, b \in \mathbb{R}$, having the first integral $H = ((x+a)^2+b)/(x^2+y^2)$. Moreover, all the possible configurations of invariant straight lines of this system which have not appeared in Propositions 13, 15, 16, 17 and 18 are given in Figure 10.

Proof. Assume that system (4) possesses one reducible conic of parabolic type, say $H_{\mathbf{p}}$ and one reducible conic of elliptic type $H_{\mathbf{e}}$. Then by Proposition 22 we may assume $H_{\mathbf{e}} = x^2 + y^2$ due to an affine transformation and $H_{\mathbf{p}} = (cx + dy + a)^2 + b$ where $a, b, c, d \in \mathbb{R}$. Moreover, due to a rotation (which keeps the form of $H_{\mathbf{e}}$) we may consider that the couple of parallel lines $H_{\mathbf{p}}$ is of the form $(x + a)^2 + b$. Thus we can assume that the system has the first integral $H = ((x + a)^2 + b)/(x^2 + y^2)$.

If $a = 0$ then the system belongs (up to a time rescaling) to the family (27) (when $a = -b$). So we do not obtain new configurations of invariant straight lines. Thus we assume that $a \neq 0$.

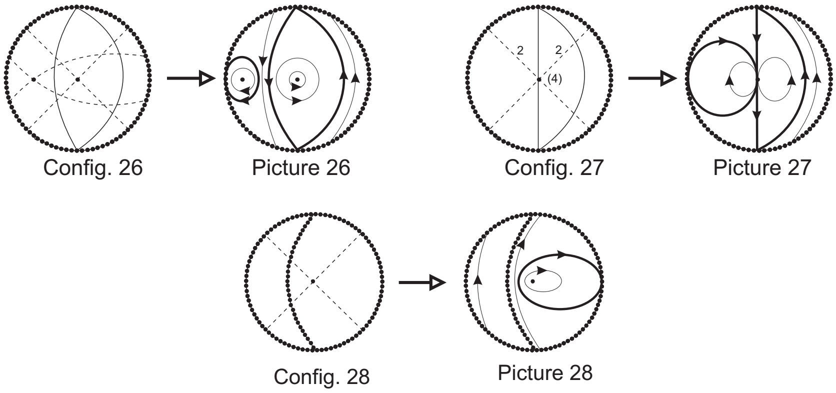

Figure 10: Configurations of invariant straight lines and their corresponding phase portraits of system (30).

To determine the type of the third reducible conic (say $\widehat{H}$) of this system according to Corollary 10 we calculate

$$\mathcal{E}_1 = 8(x^2 + y^2)((x + a)^2 + b)\widehat{H}, \tag{31}$$

where $\widehat{H} = a^2(a^2 + 2b) + b^2 + 2a(a^2 + b)x + a^2x^2 - by^2$. For the conic $\widehat{H}$ we have the invariants $\Delta = 0$ and $\delta = -ba^2$ (see the Appendix). So for $b > 0$ we have $\delta < 0$ and the third reducible conic is of hyperbolic type. We skip this case since systems of $\mathcal{S}(\mathbf{h}, \mathbf{p})$ type have already been considered. If $b < 0$ then $\delta > 0$ and the $\widehat{H}$ is of elliptic type. If $b \neq -a^2$, i.e. when the reducible conic $x^2 + y^2 = 0$ and $(x + a)^2 + b = 0$ have no points in common in the real plane we have **Config. 26** of Figure 10 . If $b = -a^2$ then we have **Config. 27**. Finally if $b = 0$ we get the degenerate system

$$\dot{x} = 2(x + a)^2 y, \qquad \dot{y} = -2(x + a)(ax - y^2), \tag{32}$$

and we have **Config. 28**. □

3.5.1 Phase portraits of systems of type $\mathcal{S}(\mathbf{p}, \mathbf{e})$

In this subsection we determine the phase portraits for each of the configuration of invariant straight lines of system (30) given in Figure 10.

We determine the phase portraits of system (30) having the configuration of invariant straight lines **Config. 26**, i.e. when $b < 0$ and $b \neq -a^2$. We have two centers: one at the origin and one at $\big((-a^2 - b)/a, 0\big)$. Moreover, we have two real invariant straight lines given by $(x + a)^2 + b = 0$. We get the phase portrait **Picture 26** of Figure 10.

