HOPF BIFURCATION FOR SOME ANALYTIC DIFFERENTIAL SYSTEMS
IN $\mathbb{R}^3$ VIA AVERAGING THEORY

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Abstract. We study the Hopf bifurcation from the singular point with eigenvalues $a \pm bi$ and $c \in \mathbb{C}$ located at the origin of an analytic differential system of the form $\dot{x} = f(x)$, where $x \in \mathbb{R}^3$. Under convenient assumptions we prove that the Hopf bifurcation can produce 1, 2 or 3 limit cycles. We also characterize the stability of these limit cycles. The main tool for proving these results is the averaging theory of first and second order.

1. Introduction and statement of the main results

The main goal of this work is to study the Hopf bifurcation in analytic differential systems in $\mathbb{R}^3$ via averaging theory. In fact the results obtained can be extended easily to $C^4$ differential systems in $\mathbb{R}^3$ but in order to simplify the notation we will present them for analytic differential systems.

More precisely, we investigate the Hopf bifurcation at the singular point located at the origin for an analytic differential systems in $\mathbb{R}^3$ of the form

$$\begin{align*}
\dot{U} & = \varepsilon aU - bV + \sum_{l+j+k \geq n} A_{ijk} U^l V^j W^k, \\
\dot{V} & = bU + \varepsilon aV + \sum_{l+j+k \geq n} B_{ijk} U^l V^j W^k, \\
\dot{W} & = \varepsilon cW + \sum_{l+j+k \geq n} C_{ijk} U^l V^j W^k,
\end{align*}$$

(1)

where $n = 2, 3$ and $\varepsilon$ is a small parameter. Note that the linear part of this system at the singular point located at the origin $(0, 0, 0)$ has eigenvalues $a \pm bi$ and $c \in \mathbb{C}$. So for $\varepsilon = 0$ the eigenvalues are $\pm bi$ and $0$, consequently we are studying a kind of zero–Hopf bifurcation.

For $n = 2$ the Hopf bifurcation of system (1) was studied in [2] were the authors obtained the following result.

Theorem 1 (First order Hopf bifurcation theorem for $n = 2$). We define the constants

$$F = A_{101} + B_{011}, \quad G = C_{020} + C_{200}, \quad D = c(cF - 4aC_{002}), \quad E = D^2 + 8ab^2 F(cF - 2aC_{002}).$$

(a) If $(E - D^2)/(FG) > 0$, then the phase portrait of system (1) has a limit cycle $\Gamma_\varepsilon$ tending to the origin as $\varepsilon \to 0$.

(b) Assume that the limit cycle $\Gamma_\varepsilon$ of statement (a) in cylindrical coordinates $(U = R \cos \theta, V = R \sin \theta, W)$ becomes $(R(\theta), W(\theta))$ with $\theta \in \mathbb{S}^1$. Then

$$R(\theta) \to \sqrt{\frac{E - D^2}{2b^2 F^3 G}} \varepsilon, \quad W(\theta) \to \frac{2a}{F} \varepsilon, \quad \text{when } \varepsilon \to 0.$$

(c) If $D^2 > E$ and $DF > 0$, then the limit cycle $\Gamma_\varepsilon$ is a local repellor; if $D^2 > E$ and $DF < 0$, then the limit cycle $\Gamma_\varepsilon$ is a local attractor; and if $D^2 < E$, then the limit cycle $\Gamma_\varepsilon$ has two invariant manifolds, one stable and the other unstable, which locally are formed by two $2$-dimensional cylinders.
Our first result is to improve Theorem 1 using the averaging theory of second order. Thus we have:

**Theorem 2** (Second order Hopf bifurcation theorem for \( n = 2 \)). We define the constants \( N_i \) for \( i = 0, 1, 2, 3, 4 \) (see (17)) and the constants \( n_k \) and \( d_k \) for \( k = 2, 4 \) (see (18)).

(a) If \( abN_0 > 0 \) and \( n_2d_2 \neq 0 \), then the phase portrait of system (1) has a limit cycle \( \Gamma_1^\varepsilon \) tending to the origin as \( \varepsilon \to 0 \).

(b) If \( N_1N_2 > 0 \), \( N_3N_4 > 0 \) and \( n_4d_4 \neq 0 \), then the phase portrait of system (1) has two limit cycles \( \Gamma_2^\varepsilon \) and \( \Gamma_3^\varepsilon \) tending to the origin as \( \varepsilon \to 0 \).

Theorem 2 will be proved in Section 3 by using the averaging theory of second order, see Section 2. We note that the estimation on the size of the limit cycles given in statements (a) and (b) of Theorem 2 can be obtained from Theorem 5 but their expressions are very long and we do not write them here. Also the stability of the limit cycles can be analyzed computing the eigenvalues of the matrix (12), but again we do not provide them because their expressions are too long.

Our second result is to extend Theorem 1 to systems (1) with \( n = 3 \). Thus we have:

**Theorem 3** (Hopf bifurcation theorem for \( n = 3 \)). We define the constants

\[
D_0 = \frac{8a}{A_{120} + 3(A_{300} + B_{030}) + B_{210}}, \quad D_1 = \frac{C_{003}((A_{120} + 3(A_{300} + B_{030}) + B_{210})c - 4a(C_{021} + C_{201}))}{(A_{120} + 3(A_{300} + B_{030}) + B_{210})C_{003} - 2(A_{102} + B_{012})(C_{021} + C_{201})}, \quad D_2 = (A_{120} + 3(A_{300} + B_{030}) + B_{210})C_{003} + 2(A_{102} + B_{012})(C_{021} + C_{201}).
\]

(a) If

\[
D_0((A_{120} + 3(A_{300} + B_{030}) + B_{210})c + 4a(C_{021} + C_{201})) 
eq 0,
\]

then the phase portrait of system (1) has a limit cycle \( \Gamma_1^\varepsilon \) tending to the origin as \( \varepsilon \to 0 \).

