

ANALYTIC INTEGRABILITY OF HAMILTONIAN SYSTEMS WITH A HOMOGENEOUS POLYNOMIAL POTENTIAL OF DEGREE 4

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ABSTRACT. In the analytic case we prove the conjecture of A.J. Maciejewski and M. Przybylska that appeared in *J. Math. Phys.* **46** (6) (2005) 062901 regarding Hamiltonian systems with a homogeneous polynomial potential of degree 4. The proof of the conjecture completes the characterization of all the analytic integrable Hamiltonian system with a homogeneous polynomial potential of degree 4.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

We consider \mathbb{C}^4 as a symplectic linear space with canonical variables $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. We are interested in Hamiltonian systems defined by the Hamiltonian function

$$(1) \quad H = \frac{1}{2} \sum_{i=1}^2 p_i^2 + V(\mathbf{q}),$$

where $V(\mathbf{q}) = V(q_1, q_2)$ is a homogeneous polynomial of degree k . To be more precise we consider the following system of four differential equations

$$(2) \quad \dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2.$$

Let $A = A(\mathbf{q}, \mathbf{p})$ and $B = B(\mathbf{q}, \mathbf{p})$ be two functions. Then their *Poisson bracket* $\{A, B\}$ is given by

$$\{A, B\} = \sum_{i=1}^2 \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

We say that functions A and B *commute* or that they are *in involution* if $\{A, B\} = 0$. We say that a non-constant function $F = F(\mathbf{q}, \mathbf{p})$ is a *first integral* for the Hamiltonian system (2) if it commutes with the Hamiltonian function H , i.e. $\{H, F\} = 0$. Since the Poisson bracket is antisymmetric it is clear that H itself is always a first integral. We will say that a 2-degree of freedom Hamiltonian system (2) is *completely* or *Liouville integrable* if it has 2 functionally independent first integrals: H , and an additional one F , which are in involution.

In the beginning of 80's all integrable Hamiltonian systems (1) with homogeneous polynomial potential of degree at most 5 and having a second polynomial first integral up to degree 4 in the variables p_1 and p_2 were found, see [11, 4, 2, 5, 1] and also [6] for the list of corresponding additional first integrals. We remark that all these first integrals are polynomials in the variables p_1, p_2, q_1 and q_2 . The main tools used there in order to identify these integrable systems were Panlevé test [3] and direct methods [7].

Key words and phrases. Analytic integrability, Hamiltonian system with 2-degrees of freedom, homogeneous polynomial potential.

An elegant result related with the integrability of Hamiltonian systems with a homogeneous polynomial potential was given by Morales and Ramis (see [10, p. 100] and references therein), which gives the necessary condition for the complete meromorphic integrability of such systems.

Using the result of Morales–Ramis, Maciejewski and Przybylska [8] gave a necessary and sufficient condition for the complete meromorphic integrability of Hamiltonian systems with the homogeneous polynomial potential of degree 3. Later on in [9] the same authors studied, among other things, the meromorphic integrability of the class of Hamiltonian systems with a homogeneous polynomial potential of degree 4. They proved that except for one family of potentials, only those systems which are already known to be integrable with a polynomial first integral, are all the meromorphically integrable systems. For this exceptional family they state:

Conjecture ([9]). *The Hamiltonian system (2) with the potential*

$$(3) \quad V = \frac{1}{2}aq_1^2(q_1 + iq_2)^2 + \frac{1}{4}(q_1^2 + q_2^2)^2,$$

is meromorphically integrable if and only if $a = 0, -2$.

For $a = 0$ and $a = -2$ the additional first integrals are given in Table 1. The case $a = 0$ is a special case of the following more general result. The Hamiltonian system (1) having a potential $V = f(q_1^2 + q_2^2)$ is integrable with the additional first integral $I = p_1q_2 - p_2q_1$.