Now we determine the phase portrait of system (30) having **Config. 27**, i.e.

$$\dot{x} = 2y\big(2ax + x^2\big), \quad \dot{y} = -2\big(ax^2 - ay^2 - xy^2\big),$$

where $a \neq 0$. There is only one singular point at the origin (which is of multiplicity four)and two real straight parallel invariant lines given by $x(x+2a) = 0$, see the **Picture 27**.

Finally we consider the phase portrait of system (30) having **Config. 28**, i.e. system (32). There is a line of singular points $x + a = 0$ and one center at the origin. For the phase portrait see **Picture 28**.

3.6 The subfamily of systems of type $\mathcal{S}(\mathbf{e}, \mathbf{e})$

Assume that system (4) possesses two distinct reducible conics of elliptic type, say $H_{\mathbf{e}}^{(i)} = L^{(i)}\bar{L}^{(i)}$ $(i = 1, 2)$. Then according to Proposition 22 we may assume $H_{\mathbf{e}}^{(2)} = x^2 + y^2$ due to an affine transformation. On the other hand we have $H_{\mathbf{e}}^{(1)} = (ax + by + c)^2 + (dx + ey + f)^2$, where $ae - bd \neq 0$ (as $H_{\mathbf{e}}^{(1)}$ is of elliptic type). Since a rotation keeps the form of $H_{\mathbf{e}}^{(2)}$ we may assume that the line $ax + by + c = 0$ is parallel to y–axis and we get $H = \big((ax+c)^2+(dx+ey+f)^2\big)/(x^2+y^2)$. As $e \neq 0$ we may assume $e = 1$ due to the rescaling $(x, y, t) \mapsto (x/e, y/e, e^2 t)$ and system (4) (after an additional time rescaling) becomes of the form

$$\begin{aligned}
\dot{x} &= 2(c^2 + f^2)y - 2fx^2 + 2(2ac + 2df)xy + 2fy^2 - 2dx^3 \\
&\quad + 2(-1 + a^2 + d^2)x^2 y + 2dxy^2, \\
\dot{y} &= -2(c^2 + f^2)x - 2(ac + df)x^2 - 2(-ac - df)y^2 - 4fxy - 2dx^2 y \\
&\quad - 2(1 - a^2 - d^2)xy^2 + 2dy^3).
\end{aligned} \tag{33}$$

To determine the type of the third reducible conic of these systems according to Corollary 10 we calculate

$$\mathcal{E}_1 = (x^2 + y^2)\big((ax + c)^2 + (dx + ey + f)^2\big)\hat{H},$$

where $\hat{H}$ is given by

$$\big(c^2{+}f^2{+}({-}c{+}ac{+}df)x{+}(cd{+}f{-}af)y\big)\big(c^2{+}f^2{+}(c{+}ac{+}df)x{-}(cd{+}f{+}af)y\big)$$

We observe that this reducible conic $\hat{H}$ is the product of two real lines, i.e. it is either of hyperbolic or parabolic type. Since systems $\mathcal{S}(\mathbf{e}, \mathbf{h})$ and $\mathcal{S}(\mathbf{e}, \mathbf{p})$ have already been considered we skip this case.

3.7 The subfamily of system (4) which possesses a single reducible conic - which is of hyperbolic type

Assume that system (4) possesses only one reducible conic of hyperbolic type. In this subsection we determine all the configurations of invariant straight lines that these kind of system can have and their corresponding phase portraits.