(b) Assume that the limit cycle \( \Gamma_1^\varepsilon \) of statement (a) in cylindrical coordinates \( (U = R \cos \theta, V = R \sin \theta, W) \) becomes \( (R(\theta), W(\theta)) \) with \( \theta \in S^1 \). Then

\[
R(\theta) \to \sqrt{-D_0\varepsilon}, \quad W(\theta) \to 0, \quad \text{when } \varepsilon \to 0.
\]

(c) If \( ab < 0 \) and \( (2c - D_0(C_{021} + C_{201}))b > 0 \), then the limit cycle \( \Gamma_1^\varepsilon \) is a local repellor; if \( ab > 0 \) and \( (2c - D_0(C_{021} + C_{201}))b < 0 \), then the limit cycle \( \Gamma_1^\varepsilon \) is a local attractor; and if \( ab > 0 \) and \( (2c - D_0(C_{021} + C_{201}))b < 0 \), or \( ab < 0 \) and \( (2c - D_0(C_{021} + C_{201}))b > 0 \), then the limit cycle \( \Gamma_1^\varepsilon \) has two invariant manifolds, one stable and the other unstable, which locally are formed by two 2-dimensional cylinders.

(d) If

\[
- \frac{2D_1((A_{102} + B_{012})c - 2aC_{003})}{b^2C_{003}} 
eq 0,
\]

then the phase portrait of the quadratic system (1) has two limit cycles \( \Gamma_1^\varepsilon \) and \( \Gamma_2^\varepsilon \) tending to the origin as \( \varepsilon \to 0 \).

(e) Assume that the limit cycle \( \Gamma_1^\varepsilon \) of statement (d) in cylindrical coordinates \( (U = R \cos \theta, V = R \sin \theta, W) \) becomes \( (R(\theta), W(\theta)) \) with \( \theta \in S^1 \). Then

\[
R(\theta) \to \frac{2\sqrt{\varepsilon}(A_{102} + B_{012})c + 2aC_{003}}{-D_2}, \quad W(\theta) \to -\frac{\sqrt{-D_1\varepsilon}}{\sqrt{C_{003}}}, \quad \text{when } \varepsilon \to 0.
\]

(f) Assume that the limit cycle \( \Gamma_2^\varepsilon \) of statement (d) in cylindrical coordinates \( (U = R \cos \theta, V = R \sin \theta, W) \) becomes \( (R(\theta), W(\theta)) \) with \( \theta \in S^1 \). Then

\[
R(\theta) \to \frac{2\sqrt{\varepsilon}(A_{102} + B_{012})c + 2aC_{003}}{-D_2}, \quad W(\theta) \to -\frac{\sqrt{-D_1\varepsilon}}{\sqrt{C_{003}}}, \quad \text{when } \varepsilon \to 0.
\]
Theorem 3 will be proved in Section 4 by using the averaging theory of first order, see Section 2. As it is mention in Section 4 we can study the stability of the limit cycles $\Gamma^1,2$ computing their eigenvalues but since they have very long expressions we have omitted them.

2. LIMIT CYCLES VIA AVERAGING THEORY

In few words we can say that the averaging method [4, 5] gives a quantitative relation between the solutions of some non–autonomous periodic differential system and the solutions of its averaged differential system, which is an autonomous one. The next theorem provides a first order approximation in $\varepsilon$ for the limit cycles of a periodic differential system, for a proof see Theorem 2.6.1 of Sanders and Verhulst [4] and Theorem 11.5 of Verhulst [5].

**Theorem 4.** We consider the following two initial value problems

(5) \[ \dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon), \quad x(0) = x_0, \]

and

(6) \[ \dot{y} = \varepsilon f^0(y), \quad y(0) = y_0, \]

where $x, y, x_0 \in \Omega$ an open subset of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, $f$ and $g$ are periodic of period $T$ in the variable $t$, and $f^0(y)$ is the averaged function of $f(t, x)$ with respect to $t$, i.e.,

(7) \[ f^0(y) = \frac{1}{T} \int_0^T f(t, y)dt. \]

Suppose: (i) $f$, its Jacobian $\partial f/\partial x$, its Hessian $\partial^2 f/\partial x^2$, $g$ and its Jacobian $\partial g/\partial x$ are defined, continuous and bounded by a constant independent on $\varepsilon$ in $[0, \infty) \times \Omega$ and $\varepsilon \in (0, \varepsilon_0]$; (ii) $T$ is a constant independent of $\varepsilon$; and (iii) $y(t)$ belongs to $\Omega$ on the interval of time $[0, 1/\varepsilon]$. Then the following statements hold.

(a) On the time scale $1/\varepsilon$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.

(b) If $p$ is a singular point of the averaged system (6) such that the determinant of the Jacobian matrix

(8) \[ \left. \frac{\partial f^0}{\partial y} \right|_{y=p} \]

is not zero, then there exists a limit cycle $\phi(t, \varepsilon)$ of period $T$ for the system (5) which is close to $p$ and such that $\phi(t, \varepsilon) \to p$ as $\varepsilon \to 0$.

(c) The stability or instability of the limit cycle $\phi(t, \varepsilon)$ is given by the stability or instability of the singular point $p$ of the averaged system (6). In fact, the singular point $p$ has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t, \varepsilon)$.

The next theorem provides a second order approximation in $\varepsilon$ for the limit cycles of a periodic differential system, for an statement in the case of a differential equation (see [3] and [1]) and for a proof see Theorem 3.5.1 of Sanders and Verhulst [4].

