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ON THE LIMIT CYCLES OF POLYNOMIAL VECTOR FIELDS

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Abstract. In this paper we study the limit cycles which can bifurcate from the periodic orbits of the center located at the origin of the quadratic polynomial differential system $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$, and of the cubic polynomial differential system $\dot{x} = -y(1-x^2-y^2)$, $\dot{y} = x(1-x^2-y^2)$, when we perturb them in the class of all polynomial vector fields with quadratic and cubic homogenous nonlinearities, respectively. For doing this study we use the averaging theory.

Keywords. limit cycle, periodic orbit, center, reversible center, averaging method.

AMS (MOS) subject classification: 34C29, 34C25, 47H11

1 Introduction and statement of the results

After the definition of limit cycle due to Poincaré [14], the statement of the 16–th Hilbert's problem [9], the discover that the limit cycles are important in the nature by Liénard [11], ... the study of the limit cycles of the planar differential systems has been one of the main problems of the qualitative theory of the differential equations.

One of the best ways of producing limit cycles is by perturbing the periodic orbits of a center. This has been studied intensively perturbing the periodic orbits of the centers of the quadratic polynomial differential systems see the book of Christopher and Li [6], and the references quoted there.

It is well known that if a quadratic polynomial differential system has a limit cycles this must surround a focus. Up to know the maximum number of known limit cycles surrounding a focus of a quadratic polynomial differential system is 3, which coincides with the maximum number of small limit cycles which can bifurcate by Hopf from a singular point of a quadratic polynomial differential system, see Bautin [1]. But as far as we know up to now there are few quadratic centers for which it is proved that the perturbation of their periodic orbits inside the class of all quadratic polynomial differential systems can produce 3 limit cycles. These are the center whose exterior boundary is formed by three invariant straight lines (see Żołądek [21]), three different families of reversible quadratic centers (see Świrszcz [18]), and the center $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$ (see Buică, Gasull and Yang [3]). The study of the perturbation of this last center has been made through the Melnikov function of third order computed using the algorithm developed by Françoise [8] and Iliev [10]. Here we can provide a new and shorter proof of this second result by using the averaging theory, see Theorem 1.

In the paper two differential systems are studied. The quadratic systems

$$\begin{array}{rcl} \dot{x} & = & -y(1+x) + \varepsilon(\lambda x + \bar{A}x^2 + \bar{B}xy + \bar{C}y^2), \\ \dot{y} & = & x(1+x) + \varepsilon(\lambda y + \bar{D}x^2 + \bar{E}xy + \bar{F}y^2), \end{array} \tag{1}$$

such that for $\varepsilon = 0$ have a straight line consisting of singular points, and the cubic systems

$$\begin{array}{rcl} \dot{x} & = & -y(1-x^2-y^2) + \varepsilon^3\lambda x + \sum\limits_{s=1}^{3} \varepsilon^s \sum\limits_{i=0}^{3} a_{i,s}x^i y^{3-i}, \\ \dot{y} & = & x(1-x^2-y^2) + \varepsilon^3\lambda y + \sum\limits_{s=1}^{3} \varepsilon^s \sum\limits_{i=0}^{3} b_{i,s}x^i y^{3-i}, \end{array} \tag{2}$$

such that for $\varepsilon = 0$ have a unit circle consisting of singular points.

We study for $\varepsilon \neq 0$ sufficiently small the number of limit cycles of systems (1) and (2) bifurcating from the periodic orbits of the centres of (1) and (2) for $\varepsilon = 0$, respectively. Our main results are the following.

Theorem 1 *For convenient λ, $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$, $\bar{F}$ system (1) has 3 limit cycles bifurcating from the periodic orbits of the center for $\varepsilon = 0$.*

Theorem 2 *The following statements hold for system (2).*

(a) Using the averaging theory of third order (see subsection 3.2) for $\varepsilon \neq 0$ sufficiently small we can obtain at most 5 limit cycles of system (2) bifurcating from the periodic orbits of the center located at the origin of system (2) with $\varepsilon = 0$.

(b) For convenient λ, $a_{i,s}$, $b_{i,s}$, $i = 0, 1, 2, 3$, $s = 1, 2, 3$ system (2) has 0, 1, 2, 3, 4 or 5 limit cycles bifurcating from the periodic orbits of the center for $\varepsilon = 0$.