Proposition 20. *Assume that a cubic system possesses a single reducible conic of hyperbolic type and does not have other reducible conics. Then this system can be written in the form*

$$\dot{x} = x\big(e + cx + ax^2 - by^2\big), \quad \dot{y} = y\big(-e - dy + ax^2 - by^2\big), \tag{34}$$

where $a^2+b^2\neq 0$ *having the first integral* $H=(ax^2+by^2+cx+dy+e)/(xy)$. *Moreover, the configurations of invariant straight lines of this system which have not appeared in Propositions 13, 15, 16, 17, 18 and 19 are given in Figure 11.*

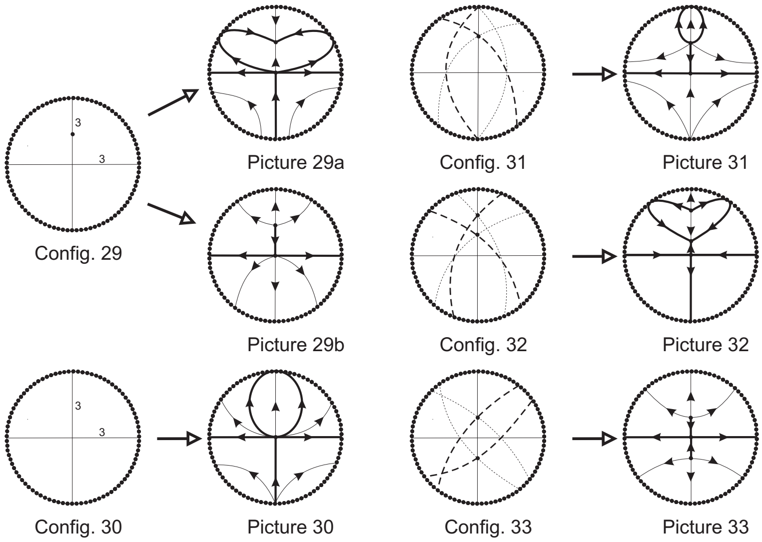

Figure 11: Configuration of invariant straight lines of system (34) and their corresponding phase portraits.

Proof. Assume that system (4) possesses a single reducible conic of hyperbolic type. Then without loss of generality we may assume that the system has a first integral of the form $H=w(x,y)/(xy)$, where $w(x,y)=ax^2+by^2+cx+dy+e$. We notice that the polynomial $w(x,y)$ must be irreducible otherwise $w(x,y)=0$ would consist of two real or complex lines and these kind of systems were studied before.

We notice that $w(x,y)=0$ needs to be a curve affinely equivalent to a real ellipse. Otherwise would have four non–real points of intersection s_1, s_2, s_3 and s_4 of $w(x,y)=0$, $xy=0$ pairwise conjugate, i.e. $s_1=\bar{s}_2$ and $s_3=\bar{s}_4$. We denote again by $\langle AB\rangle$ the straight line passing through the points A and B. The three reducible conics are $xy=0$, a product of invariant lines $\langle s_1s_3\rangle\langle s_2s_4\rangle$ and $\langle s_1s_4\rangle\langle s_2s_3\rangle$. Since the lines of each product are conjugate they intersect in the real point which has to be a center and we skip this case as we already considered the system of type $\mathcal{S}(\mathbf{h},\mathbf{e})$.

Moreover, we can assume that $w(x,y)=0$ does not intersect the conic $xy=0$ in two points $(x_0,0)$ and $(0,y_0)$ such that $x_0y_0\neq 0$ otherwise the straight line passing through these two points would be invariant and again we would be in a case studied before. We consider two cases: $w(0,0)=0$ and $w(0,0)\neq 0$.

Case: $w(0,0)=0$. This implies that $e=0$. Moreover, without loss of generality we can assume that $w(x,y)=0$ is tangent to $y=0$ (otherwise $w(x,y)$ intersects $xy=0$ in two points) so we have $c=0$. Since $c=e=0$ we get that $d\neq 0$ otherwise the conic $w(x,y)=0$ would be reducible. By a time rescaling we can assume $d=-1$. Now we consider two possibilities either $w(x,y)$ intersects the line $x=0$ at two points or only at one.

Consider that $w(x,y)=ax^2+by^2-y=0$ intersect the line $x=0$ in two different points, one the origin and the other say $(0,\tilde{y})$. By a time rescaling $y\to\alpha y$ we can assume that $\tilde{y}=1$. In short, we get the first integral $H=(ax^2+y^2-y)/(xy)$ of the system

$$\dot{x}=x(ax^2-y^2),\qquad \dot{y}=y(ax^2+y-y^2). \tag{35}$$

Finally, since $a\neq 0$, otherwise ax^2+y^2-y would not be irreducible, we can assume that $a=\pm 1$. There are two singular points, the origin and $(0,1)$. Calculating $\mathcal{E}_1=ax^3y^3$ we see that the only invariant reducible conic of system (36) is $xy=0$ having multiplicity 3. We get **Config. 29**, see Figure 11.