**Theorem 5.** We consider the following two initial value problems

(9) \[ \dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad x(0) = x_0, \]

and

(10) \[ \dot{y} = \varepsilon f^0(y) + \varepsilon^2 f^{10}(y) + \varepsilon^2 g^0(y), \quad y(0) = y_0, \]

with $f, g$: $[0, \infty) \times \Omega \times \mathbb{R}^n$, $R$: $[0, \infty) \times \Omega \times (0, \varepsilon_0] \to \mathbb{R}^n$, $\Omega$ an open set of $\mathbb{R}^n$, $f, g$ and $G$ periodic functions of period $T$ in the variable $t$,

(11) \[ f^1(t, x) = \frac{\partial f}{\partial x} y^1(t, x), \quad \text{where} \quad y^1(t, x) = \int_0^t f(s, x) ds. \]

Of course, $f^0$, $f^{10}$ and $g^0$ denote the averaged functions of $f$, $f^1$ and $g$, respectively, defined as in (7). Suppose: (i) $\partial f/\partial x$ is Lipschitz in $x$, $g$ and $R$ are Lipschitz in $x$ and all these functions...
are continuous on their domain of definition; (ii) $|R(t, x, \varepsilon)|$ is bounded by a constant uniformly in $[0, L/\varepsilon] \times 0 \times [0, \varepsilon_0]$; (iii) $T$ is a constant independent of $\varepsilon$; and (iv) $y(t)$ belongs to $\Omega$ on the interval of time $[0,1/\varepsilon]$. Then the following statements hold.

(a) On the time scale $1/\varepsilon$ we have that $x(t) = y(t) + \varepsilon y^1(t, y(t)) = O(\varepsilon^2)$.

(b) If $p$ is a singular point of the averaged system (10) such that

$$\frac{\partial(f^{10} + g^0(y))}{\partial y} \bigg|_{y=p}$$

is not zero, then there exists a limit cycle $\phi(t, \varepsilon)$ of period $T$ for the system (9) which is close to $p$ and such that $\phi(t, \varepsilon) \to p$ as $\varepsilon \to 0$.

(c) The stability or instability of the limit cycle $\phi(t, \varepsilon)$ is given by the stability or instability of the singular point $p$ of the averaged system (10). In fact, the singular point $p$ has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t, \varepsilon)$.

3. PROOF OF THEOREM 2

In the cylindrical change of coordinates $U = R \cos \theta$, $V = R \sin \theta$, $W = W$ the system (1) in the region $R > 0$ has the following form

$$\dot{R} = \alpha R + H_{11}(\theta)R^2 + H_{12}(\theta)RW + H_{13}(\theta)W^2 + O_3(R, W),$$

$$(13)$$

$$\dot{\theta} = \frac{1}{R} \left[ bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R, W) \right],$$

$$W = \alpha W + H_{31}(\theta)R^2 + H_{32}(\theta)RW + O_3(R, W),$$

where

$$H_{11}(\theta) = A_{200} \cos^3 \theta + (A_{110} + B_{200}) \cos^2 \theta \sin \theta + (A_{020} + B_{110}) \cos \theta \sin^2 \theta + B_{020} \sin^3 \theta,$$

$$H_{12}(\theta) = A_{101} \cos^2 \theta + (A_{011} + B_{101}) \cos \theta \sin \theta - A_{101} \sin^2 \theta,$$

$$H_{13}(\theta) = A_{002} \cos \theta + B_{002} \sin \theta,$$

$$H_{21}(\theta) = B_{200} \cos^3 \theta - (A_{200} - B_{110}) \cos^2 \theta \sin \theta - (A_{110} - B_{020}) \cos \theta \sin^2 \theta,$$

$$- A_{020} \sin^3 \theta,$$

$$H_{22}(\theta) = B_{101} \cos^2 \theta - 2A_{101} \cos \theta \sin \theta - A_{011} \sin^2 \theta,$$

$$H_{23}(\theta) = B_{002} \cos \theta - A_{002} \sin \theta,$$

$$H_{31}(\theta) = -C_{020} \cos^2 \theta + C_{110} \cos \theta \sin \theta + C_{020} \sin^2 \theta,$$

$$H_{32}(\theta) = C_{101} \cos \theta + C_{011} \sin \theta.$$

Therefore, system (13) in the region $\dot{\theta} \neq 0$ is equivalent to

$$\frac{dR}{d\theta} = \frac{R[\alpha R + H_{11}(\theta)R^2 + H_{12}(\theta)RW + H_{13}(\theta)W^2 + O_3(R, W)]}{bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R, W)},$$

$$\frac{dW}{d\theta} = \frac{\alpha W + H_{31}(\theta)R^2 + H_{32}(\theta)RW + O_3(R, W)}{bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R, W)}.$$

