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HOPF BIFURCATION FOR SOME ANALYTIC DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$ VIA AVERAGING THEORY

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Abstract. We study the Hopf bifurcation from the singular point with eigenvalues $a\,\varepsilon \pm b\,i$ and $c\,\varepsilon$ located at the origen of an analytic differential system of the form $\dot{\mathbf{x}} = f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^3$. Under convenient assumptions we prove that the Hopf bifurcation can produce 1, 2 or 3 limit cycles. We also characterize the stability of these limit cycles. The main tool for proving these results is the averaging theory of first and second order.

1. Introduction and statement of the main results

The main goal of this work is to study the Hopf bifurcation in analytic differential systems in $\mathbb{R}^3$ via averaging theory. In fact the results obtained can be extended easily to C^4 differential systems in $\mathbb{R}^3$ but in order to simplify the notation we will present them for analytic differential systems.

More precisely, we investigate the Hopf bifurcation at the singular point located at the origin for an analytic differential systems in $\mathbb{R}^3$ of the form

$$\begin{aligned} \dot{U} &= \varepsilon a U - bV + \sum_{l+j+k\geq n} A_{ijk} U^l V^j W^k, \\ \dot{V} &= bU + \varepsilon a V + \sum_{l+j+k\geq n} B_{ijk} U^l V^j W^k, \\ \dot{W} &= \varepsilon c W + \sum_{l+j+k\geq n} C_{ijk} U^l V^j W^k, \end{aligned} \tag{1}$$

where $n = 2, 3$ and ε is a small parameter. Note that the linear part of this system at the singular point located at the origin $(0, 0, 0)$ has eigenvalues $a\,\varepsilon \pm b\,i$ and $c\varepsilon$. So for $\varepsilon = 0$ the eigenvalues are $\pm bi$ and 0, consequently we are studying a kind of zero–Hopf bifurcation.

For $n = 2$ the Hopf bifurcation of system (1) was studied in [2] were the authors obtained the following result.

Theorem 1 (First order Hopf bifurcation theorem for $n = 2$). *We define the constants*

$$F = A_{101} + B_{011},\ G = C_{020} + C_{200},\ D = c(cF - 4aC_{002}),\ E = D^2 + 8ab^2F(cF - 2aC_{002}).$$

(a) *If $(E-D^2)/(FG) > 0$, then the phase portrait of system* (1) *has a limit cycle Γ_ε tending to the origin as $\varepsilon \to 0$.*
(b) *Assume that the limit cycle Γ_ε of statement (a) in cylindrical coordinates $(U = R\cos\theta, V = R\sin\theta, W)$ becomes $(R(\theta), W(\theta))$ with $\theta \in \mathbb{S}^1$. Then*

$$R(\theta) \to \sqrt{\frac{E - D^2}{2b^2F^3G}}\,\varepsilon, \qquad W(\theta) \to \frac{2a}{F}\,\varepsilon, \qquad \text{when } \varepsilon \to 0.$$

(c) *If $D^2 > E$ and $DF > 0$, then the limit cycle Γ_ε is a local repellor; if $D^2 > E$ and $DF < 0$, then the limit cycle Γ_ε is a local attractor; and if $D^2 < E$, then the limit cycle Γ_ε has two invariant manifolds, one stable and the other unstable, which locally are formed by two 2-dimensional cylinders.*

Our first result is to improve Theorem 1 using the averaging theory of second order. Thus we have:

Theorem 2 (Second order Hopf bifurcation theorem for $n = 2$)**.** *We define the constants N_i for $i = 0, 1, 2, 3, 4$ (see (17)) and the constants n_k and d_k for $k = 2, 4$ (see (18)).*

(a) *If $abN_0 > 0$ and $n_2 d_2 \neq 0$, then the phase portrait of system (1) has a limit cycle Γ_ε^1 tending to the origin as $\varepsilon \to 0$.*
(b) *If $N_1 N_2 > 0$, $N_3 N_4 > 0$ and $n_4 d_4 \neq 0$, then the phase portrait of system (1) has two limit cycles Γ_ε^2 and Γ_ε^3 tending to the origin as $\varepsilon \to 0$.*

Theorem 2 will be proved in Section 3 by using the averaging theory of second order, see Section 2. We note that the estimation on the size of the limit cycles given in statements (a) and (b) of Theorem 2 can be obtained from Theorem 5 but their expressions are very long and we do not write them here. Also the stability of the limit cycles can be analyzed computing the eigenvalues of the matrix (12), but again we do not provide them because their expressions are too long.

Our second result is to extend Theorem 1 to systems (1) with $n = 3$. Thus we have:

Theorem 3 (Hopf bifurcation theorem for $n = 3$)**.** *We define the constants*

$$
\begin{aligned}
D_0 &= \frac{8a}{A_{120} + 3(A_{300} + B_{030}) + B_{210}}, \\
D_1 &= \frac{C_{003}((A_{120} + 3(A_{300} + B_{030}) + B_{210})c - 4a(C_{021} + C_{201}))}{(A_{120} + 3(A_{300} + B_{030}) + B_{210})C_{003} - 2(A_{102} + B_{012})(C_{021} + C_{201})}, \\
D_2 &= (A_{120} + 3(A_{300} + B_{030}) + B_{210})C_{003} + 2(A_{102} + B_{012})(C_{021} + C_{201}).
\end{aligned} \tag{2}
$$

(a) *If*

$$\frac{D_0(-(A_{120} + 3(A_{300} + B_{030}) + B_{210})c + 4a(C_{021} + C_{201}))}{4b^2} \neq 0, \tag{3}$$

then the phase portrait of system (1) has a limit cycle Γ_ε tending to the origin as $\varepsilon \to 0$.