Now we assume that $w(x,y)=ax^2+by^2-y=0$ intersects the line $x=0$ only at the origin. This implies that $b=0$ and by the rescaling $x\to x/a$ we can assume that $a=1$. We get the first integral $H=(x^2-y)/(xy)$ of the system

$$\dot{x}=x^3,\qquad \dot{y}=y(x^2+y). \tag{36}$$

Similarly, calculating $\mathcal{E}_1=x^3y^3$ we conclude that $xy=0$ is the only reducible conic having multiplicity 3 and we get **Config. 30**.

Case: $w(0,0)\neq 0$. So $e\neq 0$ (by rescaling $e=1$) thus we can assume that $w(x,y)=ax^2+by^2+cx+dy+1$. Now we consider two cases $b=0$ and $b\neq 0$.

If $b=0$ then $d\neq 0$ otherwise $w(x,y)$ would factorize. By the rescaling $y\to\alpha y$ we can assume that $w(x,y)=0$ crosses the $x=0$ at point $(0,1)$. Thus we get that $d=-1$. Also by a rescaling of the x–axis we can assume that $a=1$. Thus we get the first integral $H=(x^2+cx-y+1)/(xy)$ of the system

$$\dot{x}=x\big(1+cx+x^2\big),\qquad \dot{y}=y\big(-1+y+x^2\big), \tag{37}$$

Since $x^2+cx-y+1=0$ must not have points in common with $y=0$ we have $|c|<2$. To determine all the six invariant straight lines we calculate

$$\mathcal{E}_1=xy(1+cx+x^2)(1+cx-2y+x^2-cxy+y^2).$$

We show that even though we have six invariant straight lines, the only real reducible conic that system (37) possesses, according to (7), is $xy=0$. Consider the conic $1+cx+x^2=0$. It is clear that does not exist α and β (as in (7)) such that $1+cx+x^2=\alpha(x^2+cx-y+1)+\beta(xy)$, so system (37) does not possess the conic $1+cx+x^2=0$. This shows that even though this reducible conic is invariant for system (37) and its two imaginary invariant straight lines lie on different level sets (complex conjugate) of the first integral H. The same can be show for the conic $1+cx-2y+x^2-cxy+y^2=0$. For this reason we cannot introduce more then one reducible invariant conic into the expression of the first integral H. We get **Config. 31**.

To make the previous argument easier to understand consider a particular case of system (37) for $c = 0$. Then calculating $\mathcal{E}_1 = 0$ we get the following six invariant straight lines for this system

$$xy(x^2+1)(1-2y+x^2+y^2) = xy(x-i)(x+i)(i+x-iy)(-i+x+iy) = 0,$$

having the first integral $H = (x^2-y+1)/(xy)$. The system we are considering possesses the reducible conic $xy = 0$, since according to (7) we can choose $\alpha = 0$ and $\beta = 1$ such that $xy = \alpha H_N + \beta H_D$, where $H_N = x^2 - y + 1$ is the numerator and $H_D = xy$ is the denominator of the first integral H. This conic $xy = 0$ corresponds to ∞ level curve of the first integral H, i.e $H = \infty$. It is easy to see that a similar equality does not hold for the conic $x^2 + 1 = 0$ or $1 - 2y + x^2 + y^2 = 0$, for any real values of α and β. So system (37) for $c = 0$ does not possess neither of these conics. In another words do not exist real level values $h_1, h_2 \in \mathbb{R}$ for which the level curves $H = h_1$ and $H = h_2$ are these reducible conics. But there are two complex conjugate level values i and $-i$ for which the level curves $\{(x,y) \in \mathbb{C}^2 : H = -i\} = \{(x,y) \in \mathbb{C}^2 : x^2 + ixy - y + 1 = (x+i)(x+iy-i)\}$ and $\{(x,y) \in \mathbb{C}^2 : H = i\} = \{(x,y) \in \mathbb{C}^2 : x^2 - ixy - y + 1 = (x-i)(x-iy+i)\}$ are the straight invariant lines.