It is known that systems of the form $\dot{x} = -y + P_3(x, y)$, $\dot{y} = x + Q_3(x, y)$, with P_3 and Q_3 homogeneous polynomials of degree 3 can have 5 small limit cycles bifurcating by Hopf from the origin, see [17, 12].

2 Polar coordinates and Cherkas transformation

We are going to use the following classical result

Lemma 3 (Cherkas [5]) *A differential equation*

$$\frac{dr}{d\varphi} = \frac{\lambda r + a(\varphi) r^k}{1 + b(\varphi) r^{k-1}}$$

can be by means of a substitution

$$\rho(\varphi) = \frac{r(\varphi)^{k-1}}{1 + b(\varphi) r(\varphi)^{k-1}}$$

converted into the Abel equation

$$\begin{aligned} \frac{d\rho}{d\varphi} &= (k-1) b(\varphi)(\lambda b(\varphi) - a(\varphi))\rho^3 + \\ & \left[(k-1)(a(\varphi) - 2\lambda b(\varphi)) - b'(\varphi)\right]\rho^2 + (k-1)\lambda\rho, \end{aligned}$$

Combining Lemma 3 with polar coordinates transformation we immediately get the next result.

Corollary 4 *Let $P(x, y)$ and $Q(x, y)$ be homogenous polynomials of degree n. Then the differential system*

$$\begin{aligned} \dot{x} &= -y + \lambda x + P_n(x, y) \\ \dot{y} &= x + \lambda y + Q_n(x, y) \end{aligned} \tag{3}$$

can be transformed into the Abel equation

$$\begin{aligned} \frac{d\rho}{d\varphi} &= (k-1) B(\varphi)(\lambda B(\varphi) - A(\varphi))\rho^3 + \\ & \left[(k-1)(A(\varphi) - 2\lambda B(\varphi)) - B'(\varphi)\right]\rho^2 + (k-1)\lambda\rho. \end{aligned}$$

where

$$A(\varphi) = \cos\varphi P_n(\cos\varphi, \sin\varphi) + \sin\varphi Q_n(\sin\varphi, \cos\varphi)$$

and

$$B(\varphi) = \cos\varphi Q_n(\cos\varphi, \sin\varphi) - \sin\varphi P_n(\sin\varphi, \cos\varphi).$$

Proof. System (3) expressed in polar coordinates becomes

$$\begin{aligned} \dot{r} &= \lambda r + A(\varphi) r^n, \\ \dot{y} &= 1 + B(\varphi) r^n. \end{aligned}$$

Dividing $\dot{r}$ by $\dot{\varphi}$ and using Lemma 3 proves the corollary. ∎

3 Averaging

In this section first we present basic results from the averaging theory that we shall need for proving the main results of this paper.

3.1 Averaging of zeroth order

We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{4}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}), \tag{5}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory. For a general introduction to the averaging theory see the books of Sanders and Verhulst [16], and of Verhulst [19].

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (5) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}. \tag{6}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (6), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Theorem 5 *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\bar{C}^2$ function. We assume that*

(i) $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)), \quad \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (5) is T–periodic;

(ii) for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (6) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi \left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right). \tag{7}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left((d\mathcal{F}/d\alpha)(a)\right) \neq 0$, then there is a T–periodic solution $\varphi(t, \varepsilon)$ of system (4) such that $\varphi(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 5 goes back to Malkin [13] and Roseau [15], for a shorter proof see [2].

3.2 Averaging of first, second and third order

The averaging theory of third order for studying specifically periodic orbits was developed in [4]. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \tag{8}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(t,\cdot) \in C^2(D)$, $F_2(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R, $D_x^2 F_1$,$D_x F_2$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2, 3$ as

$$\begin{aligned}
F_{10}(z) = {} & \frac{1}{T}\int_0^T F_1(s,z)ds, \\
F_{20}(z) = {} & \frac{1}{T}\int_0^T \left[D_z F_1(s,z)\cdot y_1(s,z) + F_2(s,z)\right] ds, \\
F_{30}(z) = {} & \frac{1}{T}\int_0^T \Big[\frac{1}{2} y_1(s,z)^T \frac{\partial^2 F_1}{\partial z^2}(s,z) y_1(s,z) + \frac{1}{2}\frac{\partial F_1}{\partial z}(s,z) y_2(s,z) \\
& + \frac{\partial F_2}{\partial z}(s,z)(y_1(s,z)) + F_3(s,z)\Big] ds,
\end{aligned}$$