Case	Potential V	Additional first integral
$a = 0$	$\frac{1}{4}(q_1^2 + q_2^2)^2$	$p_1q_2 - p_2q_1$
$a = -2$	$-q_1^2(q_1 + iq_2)^2 + \frac{1}{4}(q_1^2 + q_2^2)^2$	$p_1^2 + 3ip_1p_2 - 2p_2^2 + (q_1i - q_2)^3q_2$

TABLE 1. Two integrable cases of the potential (3) and the corresponding additional first integrals.

The main purpose of this article is to prove the conjecture in the analytic case, i.e.:

Theorem 1. *The Hamiltonian system (2) with the potential (3) is analytically integrable if and only if $a = 0, -2$.*

In short Theorem 1 completes the characterization of all analytic Hamiltonian systems (2) having a homogeneous polynomial potential of degree 4.

The way of proving Theorem 1 is to show that such Hamiltonian systems have no additional analytic first integrals when $a \neq 0, -2$. In section 2 we shall see that our differential system is weight–homogeneous. This fact will be used in section 3 for proving Theorem 1.

2. WEIGHT–HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEM

We consider polynomial differential systems of the form

$$(4) \quad \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4,$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), P_2(\mathbf{x}), P_3(\mathbf{x}), P_4(\mathbf{x}))$ and $P_i \in \mathbb{C}[x_1, x_2, x_3, x_4]$ for $i = 1, 2, 3, 4$. As usual \mathbb{N} , \mathbb{R} and \mathbb{C} denote the sets of positive integers, real and complex numbers, respectively; and $\mathbb{C}[x_1, x_2, x_3, x_4]$ denotes the polynomial ring over \mathbb{C} in the variables x_1, x_2, x_3, x_4 . Here t can be real or complex.

We say that system (4) is *weight-homogeneous* if there exist $\mathbf{s} = (s_1, s_2, s_3, s_4) \in \mathbb{N}^4$ and $d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{\alpha \in \mathbb{R}, \alpha > 0\}$,

$$P_i(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \alpha^{s_3}x_3, \alpha^{s_4}x_4) = \alpha^{s_i-1+d}P_i(x_1, x_2, x_3, x_4),$$

for $i = 1, 2, 3, 4$. We call $\mathbf{s} = (s_1, s_2, s_3, s_4)$ the *weight exponent* of system (4) and d the *weight degree* with respect to the weight exponent \mathbf{s} . In the particular case that $\mathbf{s} = (1, 1, 1, 1)$ system (4) is called a *homogeneous polynomial differential system of degree d* .

We say that a polynomial $F(x_1, x_2, x_3, x_4)$ is a *weight-homogeneous polynomial with weight exponent $\mathbf{s} = (s_1, s_2, s_3, s_4)$ and weight degree n* if

$$F(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \alpha^{s_3}x_3, \alpha^{s_4}x_4) = \alpha^n F(x_1, x_2, x_3, x_4),$$

If $\mathbf{s} = (1, 1, 1, 1)$ then we say that F is a *homogeneous polynomial of degree n* .

The following well-known proposition (easy to prove) reduces the study of the existence of analytic first integrals of a weight-homogeneous polynomial differential system (4) to the study of the existence of a weight-homogeneous polynomial first integrals.

Proposition 2. *Let H be an analytic function and let $H = \sum_i H_i$ be its decomposition into weight-homogeneous polynomials of weight degree i with respect to the weight exponent \mathbf{s} . Then H is an analytic first integral of the weight-homogeneous polynomial differential system (4) with weight exponent \mathbf{s} if and only if each weight-homogeneous part H_i is a first integral of system (4) for all i .*

We have the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^2 p_i^2 + \frac{1}{2} a q_1^2 (q_1 + i q_2)^2 + \frac{1}{4} (q_1^2 + q_2^2)^2,$$

and its associated Hamiltonian system

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= -a q_1^2 (q_1 + i q_2) - a q_1 (q_1 + i q_2)^2 - q_1 (q_1^2 + q_2^2), \\ \dot{p}_2 &= -i a q_1^2 (q_1 + i q_2) - q_2 (q_1^2 + q_2^2). \end{aligned} \tag{5}$$