Assume now that $b \neq 0$. We can assume that $a \neq 0$ since otherwise doing the change of variables $(x,y) \to (y,x)$ we would arrive to the system considered previously (system (37)). We consider two cases $ab > 0$ and $ab < 0$. If $ab > 0$ then by a rescaling of both axes $(x,y) \to (x/a, y/b)$ we get $w(x,y) = x^2+y^2+cx+dy+1$. So we have the first integral $H = (x^2+y^2+cx+dy+1)/(xy)$ of the system

$$\dot{x} = x\big(1 + cx + x^2 - y^2\big), \quad \dot{y} = y\big(-1 - dy + x^2 - y^2\big). \tag{38}$$

Since $w(x,y) = 0$ must not have points in common with $y = 0$ we get $|c| < 2$. The oval $w(x,y)$ has to cross $x = 0$ in two distinct points so $|d| > 2$ otherwise we get a system that belongs to the family $\mathcal{S}(\mathbf{h},\mathbf{e})$. To determine all the invariant straight lines we calculate

$$\begin{aligned}\mathcal{E}_1 &= xy\big(1 + 2cx + (2+c^2)x^2 + 2cx^3 + x^4 + 2dy + 3cdxy \\ &+ (2d + c^2 d)x^2 y + cdx^3 y + (-2+d^2)y^2 + (-2c + cd^2)xy^2 \\ &+ (2 - c^2 + d^2)x^2y^2 - 2dy^3 - cdxy^3 + y^4\big).\end{aligned}$$

For the fixed real values of c, d the polynomial $\mathcal{E}_1$ always factorizes since according to Corollary 10 it describes six invariant straight lines. Each invariant conic (real or complex) of system (38) pass through four points: two imaginary $I_{1,2} = (c/2 \pm (c^2-4)^{1/2}, 0)$ and two real $R_{1,2} = (0, d/2 \pm (d^2-4)^{1/2})$ defined by the system of equations $H_N := x^2 + y^2 + cx + dy + 1 = 0$, $H_D := xy = 0$. We denote again by $\langle AB \rangle$ a line passing through a point A and B. Then we have we have six invariant lines: two real xy=0, and four imaginary $\langle I_1R_1 \rangle$, $\langle I_1R_2 \rangle$, $\langle I_2R_1 \rangle$ and $\langle I_2R_2 \rangle$. Analogously to the previous case we can show that the only reducible conic that system (38) possesses is $xy = 0$ and we get **Config. 32**.

Finally consider now $ab < 0$. Without loss of generality we can assume that $a > 0$ and $b < 0$. By the rescaling $(x,y) \to (x/a, y/b)$ we get $w(x,y) =$

$x^2-y^2+cx+dy+1$. So we have the first integral $H=(x^2-y^2+cx+dy+1)/(xy)$ of the system

$$\dot{x}=x\big(1+cx+x^2+y^2\big),\quad \dot{y}=y\big(-1-dy+x^2+y^2\big). \tag{39}$$

Again since $w(x,y)=0$ must not have points in common with $y=0$ we get $|c|<2$. So to determine the configuration we calculate

$$\begin{aligned}\mathcal{E}_1 &= xy\big(1+2cx+(2+c^2)x^2+2cx^3+x^4+2dy+3cdxy\\ &+ (2d+c^2d)x^2y+cdx^3y+(2+d^2)y^2+(2c+cd^2)xy^2\\ &+ (-2+c^2+d^2)x^2y^2+2dy^3+cdxy^3+y^4\big).\end{aligned}$$

Here also the only reducible conic that system (39) possesses is $xy=0$. The four other invariant straight are $\langle I_1R_1\rangle$, $\langle I_1R_2\rangle$, $\langle I_2R_1\rangle$ and $\langle I_2R_2\rangle$, where $I_{1,2}=(c/2\pm(c^2-4)^{1/2},0)$ and $R_{1,2}=(0,d/2\pm(d^2+4)^{1/2})$. Thus we have **Config. 33**.

□

3.7.1 Phase portraits of system (34)

In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (34).