where

$$\begin{aligned}
y_1(s,z) &= \int_0^s F_1(t,z)dt, \\
y_2(s,z) &= \int_0^s \left[\frac{\partial F_1}{\partial z}(t,z)\int_0^t F_1(r,z)dr + F_2(t,z)\right] dt.
\end{aligned}$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\varphi(\cdot,\varepsilon)$ of the system such that $\varphi(0,\varepsilon) = a_\varepsilon$.

The expression $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order*.

If F_{10} and F_{20} are identically zero and F_{30} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ are mainly the zeros of F_{30} for ε sufficiently small. In this case the previous result provides the *averaging theory of third order*.

4 Quadratic case

Proof of Theorem 1. From Corollary 4 applied to system (1) it follows that finding limit cycles of (1) is equivalent to finding periodic solutions of

$$\begin{aligned}\frac{d\rho}{d\varphi} = {} & (\sin\varphi)\rho^2+\epsilon\left[-\tfrac{1}{4}\cos\varphi((3\bar{A}+\bar{C}+\bar{E}-4\lambda)\cos\varphi+\right.\\ & (\bar{A}-\bar{C}-\bar{E})\cos 3\varphi+\\ & 2(\bar{B}+\bar{D}+\bar{F}+(\bar{B}+\bar{D}-\bar{F})\cos 2\varphi)\sin\varphi)\rho^3+\\ & ((\bar{A}+\bar{C}-2\lambda)\cos\varphi+(\bar{A}-\bar{C}-\bar{E})\cos 3\varphi+\\ & \left.(\bar{D}+\bar{F})\sin\varphi+(\bar{B}+\bar{D}-\bar{F})\sin 3\varphi)\rho^2+\lambda\rho\right].\end{aligned} \tag{9}$$

We are going to apply Theorem 5 to system (9). We first solve differential equation

$$\frac{d\rho}{d\varphi}=(\sin\varphi)\rho^2,$$

with initial condition $\rho(0)=R/(1+R)$ and we get $\rho(\varphi,R)=R/(1+R\cos\varphi)$. Thus $M_R(\varphi)$ in (7) will be a solution of a differential equation $M_R'(\varphi)=(2R\sin\varphi)/(1+R\cos\varphi)$, namely, $M_R(\varphi)=1+2\ln(1+R)-2\ln(1+r\cos\varphi)$. Thus formula (7) yields

$$\begin{aligned}\mathcal{F}(R) = {} & \int_0^{2\pi}\left(\lambda\frac{R}{\Xi(\varphi,R)}+\right.\\ & \bar{A}\frac{\cos\varphi(R\cos\varphi+8\cos(2\varphi)+3R\cos(3\varphi))R^2}{4\Xi(\varphi,R)}+\\ & \bar{B}\frac{(2R\sin 2\varphi+8\sin 3\varphi+3R\sin 4\varphi)R^2}{8\Xi(\varphi,R)}-\\ & \bar{C}\frac{\cos\varphi(3R\cos\varphi+4)\sin^2\varphi R^2}{\Xi(\varphi,R)}+\\ & \bar{D}\frac{\cos^2\varphi(3R\cos\varphi+4)\sin\varphi R^2}{\Xi(\varphi,R)}-\\ & \bar{E}\frac{\cos\varphi(R\cos\varphi+8\cos 2\varphi+3R\cos 3\varphi-4)R^2}{4\Xi(\varphi,R)}+\\ & \left.\bar{F}\frac{(5R\cos\varphi+8\cos 2\varphi+3R\cos 3\varphi)\sin\varphi R^2}{4\Xi(\varphi,R)}\right)d\varphi,\end{aligned} \tag{10}$$

where $\Xi(\varphi, R) = (R\cos\varphi + 1)^3(2\log(R+1) - 2\log(R\cos\varphi+1)+1)$. Now observe that the terms in front of $\bar{B}$, $\bar{D}$ and $\bar{F}$ are odd π-periodic functions of φ, thus their integrals from 0 to 2π are equal to zero. Therefore