By the change of variables $w_1 = q_1 + i q_2$, $w_2 = q_1 - i q_2$, $v_1 = p_1 + i p_2$, $v_2 = p_1 - i p_2$ the differential system (5) becomes

$$\begin{aligned} \dot{w}_1 &= v_1, \\ \dot{w}_2 &= v_2, \\ \dot{v}_1 &= -\frac{1}{2} w_1^2 [(a+2)w_2 + a w_1], \\ \dot{v}_2 &= -\frac{1}{2} w_1 [(a+2)w_2^2 + a(2w_1^2 + 3w_1 w_2)]. \end{aligned} \tag{6}$$

It has the first integral

$$\bar{H} = 4v_1 v_2 + w_1^2 [2w_2^2 + a(w_1 + w_2)^2]. \tag{7}$$

It is easy to check that system (6) is a weight-homogeneous polynomial differential system with weight exponent $(1, 1, 2, 2)$ and weight degree 2. Therefore, in view of Proposition 2 to study the existence of a second independent analytic first integral of system (6) it is enough to study the existence of a weight-homogeneous polynomial first integral of system (6) with weight exponent $(1, 1, 2, 2)$.

3. PROOF OF THE MAIN RESULT

Before going into the technicalities of the proof of our main result (Theorem 1), we would like to highlight the main idea behind it. First, we shall restrict system (6) to the zero level of the first integral \bar{H} , which is a polynomial function. The restriction to this level set gives rise to a non-trivial rational first integral \bar{F} of the restricted system. To be more precise $\bar{F}(w_1, w_2, v_2) \in C(w_1, w_2, v_2)$ is a Laurent polynomial with respect to v_2 , i.e., $\bar{F}(w_1, w_2, v_2) \in C[w_1, w_2][v_2^{-1}, v_2]$. So, it can be written in the following form

$$\bar{F} = \sum_{i=-k}^m g_i(w_1, w_2) v_2^i, \quad g_m(w_1, w_2) \neq 0,$$

and $g_{m+j}(w_1, w_2) = 0$ for $j \geq 1$. We claim that knowing the two highest order terms of $g_i(w_1, w_2)$ with respect to w_2 for $-k+2 \leq i \leq m$, we achieve a contradiction considering the terms $g_{-k}(w_1, w_2)$ and $g_{-k+1}(w_1, w_2)$.

From sections 1 and 2 it follows that for proving Theorem 1 it is sufficient to show that if $a(a+2) \neq 0$, then system (6) has no weight-homogeneous polynomial first integrals with weight exponent $(1, 1, 2, 2)$. Let $F = F(w_1, w_2, v_1, v_2) \in \mathbb{C}[w_1, w_2, v_1, v_2]$ be a weight-homogeneous polynomial first integral of system (6) with weight exponent $(1, 1, 2, 2)$ and weight degree $n \geq 1$. For each $n \geq 1$, we can express it as

$$F = \sum_{l_1+l_2+2l_3+2l_4=n} F_{l_1, l_2, l_3, l_4} w_1^{l_1} w_2^{l_2} v_1^{l_3} v_2^{l_4}.$$

The function F cannot depend only on w_1 and w_2 . Indeed, if $F = F(w_1, w_2)$ then from (6) we get

$$v_1 \frac{\partial F}{\partial w_1} + v_2 \frac{\partial F}{\partial w_2} = 0,$$

and consequently F is a constant. So F depends on v_1 or v_2 and thus $n \geq 2$.