We note that this system is $2\pi$-periodic in the variable $\theta$. Performing the rescaling $(R, W) = (\sqrt{\varepsilon}, \zeta \sqrt{\varepsilon})$ the system (14) becomes into the normal form for applying the averaging theory. That is, in the variables $(\varphi, \zeta)$ system (14) writes

$$\frac{d\varphi}{d\theta} = \varepsilon^{1/2} f_1(\theta, \varphi, \zeta) + \varepsilon g_1(\theta, \varphi, \zeta) + \varepsilon^{3/2} R_1(\theta, \varphi, \zeta, \varepsilon),$$

$$\frac{d\zeta}{d\theta} = \varepsilon^{1/2} f_2(\theta, \varphi, \zeta) + \varepsilon g_2(\theta, \varphi, \zeta) + \varepsilon^{3/2} R_2(\theta, \varphi, \zeta, \varepsilon),$$

$$\frac{dR}{d\theta} = \frac{R[\alpha R + H_{11}(\theta)R^2 + H_{12}(\theta)RW + H_{13}(\theta)W^2 + O_3(R, W)]}{bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R, W)},$$

$$\frac{dW}{d\theta} = \frac{\alpha W + H_{31}(\theta)R^2 + H_{32}(\theta)RW + O_3(R, W)}{bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R, W)}.$$
where

\[
\begin{align*}
  f_1(\theta, \varrho, \zeta) &= \frac{1}{b} [H_{11}(\theta)\varrho^2 + H_{12}(\theta)\varrho\zeta + H_{13}(\theta)\zeta^2], \\
  g_1(\theta, \varrho, \zeta) &= \frac{a}{b} - \frac{1}{b^2} [H_{11}(\theta)\varrho^2 + H_{12}(\theta)\varrho\zeta + H_{13}(\theta)\zeta^2] [H_{21}(\theta)\varrho^2 + H_{22}(\theta)\varrho\zeta + H_{23}(\theta)\zeta^2], \\
  f_2(\theta, \varrho, \zeta) &= \frac{\theta}{b} [H_{31}(\theta)\varrho + H_{32}(\theta)\zeta], \\
  g_2(\theta, \varrho, \zeta) &= \frac{c}{b} - \frac{1}{b^2} [H_{31}(\theta)\varrho + H_{32}(\theta)\zeta][H_{21}(\theta)\varrho^2 + H_{22}(\theta)\varrho\zeta + H_{23}(\theta)\zeta^2].
\end{align*}
\]

So system (15) is equivalent to system (9) taking \(x = (\varrho, \zeta), \ t = \theta, \ f(t, x) = (f_1(\theta, \varrho, \zeta), f_2(\theta, \varrho, \zeta)), \ g(t, x) = (g_1(\theta, \varrho, \zeta), g_2(\theta, \varrho, \zeta)), \ T = 2\pi \) and \(\epsilon = \epsilon^{1/2}\).

Let \(\Omega\) be the open subset and \(\epsilon_0\) the positive number which appear in the statement of Theorem 5. Then, it is easy to verify that, system (15) satisfies the assumptions of Theorem 5 if we take as \(\Omega\) an open disc centered at the origin in \(\mathbb{R}^2\) and a sufficiently small \(\epsilon_0\). Let

\[
g_i^{(1)}(\theta, \varrho, \zeta) = \int_0^\theta f_i(s, \varrho, \zeta) \, ds \quad \text{for } i = 1, 2.
\]

Then

\[
y_i^{(1)}(\theta, \varrho, \zeta) = \frac{1}{12b} \left\{ 12(b_{000} - b_{000} \cos \theta + a_{002} \sin \theta)\zeta^2 \\
+ 3(a_{011} + b_{101} - a_{011} \cos^2 \theta - B_{101} \sin^2 \theta + 4A_{011} \cos \theta \sin \theta) \\
+ A_{011} \sin^2 \theta + B_{101} \sin^2 \theta \zeta \theta \\
+ [4A_{110} + 8B_{020} + 4B_{200} - 3A_{110} \cos \theta - 9B_{020} \cos \theta - 3B_{200} \cos \theta \\
- A_{110} \cos^3 \theta + 3B_{020} \cos^3 \theta + 3A_{110} \cos \sin \theta + 9A_{110} \sin \theta \\
+ 3B_{110} \sin \theta - 3A_{020} \cos^2 \theta \sin \theta + 3A_{200} \cos^2 \theta \sin \theta - 3B_{110} \cos^2 \theta \sin \theta \\
+ 3A_{110} \cos \theta \sin^2 \theta - 3B_{020} \cos \theta \sin^2 \theta + 3B_{200} \cos \theta \sin^2 \theta \\
+ A_{020} \sin^3 \theta - A_{200} \sin^3 \theta + B_{110} \sin^3 \theta \theta^2 \right\},
\]

\[
y_i^{(2)}(\theta, \varrho, \zeta) = \frac{\theta}{2b} \left\{ (C_{110} \sin^2 \theta - 2C_{020} \sin \theta \cos \theta) \theta + [2C_{011} + 2C_{101} \sin \theta - 2C_{011} \cos \theta] \zeta \right\}.
\]

We have

\[
\begin{align*}
  \begin{pmatrix} f_1^{(1)}(\theta, \varrho, \zeta) \\ f_2^{(1)}(\theta, \varrho, \zeta) \end{pmatrix} &= \begin{pmatrix} \frac{\partial f_1}{\partial \varrho}(\theta, \varrho, \zeta) & \frac{\partial f_1}{\partial \zeta}(\theta, \varrho, \zeta) \\ \frac{\partial f_2}{\partial \varrho}(\theta, \varrho, \zeta) & \frac{\partial f_2}{\partial \zeta}(\theta, \varrho, \zeta) \end{pmatrix} \begin{pmatrix} y_1^{(1)}(\theta, \varrho, \zeta) \\ y_2^{(1)}(\theta, \varrho, \zeta) \end{pmatrix}.
\end{align*}
\]