$$\begin{aligned}\mathcal{F}(R) &= \int_0^{2\pi}\Bigg(\lambda\frac{R}{\Xi(\varphi,R)}+\\ &\quad \bar{A}\frac{\cos\varphi(R\cos\varphi+8\cos(2\varphi)+3R\cos(3\varphi))R^2}{4\Xi(\varphi,R)}+\\ &\quad \bar{C}\frac{\cos\varphi(3R\cos\varphi+4)\sin^2\varphi R^2}{\Xi(\varphi,R)}+\\ &\quad \bar{E}\frac{\cos\varphi(R\cos\varphi+8\cos 2\varphi+3R\cos 3\varphi-4)R^2}{4\Xi(\varphi,R)}\Bigg)\,d\varphi\\ &= \lambda f_1(R)+\bar{A}f_2(R)+\bar{C}f_3(R)-\bar{E}f_4(R).\end{aligned} \tag{11}$$

We claim that the four functions f_1, f_2, f_3 and f_4 are linearly independent. Now we prove the claim. By straightforward calculation we obtain the following Taylor expansions:

$$\begin{aligned}f_1(R) &= \frac{1}{24}\pi R\left(2615R^4-800R^3+312R^2-96R+48\right)+\mathcal{O}(R^6),\\ f_2(R) &= \frac{1}{24}\pi R^3\left(313R^2-60,R-18\right)+\mathcal{O}(R^6),\\ f_3(R) &= \frac{1}{24}\pi R^3\left(401R^2-84R-6\right)+\mathcal{O}(R^6),\\ f_4(R) &= -\frac{1}{24}\pi R^3\left(43R^2-12R+6\right)+\mathcal{O}(R^6).\end{aligned}$$

The determinant of the coefficient matrix of terms $R^2,\ldots,R^5$ is $\pi^4/3$ and the claim follows.

A well-known classical result states that if a family n functions is linearly independent, then there exists their linear combination with at least $n-1$ zeroes. We provide a short proof of this fact in Appendix A. Thus Theorem 1 follows. ∎

We have provided the proof that a family of n functions linearly independent can have $n-1$ zeroes, because we do not find a reference where this result was proved. We have a very close result due to Coll, Gasull and Prohens [7] proving the same conclusion but with the additional assumption that a function cannot change of sign.

Remark 6 *System (1) is a perturbation of a reversible quadratic center with an invariant straight line (compare [20]). The unperturbed system is invariant under the change of coordinates $(y,t)\longrightarrow(-y,-t)$. This is a reason why the terms in averaging formula coming from xy in $\dot{x}$ and x^2 and y^2 in $\dot{y}$ vanish.*

5 Cubic case

Proof of Theorem 2. First we prove statement (b). We shall use third order averaging to show that the system

$$\begin{aligned}
\dot{x} &= -y(1-x^2-y^2)+\varepsilon^3\lambda x- \\
&\tfrac{1}{1200}(75\mathcal{B}\varepsilon+108\mathcal{E}+19840)\varepsilon x^3+(j+24)\varepsilon x^2y+ \\
&\left(4\varepsilon^3(\mathcal{A}-4\lambda)+\varepsilon^2\left(\tfrac{27\mathcal{B}}{128}-\mathcal{C}\right)+\tfrac{(81\mathcal{E}+16480)\varepsilon}{300}\right)xy^2+ \\
&\tfrac{1}{2}\varepsilon(2j+\mathcal{D}\varepsilon)y^3, \\
\dot{y} &= x(1-x^2-y^2)+\varepsilon^3\lambda y+ \\
&\tfrac{1}{2}(\mathcal{D}\varepsilon-2j)\varepsilon x^3+\left(\varepsilon^2\left(\mathcal{C}-\tfrac{3\mathcal{B}}{128}\right)+\tfrac{(81\mathcal{E}+18080)\varepsilon}{300}\right)x^2y- \\
&(j+40)\varepsilon xy^2-\tfrac{1}{300}(27\mathcal{E}+6560)\varepsilon y^3,
\end{aligned} \tag{12}$$

can have 0, 1, 2, 3, 4 or 5 limit cycles for an appropriate choice of the parameters λ, $\mathcal{A}$, $\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$ and $\mathcal{E}$. System (12) is clearly a special case of system (2), thus once we show it, statement (b) will be proved.