We study the first integral F on the level $\bar{H} = 0$ by eliminating, for example, v_1 as follows

$$v_1 = -\frac{w_1^2[2w_2^2 + a(w_1 + w_2)^2]}{4v_2}.$$

Thus we end up with the following system

$$\begin{aligned} \dot{w}_1 &= -\frac{w_1^2}{4v_2} \left[aw_1(w_1 + 2w_2) + (a+2)w_2^2 \right], \\ \dot{w}_2 &= v_2, \\ \dot{v}_2 &= -\frac{w_1}{2} \left[aw_1(2w_1 + 3w_2) + (a+2)w_2^2 \right]. \end{aligned} \tag{8}$$

Note that the restriction of the polynomial first integral F to the level set $\bar{H} = 0$ can be written as

$$\begin{aligned} \bar{F} &= \sum_{l_1+l_2+2l_3+2l_4=n} \frac{(-1)^{l_3}}{4^{l_3}} \bar{F}_{l_1, l_2, l_3, l_4} w_1^{l_1} w_2^{l_2} \left[\frac{w_1^2(2w_2^2 + a(w_1 + w_2)^2)}{v_2} \right]^{l_3} v_2^{l_4} \\ &:= \sum_{i=-[n/2]}^{[n/2]} \bar{F}_i(w_1, w_2) v_2^i, \end{aligned} \tag{9}$$

where each $\bar{F}_i(w_1, w_2)$ is a homogeneous polynomial of weight degree $n - 2i$. As usual $[x]$ denotes the integer part function of x . Setting

$$A = \frac{w_1}{2} [aw_1(2w_1 + 3w_2) + (a + 2)w_2^2],$$

$$B = \frac{w_1^2}{4} [aw_1(w_1 + 2w_2) + (a + 2)w_2^2],$$

we have that \bar{F} is a first integral of system (8) if and only if it satisfies

$$(10) \quad -\frac{B}{v_2} \frac{\partial \bar{F}}{\partial w_1} + v_2 \frac{\partial \bar{F}}{\partial w_2} - A \frac{\partial \bar{F}}{\partial v_2} = 0.$$

Computing in (10) the different coefficients of v_2^j with $-[n/2] - 1 \leq j \leq [n/2] + 1$, we conclude that \bar{F} is a first integral of system (8) if and only if

$$(11) \quad \begin{aligned} \frac{\partial \bar{F}_i}{\partial w_2} &= 0, \quad \text{for } i = [n/2], [n/2] - 1, \\ \frac{\partial \bar{F}_i}{\partial w_2} - B \frac{\partial \bar{F}_{i+2}}{\partial w_1} - (i + 2)A \bar{F}_{i+2} &= 0, \quad \text{for } i = [n/2] - 2, \dots, -[n/2], \\ B \frac{\partial \bar{F}_i}{\partial w_1} + iA \bar{F}_i &= 0, \quad \text{for } i = -[n/2] + 1, -[n/2]. \end{aligned}$$

It is clear that we can express the function \bar{F} given in (9) as

$$\bar{F} = \sum_{i=-[n/2]}^m \bar{F}_i(w_1, w_2) v_2^i, \quad \bar{F}_m(w_1, w_2) \neq 0,$$

and such that $\bar{F}_{m+j}(w_1, w_2) = 0$ for $j \geq 1$.

In the following lemma we calculate the two terms of highest degree in w_2 of the functions $F_i(w_1, w_2)$. In what follows l.o.t. (w_2) will denote the lower order terms in the variable w_2 .

Lemma 3. *For $l \geq 1$ and $m - 2l \geq -[n/2]$ we have*

$$\bar{F}_m = C_0 w_1^{n-2m},$$

$$\bar{F}_{m-2l} = C_l^n C_0 w_2^{3l} w_1^{n-2m+2l} (a+2)^l + D_l^{n,m} C_0 w_2^{3l-1} w_1^{n-2m+2l+1} a (a+2)^{l-1} + \text{l.o.t.}(w_2),$$

where $C_0 \neq 0$. Moreover the coefficients of the above polynomials are

$$(12) \quad \begin{aligned} C_l^n &= \prod_{k=1}^l \frac{n-3k+3}{12k}, \\ D_l^{n,m} &= \sum_{i=1}^l \frac{n+m-5i+5}{2(3i-1)} \prod_{j=i+1}^l \frac{n-3j+4}{4(3j-1)} \prod_{r=1}^{i-1} \frac{n-3r+3}{12r}. \end{aligned}$$

We note that \bar{F}_{m-2l} are the homogeneous polynomial of weight degree $n - 2m + 4l$.