(b) *Assume that the limit cycle Γ_ε of statement (a) in cylindrical coordinates $(U = R\cos\theta, V = R\sin\theta, W)$ becomes $(R(\theta), W(\theta))$ with $\theta \in \mathbb{S}^1$. Then*

$$R(\theta) \to \sqrt{-D_0\varepsilon}, \qquad W(\theta) \to 0, \qquad \text{when } \varepsilon \to 0.$$

(c) *If $ab < 0$ and $(2c - D_0(C_{021} + C_{201}))b > 0$, then the limit cycle Γ_ε is a local repellor; if $ab > 0$ and $(2c - D_0(C_{021} + C_{201}))b < 0$, then the limit cycle Γ_ε is a local attractor; and if $ab > 0$ and $(2c - D_0(C_{021} + C_{201}))b < 0$, or $ab < 0$ and $(2c - D_0(C_{021} + C_{201}))b > 0$, then the limit cycle Γ_ε has two invariant manifolds, one stable and the other unstable, which locally are formed by two 2-dimensional cylinders.*

(d) *If*

$$-\frac{2D_1((A_{102} + B_{012})c - 2aC_{003})}{b^2 C_{003}} \neq 0, \tag{4}$$

then the phase portrait of the quadratic system (1) has two limit cycles Γ_ε^1 and Γ_ε^2 tending to the origin as $\varepsilon \to 0$.

(e) *Assume that the limit cycle Γ_ε^1 of statement (d) in cylindrical coordinates $(U = R\cos\theta, V = R\sin\theta, W)$ becomes $(R(\theta), W(\theta))$ with $\theta \in \mathbb{S}^1$. Then*

$$R(\theta) \to \frac{2\sqrt{\varepsilon}\sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \qquad W(\theta) \to -\frac{\sqrt{-D_1\varepsilon}}{\sqrt{C_{003}}}, \qquad \text{when } \varepsilon \to 0.$$

(f) *Assume that the limit cycle Γ_ε^2 of statement (d) in cylindrical coordinates $(U = R\cos\theta, V = R\sin\theta, W)$ becomes $(R(\theta), W(\theta))$ with $\theta \in \mathbb{S}^1$. Then*

$$R(\theta) \to \frac{2\sqrt{\varepsilon}\sqrt{-(A_{102} + B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \qquad W(\theta) \to \frac{\sqrt{-D_1\varepsilon}}{\sqrt{C_{003}}}, \qquad \text{when } \varepsilon \to 0.$$

Theorem 3 will be proved in Section 4 by using the averaging theory of first order, see Section 2. As it is mention in Section 4 we can study the stability of the limit cycles $\Gamma_\varepsilon^{1,2}$ computing their eigenvalues but since they have very long expressions we have omitted them.

2. Limit cycles via averaging theory

In few words we can say that the averaging method [4, 5] gives a quantitative relation between the solutions of some non–autonomous periodic differential system and the solutions of its averaged differential system, which is an autonomous one. The next theorem provides a first order approximation in ε for the limit cycles of a periodic differential system, for a proof see Theorem 2.6.1 of Sanders and Verhulst [4] and Theorem 11.5 of Verhulst [5].

Theorem 4. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon\, f(t,x) + \varepsilon^2\, g(t,x,\varepsilon), \qquad x(0) = x_0, \tag{5}$$

and

$$\dot{y} = \varepsilon\, f^0(y), \qquad y(0) = x_0. \tag{6}$$

where x, y, $x_0 \in \Omega$ an open subset of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f and g are periodic of period T in the variable t, and $f^0(y)$ is the averaged function of $f(t,x)$ with respect to t, i.e.,

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt. \tag{7}$$

Suppose: (i) f, its Jacobian $\partial f/\partial x$, its Hessian $\partial^2 f/\partial x^2$, g and its Jacobian $\partial g/\partial x$ are defined, continuous and bounded by a constant independent on ε in $[0,\infty)\times\Omega$ and $\varepsilon \in (0,\varepsilon_0]$; (ii) T is a constant independent of ε; and (iii) $y(t)$ belongs to Ω on the interval of time $[0,1/\varepsilon]$. Then the following statements hold.

(a) *On the time scale $1/\varepsilon$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.*
(b) *If p is a singular point of the averaged system (6) such that the determinant of the Jacobian matrix*

$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \tag{8}$$

is not zero, then there exists a limit cycle $\phi(t,\varepsilon)$ of period T for the system (5) which is close to p and such that $\phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.
(c) *The stability or instability of the limit cycle $\phi(t,\varepsilon)$ is given by the stability or instability of the singular point p of the averaged system (6). In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t,\varepsilon)$.*

The next theorem provides a second order approximation in ε for the limit cycles of a periodic differential system, for an statement in the case of a differential equation (see [3] and [1]) and for a proof see Theorem 3.5.1 of Sanders and Verhulst [4].