So

\[
\begin{align*}
  f_i^{(1)}(\varrho, \zeta) &= \frac{1}{2\pi} \int_0^{2\pi} f_i^{(1)}(\theta, \varrho, \zeta) \, d\theta, \quad \text{for } i = 1, 2, \\
  g_i^{(1)}(\varrho, \zeta) &= \frac{1}{2\pi} \int_0^{2\pi} g_i(\theta, \varrho, \zeta) \, d\theta, \quad \text{for } i = 1, 2,
\end{align*}
\]
then

\[
f^{10}_1(\varrho, \zeta) = \frac{1}{8b^2}[-((A_{011} + B_{101})C_{020} + A_{101}C_{110})\varrho^3 + 2(A_{002}(A_{110} + 3B_{020} + B_{200}) - 4C_{011}) - B_{002}(A_{020} + 3A_{200} + B_{110} - 4C_{101}))\zeta^2],
\]

\[
f^{10}_2(\varrho, \zeta) = \frac{\zeta}{8b^2}([(A_{020} + 3A_{200} + B_{110})C_{011} + 2(A_{011} + B_{101})C_{020} - (A_{110} + 3B_{020} + B_{200})C_{101} + 2A_{101}C_{110})\varrho^3 + 4(A_{002}C_{011} - B_{002}C_{101})\zeta^2],
\]

\[
g^0_1(\varrho, \zeta) = \frac{\varrho}{8b^2}[8ab + (A_{110}A_{200} + A_{020}(A_{110} + 2B_{020}) - 2A_{200}B_{200} - B_{110}(B_{200} + B_{200}))\varrho^2 + 2(B_{002}(A_{020} - A_{200} - B_{110}) + A_{002}(A_{110} + B_{020} - B_{200}))\zeta^2],
\]

\[
g^0_2(\varrho, \zeta) = \frac{\zeta}{8b^2}[8bc + ((3A_{020} + A_{200} - B_{110})C_{011} + 2(A_{011} + B_{101})C_{020} + A_{110}C_{101} - (B_{020} + 3B_{200})C_{101} + 2A_{101}C_{110})\varrho^2 + 4(A_{002}C_{011} - B_{002}C_{101})\zeta^2].
\]

The averaged system \( f^{10}_i(\varrho, \zeta) + g^0_i(\varrho, \zeta), i = 1, 2 \) has six singular points:

\[
q_1 = \left(-\frac{2\sqrt{2ab}}{\sqrt{N_0}}, 0\right), \quad q_2 = \left(\frac{2\sqrt{2ab}}{\sqrt{N_0}}, 0\right),
\]

\[
q_3 = \left(-\frac{\sqrt{N_1}}{\sqrt{N_2}} - \frac{\sqrt{N_3}}{\sqrt{N_4}}, \sqrt{N_2} - \frac{\sqrt{N_3}}{\sqrt{N_4}}\right), \quad q_4 = \left(\frac{\sqrt{N_1}}{\sqrt{N_2}} - \frac{\sqrt{N_3}}{\sqrt{N_4}}, \sqrt{N_2} - \frac{\sqrt{N_3}}{\sqrt{N_4}}\right),
\]

\[
q_5 = \left(-\frac{\sqrt{N_1}}{\sqrt{N_2}} - \frac{\sqrt{N_3}}{\sqrt{N_4}}, \sqrt{N_2} - \frac{\sqrt{N_3}}{\sqrt{N_4}}\right), \quad q_6 = \left(\frac{\sqrt{N_1}}{\sqrt{N_2}} - \frac{\sqrt{N_3}}{\sqrt{N_4}}, \sqrt{N_2} - \frac{\sqrt{N_3}}{\sqrt{N_4}}\right),
\]

where

\[
N_0 = -A_{110}A_{200} - A_{020}(A_{110} + 2B_{020}) + 2A_{200}B_{200} + B_{110}(B_{020} + B_{200}) + (A_{011} + B_{101})C_{020} + A_{101}C_{110},
\]

\[
N_1 = 4b((A_{020}(A_{110} + 2B_{020})c + B_{002}(2A_{200} + B_{110})c + 2A_{002}(a + c)C_{011} - 2B_{002}(a + c)C_{101}),
\]

\[
N_2 = B_{002}(-4b^2C_{011} - 2A_{020}(2A_{200} + B_{110})C_{011} - 2A_{011}B_{110}C_{020} - 2B_{101}B_{110}C_{020} + B_{020}B_{110}C_{101} + B_{110}B_{200}C_{101} + A_{020}C_{101} - A_{110} + 2B_{020} + 4C_{011}) + 3A_{011}C_{020}C_{101} + 3B_{101}C_{020}C_{101} - 4B_{200}C_{011} - 4B_{200}C_{110} - 2A_{101}B_{110}C_{110} + 3A_{101}C_{101} + A_{200}(-2B_{110}C_{011} - 4(A_{011} + B_{101})C_{020} + (A_{110} + 4B_{020} + 2B_{200} + 4C_{011})C_{110} - 4A_{011}C_{110}) + A_{002}(A_{020}(A_{110} + 2B_{020} - 4C_{011})C_{110} + 2A_{200}(2B_{020} + B_{200} - 2C_{011})C_{011} - 4B_{020}C_{101} + C_{011}(B_{200}(B_{110} + 4C_{101}) - 3((A_{011} + B_{101})C_{020} + A_{101}C_{110}) + A_{110}A_{200}C_{011} + 2((A_{011} + B_{101})C_{020} - (B_{020} + 2B_{020})C_{101} + A_{101}C_{110}) + B_{020}(B_{110}C_{011} + 4((A_{011} + B_{101})C_{020} - 2B_{020}C_{110} + C_{011}C_{101} + A_{101}C_{110})),
\]

\[
N_3 = -b(A_{002}C_{011} - B_{002}C_{101})(-A_{110}A_{200} + A_{020}(A_{110} + 2B_{020})c + B_{020}B_{110}c + 2A_{200}B_{200}c + B_{110}B_{200}c + 4aA_{200}C_{011} + 4aA_{011}C_{020} + 4aB_{011}C_{020} + A_{011}C_{020} + B_{110}C_{020} - 4aB_{020}C_{101} - 4aB_{200}C_{110} + A_{101}(4a + c)C_{110}),
\]