Before proceeding with the proof of Lemma 3, we would like to highlight some basic facts about the expansion of the function of several variables. First, if we integrate an expansion with respect to, e.g., the variable w_2 , then the additive constant depends generally on the remaining variables. Second, if we differentiate an expansion with respect to, e.g., variable w_2 , then we can lose control on the leading term of expansion.

Proof of Lemma 3. Using (11) we get that

$$\frac{\partial \bar{F}_m}{\partial w_2} = 0 \quad \text{that is} \quad \bar{F}_m = \bar{F}_m(w_1).$$

Since \bar{F}_m has weight degree $n - 2m$ and we are assuming that is different from zero, we obtain $\bar{F}_m = C_0 w_1^{n-2m}$ with $C_0 \neq 0$.

Now we use (11) with $i = m - 2$, that is, we compute the coefficient of v_2^{m-1} in (9). We get

$$\begin{aligned}
 (13) \quad \frac{\partial \bar{F}_{m-2}}{\partial w_2} &= B \frac{\partial \bar{F}_m}{\partial w_1} + m A \bar{F}_m \\
 &= \frac{w_1^2}{4} \left[a w_1 (w_1 + 2w_2) + (a+2)w_2^2 \right] C_0 (n-2m) w_1^{n-2m-1} \\
 &\quad + \frac{m w_1}{2} \left[a w_1 (2w_1 + 3w_2) + (a+2)w_2^2 \right] C_0 w_1^{n-2m} \\
 &= \frac{C_0}{4} w_1^{n-2m+1} \left[n(a+2)w_2^2 + 2(n+m)aw_1w_2 + \text{l.o.t.}(w_2) \right].
 \end{aligned}$$

Integrating (13) in w_2 we get

$$\begin{aligned}
 \bar{F}_{m-2} &= \frac{C_0}{12} n(a+2)w_1^{n-2m+1}w_2^3 + \frac{C_0}{4} (n+m)aw_1^{n-2m+2}w_2^2 + \text{l.o.t.}(w_2) \\
 &= C_1^n C_0 (a+2)w_1^{n-2m+1}w_2^3 + C_0 D_1^{n,m} aw_1^{n-2m+2}w_2^2 + \text{l.o.t.}(w_2).
 \end{aligned}$$

This proves (3) with $l = 1$ where $C_1^n = n/12$ and $D_1^{n,m} = (n+m)/4$.

Now we assume (3) is true for $l = 1, \dots, k$ and we will prove it for $k+1$ (assuming that $m - 2(k+1) \notin \{-[n/2] + 1, -[n/2]\}$). Computing the coefficient of v_2^{m-2k-1} in (9), that is considering (11) with $i = m - 2k - 2$, we get

$$(14) \quad \frac{\partial \bar{F}_{m-2(k+1)}}{\partial w_2} = B \frac{\partial \bar{F}_{m-2k}}{\partial w_1} + (m-2k)A\bar{F}_{m-2k}.$$

By the induction hypothesis we have

$$\begin{aligned}
 (15) \quad &\frac{\partial \bar{F}_{m-2(k+1)}}{\partial w_2} \\
 &= \frac{w_1^2}{4} (aw_1(w_1 + 2w_2) + (a+2)w_2^2) \left[C_k^n C_0 w_2^{3k} (n-2m+k) w_1^{n-2m+k-1} (a+2)^k \right. \\
 &\quad \left. + D_k^{n,m} C_0 w_2^{3k-1} (n-2m+k+1) w_1^{n-2m+k} a(a+2)^{k-1} + \text{l.o.t.}(w_2) \right] \\
 &\quad + \frac{(m-2k)w_1}{2} (aw_1(2w_1 + 3w_2) + (a+2)w_2^2) \left[C_k^n C_0 w_2^{3k} w_1^{n-2m+k} (a+2)^k \right. \\
 &\quad \left. + D_k^{n,m} C_0 w_2^{3k-1} w_1^{n-2m+k+1} a(a+2)^{k-1} + \text{l.o.t.}(w_2) \right] \\
 &= \tilde{C}_{k+1}^n C_0 w_2^{3k+2} w_1^{n-2m+k+1} (a+2)^{k+1} + \tilde{D}_{k+1}^{n,m} C_0 w_2^{3k+1} w_1^{n-2m+k+2} a(a+2)^k \\
 &\quad + \text{l.o.t.}(w_2),
 \end{aligned}$$