Using Cherkas Transformation (Lemma 3) we transform system (12) into the Abel equation

$$\frac{d\rho}{d\varphi}=\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3, \tag{13}$$

where

$$\begin{aligned}
F_1 &= \rho^3\left(\frac{3}{50}(3\mathcal{E}+640)\cos(4\varphi)+8(\sin(2\varphi)-2\sin(4\varphi))-\frac{16}{3}\cos(2\varphi)\right)+ \\
&\rho^2\left(-\frac{9}{50}(3\mathcal{E}+640)\cos(4\varphi)-8\sin(2\varphi)+48\sin(4\varphi)+\frac{16}{3}\cos(2\varphi)\right), \\
F_2 &= \frac{\rho^3}{30000}\left[25(6400j+75\mathcal{B}+432\mathcal{E}+117760)\cos(2\varphi)-\right. \\
&75\cos(4\varphi)(72(j+8)\mathcal{E}+15360(j+8)-25\mathcal{B})- \\
&600\sin(2\varphi)(400j+25\mathcal{D}+12\mathcal{E}+7360)+ \\
&480000(j+8)\sin(4\varphi)-7200(\mathcal{E}+80)\sin(6\varphi)+ \\
&3(9\mathcal{E}+1120)(9\mathcal{E}+2720)\sin(8\varphi)- \\
&\left.400(27\mathcal{E}+7360)\cos(6\varphi)+14400(3\mathcal{E}+640)\cos(8\varphi)\right]+ \\
&\rho^2\left(\left(\frac{3\mathcal{B}}{128}-\mathcal{C}\right)\cos(2\varphi)-\frac{3}{16}\mathcal{B}\cos(4\varphi)+3\mathcal{D}\sin(\varphi)\cos(\varphi)\right), \\
F_3 &= -2\lambda\rho+ \\
&\rho^2\left((\mathcal{A}-4\lambda)(2\cos(2\varphi)-3\cos(4\varphi))+\mathcal{A}\right)+ \\
&\rho^3\left\{\mathcal{A}\cos 4\varphi-\mathcal{A}-\frac{11\mathcal{B}}{64}+2\mathcal{C}-\frac{4\mathcal{D}}{3}+2\lambda+\right. \\
&\frac{1}{76800}\left[\sin(2\varphi)(384(100(j+4)\mathcal{D}-3\mathcal{C}(3\mathcal{E}+640))+\mathcal{B}(513\mathcal{E}+103040))-\right. \\
&96\cos(2\varphi)(25(2j-7)\mathcal{B}+3200\mathcal{C}-6\mathcal{D}(3\mathcal{E}+640))-
\end{aligned}$$

$$400\cos(4\varphi)(3(4j+21)\mathcal{B}+128(3\mathcal{C}+2\mathcal{D}+6\lambda))+$$
$$\sin(6\varphi)(1152(3\mathcal{C}\mathcal{E}+640\mathcal{C}-400\mathcal{D})-\mathcal{B}(81\mathcal{E}+23680))-$$
$$96\cos(6\varphi)(175\mathcal{B}-640(5\mathcal{C}+18\mathcal{D})-54\mathcal{D}\mathcal{E})+$$
$$800\sin(4\varphi)(11\mathcal{B}+64(3\mathcal{D}-2\mathcal{C}))+144\mathcal{B}(3\mathcal{E}+640)\sin(8\varphi)+$$
$$38400\mathcal{B}\cos(8\varphi)]\}\,.$$