(17)
HOPF BIFURCATION FOR SOME ANALYTIC DIFFERENTIAL SYSTEMS IN R^3

\[ N_4 = (-A_{02}C_{011} + B_{002}C_{101})B_{002}(4A_{200}^2C_{011} + 2B_{101}B_{110}C_{020} + A_{011}C_{020}(2B_{110} - 3C_{101}) - B_{020}B_{110}C_{101} - B_{110}B_{200}C_{101} - 3B_{101}C_{020}C_{101} + 4B_{020}C_{101}^2 + 4B_{200}C_{101}^2 + A_{020}(4A_{200}C_{011} + 2B_{110}C_{011} - (A_{110} + 2B_{020} + 4C_{011})C_{101}) + 2A_{101}B_{110}C_{110} - 3A_{101}C_{101}C_{110} + A_{200}(2B_{110}C_{011} + 4(A_{110} + B_{101})C_{020} - (A_{110} + 4B_{020} + 2B_{020} + 4C_{011})C_{101} + 4A_{101}C_{110})) - A_{002}(A_{110} + 2B_{020} - 4C_{011})C_{101} + 2A_{200}(2B_{200} + B_{200} - 2C_{011})C_{101} - 4B_{200}C_{101} + C_{011}(B_{200}(B_{110} + 4C_{101}) - 3(A_{101} + B_{110}C_{020} + A_{101}C_{110})) + A_{110}(A_{200}C_{110} + 2((A_{110} + B_{101})C_{020} - (B_{200} + B_{200}C_{101} + A_{101}C_{110}) + B_{020}(B_{110}C_{011} + 4((A_{110} + B_{101})C_{020} - B_{200}C_{101} + C_{011}(C_{011} + A_{101}C_{110})))).
\]

We observe that the averaged theory can be conclusive only at points \( q_2, q_4 \) and \( q_6 \). In fact, at \( q_1, q_3 \) and \( q_5 \) we have \( c < 0 \) and it means that \( R < 0 \) which does not make sense.

For \( k = 2, 4, 6 \) the Jacobian matrix (12) at point \( q_k \) has determinant \( n_k/d_k \) where

\[
\begin{align*}
n_2 &= -2a(-A_{110}A_{200}c + B_{020}B_{110}c + 2A_{200}B_{200}c + B_{110}B_{200}c + A_{110}C_{020}c + B_{010}C_{020}c + A_{101}C_{101}c + 4aA_{200}C_{011} - A_{020}(A_{110} + 2B_{020} - 4aC_{011}) + 4aA_{011}C_{020} + 4aB_{101}C_{020} - 4aB_{200}C_{101} - 4aA_{101}C_{110}), \\
d_2 &= \frac{b^2}{2}(-A_{110}A_{200} + 2B_{200}A_{200} - A_{020}(A_{110} + 2B_{200}) + B_{020}B_{110} + B_{110}B_{200} + A_{011}C_{020} + B_{010}C_{020} + A_{101}C_{110}), \\
n_4 &= 2(A_{002}(A_{110} - 2(a + c)C_{011} - B_{020}) + B_{020}(-2A_{200}c + B_{200}) + 2(a + c)C_{101})(A_{110}A_{200}c - B_{020}B_{110}c - 2A_{200}B_{200}c - B_{110}B_{200}c - A_{011}C_{020}c - B_{010}C_{020}c - A_{101}C_{101}c - 4aA_{200}C_{011} + A_{020}(A_{110}c + 2B_{200}c - 4aC_{011}) - 4aA_{011}C_{020} - 4aB_{101}C_{020} + 4aB_{200}C_{101} + 4aA_{101}C_{110}), \\
d_4 &= \frac{b^2}{2}(B_{002}(-4C_{011}A_{200}^2 + (-2B_{110}C_{011} + 4C_{110}C_{011} - 4A_{101}C_{020} - 4B_{101}C_{020} + A_{110}C_{011} + 4B_{200}C_{101} + 2B_{200}C_{110} - 4A_{101}C_{110})(A_{200} - 4B_{200}C_{011} - 2A_{011}B_{110}C_{020} - 2B_{010}B_{101}C_{020} + B_{020}B_{110}C_{011} + 3A_{011}C_{020}C_{101} + 3B_{010}C_{020}C_{110} + A_{002}(4A_{020}C_{011} - 2B_{010}C_{011} + (A_{110} + 2B_{020} + 4C_{011})C_{101}) - 2A_{101}B_{110}C_{110} + 3A_{101}C_{011}C_{110} + A_{002}(4C_{011}B_{200} + 4A_{011}C_{020}B_{200} + 4B_{010}C_{020}B_{200} - 4A_{200}C_{101}B_{020} + 4C_{011}C_{020}B_{101}B_{020} + 4A_{110}C_{110}B_{020} - 4A_{200}C_{011}B_{200} + 2A_{011}B_{010}B_{200} + A_{101}B_{010}B_{101} - 2A_{101}B_{110}C_{020} - 2A_{010}B_{101}C_{020} - 4C_{011}B_{010}B_{200} - 4A_{110}C_{011}C_{020}B_{200} - 3A_{011}C_{011}C_{110} - 3A_{010}C_{011}C_{110} + 3A_{010}C_{011}C_{110} + A_{101}(A_{200}C_{011} + 2A_{011}C_{020} + 2B_{101}C_{020} - 2B_{020}C_{101} - 2B_{020}C_{110} + 2A_{101}C_{110}))))).
\end{align*}
\]