First consider system (34) having the configuration of invariant straight lines **Config. 29**, i.e. the system

$$\dot{x}=x\big(ax^2-y^2\big),\quad \dot{y}=y\big(y+ax^2-y^2\big),$$

where $a\neq 0$. We have two singular points: one at the origin and one at $(0,1)$. For $a>0$ we get **Picture 29a** and for $a<0$ we get **Picture 29b**.

We determine the phase portrait of system (34) having **Config. 30**. Thus we consider system

$$\dot{x}=ax^3,\quad \dot{y}=y\big(y+ax^2\big),$$

where $a\neq 0$. We have only one degenerate singular point at the origin and no other singular points. We get the phase portrait **Picture 30**.

We consider now system (34) having **Config. 31**, i.e. system (37). The system has two singular points: one saddle at the origin and a node at $(0,1)$. The phase portrait is given in **Picture 31**.

To determine the phase portrait of system (34) having **Config. 32** we consider the system (38) There are three singular points: one saddle at the origin, and two nodes $(0,-d/2\pm\sqrt{d^2-4}/2)$. We have the phase portrait **Picture 32**.

Finally we study the phase portrait of system (34) having **Config. 33**. Thus we consider the system (39). There are three singular points: one saddle at the origin and two nodes $(0,-d/2\pm\sqrt{d^2+4}/2)$. We get **Picture 33**.

3.8 The subfamily of systems which possesses a single reducible conic - which is of parabolic type

Assume that system (4) possesses only one reducible conic of parabolic type. In this subsection we shall determine all configurations of invariant straight lines that these kind of system can have and their corresponding phase portraits.

As we know (see the Appendix) there are three normal forms of the reducible conic of parabolic type, namely real parallel lines, complex parallel lines and double line.

Proposition 21. *Assume that a cubic system possesses a single reducible conic of parabolic type and does not have other reducible conics. Then this system can be written in the form*

$$\begin{aligned} \dot{x} &= -(p+x^2)(e+bx+2cy), \\ \dot{y} &= dp - 2fx - dx^2 + bpy - 2exy - bx^2y - 2cxy^2, \end{aligned} \tag{40}$$

having the first integral $H = (bxy + cy^2 + dx + ey + f)/(x^2 + p)$*. Moreover, all the configuration of invariant straight lines of this system which have not appeared in Propositions 13, 15, 16, 17, 18, 19 and 20 are given in Figure 12.*

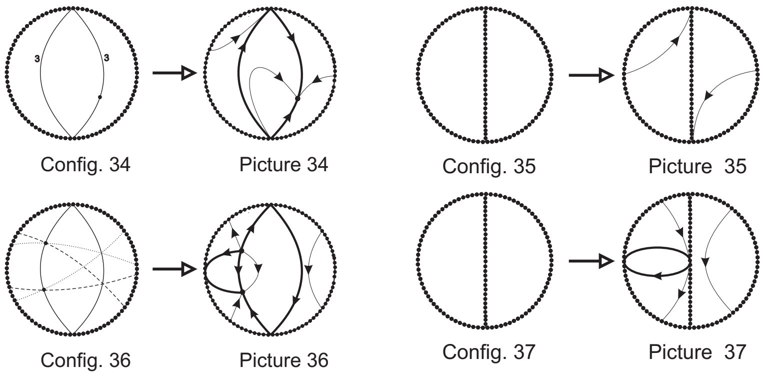

Figure 12: Configurations of invariant straight lines of system (34) and their corresponding phase portraits.

Proof. We assume that system (3) has a single reducible conic of parabolic type $H_{\mathbf{p}} = 0$. By an affine change of variable we assume that $H_{\mathbf{p}} = x^2 + p$ where $p \in \{0, \pm 1\}$. Thus the first integral of the systems possessing only one reducible conic of parabolic type can be written as $H = w(x,y)/(x^2+p)$, where $w(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$. We notice that $w(x,y)$ must be irreducible otherwise $w(x,y) = 0$ would consist of irreducible lines (complex or real) what would contradict our assumption of having only one reducible conic. Since in a real linear system of conics there is at least one conic which contains a real line, see for more details [5], and recall our case the infinity is always degenerate, we can assume that $p \in \{0, -1\}$. Without loss of generality we can also assume that $a = 0$. Now we consider two cases $c = 0$ and $c \neq 0$.