Theorem 5. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon), \quad x(0) = x_0, \tag{9}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2 f^{10}(y) + \varepsilon^2 g^0(y), \quad y(0) = x_0, \tag{10}$$

with $f, g\colon [0,\infty)\times\Omega\times\mathbb{R}^n$, $R\colon [0,\infty)\times\Omega\times(0,\varepsilon_0] \to \mathbb{R}^n$, Ω an open set of $\mathbb{R}^n$, f, g and G periodic functions of period T in the variable t,

$$f^1(t,x) = \frac{\partial f}{\partial x} y^1(t,x), \quad \text{where} \quad y^1(t,x) = \int_0^t f(s,x)\,ds. \tag{11}$$

Of course, f^0, f^{10} and g^0 denote the averaged functions of f, f^1 and g, respectively, defined as in (7). Suppose: (i) $\partial f/\partial x$ is Lipschitz in x, g and R are Lipschitz in x and all these functions *are continuous on their domain of definition; (ii)* $|R(t,x,\varepsilon)|$ *is bounded by a constant uniformly in* $[0, L/\varepsilon) \times \Omega \times (0, \varepsilon_0]$*; (iii)* T *is a constant independent of* ε*; and (iv)* $y(t)$ *belongs to* Ω *on the interval of time* $[0, 1/\varepsilon]$*. Then the following statements hold.*

(a) *On the time scale* $1/\varepsilon$ *we have that* $x(t) = y(t) + \varepsilon y^1(t, y(t)) = O(\varepsilon^2)$.

(b) *If* p *is a singular point of the averaged system* (10) *such that*

$$\left.\frac{\partial(f^{10} + g^0(y))}{\partial y}\right|_{y=p} \tag{12}$$

is not zero, then there exists a limit cycle $\phi(t,\varepsilon)$ *of period* T *for the system* (9) *which is close to* p *and such that* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

(c) *The stability or instability of the limit cycle* $\phi(t,\varepsilon)$ *is given by the stability or instability of the singular point* p *of the averaged system* (10)*. In fact, the singular point* p *has the stability behavior of the Poincaré map associated to the limit cycle* $\phi(t,\varepsilon)$.

3. Proof of Theorem 2

In the cylindrical change of coordinates $U = R\cos\theta$, $V = R\sin\theta$, $W = W$ the system (1) in the region $R > 0$ has the following form

$$\begin{aligned} \dot R &= a\epsilon R + H_{11}(\theta)R^2 + H_{12}(\theta)RW + H_{13}(\theta)W^2 + O_3(R,W), \\ \dot\theta &= \frac{1}{R}\left[bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R,W)\right], \\ \dot W &= c\epsilon W + H_{31}(\theta)R^2 + H_{32}(\theta)RW + O_3(R,W), \end{aligned} \tag{13}$$

where

$$\begin{aligned} H_{11}(\theta) &= A_{200}\cos^3\theta + (A_{110}+B_{200})\cos^2\theta\sin\theta + (A_{020}+B_{110})\cos\theta\sin^2\theta \\ &\quad + B_{020}\sin^3\theta, \\ H_{12}(\theta) &= A_{101}\cos^2\theta + (A_{011}+B_{101})\cos\theta\sin\theta - A_{101}\sin^2\theta, \\ H_{13}(\theta) &= A_{002}\cos\theta + B_{002}\sin\theta, \\ H_{21}(\theta) &= B_{200}\cos^3\theta - (A_{200}-B_{110})\cos^2\theta\sin\theta - (A_{110}-B_{020})\cos\theta\sin^2\theta \\ &\quad - A_{020}\sin^3\theta, \\ H_{22}(\theta) &= B_{101}\cos^2\theta - 2A_{101}\cos\theta\sin\theta - A_{011}\sin^2\theta, \\ H_{23}(\theta) &= B_{002}\cos\theta - A_{002}\sin\theta, \\ H_{31}(\theta) &= -C_{020}\cos^2\theta + C_{110}\cos\theta\sin\theta + C_{020}\sin^2\theta, \\ H_{32}(\theta) &= C_{101}\cos\theta + C_{011}\sin\theta. \end{aligned}$$

Therefore, system (13) in the region $\dot\theta \neq 0$ is equivalent to

$$\begin{aligned} \frac{dR}{d\theta} &= \frac{R[a\epsilon R + H_{11}(\theta)R^2 + H_{12}(\theta)RW + H_{13}(\theta)W^2 + O_3(R,W)]}{bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R,W)} \\ \frac{dW}{d\theta} &= \frac{R[c\epsilon W + H_{31}(\theta)R^2 + H_{32}(\theta)RW + O_3(R,W)]}{bR + H_{21}(\theta)R^2 + H_{22}(\theta)RW + H_{23}(\theta)W^2 + O_3(R,W)}. \end{aligned} \tag{14}$$

We note that this system is 2π-periodic in the variable θ. Performing the rescaling $(R,W) = (\varrho\sqrt{\epsilon}, \zeta\sqrt{\epsilon})$ the system (14) becomes into the normal form for applying the averaging theory. That is, in the variables (ϱ,ζ) system (14) writes

$$\begin{aligned} \frac{d\varrho}{d\theta} &= \epsilon^{1/2} f_1(\theta,\varrho,\zeta) + \epsilon g_1(\theta,\varrho,\zeta) + \epsilon^{3/2} R_1(\theta,\varrho,\zeta,\epsilon), \\ \frac{d\zeta}{d\theta} &= \epsilon^{1/2} f_2(\theta,\varrho,\zeta) + \epsilon g_2(\theta,\varrho,\zeta) + \epsilon^{3/2} R_2(\theta,\varrho,\zeta,\epsilon), \end{aligned} \tag{15}$$

where

$$
\begin{aligned}
f_1(\theta,\varrho,\zeta) &= \frac{1}{b}[H_{11}(\theta)\varrho^2 + H_{12}(\theta)\varrho\zeta + H_{13}(\theta)\zeta^2],\\
g_1(\theta,\varrho,\zeta) &= \frac{a\varrho}{b} - \frac{1}{b^2\varrho}[H_{11}(\theta)\varrho^2 + H_{12}(\theta)\varrho\zeta + H_{13}(\theta)\zeta^2][H_{21}(\theta)\varrho^2 + H_{22}(\theta)\varrho\zeta\\
&\quad + H_{23}(\theta)\zeta^2],\\
f_2(\theta,\varrho,\zeta) &= \frac{\varrho}{b}[H_{31}(\theta)\varrho + H_{32}(\theta)\zeta],\\
g_2(\theta,\varrho,\zeta) &= \frac{c\zeta}{b} - \frac{1}{b^2}[H_{31}(\theta)\varrho + H_{32}(\theta)\zeta][H_{21}(\theta)\varrho^2 + H_{22}(\theta)\varrho\zeta + H_{23}(\theta)\zeta^2].
\end{aligned}
$$