4. Proof of Theorem 3

In the cylindrical change of coordinates \( U = R \cos \theta, V = R \sin \theta, W = W \) the system (1) with \( n = 3 \) in the region \( R > 0 \) has the following form

\[
\begin{align*}
\dot{R} &= \alpha R + h_{11}(\theta)R^3 + h_{12}(\theta)R^2W + h_{13}(\theta)RW^2 + h_{14}(\theta)W^3 + O_4(R, W), \\
\dot{\theta} &= \frac{1}{R} \left[ bR + h_{21}(\theta)R^3 + h_{22}(\theta)R^2W + h_{23}(\theta)RW^2 + h_{24}(\theta)W^3 + O_4(R, W) \right], \\
\dot{W} &= \alpha W + h_{31}(\theta)R^3 + h_{32}(\theta)R^2W + h_{33}(\theta)RW^2 + h_{34}(\theta)W^3 + O_4(R, W),
\end{align*}
\]
where
\[
\begin{align*}
h_{11}(\theta) &= A_{300} \cos^4 \theta + (A_{210} + B_{300}) \cos^3 \theta \sin \theta + (A_{120} + B_{210}) \cos^2 \theta \sin^2 \theta \\
&\quad + (A_{030} + B_{120}) \cos \theta \sin^3 \theta + B_{030} \sin^4 \theta, \\
h_{12}(\theta) &= A_{201} \cos^3 \theta + (A_{111} + B_{201}) \cos^2 \theta \sin \theta + (A_{021} + B_{111}) \cos \theta \sin^2 \theta \\
&\quad + B_{021} \sin^3 \theta, \\
h_{13}(\theta) &= A_{102} \cos^2 \theta + (A_{012} + B_{102}) \cos \theta \sin \theta + B_{012} \sin^2 \theta, \\
h_{14}(\theta) &= A_{003} \cos \theta + B_{003} \sin \theta, \\
h_{21}(\theta) &= B_{300} \cos^4 \theta + (A_{300} - B_{210}) \cos^3 \theta \sin \theta + (A_{210} + B_{120}) \cos^2 \theta \sin^2 \theta \\
&\quad + (A_{120} - B_{030}) \sin^3 \theta \cos \theta - B_{030} \sin^4 \theta, \\
h_{22}(\theta) &= B_{201} \cos^3 \theta + (A_{201} - B_{111}) \cos^2 \theta \sin \theta + (A_{111} + B_{021}) \cos \theta \sin^2 \theta \\
&\quad - A_{021} \sin^3 \theta, \\
h_{23}(\theta) &= B_{102} \cos^2 \theta + (A_{102} - B_{012}) \cos \theta \sin \theta - A_{012} \sin^2 \theta, \\
h_{24}(\theta) &= B_{003} \cos \theta - A_{003} \sin \theta, \\
h_{31}(\theta) &= C_{300} \cos^3 \theta + C_{210} \cos^2 \theta \sin \theta + C_{120} \cos \theta \sin^2 \theta + C_{030} \sin^3 \theta, \\
h_{32}(\theta) &= C_{201} \cos^2 \theta + C_{111} \cos \theta \sin \theta + C_{021} \sin^2 \theta, \\
h_{33}(\theta) &= C_{102} \cos \theta + C_{012} \sin \theta, \\
h_{34}(\theta) &= C_{003}.
\end{align*}
\]

Therefore, system (19) in the region $\dot{\theta} \neq 0$ is equivalent to
\[
\begin{align*}
\frac{dR}{d\theta} &= R(\alpha R + h_{11}(\theta)R^3 + h_{12}(\theta)R^2W + h_{13}(\theta)RW^2 + h_{14}(\theta)W^3 + O_4(R, W)), \\
\frac{dW}{d\theta} &= R(\epsilon \sqrt{\alpha} R + h_{31}(\theta)R^3 + h_{32}(\theta)R^2W + h_{33}(\theta)RW^2 + h_{34}(\theta)W^3 + O_4(R, W)).
\end{align*}
\] (20)

We note that this system is $2\pi$-periodic in the variable $\theta$. Performing the rescaling $(R, W) = (\sqrt{\epsilon} R, \sqrt{\epsilon} W)$ the system (20) becomes into the normal form for applying the averaging theory. That is, in the variables $(\rho, \zeta)$ system (20) writes
\[
\begin{align*}
\frac{d\rho}{d\theta} &= \epsilon f_1(\rho, \varphi, \zeta) + \epsilon^2 g_1(\theta, \rho, \varphi, \epsilon), \\
\frac{d\zeta}{d\theta} &= \epsilon f_2(\rho, \varphi, \zeta) + \epsilon^2 g_2(\theta, \rho, \varphi, \epsilon),
\end{align*}
\] (21)

where
\[
\begin{align*}
f_1(\theta, \rho, \varphi, \zeta) &= \frac{1}{b} [a \rho + h_{11}(\theta) \rho^3 + h_{12}(\theta) \rho \varphi \zeta + h_{13}(\theta) \rho \varphi^2 + h_{14}(\theta) \zeta^3], \\
f_2(\theta, \rho, \varphi, \zeta) &= \frac{1}{b} [c \zeta + h_{31}(\theta) \rho^3 + h_{32}(\theta) \rho \varphi \zeta + h_{33}(\theta) \rho \varphi^2 + h_{34}(\theta) \varphi^3].
\end{align*}
\]

So, system (21) is equivalent to system (5) taking $x = (\rho, \zeta)$, $t = \theta$, $f(t, x) = (f_1(\theta, \rho, \varphi, \zeta), f_2(\theta, \rho, \varphi, \zeta))$, $T = 2\pi$ and $\epsilon = \epsilon$.