where

$$\begin{aligned}
 \tilde{C}_{k+1}^n &= \frac{n-3k}{4} C_k^n = \frac{n-3(k+1)+3}{4} \prod_{i=1}^k \frac{n-3i+3}{12i} = 3(k+1) \prod_{i=1}^{k+1} \frac{n-3i+3}{12i} \\
 &= 3(k+1) C_{k+1}^n,
 \end{aligned}$$

and

$$\begin{aligned}
\tilde{D}_{k+1}^{n,m} &= \frac{n+m-5k}{2} C_k^n + \frac{n-3k+1}{4} D_k^{n,m} \\
&= \frac{n+m-5(k+1)+5}{2} \prod_{i=1}^k \frac{n-3i+3}{12i} \\
&\quad + \frac{n-3(k+1)+4}{4} \sum_{i=1}^k \frac{n+m-5i+5}{2(3i-1)} \prod_{j=i+1}^k \frac{n-3j+4}{4(3j-1)} \prod_{r=1}^{i-1} \frac{n-3r+3}{12r} \\
&= (3(k+1)-1) \frac{n+m-5(k+1)+5}{2(3(k+1)-1)} \prod_{i=1}^k \frac{n-3i+3}{12i} \\
&\quad + (3(k+1)-1) \sum_{i=1}^k \frac{n+m-5i+5}{2(3i-1)} \prod_{j=i+1}^{k+1} \frac{n-3j+4}{4(3j-1)} \prod_{r=1}^{i-1} \frac{n-3r+3}{12r} \\
&= (3(k+1)-1) D_{k+1}^{n,m}.
\end{aligned}$$

Integrating (15) with respect to w_2 we have

$$\begin{aligned}
\bar{F}_{n-2(k+1)} &= \frac{\tilde{C}_{k+1}^n}{3(k+1)} C_0 w_2^{3(k+1)} w_1^{n-2m+k+1} (a+2)^{k+1} \\
&\quad + \frac{\tilde{D}_{k+1}^{n,m}}{3k+2} C_0 w_2^{3(k+1)-1} w_1^{n-2m+k+2} a(a+2)^k + \text{l.o.t.}(w_2) \\
&= C_{k+1}^n C_0 w_2^{3(k+1)} w_1^{n-2m+k+1} (a+2)^{k+1} \\
&\quad + D_{k+1}^{n,m} C_0 w_2^{3(k+1)-1} w_1^{n-2m+k+2} a(a+2)^k + \text{l.o.t.}(w_2).
\end{aligned}$$

This proves (3) for $k+1$. □

Lemma 4. Assume that n is a multiple of 3. Then

- (a) if $m \geq \lfloor \frac{n}{2} \rfloor - \frac{n}{3} + 2$, then $D_l^{n,m} \neq 0$ for $l \geq n/3 + 1$,
- (b) if $m < \lfloor \frac{n}{2} \rfloor - \frac{n}{3} + 2$, then $C_l^n \neq 0$ for $l \geq 1$.