So system (15) is equivalent to system (9) taking $x=(\varrho,\zeta)$, $t=\theta$, $f(t,x)=(f_1(\theta,\varrho,\zeta),f_2(\theta,\varrho,\zeta))$, $g(t,x)=(g_1(\theta,\varrho,\zeta),g_2(\theta,\varrho,\zeta))$, $T=2\pi$ and $\varepsilon=\epsilon^{1/2}$.

Let Ω be the open subset and ϵ_0 the positive number which appear in the statement of Theorem 5. Then, it is easy to verify that, system (15) satisfies the assumptions of Theorem 5 if we take as Ω an open disc centered at the origin in $\mathbb{R}^2$ and a sufficiently small ϵ_0. Let

$$
y_i^1(\theta,\varrho,\zeta) = \int_0^\theta f_i(s,\varrho,\zeta)\,ds \quad \text{for } i=1,2.
$$

Then

$$
\begin{aligned}
y_1^1(\theta,\varrho,\zeta) = \frac{1}{12b}\big(&12[B_{002} - B_{002}\cos\theta + A_{002}\sin\theta]\zeta^2\\
&+ 3[A_{011} + B_{101} - A_{011}\cos^2\theta - B_{101}\cos^2\theta + 4A_{101}\cos\theta\sin\theta\\
&+ A_{011}\sin^2\theta + B_{101}\sin^2\theta]\zeta\varrho\\
&+ [4A_{110} + 8B_{020} + 4B_{200} - 3A_{110}\cos\theta - 9B_{020}\cos\theta - 3B_{200}\cos\theta\\
&- A_{110}\cos^3\theta + B_{020}\cos^3\theta - B_{200}\cos^3\theta + 3A_{020}\sin\theta + 9A_{200}\sin\theta\\
&+ 3B_{110}\sin\theta - 3A_{020}\cos^2\theta\sin\theta + 3A_{200}\cos^2\theta\sin\theta - 3B_{110}\cos^2\theta\sin\theta\\
&+ 3A_{110}\cos\theta\sin^2\theta - 3B_{020}\cos\theta\sin^2\theta + 3B_{200}\cos\theta\sin^2\theta\\
&+ A_{020}\sin^3\theta - A_{200}\sin^3\theta + B_{110}\sin^3\theta]\varrho^2\big),\\
y_2^1(\theta,\varrho,\zeta) = \frac{\varrho}{2b}\big(&[C_{110}\sin^2\theta - 2C_{020}\sin\theta\cos\theta]\varrho + [2C_{011} + 2C_{101}\sin\theta - 2C_{011}\cos\theta]\zeta\big).
\end{aligned}
$$

We have

$$
\begin{pmatrix} f_1^1(\theta,\varrho,\zeta)\\ f_2^1(\theta,\varrho,\zeta)\end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial\varrho}(\theta,\varrho,\zeta) & \dfrac{\partial f_1}{\partial\zeta}(\theta,\varrho,\zeta)\\ \dfrac{\partial f_2}{\partial\varrho}(\theta,\varrho,\zeta) & \dfrac{\partial f_2}{\partial\zeta}(\theta,\varrho,\zeta)\end{pmatrix} \begin{pmatrix} y_1^1(\theta,\varrho,\zeta)\\ y_2^1(\theta,\varrho,\zeta)\end{pmatrix}. \tag{16}
$$

So

$$
\begin{aligned}
f_i^{10}(\varrho,\zeta) &= \frac{1}{2\pi}\int_0^{2\pi} f_i^1(\theta,\varrho,\zeta)\,d\theta, \quad \text{for } i=1,2,\\
g_i^{0}(\varrho,\zeta) &= \frac{1}{2\pi}\int_0^{2\pi} g_i(\theta,\varrho,\zeta)\,d\theta, \quad \text{for } i=1,2,
\end{aligned}
$$

then

$$
\begin{aligned}
f_1^{10}(\varrho,\zeta) =& \frac{1}{8b^2}[-((A_{011}+B_{101})C_{020}+A_{101}C_{110})\varrho^3+2(A_{002}(A_{110}+3B_{020}+B_{200}\\
&-4C_{011})-B_{002}(A_{020}+3A_{200}+B_{110}-4C_{101}))\varrho\zeta^2],\\
f_2^{10}(\varrho,\zeta) =& \frac{\zeta}{8b^2}[((A_{020}+3A_{200}+B_{110})C_{011}+2(A_{011}+B_{101})C_{020}\\
&-(A_{110}+3B_{020}+B_{200})C_{101}+2A_{101}C_{110})\varrho^2+4(A_{002}C_{011}-B_{002}C_{101})\zeta^2],\\
g_1^{0}(\varrho,\zeta) =& \frac{\varrho}{8b^2}[8ab+(A_{110}A_{200}+A_{020}(A_{110}+2B_{020})-2A_{200}B_{200}-B_{110}(B_{020}+B_{200}))\varrho^2\\
&+2(B_{002}(A_{020}-A_{200}-B_{110})+A_{002}(A_{110}+B_{020}-B_{200}))\zeta^2],\\
g_2^{0}(\varrho,\zeta) =& \frac{\zeta}{8b^2}[8bc+((3A_{020}+A_{200}-B_{110})C_{011}+2(A_{011}+B_{101})C_{020}+A_{110}C_{101}\\
&-(B_{020}+3B_{200})C_{101}+2A_{101}C_{110})\varrho^2+4(A_{002}C_{011}-B_{002}C_{101})\zeta^2].
\end{aligned}
$$