Let $\Omega$ be the open subset which appear in the statement of Theorem 4. Then it is easy to verify that system (21) satisfies the assumptions of Theorem 4 if we take as $\Omega$ an open disc centered at the origin in $\mathbb{R}^2$ and a sufficiently small $\epsilon_0$. Since
\[
f_1^0(\rho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f_1(\theta, \rho, \varphi) d\theta,
\]
for \( i = 1, 2 \), we get that
\[
\begin{align*}
I_1^{\theta}(\varrho, \zeta) &= \frac{g}{8b}[8a + (A_{120} + 3(A_{300} + B_{030}) + B_{210})\varrho^2 + 4(A_{102} + B_{012})\zeta^2], \\
I_2^{\theta}(\varrho, \zeta) &= \frac{c}{2b}[2c + (C_{021} + C_{201})\varrho^2 + 2C_{003}\zeta^2].
\end{align*}
\]

Hence with the notation of \( D_1 \) and \( D_2 \) introduced in (2), the averaged system (6) has nine singular points:
\[
\begin{align*}
p_1 &= (0, 0), \quad p_2 = (-\sqrt{-D_0}, 0), \quad p_3 = (\sqrt{-D_0}, 0), \\
p_4 &= \left(0, -\frac{i\sqrt{c}}{\sqrt{C_{003}}}\right), \quad p_5 = \left(0, \frac{i\sqrt{c}}{\sqrt{C_{003}}}\right), \\
p_6 &= \left(\frac{2\sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{-\sqrt{-D_1}}{\sqrt{C_{003}}}\right), \\
p_7 &= \left(\frac{2\sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{-\sqrt{-D_1}}{\sqrt{C_{003}}}\right), \\
p_8 &= \left(\frac{2\sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{-\sqrt{-D_1}}{\sqrt{C_{003}}}\right), \\
p_9 &= \left(\frac{2\sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{-\sqrt{-D_1}}{\sqrt{C_{003}}}\right).
\end{align*}
\]

We observe that the averaged theory can be conclusive only at points \( p_3, p_7 \) and \( p_9 \). In fact at \( p_2, p_6 \) and \( p_8 \) we have \( g < 0 \) and it means that \( R < 0 \) which does not make sense. Moreover at \( p_1, p_4 \) and \( p_5 \) we have \( g = 0 \), i.e., \( R = 0 \) and the cylindrical coordinates change \( R = 0 \) is not a diffeomorphism.

The Jacobian matrix (8) at point \( p_3 \) has determinant
\[
\frac{D_0(- (A_{120} + 3(A_{300} + B_{030}) + B_{210})c + 4a(C_{021} + C_{201}))}{4b^2},
\]
which is different from zero according to our hypothesis (see (3)). So, by statement (b) of Theorem 4, there exists a limit cycle for system (21) which converges to \( 0 \) as \( \epsilon \to 0 \).

To prove statement (b) we just observe the following. The limit cycle \( \Gamma_\epsilon \) can be written in the system (20) as \( \{R(\theta), W(\theta)) : \theta \in S) \). The point \( p_3 \) in the coordinates \( (R, W) \) is written as
\[
(\sqrt{-D_0}, 0).
\]

Thus statement (b) of Theorem 3 follows immediately from statement (b) of Theorem 4. It completes the proof of (a) and (b).

The stability of the limit cycle is determined by the eigenvalues of system (6) at the point \( p_3 \). Taking the notation of \( D_0 \) given in (2), the eigenvalues are
\[
-\frac{2a}{b} \quad \text{and} \quad \frac{2c - D_0(C_{021} + C_{201})}{2b}.
\]

By statement (c) of Theorem 4 it follows immediately the different kind of stability for the limit cycle stated in (b) of Theorem 3. So statement (c) of Theorem 3 is proved.

The Jacobian matrix (8) at points \( p_7 \) and \( p_8 \) have determinant
\[
-\frac{2D_1((A_{102} + B_{012})c - 2aC_{003})}{b^2C_{003}},
\]
which is different from zero according to our hypothesis (see (4)). So, by statement (b) of Theorem 4, there exists two limit cycles for system (21) one that converges to \( p_7 \) as \( \epsilon \to 0 \) and one that converges to \( p_9 \) as \( \epsilon \to 0 \).
To prove statement (e) we just observe the following. The limit cycle $\Gamma_1$ can be written in the system (20) as \( \{ R(\theta), W(\theta) : \theta \in S \} \). The point $p_7$ at the coordinates $(R, W)$ is written as
\[
\left( \frac{2\sqrt{\epsilon} \sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{-\sqrt{-D_1 \epsilon}}{\sqrt{C_{003}}} \right).
\]
Thus statement (e) of Theorem 3 follows immediately from statement (b) of Theorem 4.

To prove statement (f) we just observe the following. The limit cycle $\Gamma_2$ can be written in the system (20) as \( \{ R(\theta), W(\theta) : \theta \in S \} \). The point $p_9$ at the coordinates $(R, W)$ is written as
\[
\left( \frac{2\sqrt{\epsilon} \sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{\sqrt{-D_1 \epsilon}}{\sqrt{C_{003}}} \right).
\]
Thus statement (f) of Theorem 3 follows immediately from statement (b) of Theorem 4. In short this completes the proofs of statements (d), (e) and (f) of Theorem 3.

The stability of those limit cycles is determined by the eigenvalues of system (6) at the points $p_7$ and $p_9$, respectively. Then, by statement (c) of Theorem 4, it follows immediately the different kind of stability for the limit cycles. Since those eigenvalues can be computed easily using Mathematica or any other algebraic manipulator, but their expressions are very long, here they have been omitted.

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References


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