Proof. First we prove statement (a). It follows from the proof of Lemma 3 that for $l \geq 1$, $D_l^{n,m}$ satisfies

$$(16) \quad D_{l+1}^{n,m} = \frac{n+m-5l}{2(3l+2)} C_l^n + \frac{n-3l+1}{4(3l+2)} D_l^{n,m}.$$

We note that from (12) $C_l^n = 0$ if $l \geq n/3 + 1$ and hence, for $l \geq n/3 + 2$ we have from (16) that

$$(17) \quad D_l^{n,m} = D_{n/3+1}^{n,m} \prod_{i=n/3+2}^l \frac{n-3i+4}{4(3i-1)}.$$

In particular, since n is a multiple of 3, if we show that $D_{n/3+1}^{n,m} \neq 0$ then from (17) we have that $D_l^{n,m} \neq 0$ for any $l \geq n/3 + 1$. We recall that since $\lfloor \frac{n}{2} \rfloor \geq m \geq \lfloor \frac{n}{2} \rfloor - \frac{n}{3} + 2$ and n is a multiple of 3, we have $n \geq 6$. In what follows we will show that $D_{n/3+1}^{n,m} \neq 0$. We proceed by contradiction. Assume that $D_{n/3+1}^{n,m} = 0$. Then it follows from (16) that

$$(18) \quad D_{n/3}^{n,m} = \left(\frac{4n}{3} - 2m \right) \prod_{k=1}^{n/3} \frac{n-3k+3}{12k} = \frac{1}{4^{n/3}} \left(\frac{4n}{3} - 2m \right).$$

We rewrite (16) in the following way

$$\begin{aligned}
 D_l^{n,m} &= \frac{4(3l+2)}{n-3l+1} \left[D_{l+1}^{n,m} + \frac{5l-n-m}{2(3l+2)} C_l^m \right] \\
 (19) \quad &= \frac{4(3l+2)}{n-3l+1} D_{l+1}^{n,m} + \frac{2(5l-n-m)}{n-3l+1} \prod_{j=1}^l \frac{n-3j+3}{12j}.
 \end{aligned}$$

We claim that given a sequence of the form $x_M = \bar{x}$ and for $k = M-1, \dots, N$

$$x_k = A_{k+1}x_{k+1} + f_{k+1},$$

then

$$(20) \quad x_N = \bar{x} \prod_{l=N+1}^M A_l + \sum_{l=N+1}^M f_l \prod_{j=N+1}^{l-1} A_j.$$

Now we prove the claim by backwards induction. For $k = M-1$ we have $x_{M-1} = A_M x_M + f_M = A_M \bar{x} + f_M$ that is equation (20) with $N = M-1$. Now we assume is true for $N = M-1, \dots, L$ and we will prove it for $N = L-1$. By backwards induction we can write

$$\begin{aligned}
 x_{L-1} &= A_L x_L + f_L = A_L \bar{x} \prod_{l=L+1}^M A_l + A_L \sum_{l=L+1}^M f_l \prod_{j=L+1}^{l-1} A_j + f_L \\
 &= \bar{x} \prod_{l=L}^M A_l + \sum_{l=L+1}^M f_l \prod_{j=L}^{l-1} A_j + f_L = \bar{x} \prod_{l=L}^M A_l + \sum_{l=L}^M f_l \prod_{j=L}^{l-1} A_j,
 \end{aligned}$$

which is (20) for $N = L-1$. Therefore, it follows from (20) with $N = 1$, $x_1 = D_1^{n,m}$, $M = n/3$, $\bar{x} = x_M = D_{n/3}^{n,m}$ (see (18)),

$$A_l = \frac{4(3l-1)}{n-3l+4} \quad \text{and} \quad f_l = \frac{2(5l-n-m-5)}{n-3l+4} \prod_{j=1}^{l-1} \frac{n-3j+3}{12j}$$

that

$$\begin{aligned}
 D_1^{n,m} &= \frac{1}{4^{n/3}} \left(\frac{4n}{3} - 2m \right) \prod_{l=2}^{n/3} \frac{4(3l-1)}{n-3l+4} \\
 (21) \quad &+ \sum_{l=2}^{n/3} \frac{2(5l-n-m-5)}{n-3l+4} \prod_{j=1}^{l-1} \frac{n-3j+3}{12j} \prod_{r=2}^{l-1} \frac{4(3r-1)}{n-3r+4}.
 \end{aligned}$$