The averaged system $f_i^{10}(\varrho,\zeta)+g_i^{0}(\varrho,\zeta)$, $i=1,2$ has six singular points:

$$
\begin{aligned}
q_1 &= \left(-\frac{2\sqrt{2ab}}{\sqrt{N_0}},0\right), \quad q_2 = \left(\frac{2\sqrt{2ab}}{\sqrt{N_0}},0\right),\\
q_3 &= \left(-\frac{\sqrt{N_1}}{\sqrt{N_2}},-\frac{\sqrt{N_3}}{\sqrt{N_4}}\right), \quad q_4 = \left(\frac{\sqrt{N_1}}{\sqrt{N_2}},-\frac{\sqrt{N_3}}{\sqrt{N_4}}\right),\\
q_5 &= \left(-\frac{\sqrt{N_1}}{\sqrt{N_2}},\frac{\sqrt{N_3}}{\sqrt{N_4}}\right), \quad q_6 = \left(\frac{\sqrt{N_1}}{\sqrt{N_2}},\frac{\sqrt{N_3}}{\sqrt{N_4}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
N_0 =& -A_{110}A_{200}-A_{020}(A_{110}+2B_{020})+2A_{200}B_{200}+\\
&B_{110}(B_{020}+B_{200})+(A_{011}+B_{101})C_{020}+A_{101}C_{110},\\
N_1 =& 4b(-A_{002}(A_{110}+2B_{020})c+B_{002}(2A_{200}+B_{110})c+\\
&2A_{002}(a+c)C_{011}-2B_{002}(a+c)C_{101}),\\
N_2 =& B_{002}(-4A_{200}^2C_{011}-2A_{020}(2A_{200}+B_{110})C_{011}-2A_{011}B_{110}C_{020}-\\
&2B_{101}B_{110}C_{020}+B_{020}B_{110}C_{101}+B_{110}B_{200}C_{101}+A_{020}C_{101}\\
&(A_{110}+2B_{020}+4C_{011})+3A_{011}C_{020}C_{101}+3B_{101}C_{020}C_{101}-\\
&4B_{020}C_{101}^2-4B_{200}C_{101}^2-2A_{101}B_{110}C_{110}+3A_{101}C_{101}C_{110}+\\
&A_{200}(-2B_{110}C_{011}-4(A_{011}+B_{101})C_{020}+(A_{110}+4B_{020}+2B_{200}+\\
&4C_{011})C_{101}-4A_{101}C_{110}))+A_{002}(A_{020}(A_{110}+2B_{020}-4C_{011})C_{011}+\\
&2A_{200}(2B_{020}+B_{200}-2C_{011})C_{011}-4B_{020}^2C_{101}+\\
&C_{011}(B_{200}(B_{110}+4C_{101})-3((A_{011}+B_{101})C_{020}+A_{101}C_{110}))+\\
&A_{110}(A_{200}C_{011}+2((A_{011}+B_{101})C_{020}-(B_{020}+B_{200})C_{101}+\\
&A_{101}C_{110}))+B_{020}(B_{110}C_{011}+4((A_{011}+B_{101})C_{020}-B_{200}C_{101}+\\
&C_{011}C_{101}+A_{101}C_{110}))),\\
N_3 =& -b(A_{002}C_{011}-B_{002}C_{101})(-A_{110}A_{200}c-A_{020}(A_{110}+2B_{020})c+\\
&B_{020}B_{110}c+2A_{200}B_{200}c+B_{110}B_{200}c+4aA_{020}C_{011}+4aA_{200}C_{011}+\\
&4aA_{011}C_{020}+4aB_{101}C_{020}+A_{011}cC_{020}+B_{101}cC_{020}-4aB_{020}C_{101}-\\
&4aB_{200}C_{101}+A_{101}(4a+c)C_{110}),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
N_4 =&(-A_{002}C_{011} + B_{002}C_{101})B_{002}(4A_{200}^2C_{011} + 2B_{101}B_{110}C_{020}+\\
&A_{011}C_{020}(2B_{110} - 3C_{101}) - B_{020}B_{110}C_{101} - B_{110}B_{200}C_{101}-\\
&3B_{101}C_{020}C_{101} + 4B_{020}C_{101}^2 + 4B_{200}C_{101}^2 + A_{020}(4A_{200}C_{011}+\\
&2B_{110}C_{011} - (A_{110} + 2B_{020} + 4C_{011})C_{101}) + 2A_{101}B_{110}C_{110}-\\
&3A_{101}C_{101}C_{110} + A_{200}(2B_{110}C_{011} + 4(A_{011} + B_{101})C_{020}-\\
&(A_{110} + 4B_{020} + 2B_{200} + 4C_{011})C_{101} + 4A_{101}C_{110}))-\\
&A_{002}(A_{020}(A_{110} + 2B_{020} - 4C_{011})C_{011} + 2A_{200}(2B_{020} + B_{200}-\\
&2C_{011})C_{011} - 4B_{020}^2C_{101} + C_{011}(B_{200}(B_{110} + 4C_{101})-\\
&3((A_{011} + B_{101})C_{020} + A_{101}C_{110})) + A_{110}(A_{200}C_{011}+\\
&2((A_{011} + B_{101})C_{020} - (B_{020} + B_{200})C_{101} + A_{101}C_{110}))+\\
&B_{020}(B_{110}C_{011} + 4((A_{011} + B_{101})C_{020} - B_{200}C_{101}+\\
&C_{011}C_{101} + A_{101}C_{110}))).
\end{aligned}
$$

We observe that the averaged theory can be conclusive only at points q_2, q_4 and q_6. In fact, at q_1, q_3 and q_5 we have $\varrho < 0$ and it means that $R < 0$ which does not make sense.