By straightforward calculation we verify that $F_{10}=0$,

$$\begin{aligned} y_1(\rho,\varphi) &= \frac{\rho^3}{300}\sin\varphi((27\mathcal{E}+4160)\cos\varphi+3(3(3\mathcal{E}+640)\cos 3\varphi-800\sin 3\varphi))- \\ &\quad \frac{\rho^2}{600}\left(2\sin(2\varphi)(27(3\mathcal{E}+640)\cos 2\varphi-800(9\sin 2\varphi+1))+4800\sin^2\varphi\right), \end{aligned}$$

and $F_{20}=0$ (see subsection 3.2). Next

$$\begin{aligned} y_2(\rho,\varphi) &= \frac{1}{128}\rho^2(9\mathcal{B}\cos\varphi+12\mathcal{B}\cos(3\varphi)+128\mathcal{C}\cos\varphi-192\mathcal{D}\sin\varphi)\sin\varphi+ \\ &\quad \rho^3\left[\left(\frac{8j}{3}+\frac{\mathcal{B}}{32}-\frac{9\mathcal{E}}{25}+\frac{128}{15}\right)\sin(2\varphi)-\right. \\ &\quad \frac{1}{50}(400j+25\mathcal{D}-24\mathcal{E}+1280)\sin^2\varphi- \\ &\quad \frac{9}{200}j\mathcal{E}\sin(4\varphi)+\frac{8}{9}(9j+494)\sin^2(2\varphi)-\frac{48}{5}j\sin(4\varphi)+ \\ &\quad \frac{1}{64}\mathcal{B}\sin(4\varphi)+\frac{81\mathcal{E}^2\sin^2(4\varphi)}{4000}-\frac{4}{5}\mathcal{E}\sin^2(3\varphi)+\frac{216}{25}\mathcal{E}\sin^2(4\varphi)- \\ &\quad \frac{63}{25}\mathcal{E}\sin(4\varphi)-\frac{3}{5}\mathcal{E}\sin(6\varphi)+\frac{9}{5}\mathcal{E}\sin(8\varphi)-64\sin^2(3\varphi)+ \\ &\quad \left.\frac{3808}{5}\sin^2(4\varphi)-\frac{7904}{15}\sin(4\varphi)-\frac{1472}{9}\sin(6\varphi)+384\sin(8\varphi)\right]+ \\ &\quad \rho^4\left[-\frac{243\mathcal{E}^2\sin^2(4\varphi)}{16000}-\frac{1}{25}(21\mathcal{E}+2480)\sin^2\varphi+\frac{29}{25}\mathcal{E}\sin^2(3\varphi)-\right. \\ &\quad \frac{162}{25}\mathcal{E}\sin^2(4\varphi)+\frac{1}{300}(189\mathcal{E}+9920)\sin(2\varphi)+\frac{27}{25}\mathcal{E}\sin(4\varphi)+ \\ &\quad \frac{87}{100}\mathcal{E}\sin(6\varphi)-\frac{27}{20}\mathcal{E}\sin(8\varphi)-\frac{1528}{9}\sin^2(2\varphi)+\frac{464}{5}\sin^2(3\varphi)- \\ &\quad \left.\frac{2856}{5}\sin^2(4\varphi)+\frac{3056}{15}\sin(4\varphi)+\frac{10672}{45}\sin(6\varphi)-288\sin(8\varphi)\right]+ \\ &\quad \rho^5\frac{((27\mathcal{E}+4160)\cos\varphi+3(3(3\mathcal{E}+640)\cos(3\varphi)-800\sin(3\varphi)))^2\sin^2\varphi}{60000} \end{aligned}$$

and

$$\begin{aligned} F_{30}(\rho) &= -2\lambda\rho+\mathcal{A}\rho^2-\left(\mathcal{A}-\mathcal{B}-\frac{2\mathcal{D}}{3}-2\lambda\right)\rho^3- \\ &\quad \left(\frac{91\mathcal{B}}{128}-\mathcal{C}+\frac{7\mathcal{D}}{3}-\frac{4\mathcal{E}}{5}\right)\rho^4+\left(\mathcal{D}-\frac{9\mathcal{E}}{5}\right)\rho^5+\mathcal{E}\rho^6. \end{aligned}$$

The coefficients of F_{30} are linearly independent (linear) functions of λ, $\mathcal{A}$, $\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$ and $\mathcal{E}$. Therefore for any $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5 \in \mathbb{R}$ there exist λ, $\mathcal{A}$, $\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$,$\mathcal{E}$ such that $F_{30}(\rho_i) = 0$ for $i = 1, 2, 3, 4, 5$. Thus, by using subsection 3.2 ends the proof of statement (b).