Now we know from Lemma 3 that $D_1^{n,m} = (n+m)/4$. In particular the function

$$G_{n,m} = D_1^{n,m+1} - D_1^{n,m} = \frac{n+m+1}{4} - \frac{n+m}{4} = \frac{1}{4}.$$

Hence $G_{n,m}$ does not depend on neither n nor m and is equal to the constant $1/4$. From (21) we have

(22)

$$G_{n,m} = -\frac{2}{4^{n/3}} \prod_{l=2}^{n/3} \frac{4(3l-1)}{n-3l+4} - 2 \sum_{l=2}^{n/3} \frac{1}{n-3l+4} \prod_{j=1}^{l-1} \frac{n-3j+3}{12j} \prod_{r=2}^{l-1} \frac{4(3r-1)}{n-3r+4}.$$

In particular $G_{n,m}$ given in (22) must be equal to $1/4$ for any value of n . When $n = 6$ we get from (22) that $G_{6,m} = -7/8 \neq 1/4$, a contradiction. This concludes the proof of the statement (a).

Now we prove the statement (b). Note that $C_l^n = 0$ if $l \geq (n+3)/3$. So we shall prove that $l < (n+3)/3$. Since $m-2l \geq -[n/2]$ and $m < [\frac{n}{2}] - \frac{n}{3} + 2$, we have that $2l < [n/2] - n/3 + 2$ which concludes the proof. \square

Proof of Theorem 1. We will show that $\bar{F} = 0$. In fact we shall see that $\bar{F}_i = 0$ for $i = -[n/2], \dots, [n/2]$, and consequently $\bar{F} = 0$. We note that by Lemma 3 if n is not a multiple of 3, then $C_l^n \neq 0$. On the other hand it follows from Lemma 4 that if n is a multiple of 3 and if $m \geq [\frac{n}{2}] - \frac{n}{3} + 2$ then $D_l^{n,m} \neq 0$ for $l \geq n/3 + 1$; and if $m < [\frac{n}{2}] - \frac{n}{3} + 2$, then $C_l^n \neq 0$ for $l \geq 1$.

Applying Lemma 3 and using (11) for $i = -[n/2]$, we get

$$\begin{aligned} 0 &= B \frac{\partial \bar{F}_{-[n/2]}}{\partial w_1} - [n/2] A \bar{F}_{-[n/2]} \\ &= 3(l+1) C_{l+1}^n C_0 w_2^{3l+2} w_1^{n-2m+l+1} (a+2)^{l+1} + \\ &\quad (3l+2) D_{l+1}^{n,m} C_0 w_2^{3l+1} w_1^{n-2m+l+2} a (a+2)^l + \text{l.o.t.}(w_2). \end{aligned}$$

Again applying Lemma 3 and using (11) for $i = -[n/2] + 1$, we have

$$\begin{aligned} 0 &= B \frac{\partial \bar{F}_{-[n/2]+1}}{\partial w_1} + (-[n/2] + 1) A \bar{F}_{-[n/2]+1} \\ &= 3(l+1) C_{l+1}^n C_0 w_2^{3l+2} w_1^{n-2m+l+1} (a+2)^{l+1} + \\ &\quad (3l+2) D_{l+1}^{n,m} C_0 w_2^{3l+1} w_1^{n-2m+l+2} a (a+2)^l + \text{l.o.t.}(w_2). \end{aligned}$$

Since $C_0 a(a+2) \neq 0$, we obtain that $C_{l+1}^n = D_{l+1}^{n,m} = 0$. But $C_l^n \neq 0$ if n is not a multiple of 3, or n is a multiple of 3 and $m < [n/2] - n/3 + 2$, we have that n is a multiple of 3 and $m \geq [n/2] - n/3 + 2$. Therefore, we have a contradiction since by Lemma 4 in this case $D_{l+1}^{n,m} \neq 0$. This completes the proof of the theorem. \square

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