For $k = 2, 4, 6$ the Jacobian matrix (12) at point q_k has determinant n_k/d_k where

$$
\begin{aligned}
n_2 =\; & -2a(-A_{110}A_{200}c + B_{020}B_{110}c + 2A_{200}B_{200}c + B_{110}B_{200}c+\\
& A_{011}C_{020}c + B_{101}C_{020}c + A_{101}C_{110}c + 4aA_{200}C_{011}-\\
& A_{020}(A_{110}c + 2B_{020}c - 4aC_{011}) + 4aA_{011}C_{020} + 4aB_{101}C_{020}-\\
& 4aB_{020}C_{101} - 4aB_{200}C_{101} + 4aA_{101}C_{110}),\\
d_2 =\; & b^2(-A_{110}A_{200} + 2B_{200}A_{200} - A_{020}(A_{110} + 2B_{020})+\\
& B_{020}B_{110} + B_{110}B_{200} + A_{011}C_{020} + B_{101}C_{020} + A_{101}C_{110}),\\
n_4 =\; & 2(A_{002}(A_{110}c - 2((a + c)C_{011} - B_{020}c)) + B_{002}(-2A_{200}c - B_{110}c+\\
& 2(a + c)C_{101}))(A_{110}A_{200}c - B_{020}B_{110}c - 2A_{200}B_{200}c - B_{110}B_{200}c-\\
& A_{011}C_{020}c - B_{101}C_{020}c - A_{101}C_{110}c - 4aA_{200}C_{011} + A_{020}(A_{110}c+\\
& 2B_{020}c - 4aC_{011}) - 4aA_{011}C_{020} - 4aB_{101}C_{020} + 4aB_{020}C_{101}+\\
& 4aB_{200}C_{101} - 4aA_{101}C_{110}),\\
d_4 =\; & b^2(B_{002}(-4C_{011}A_{200}^2 + (-2B_{110}C_{011} + 4C_{101}C_{011} - 4A_{011}C_{020}-\\
& 4B_{101}C_{020} + A_{110}C_{101} + 4B_{020}C_{101} + 2B_{200}C_{101} - 4A_{101}C_{110})A_{200}-\\
& 4B_{020}C_{101}^2 - 4B_{200}C_{101}^2 - 2A_{011}B_{110}C_{020} - 2B_{101}B_{110}C_{020}+\\
& B_{020}B_{110}C_{101} + B_{110}B_{200}C_{101} + 3A_{011}C_{020}C_{101} + 3B_{101}C_{020}C_{101}+\\
& A_{020}(-4A_{200}C_{011} - 2B_{110}C_{011} + (A_{110} + 2B_{020} + 4C_{011})C_{101})-\\
& 2A_{101}B_{110}C_{110} + 3A_{101}C_{101}C_{110}) + A_{002}(-4C_{101}B_{020}^2+\\
& 4A_{200}C_{011}B_{020} + B_{110}C_{011}B_{020} + 4A_{011}C_{020}B_{020} + 4B_{101}C_{020}B_{020}-\\
& 4B_{200}C_{101}B_{020} + 4C_{011}C_{101}B_{020} + 4A_{101}C_{110}B_{020} - 4A_{200}C_{011}^2+\\
& 2A_{200}B_{200}C_{011} + B_{110}B_{200}C_{011} + A_{020}(A_{110} + 2B_{020} - 4C_{011})C_{011}-\\
& 3A_{011}C_{011}C_{020} - 3B_{101}C_{011}C_{020} + 4B_{200}C_{011}C_{101} - 3A_{101}C_{011}C_{110}+\\
& A_{110}(A_{200}C_{011} + 2A_{011}C_{020} + 2B_{101}C_{020} - 2B_{020}C_{101} - 2B_{200}C_{101}+\\
& 2A_{101}C_{110}))).
\end{aligned} \tag{18}
$$

4. Proof of Theorem 3

In the cylindrical change of coordinates $U = R\cos\theta$, $V = R\sin\theta$, $W = W$ the system (1) with $n = 3$ in the region $R > 0$ has the following form

$$
\begin{aligned}
\dot{R} &= a\epsilon R + h_{11}(\theta)R^3 + h_{12}(\theta)R^2W + h_{13}(\theta)RW^2 + h_{14}(\theta)W^3 + O_4(R, W),\\
\dot{\theta} &= \frac{1}{R}\left[bR + h_{21}(\theta)R^3 + h_{22}(\theta)R^2W + h_{23}(\theta)RW^2 + h_{24}(\theta)W^3 + O_4(R, W)\right],\\
\dot{W} &= c\epsilon W + h_{31}(\theta)R^3 + h_{32}(\theta)R^2W + h_{33}(\theta)RW^2 + h_{34}(\theta)W^3 + O_4(R, W),
\end{aligned} \tag{19}
$$