Now we sketch the proof of statement (a). If instead of doing the computations of the proof of statement (b) for system (12) we did them for the general system (2) we would obtain a function $F_{30}(\rho)$ which again is a polynomial of degree 6 in ρ without independent term. Thus the averaging theory of third order can only produce for $\varepsilon \neq 0$ sufficiently small at most 5 limit cycles of system (2) bifurcating from the periodic orbits at the origin of system (2) with $\varepsilon = 0$. ∎

Remark 7 *There is much freedom in the choice of system (12), it was chosen for simplicity of calculations.*

6 Appendix A

Let A be a set and let $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$. We say that $f_1, \ldots f_n$ are *linearly independent* functions if and only if there holds

$$\forall_{a \in A} \sum_{i=1}^{n} \alpha_i f_i(a) = 0 \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

Proposition 1 *If $f_1, \ldots f_n : A \to \mathbb{R}$ are linearly independent then there exist $a_1, \ldots, a_{n-1} \in A$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that for every $i \in \{1, \ldots, n-1\}$*

$$\sum_{k=1}^{n} \alpha_k f_k(a_i) = 0.$$

Lemma 8 *There exist $a_1, \ldots, a_n$ such that n vectors*

$$\begin{pmatrix} f_1(a_1) \\ f_1(a_2) \\ \vdots \\ f_1(a_n) \end{pmatrix} \quad \begin{pmatrix} f_2(a_1) \\ f_2(a_2) \\ \vdots \\ f_2(a_n) \end{pmatrix} \quad \cdots \quad \begin{pmatrix} f_n(a_1) \\ f_n(a_2) \\ \vdots \\ f_n(a_n) \end{pmatrix}$$

are linearly independent.

Proof. By induction. For $n = 1$ it is trivially true. Let us assume that Lemma 8 is true for $n-1$ and suppose that it is not true for n. That would mean that for every $a \in A$ there exist $\alpha_1(a), \ldots, \alpha_n(a)$ not all equal to zero such that

$$\alpha_1(a) \begin{pmatrix} f_1(a_1) \\ f_1(a_2) \\ \vdots \\ f_1(a_{n-1}) \\ f_1(a) \end{pmatrix} + \alpha_2(a) \begin{pmatrix} f_2(a_1) \\ f_2(a_2) \\ \vdots \\ f_2(a_{n-1}) \\ f_2(a) \end{pmatrix} + \ldots + \alpha_n(a) \begin{pmatrix} f_n(a_1) \\ f_n(a_2) \\ \vdots \\ f_n(a_{n-1}) \\ f_n(a) \end{pmatrix} = 0.$$

By induction hypothesis $\alpha_n(a) \neq 0$ and we have two possibilities:

i) There exists $i \in \{1, \dots, n-1\}$ such that $f_n(a_i) \neq 0$. In this case $\alpha_k(a)/\alpha_n(a)$ do not depend on a for $k = 1, 2, \dots, n-1$ (induction hypothesis). But then for every a $f_n(a) = \sum_{k=1}^{n-1} \alpha_k(a)/\alpha_n(a) f_k(a)$, contradicting independence of $f_1, \dots, f_n$.

ii) For every $i \in \{1, \dots, n-1\}$ $f_n(a_i) = 0$. In this case by induction hypothesis In this case $\alpha_k(a) \equiv 0$ for $k = 1, 2, \dots, n-1$ and therefore $f_n(a) \equiv 0$ - contradiction.

∎

Proof of Proposition 1. Take $a_1, \dots, a_n$ from Lemma 8 then the matrix

$$\mathcal{A} = \begin{bmatrix} f_1(a_1) & f_2(a_1) & \dots & f_n(a_1) \\ f_1(a_2) & f_2(a_2) & \dots & f_n(a_2) \\ \vdots & \vdots & \dots & \vdots \\ f_1(a_n) & f_2(a_n) & \dots & f_n(a_n) \end{bmatrix}$$

is invertible, therefore the equation $\mathcal{A} \cdot \overrightarrow{\alpha} = [0, 0, \dots, 0, 1]^T$ has a solution $\overrightarrow{\alpha}$. This means in particular there exist $\alpha_1, \dots, \alpha_n$ such that $[f_1(a_i), f_2(a_i), \dots, f_n(a_i)] \cdot [\alpha_1, \dots, \alpha_n]^T = 0$ for $i = 1, 2, \dots, n-1$. ∎

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