Case $c = 0$. We have the first integral $H = (bxy + dx + ey + f)/(x^2 + p)$, and we have to assume that $b \neq 0$ otherwise system (3) would be quadratic.

So by a time rescaling we have $b = 1$ as well as $d = 0$ by the change of variable $y \to y - d$. Clearly $f \neq 0$ otherwise the numerator factorizes so by the time rescaling $f = 1$. We end up with the first integral $H = (xy + ey + 1)/(x^2 + p)$.

If $p = -1$ then we have to assume that $e = \pm 1$, otherwise we would have another invariant straight line passing through points $(1, -1/(e+1))$ and $(-1, -1/(e-1))$. We can assume that $e = 1$ since if it is negative we change the variables $(x, y) \to (-x, -y)$ and we are done. Finally we get the following system

$$\dot{x} = -1 + x + x^2 - x^3, \quad \dot{y} = -2x - y + 2xy - x^2 y,$$

having the first integral $H = (xy + y + 1)/(x^2)$. To determine the configuration of invariant straight lines according to Corollary 10 we calculate $\mathcal{E}_1 = 2(x-1)^3(x+1)^3$, and we get **Config. 34**.

Now if $p = 0$ we have the first integral $H = (xy + ey + 1)/x^2$. Clearly $f \neq 0$ otherwise the numerator of the first integral would be $xy + ey = y(x + e)$ and two invariant straight line would appear contradicting our assumption. Now if $e = 0$ then we have a degenerate system

$$\dot{x} = -x^2, \quad \dot{y} = -x(2 + xy).$$

To determine the configuration of invariant straight lines we calculate $\mathcal{E}_1 = 2x^6$, and we get **Config. 35**. If $e \neq 0$ then by an appropriate time and axes rescaling we arrive at $H = (xy + y + 1)/x^2$ thus system (3) is

$$\dot{x} = -x^2(1 + x), \quad \dot{y} = -x(2 + 2y + xy),$$

and we have a configuration that is topologically equivalent to **Config. 20**.

Case $c \neq 0$. Here by a time rescaling we have $c = 1$. So we get the first integral $H = (bxy + y^2 + dx + ey + f)/(x^2 + p)$. Now we can assume that $b = 0$, first by the change of variable $y \to -b/2x + y$ and then by canceling the coefficient of x^2 in the numerator of the first integral which appears after the change of variable. We have the first integral $H = (y^2 + dx + ey + f)/(x^2 + p)$. Again by the change $y \to y - e/2$ we have $e = 0$. We can also assume that $d > 0$ since if it was zero we would have additional invariant straight lines and if negative we do the change of variable $x \to -x$. Thus we can assume that $d = 1$. We end up with the first integral $H = (y^2 + x + f)/(x^2 + p)$. Let $p = -1$ then we get

$$\dot{x} = 2y - 2x^2 y, \quad \dot{y} = -1 - 2fx - x^2 - 2xy^2, \tag{41}$$

having the first integral $H = (y^2 + x + f)/(x^2 - 1)$, where $|f| < 1$. Calculating $\mathcal{E}_1 = 8(x-1)(x+1)\tilde{H}$, where $\tilde{H} = 1 + 4fx + (2 + 4f^2)x^2 + 4fx^3 + x^4 + 4fy^2 + 8xy^2 + 4fx^2y^2 + 4y^4$ we get the configuration of invariant straight lines. These lines pass through four points: two imaginary $I_{1,2} = (\pm i(-f-1)^{1/2}, 0)$ and two real $R_{1,2} = (0, \pm(1-f)^{1/2})$ defined by the system of equations $y^2 + x + f = 0$, $(x^2 - 1) = 0$. We have two real invariant straight lines $xy = 0$ and four imaginary $\langle I_1 R_1 \rangle$, $\langle I_1 R_2 \rangle$, $\langle I_2 R_1 \rangle$ and $\langle I_2 R_2 \rangle$. The only reducible conic that system (41) possesses is $xy = 0$. We get **Config. 36**.