where

$$
\begin{aligned}
h_{11}(\theta) &= A_{300}\cos^4\theta + (A_{210}+B_{300})\cos^3\theta\sin\theta + (A_{120}+B_{210})\cos^2\theta\sin^2\theta \\
&\quad + (A_{030}+B_{120})\cos\theta\sin^3\theta + B_{030}\sin^4\theta, \\
h_{12}(\theta) &= A_{201}\cos^3\theta + (A_{111}+B_{201})\cos^2\theta\sin\theta + (A_{021}+B_{111})\cos\theta\sin^2\theta \\
&\quad + B_{021}\sin^3\theta, \\
h_{13}(\theta) &= A_{102}\cos^2\theta + (A_{012}+B_{102})\cos\theta\sin\theta + B_{012}\sin^2\theta, \\
h_{14}(\theta) &= A_{003}\cos\theta + B_{003}\sin\theta, \\
h_{21}(\theta) &= B_{300}\cos^4\theta + (A_{300}-B_{210})\cos^3\theta\sin\theta + (-A_{210}+B_{120})\cos^2\theta\sin^2\theta \\
&\quad + (A_{120}-B_{030})\sin^3\theta\cos\theta - A_{030}\sin^4\theta, \\
h_{22}(\theta) &= B_{201}\cos^3\theta + (A_{201}-B_{111})\cos^2\theta\sin\theta + (-A_{111}+B_{021})\cos\theta\sin^2\theta \\
&\quad - A_{021}\sin^3\theta, \\
h_{23}(\theta) &= B_{102}\cos^2\theta + (A_{102}-B_{012})\cos\theta\sin\theta - A_{012}\sin^2\theta, \\
h_{24}(\theta) &= B_{003}\cos\theta - A_{003}\sin\theta, \\
h_{31}(\theta) &= C_{300}\cos^3\theta + C_{210}\cos^2\theta\sin\theta + C_{120}\cos\theta\sin^2\theta + C_{030}\sin^3\theta, \\
h_{32}(\theta) &= C_{201}\cos^2\theta + C_{111}\cos\theta\sin\theta + C_{021}\sin^2\theta, \\
h_{33}(\theta) &= C_{102}\cos\theta + C_{012}\sin\theta, \\
h_{34}(\theta) &= C_{003}.
\end{aligned}
$$

Therefore, system (19) in the region $\dot\theta \neq 0$ is equivalent to

$$
(20)\qquad
\begin{aligned}
\frac{dR}{d\theta} &= \frac{R[a\epsilon R + h_{11}(\theta)R^3 + h_{12}(\theta)R^2W + h_{13}(\theta)RW^2 + h_{14}(\theta)W^3 + O_4(R,W)]}{bR + h_{21}(\theta)R^3 + h_{22}(\theta)R^2W + h_{23}(\theta)RW^2 + h_{24}(\theta)W^3 + O_4(R,W)}, \\
\frac{dW}{d\theta} &= \frac{R[c\epsilon W + h_{31}(\theta)R^3 + h_{32}(\theta)R^2W + h_{33}(\theta)RW^2 + h_{34}(\theta)W^3 + O_4(R,W)]}{bR + h_{21}(\theta)R^3 + h_{22}(\theta)R^2W + h_{23}(\theta)RW^2 + h_{24}(\theta)W^3 + O_4(R,W)}.
\end{aligned}
$$

We note that this system is 2π-periodic in the variable θ. Performing the rescaling $(R,W) = (\varrho\sqrt{\epsilon}, \zeta\sqrt{\epsilon})$ the system (20) becomes into the normal form for applying the averaging theory. That is, in the variables (ϱ,ζ) system (20) writes

$$
(21)\qquad
\begin{aligned}
\frac{d\varrho}{d\theta} &= \epsilon f_1(\theta,\varrho,\zeta) + \epsilon^2 g_1(\theta,\varrho,\zeta,\epsilon), \\
\frac{d\zeta}{d\theta} &= \epsilon f_2(\theta,\varrho,\zeta) + \epsilon^2 g_2(\theta,\varrho,\zeta,\epsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(\theta,\varrho,\zeta) &= \frac{1}{b}[a\varrho + h_{11}(\theta)\varrho^3 + h_{12}(\theta)\varrho^2\zeta + h_{13}(\theta)\varrho\zeta^2 + h_{14}(\theta)\zeta^3], \\
f_2(\theta,\varrho,\zeta) &= \frac{1}{b}[c\zeta + h_{31}(\theta)\varrho^3 + h_{32}(\theta)\varrho^2\zeta + h_{33}(\theta)\varrho\zeta^2 + h_{34}(\theta)\zeta^3].
\end{aligned}
$$

So, system (21) is equivalent to system (5) taking $x = (\varrho,\zeta)$, $t = \theta$, $f(t,x) = (f_1(\theta,\varrho,\zeta), f_2(\theta,\varrho,\zeta))$, $T = 2\pi$ and $\varepsilon = \epsilon$.

Let Ω be the open subset which appear in the statement of Theorem 4. Then it is easy to verify that system (21) satisfies the assumptions of Theorem 4 if we take as Ω an open disc centered at the origin in $\mathbb{R}^2$ and a sufficiently small ϵ_0. Since

$$
f_i^0(\varrho,\zeta) = \frac{1}{2\pi}\int_0^{2\pi} f_i(\theta,\varrho,\zeta)\,d\theta,
$$

for $i = 1, 2$, we get that

$$f_1^0(\varrho,\zeta) = \frac{\varrho}{8b}[8a + (A_{120} + 3(A_{300} + B_{030}) + B_{210})\varrho^2 + 4(A_{102} + B_{012})\zeta^2],$$
$$f_2^0(\varrho,\zeta) = \frac{\zeta}{2b}[2c + (C_{021} + C_{201})\varrho^2 + 2C_{003}\zeta^2].$$

