This is a preprint of: "A note on a rational difference equation", Lluís Alsedà, Michal Misiurewicz, *J. Difference Equ. Appl.*, 1711–1713, 2011.
DOI: [10.1080/10236191003657261]

RESEARCH ARTICLE

A note on a rational difference equation

Lluís Alsedà[a*] and Michał Misiurewicz[b]

[a]*Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain. Email:* alseda@mat.uab.es;
[b]*Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202-3216, USA. Email:* mmisiure@math.iupui.edu

We answer a question raised by G. Ladas at the **ICDEA 2009** conference, by showing that the nonautonomous difference equation $x_{n+1} = \frac{1}{x_n+A_n}$ with $x_n, A_n > 0$ and $A_n \to 0$ with the ratios A_{n+1}/A_n bounded can have solutions whose set of accumulation points contain a nondegenerate interval.

Keywords: Difference equation, rational, nonautonomous, solutions, non-convergence

AMS Subject Classification: Primary: 39A33, 39A20.

1. Introduction

At the **ICDEA 2009** conference, G. Ladas raised the question whether the solutions of the nonautonomous difference equation

$$x_{n+1} = \frac{1}{x_n + A_n}, \tag{1}$$

where

$$x_n, A_n > 0, \quad A_n \to 0$$

with the ratios A_{n+1}/A_n bounded, have to converge to a period 2 or 1 cycle.

We will show that the answer to this question is negative. Namely, the following theorem holds.

Theorem 1.1 . *Let t, s be real numbers such that $1 < s < t < \sqrt{2}$. Then there exists a sequence (A_n) of positive numbers, with $A_n \to 0$ and the ratios A_{n+1}/A_n taking only values $1/2$ and 2, such that the difference equation (1) has a positive solution (x_n) for which the set of accumulation points of the sequence (x_{2n}) is equal to the interval $[s, t]$.*

2. Proof of the theorem

We fix s and t as above, $x_0 \in [s,t]$, and $A_0 \in (0,1/9)$ such that $t(A_0+t) < 2$. Then, for each n, we define A_{2n+1} and A_{2n+2} by induction, setting:

(a) $A_{2n+1} = A_{2n}/2$ and $A_{2n+2} = 2A_{2n+1}$, or
(b) $A_{2n+1} = A_{2n}/2$ and $A_{2n+2} = A_{2n+1}/2$, or
(c) $A_{2n+1} = 2A_{2n}$ and $A_{2n+2} = A_{2n+1}/2$.

We will have $A_k \to 0$ as long as Case (b) appears infinitely many times. Moreover, $A_{2n+2} \le A_{2n}$, so for every n we get $A_{2n} \le A_0$. Thus, $t(A_{2n}+t) \le t(A_0+t) < 2$ and $A_{2n} < 1/9$.

In order to determine when to use which case, let us find $x_{2n+2} - x_{2n}$ in all three cases. In all cases we have $x_{2n+1} = 1/(x_{2n}+A_{2n})$. Now, in cases (a) and (b) we have

$$x_{2n+2} = \frac{1}{\frac{1}{x_{2n}+A_{2n}} + \frac{A_{2n}}{2}} = \frac{2(x_{2n}+A_{2n})}{2+A_{2n}(x_{2n}+A_{2n})},$$

so

$$\begin{aligned} x_{2n+2} - x_{2n} &= \frac{2x_{2n} + 2A_{2n} - 2x_{2n} - A_{2n}x_{2n}(x_{2n}+A_{2n})}{2+A_{2n}(x_{2n}+A_{2n})} \\ &= A_{2n}\frac{2 - x_{2n}(x_{2n}+A_{2n})}{2+A_{2n}(x_{2n}+A_{2n})}. \end{aligned}$$

If $x_{2n} \le t$, then

$$2 \ge 2 - x_{2n}(x_{2n}+A_{2n}) \ge 2 - t(A_0+t) > 0.$$

Moreover,

$$2 \le 2 + A_{2n}(x_{2n}+A_{2n}) \le 2 + A_0(A_0+t).$$

Hence,

$$c_1 = \frac{2-t(A_0+t)}{2+A_0(A_0+t)} \in (0,1)$$

and in cases (a) and (b) we get

$$A_{2n} \ge x_{2n+2} - x_{2n} \ge c_1 A_{2n}, \tag{2}$$

provided $x_{2n} \le t$.

Consider now Case (c). We have

$$x_{2n+2} = \frac{1}{\frac{1}{x_{2n}+A_{2n}} + 2A_{2n}} = \frac{x_{2n}+A_{2n}}{1+2A_{2n}(x_{2n}+A_{2n})},$$

so

$$x_{2n+2} - x_{2n} = \frac{x_{2n} + A_{2n} - x_{2n} - 2A_{2n}x_{2n}(x_{2n} + A_{2n})}{1 + 2A_{2n}(x_{2n} + A_{2n})}$$
$$= A_{2n}\frac{1 - 2x_{2n}(x_{2n} + A_{2n})}{1 + 2A_{2n}(x_{2n} + A_{2n})}.$$

If $1 \leq x_{2n} \leq 2$ then

$$-8 \leq 1 - 4(2 + A_0) \leq 1 - 2x_{2n}(x_{2n} + A_{2n}) \leq 1 - 2 = -1.$$

Moreover,

$$1 \leq 1 + 2A_{2n}(x_{2n} + A_{2n}) \leq 1 + 2A_0(2 + A_0) \leq 2.$$

Hence, in Case (c) we get

$$-8A_{2n} \leq x_{2n+2} - x_{2n} \leq -A_{2n}/2, \tag{3}$$

provided $1 \leq x_{2n} \leq 2$.

Now we can specify our sequence A_k. We start with arbitrary $A_0 \in (0, 1/9)$ and $x_0 \in (s, t)$. Then we use Case (a) as long as $x_{2n} \leq t$. In view of (2) and the fact that when using Case (a) A_{2n} remains constant, we see that x_{2n} grows, and at a certain moment we will have $x_{2n} \leq t$ and $x_{2n+2} > t$. For this n we use Case (b) instead of Case (a). By (2), $x_{2n+2} \leq t + 1/9 < 2$. Now we start using Case (c). Again A_{2n} remains constant, but this time by (3) x_{2n} decreases, and at a certain moment we get $x_{2n} \geq s$ and $x_{2n+2} \leq s$. Then $x_{2n+2} \geq s - 8/9 > 0$, and we start using Case (a) again, and so on.

In such a way we use Case (b) infinitely many times, so $A_k \to 0$. This, together with (2) and (3), implies that at the moments when $x_{2n} \geq t$ or $x_{2n} \leq s$ the distance of x_{2n} from the interval $[s, t]$ goes to 0. Thus, by using again $A_k \to 0$ together with (2) and (3), we see that the set of accumulation points of the sequence (x_{2n}) is equal to the interval $[s, t]$.

3. Acknowledgements

The first author was partially supported by MEC grant number MTM2008-01486.