If $p = 0$ we have the first integral $H = (y^2 + x + f)/x^2$. We consider two possibilities: $f = 0$ or $f \neq f$. If $f = 0$ then we the system

$$\dot{x} = -2x^2 y, \quad \dot{y} = -x^2 - 2xy^2, \tag{42}$$

having the first integral $H = (y^2 + x)/x^2$ and we get the configuration of the invariant straight lines **Config. 37**.

By the change of variables $x \to fx$, $y \to |f|^{1/2}y$, and the time rescaling we get $H = (\pm y^2 + x + 1)/x^2$. We skip the system

$$\dot{x} = -2x^2 y, \quad \dot{y} = -x(2 + x + 2y^2),$$

having the first integral $H = (y^2+x+1)/x^2$, since it belongs to family $\mathcal{S}(\mathbf{p}, \mathbf{e})$. We also skip the system

$$\dot{x} = 2x^2 y, \quad \dot{y} = -x(2 + x - 2y^2),$$

having the first integral $H = (-y^2 + x + 1)/x^2$, because it has been studied before (**Config. 13**). □

3.8.1 Phase portraits of system (40)

In this subsection we determine the phase portraits for each configuration of invariant straight lines of system (40).

Consider system (40) having **Config. 34**, i.e.

$$\dot{x} = -(x^2 - 1)(1 + x), \quad \dot{y} = -2x - y - 2xy - x^2 y.$$

There is only one singular point, a node, at $(1, -1/2)$. We have the phase portrait **Picture 34**.

Now we consider (40) having **Config. 35**. Thus we study the system

$$\dot{x} = -x^3, \quad \dot{y} = -x(2 + xy).$$

There is a line of singular points $x = 0$ and no other singular points. We get the phase portrait **Picture 35**.

We consider the phase portrait of system (40) having the configuration of invariant straight lines **Config. 36**. Thus we study

$$\dot{x} = -2y(x^2 - 1), \quad \dot{y} = -1 - 2fx - x^2 - 2xy^2,$$

where $|f| < 1$. The only singular points of the system are two nodes $(0, \pm\sqrt{1 - f})$ and we get the phase portrait **Picture 36**.

Finally we determine the phase portrait of system (42). We notice that there is a line of singular points $x = 0$ and no other singular points. We get the phase portrait **Picture 37**.

3.9 Appendix 1: The conics

We want to recall basic information about the conics. The starting point is to note that every conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{43}$$

can be written in matrix form as $vAv^T = 0$ where

$$A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \qquad v^T = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

and A is called the *matrix* of the conic (43). By B we denote the *leading matrix* of A, i.e.

$$B = \begin{pmatrix} a & h \\ h & b \end{pmatrix}.$$

We define two numbers $\Delta = \det A$ and $\delta = \det B$.

After an affine change of coordinates any conic can be represented as one of the nine canonical forms shown in Proposition 22. By calculating the Δ and δ invariants for a given conic we can tell to which class it belongs as it is shown in the next proposition. We do not distinguish between real parallel lines, complex parallel lines and a double line. We also do not distinguish between real and complex ellipse.

Proposition 22. *Let $g = 0$ be a conic in $\mathbb{R}^2$, then g is affinely equivalent to one of the following nine normal forms,*

NormalForms	Conic	Δ invariant	δ invariant
$x^2 + y^2 - 1 = 0$	real ellipse	$\Delta \neq 0$	$\delta > 0$
$x^2 + y^2 + 1 = 0$	complex ellipse	$\Delta \neq 0$	$\delta > 0$
$x^2 - y^2 - 1 = 0$	hyperbola	$\Delta \neq 0$	$\delta < 0$
$x^2 - y = 0$	parabola	$\Delta \neq 0$	$\delta = 0$
$x^2 - y^2 = 0$	real line-pair	$\Delta = 0$	$\delta < 0$
$x^2 + y^2 = 0$	complex line-pair	$\Delta = 0$	$\delta > 0$
$x^2 - 1 = 0$	real parallel lines	$\Delta = 0$	$\delta = 0$
$x^2 + 1 = 0$	complex parallel lines	$\Delta = 0$	$\delta = 0$
$x^2 = 0$	double line	$\Delta = 0$	$\delta = 0.$

(44)

Proof. For the proof see [4]. □

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