Hence with the notation of D_1 and D_2 introduced in (2), the averaged system (6) has nine singular points:

$$p_1 = (0,0), \quad p_2 = (-\sqrt{-D_0}, 0), \quad p_3 = (\sqrt{-D_0}, 0),$$
$$p_4 = \left(0, \frac{-i\sqrt{c}}{\sqrt{C_{003}}}\right), \quad p_5 = \left(0, \frac{i\sqrt{c}}{\sqrt{C_{003}}}\right),$$
$$p_6 = \left(-\frac{2\sqrt{-(A_{102}+B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{-\sqrt{-D_1}}{\sqrt{C_{003}}}\right),$$
$$p_7 = \left(\frac{2\sqrt{-(A_{102}+B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{-\sqrt{-D_1}}{\sqrt{C_{003}}}\right),$$
$$p_8 = \left(-\frac{2\sqrt{-(A_{102}+B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{\sqrt{-D_1}}{\sqrt{C_{003}}}\right),$$
$$p_9 = \left(\frac{2\sqrt{-(A_{102}+B_{012})c + 2aC_{003}}}{\sqrt{-D_2}}, \frac{\sqrt{-D_1}}{\sqrt{C_{003}}}\right).$$

We observe that the averaged theory can be conclusive only at points p_3, p_7 and p_9. In fact at p_2, p_6 and p_8 we have $\varrho < 0$ and it means that $R < 0$ which does not make sense. Moreover at p_1, p_4 and p_5 we have $\varrho = 0$, i.e., $R = 0$ and the cylindrical coordinates change $R = 0$ is not a diffeomorphism.

The Jacobian matrix (8) at point p_3 has determinant

$$\frac{D_0(-(A_{120} + 3(A_{300} + B_{030}) + B_{210})c + 4a(C_{021} + C_{201}))}{4b^2},$$

which is different from zero according to our hypothesis (see (3)). So, by statement (b) of Theorem 4, there exists a limit cycle for system (21) which converges to p_3 as $\epsilon \to 0$.

To prove statement (b) we just observe the following. The limit cycle Γ_ϵ can be written in the system (20) as $\{R(\theta), W(\theta)) : \theta \in S\}$. The point p_3 in the coordinates (R, W) is written as

$$(\sqrt{-D_0\epsilon}, 0).$$

Thus statement (b) of Theorem 3 follows immediately from statement (b) of Theorem 4. It completes the proof of (a) and (b).

The stability of the limit cycle is determined by the eigenvalues of system (6) at the point p_3. Taking the notation of D_0 given in (2), the eigenvalues are

$$-\frac{2a}{b} \quad \text{and} \quad \frac{2c - D_0(C_{021} + C_{201})}{2b}.$$

By statement (c) of Theorem 4 it follows immediately the different kind of stability for the limit cycle stated in (b) of Theorem 3. So statement (c) of Theorem 3 is proved.

The Jacobian matrix (8) at points p_7 and p_8 have determinant

$$-\frac{2D_1((A_{102} + B_{012})c - 2aC_{003})}{b^2 C_{003}},$$

which is different from zero according to our hypothesis (see (4)). So, by statement (b) of Theorem 4, there exists two limit cycles for system (21) one that converges to p_7 as $\epsilon \to 0$ and one that converges to p_9 as $\epsilon \to 0$.

To prove statement (e) we just observe the following. The limit cycle Γ_ϵ^1 can be written in the system (20) as $\{R(\theta), W(\theta)) : \theta \in S\}$. The point p_7 at the coordinates (R, W) is written as

$$\left(\frac{2\sqrt{\epsilon}\sqrt{-(A_{102}+B_{012})c+2aC_{003}}}{\sqrt{-D_2}}, -\frac{\sqrt{-D_1\epsilon}}{\sqrt{C_{003}}} \right).$$

Thus statement (e) of Theorem 3 follows immediately from statement (b) of Theorem 4.

To prove statement (f) we just observe the following. The limit cycle Γ_ϵ^2 can be written in the system (20) as $\{R(\theta), W(\theta)) : \theta \in S\}$. The point p_9 at the coordinates (R, W) is written as

$$\left(\frac{2\sqrt{\epsilon}\sqrt{-(A_{102}+B_{012})c+2aC_{003}}}{\sqrt{-D_2}}, \frac{\sqrt{-D_1\epsilon}}{\sqrt{C_{003}}} \right).$$

Thus statement (f) of Theorem 3 follows immediately from statement (b) of Theorem 4. In short this completes the proofs of statements (d), (e) and (f) of Theorem 3.

The stability of those limit cycles is determined by the eigenvalues of system (6) at the points p_7 and p_9, respectively. Then, by statement (c) of Theorem 4, it follows immediately the different kind of stability for the limit cycles. Since those eigenvalues can be computed easily using Mathematica or any other algebraic manipulator, but their expressions are very long, here they have been omitted.

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References

[1] A. Biuca and J. Llibre, *Averaging methods for finding periodic orbits via Brownew degree*, Bull. Sci. Math. **128** (2004), 7–22.

[2] C. A. Buzzi, J. Llibre and P. R. Da Silva, *Generalized 3–dimensional Hopf bifurcation via averaging theory*, Discrete, Continuous Dynamical Systems, Series A **17** (2007), 529–540.

[3] J. Llibre *Averaging theory and limit cycles for quadratic systems*, Radovi Matematicki **11** (2002), 215–228.

[4] J.A. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sci., Vol. 59, Springer-Verlag, New York, 1985.

[5] F. Verhulst, *Nonlinear differential equations and dynamical systems*, Universitext. Springer-Verlag, Berlin, 1996.

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